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HIERARCHIES AND COMPATIBILITY ON COURANT ALGEBROIDS

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Abstract: We extend to the context of Courant algebroids several hierarchies that can be constructed on Poisson-Nijenhuis manifolds. More precisely, we introduce several notions (Poisson-Nijenhuis, deformation-Nijenhuis and Nijenhuis pairs) that extend to Courant algebroids the notion of a Poisson-Nijenhuis manifold, by using the idea that both the Poisson and the Nijenhuis structures, although they seem to be different in nature when considered on manifolds, are just $(1,1)$-tensors on the usual Courant algebroid $TM \oplus T^*M$ satisfying several constraints. For each of the generalizations mentioned, we show that there are natural hierarchies obtained by successive deformation by one of the $(1,1)$-tensor.

Introduction

The purpose of the present article is to explain how $(1,1)$-tensors with vanishing Nijenhuis torsion on a Courant structure naturally give rise to several type of hierarchies - and to show it using as much as possible of supergeometric formalism. To start with, we say a few words on, respectively, Courant structures, supergeometric formalism, Leibniz algebroids, Nijenhuis torsion and hierarchies. Having recalled these notions, we explain the purpose of our study. We finish this introduction by a more detailed summary of the content of the present work.

0.1. On Courant structures, Nijenhuis torsion, supergeometry and hierarchies.

Courant structures. It has been noticed by Courant [5] that the following bilinear assignment on the space of sections of $TM \oplus T^*M$, for M a manifold:

$$[(X,\alpha),(Y,\beta)] := ([X,Y], L_X\beta - i_Y \mathrm{d}\alpha)$$

(with $X, Y \in \Gamma(TM) = \mathfrak{X}(M), \alpha, \beta \in \Gamma(T^*M) = \Omega^1(M)$) still satisfies the Jacobi identity, and that its default of being skew-symmetric is given, for all $u, v \in \Gamma(TM \oplus T^*M)$, by:

$$[u,v] - [v,u] \,=\, \mathrm{d}\,\langle u,v\rangle,$$

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where ρ is given by the projection on TM and $\langle\cdot,\cdot\rangle$ stands for the canonical non-degenerate bilinear form on $TM \oplus T^*M$. When made abstract, this example yields to the definition of Courant algebroids [15], which are pseudo-euclidian vector spaces, equipped with a bracket which satisfies the Jacobi identity, an anchor map encoding the default of being C^∞-linear, and (together with the pseudo-euclidian product) the default of being skew-symmetric. Relaxing the Jacobi identity yields a weaker notion of pre-Courant algebroid (see Definition 1.1 below).

A general idea about Courant structures is that it allows one to deal with *two* Lie algebroid-like brackets in the same time : one on a vector bundle, and one on its dual, like it happens for Lie bialgebroids [16, 11].

Supergeometric formalism. To say the least, to deal with Courant bracket can be an heavy task when it comes about computation, due to the many structures that make it, and to the un-natural aspects of some of its operations. Fortunately, in supergeometric formalism, all these structures and conditions are encoded in two objets and one condition. The idea goes as follows. To every vector bundle equipped with a non-degenerate bilinear form is associated a graded commutative algebra, equipped with a Poisson bracket denoted by $\{\cdot,\cdot\}$ (which coincides with the big bracket in some particular cases) [21]. It happens that pre-Courant structures are in one-to-one correspondence with functions of degree 3, and pre-Courant structures which are indeed Courant are precisely those that satisfy:

$$\{\Theta,\Theta\} = 0.$$

A general idea about supergeometric language is that it enables to encode several structures by a simple function (hence allowing to encode a Courant structure by a simple letter Θ), and that it is a tool of remarkable efficiency for some computations, see, for instance [1] and [23].

Leibniz algebroids. Courant structures on vector bundles can be viewed as special cases of Leibniz algebroids [9]. These are vector bundles $E \to M$ equipped with a $\mathbb{R}$-bilinear bracket on its space of sections and a vector bundle morphism $\rho : E \to TM$ satisfying the Leibniz rule:

$$[X, fY] = f[X,Y] + (\rho(X).f)Y$$

and the Jacobi identity:

$$[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]],$$

for all $X, Y, Z \in \Gamma(E)$ and $f \in C^\infty(M)$. Relaxing the Jacobi identity yields the weaker notion of pre-Leibniz algebroid. When the base manifold reduces to a point, a Leibniz algebroid is just a Leibniz (or Loday) algebra, while a pre-Leibniz algebroid is simply an algebra. It is easy to check (see [12]) that pre-Courant algebroids are pre-Leibniz algebroids. But it is important to stress that the supergeometric approach, referred above for pre-Courant and Courant structures, is not valid in the pre-Leibniz and Leibniz algebroid framework.

Nijenhuis torsion. The Nijenhuis torsion of a $(1,1)$-tensor, i.e a fiberwise linear endomorphism of TM, is the $(2,1)$-tensor given by:

$$X, Y \mapsto [NX, NY] - N[X,Y]_N, \text{ where } [X,Y]_N := [NX,Y] + [X,NY] - N[X,Y].$$

For a $(1,1)$-tensor, being Nijenhuis torsion-free is in general meaningful (we shall just say "Nijenhuis tensors" for torsion-free tensors). For an almost complex structure, it means for instance that it comes from a complex one [18]. For an operator squaring to the identity, it means that its eigenspaces are complementary integrable distributions. The previous definition can be extended without any change from TM to arbitrary Lie algebroids [8], then from Lie algebroids to Courant algebroids [3] and Leibniz algebroids.

A general idea about Nijenhuis $(1,1)$-tensors is that it allows to deform an object into an object of the same type, for instance, to deform a Lie algebroid bracket $[.,.]$ into the bracket $[.,.]_N$ above, which can be shown to be a Lie algebroid bracket again, or to deform a Poisson structure into another one.

Hierarchies There is no mathematical definition of what a *hierarchy* is, but, within the context of integrable systems, the name has been commonly given either to families (indexed by $\mathbb{N}$ or $\mathbb{Z}$) of Hamiltonian functions that commute for a fixed Poisson structure, or of Poisson structures/Lie algebroids which commute between themselves - and sometimes families of both Poisson structures and Hamiltonian functions such that two functions in that family commute with respect to any Poisson structure, see chapter 7 in [19]. We use that name in the same spirit: i.e., for us a hierarchy is either a family of commuting Courant structures, either a family of Nijenhuis tensors that commute w.r.t. to some Courant structure - or a family of both Courant and (pairs of) Nijenhuis tensors.

A general idea [14], [17], [6], [4] about hierarchies is that we start with a few objects, compatible between themselves, then we give ourself a Nijenhuis tensor with the help of which we deform the objects in question, yielding sequences of objects of various types, which are all compatible between themselves.

0.2. Purpose and content of the present article. Our goal is, as we already stated, to construct hierarchies as follows:

(1) hierarchies of Courant structures, given a Nijenhuis tensor on a Courant algebroid,
(2) hierarchies of Poisson structures, given a Nijenhuis tensor compatible with a given Poisson structure on a Courant algebroid. For this point, the Courant structure does not need to satisfy the Jacobi identity : it just needs to be what we called a pre-Courant structure.
(3) hierarchies of Courant structures and pairs of tensors that we call deforming-Nijenhuis pairs or Nijenhuis pairs. Again, pre-Courant structures are enough for most results presented here.

The idea behind item 1) above is simply that what holds true for manifolds and Lie algebroids should hold true for Courant structures as well, and that, in particular, deforming a Courant structure by a Nijenhuis tensor should give a hierarchy of compatible Courant structures. The idea behind items 2) and 3) is more involved. We invite the reader to have in mind the case of Poisson-Nijenhuis structures to get some intuitive picture, but we insist that our constructions apply to much more general contexts. The idea is that, in terms of Courant algebroids, Poisson-Nijenhuis structures [17, 14, 3] can be seen as follows:

- we consider the Courant algebroid Θ on $TM \oplus T^*M$ already evoked,
- we see a Poisson structure π on the manifold M as a skew-symmetric $(1,1)$-tensor $J_\pi : TM \oplus T^*M \to T^*M \oplus TM$ (see Example 1.6 a),
- we see a $(1,1)$-tensor N on the manifold M as a skew-symmetric $(1,1)$-tensor $I_N : TM \oplus T^*M \to TM \oplus T^*M$ (see Example 1.6 c),

then we check that the conditions of compatibility required on (π, N) to be Poisson-Nijenhuis mean that J_π and I_N anti-commute and anti-commute w.r.t. the Courant structure, see Example 3.23. When made abstract, these conditions yield our Definition 3.21 of Poisson-Nijenhuis pair, where we are given, as in the three items just above, a pre-Courant structure Θ, a Poisson tensor J, and a Nijenhuis tensor I, supposed to be compatible in the sense that I and J anti-commute and anti-commute w.r.t. the pre-Courant structure Θ. Having established this definition, we can address the purposes of items 2 and 3 above, by generalizing the hierarchies of [17]. Indeed, it happens that the notion of Poisson-Nijenhuis is slightly too restrictive, and that hierarchies can be constructed in the more general context of deforming-Nijenhuis pairs and Nijenhuis pairs.

The statements of most results in this article are written in the pre-Courant algebroid framework and are proved using the supergeometric formalism. However, for some of them, the proofs only use the pre-Leibniz structure induced by the pre-Courant structure, so that these results hold not only for pre-Courant algebroids, but also for the more general setting of pre-Leibniz algebroids. This happens, for example, with most results in sections 2.1 and 2.2 and the whole section 4. Indeed, most results of that section remain true for every vector space endowed with a quadratic form, provided that it admits the property that the deformed operator by a Nijenhuis torsion-free linear operator is again of the same type - which is true, without much trouble for operators that satisfy only linear or quadratic relations (like skew-symmetry and Jacobi). The lack of convincing examples prevented us from going to such a unnecessary level of generality.

Let us give a more precise content, by giving the explicit statements of the most important results of the present article.

Given a skew-symmetric $(1,1)$-tensor I on (E,Θ), by a deformation of a given superfunction K by I, we mean the superfunction $\{I,K\}$. When $K:=\Theta$ is a pre-Courant structure, then $\{I,\Theta\}$ is a pre-Courant structure again. When $K:=J$ is a skew-symmetric $(1,1)$-tensor, then $\{I,J\}$ is the skew-symmetric $(1,1)$-tensor $J\circ I - I\circ J$, which is equal to $2\,J\circ I$ when I and J anti-commute. Under this last assumption, deforming n times J by I yields to $J\circ I^n$ (up to a non-zero scalar).

Definition 0.1. Let (E,Θ) be a pre-Courant algebroid.

(1) A skew-symmetric $(1,1)$-tensor I on (E,Θ) is said to be *Nijenhuis* for Θ if its Nijenhuis torsion vanishes, *deforming for* Θ if $\Theta_{J,J}=\lambda\Theta$, with $\lambda\in\mathbb{R}$ and *Poisson* if it is deforming with $\lambda=0$.

(2) A pair (J, I) of skew-symmetric $(1,1)$-tensors is said to be a *deforming-Nijenhuis pair* for Θ if
- I and J anti-commute
- I and J anti-commute w.r.t. Θ, i.e.:

$$\{J, \{I, \Theta\}\} + \{I, \{J, \Theta\}\} = 0,$$

- J is deforming for Θ;
- I is Nijenhuis for Θ.

It is said to be a *compatible pair w.r.t.* Θ when it only satisfies the two first items above.

In section 2.1, we assume that Θ is indeed a Courant structure, and show that the Courant structure can be deformed n times by a Nijenhuis tensor I, and that the henceforth obtained objects $(\Theta_n)_{n\in\mathbb{N}}$ are compatible.

Theorem 1. *(see Theorem 2.7). If I is a Nijenhuis tensor for a Courant algebroid (E, Θ), then Θ_m and Θ_n are compatible Courant structures, for all $m, n \in \mathbb{N}$.*

Then, we show that the property of being compatible is, for a given compatible pair (I, J) also preserved when deforming n times J by I, provided that I is Nijenhuis (or at least satisfies the weaker condition indicated below), and that this result still holds true with respect to pre-Courant structures obtained when deforming Θ by I.

Theorem 2. *(see Theorems 2.18 and 2.21). Let (E, Θ) be a pre-Courant algebroid. Let I and J be two skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E, Θ) that are compatible w.r.t. Θ.*

a) *If $\mathcal{T}_\Theta I(JX, Y) = \mathcal{T}_\Theta I(X, JY) = 0$, for all sections X and Y on E, then $(I, I^n \circ J)$ is a compatible pair w.r.t. Θ_m, for all $m, n \in \mathbb{N}$.*

b) *If I is a Nijenhuis tensor, then $(I^{2s+1}, I^t \circ J)$ is a compatible pair w.r.t. Θ_n, for all $n, s, t \in \mathbb{N}$.*

c) *If I and J are Nijenhuis tensors, then $(I^{2s+1}, I^t \circ J^{2k+1})$ is a compatible pair w.r.t. Θ_n, for all $n, s, t, k \in \mathbb{N}$.*

We then turn our attention to deforming-Nijenhuis pairs, and prove several results culminating to the following one.

Theorem 3. *(see Theorem 3.8). Let I and J be two skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E, Θ). If (J, I) is a deforming-Nijenhuis* *pair for* Θ*, then* (J, I^{2n+1}) *is a deforming-Nijenhuis pair for* Θ_m*, for all* $m, n \in \mathbb{N}$.

For Poisson-Nijenhuis pairs (J, I), (i.e deforming-Nijenhuis pairs where the deforming tensor J is supposed to be Poisson for Θ), the result goes as follows.

Theorem 4. *(see Theorems 3.18 and 3.24). Let* (J, I) *be a Poisson-Nijenhuis pair on a pre-Courant algebroid* (E, Θ) *such that* $\Theta_{\{J,\{I,J\}\}} = 0$*. Then, for all* $m, n, k \in \mathbb{N}$*,*

a) $I^n \circ J$ *and* $I^m \circ J$ *are compatible Poisson tensors for* Θ_k*;*
b) $(I^n \circ J, I^{2m+1})$ *is a Poisson-Nijenhuis pair for* Θ_k*.*

Last, we conclude with the case of Nijenhuis pairs, i.e. pairs (I, J) of Nijenhuis tensors compatible w.r.t. to Θ, to wit the following general result.

Theorem 5. *(see Theorem 4.12). Let* I *and* J *be two skew-symmetric* $(1, 1)$*-tensors on a pre-Courant algebroid* (E, Θ)*. If* (I, J) *is a Nijenhuis pair for* Θ*, then for all* $m, n, t \in \mathbb{N}$*,* $(I^{2m+1} \circ J^n, J^{2t+1})$ *is a Nijenhuis pair for* Θ*, and, more generally, for all the Courant structures obtained by deforming* Θ *several times, either by* I *or by* J*.*

1. Skew-symmetric tensors on Courant algebroids

1.1. Courant algebroids in supergeometric terms. We begin this section by introducing the supergeometric setting, following the same approach as in [23, 21, 20]. Given a vector bundle $A \to M$, we denote by $A[n]$ the graded manifold obtained by shifting the fibre degree by n. The graded manifold $T^*[2]A[1]$ is equipped with a canonical symplectic structure which induces a Poisson bracket on its algebra of functions $\mathcal{F} := C^\infty(T^*[2]A[1])$. This Poisson bracket is sometimes called the *big bracket* (see [12]).

Let us describe locally this Poisson algebra. Fix local coordinates $x_i, p^i, \xi_a, \theta^a$, $i \in \{1, \ldots, n\}, a \in \{1, \ldots, d\}$, in $T^*[2]A[1]$, where x_i, ξ_a are local coordinates on $A[1]$ and p^i, θ^a are their associated moment coordinates. In these local coordinates, the Poisson bracket is given by

$$\{p^i, x_i\} = \{\theta^a, \xi_a\} = 1, \quad i = 1, \ldots, n, \; a = 1, \ldots, d,$$

while all the remaining brackets vanish.

The Poisson algebra of functions $\mathcal{F}$ is endowed with a $(\mathbb{N} \times \mathbb{N})$-valued bidegree. We define this bidegree locally but it is well defined globally (see [23, 21] for more details). The bidegrees are locally set as follows: the coordinates on the base manifold M, x_i, $i \in \{1, \dots, n\}$, have bidegree $(0,0)$, while the coordinates on the fibres, ξ_a, $a \in \{1, \dots, d\}$, have bidegree $(0,1)$ and their associated moment coordinates, p^i and θ^a, have bidegrees $(1,1)$ and $(1,0)$, respectively. The algebra of functions $\mathcal{F}$ inherits this bidegree and we set

$$\mathcal{F} = \bigoplus_{k,l \in \mathbb{N} \times \mathbb{N}} \mathcal{F}^{k,l}$$

We can verify that the big bracket has bidegree $(-1,-1)$, i.e.,

$$\{\mathcal{F}^{k_1,l_1}, \mathcal{F}^{k_2,l_2}\} \subset \mathcal{F}^{k_1+k_2-1,l_1+l_2-1}.$$

This construction is a particular case of a more general one in which we consider a vector bundle E equipped with a fibrewise non-degenerate symmetric bilinear form $\langle .,. \rangle$. In this more general setting, we consider the graded symplectic manifold $\mathcal{E} := p^*(T^*[2]E[1])$, which is the pull-back of $T^*[2]E[1]$ by the application $p : E[1] \to E[1] \oplus E^*[1]$ defined by $X \mapsto (X, \frac{1}{2}\langle X, . \rangle)$. We denote by $\mathcal{F}_E$ the graded algebra of functions on $\mathcal{E}$, i.e., $\mathcal{F}_E := C^\infty(\mathcal{E})$. The algebra of functions $\mathcal{F}_E$ is equipped with the canonical Poisson bracket, denoted by $\{.,.\}$, which has degree -2. Notice that $\mathcal{F}_E^0 = C^\infty(M)$ and $\mathcal{F}_E^1 = \Gamma(E)$. Under these identifications, the Poisson bracket is given, in degrees 0 and 1, by

$$\begin{aligned}
\{f, g\} &= 0; \\
\{f, X\} &= 0; \\
\{X, Y\} &= \langle X, Y \rangle,
\end{aligned}$$

for all $X, Y \in \Gamma(E)$ and $f, g \in C^\infty(M)$.

The construction above corresponds to the case where $E := A \oplus A^*$ and $\langle .,. \rangle$ is the usual symmetric bilinear form. Notice that, with the notation introduced so far, the algebra of functions $\mathcal{F} = C^\infty(T^*[2]A[1])$ should be denoted by $\mathcal{F}_{A \oplus A^*}$.

Let us define the notion of (pre-)Courant structure on a vector bundle E equipped with a fibrewise non-degenerate symmetric bilinear form $\langle .,. \rangle$.

Definition 1.1. A *pre-Courant* structure on $(E, \langle .,. \rangle)$ is a pair $(\rho, [.,.])$, where the *anchor* ρ is a bundle map from E to TM and the *Dorfman bracket* $[.,.]$ is a $\mathbb{R}$-bilinear (non necessarily skew-symmetric) assignment on $\Gamma(E)$

satisfying the relations

$$\rho(X) \cdot \langle Y, Z \rangle = \langle [X, Y], Z \rangle + \langle Y, [X, Z] \rangle \tag{1}$$

and

$$\rho(X) \cdot \langle Y, Z \rangle = \langle X, [Y, Z] + [Z, Y] \rangle, \tag{2}$$

for all $X, Y, Z \in \Gamma(E)$. *

Moreover, if the Jacobi identity,

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]],$$

is satisfied for all $X, Y, Z \in \Gamma(E)$, then the *Dorfman bracket* $[.,.]$ is a Leibniz bracket and the pair $(\rho, [.,.])$ is called a *Courant* structure on $(E, \langle ., . \rangle)$.

There is a one-to-one correspondence between pre-Courant structures on $(E, \langle ., . \rangle)$ and functions of $\mathcal{F}_E^3$. The anchor and Dorfman bracket associated to a given $\Theta \in \mathcal{F}_E^3$ are defined, for all $X, Y \in \Gamma(E)$ and $f \in C^\infty(M)$, by

$$\rho(X) \cdot f = \{\{X, \Theta\}, f\} \quad \text{and} \quad [X, Y] = \{\{X, \Theta\}, Y\}.$$

The following theorem addresses how the Jacobi identity is expressed in this supergeometric setting.

Theorem 1.2. *There is a one-to-one correspondence between Courant structures on $(E, \langle ., . \rangle)$ and functions $\Theta \in \mathcal{F}_E^3$ such that $\{\Theta, \Theta\} = 0$.*

If Θ is a (pre-)Courant structure on $(E, \langle ., . \rangle)$, then the triple $(E, \langle ., . \rangle, \Theta)$ is called a *(pre-)Courant algebroid*. For the sake of simplicity, we will often denote a (pre-)Courant algebroid by the pair (E, Θ) instead of the triple $(E, \langle ., . \rangle, \Theta)$.

When $E = A \oplus A^*$ and $\langle ., . \rangle$ is the usual symmetric bilinear form, a pre-Courant structure $\Theta \in \mathcal{F}_E^3$ can be decomposed according to its bidegrees:

$$\Theta = \mu + \gamma + \phi + \psi,$$

with $\mu \in \mathcal{F}_{A\oplus A^*}^{1,2}, \gamma \in \mathcal{F}_{A\oplus A^*}^{2,1}, \phi \in \mathcal{F}_{A\oplus A^*}^{0,3} = \Gamma(\wedge^3 A^*)$ and $\psi \in \mathcal{F}_{A\oplus A^*}^{3,0} = \Gamma(\wedge^3 A)$.

We recall from [23] that, when $\gamma = \phi = \psi = 0$, Θ is a Courant structure on $(A \oplus A^*, \langle ., . \rangle)$ if and only if (A, μ) is a Lie algebroid.

* From (1) and (2), we get [12]

$$[X, fY] = f[X, Y] + (\rho(X).f)Y,$$

for all $X, Y \in \Gamma(E)$ and $f \in C^\infty(M)$. Thus, as we already mentioned in the Introduction, a pre-Courant algebroid is always a pre-Leibniz algebroid.

Also, when $\phi = \psi = 0$, Θ is a Courant structure on $(A \oplus A^*, \langle ., . \rangle)$ if and only if $((A, \mu), (A^*, \gamma))$ is a Lie bialgebroid.

1.2. Deformation of Courant structures. Let $(E, \langle ., . \rangle, \Theta)$ be a Courant algebroid and J a vector bundle endomorphism of E, $J : E \to E$. If,

$$\langle Ju, v \rangle + \langle u, Jv \rangle = 0,$$

for all $u, v \in E$, J is said to be *skew-symmetric*. If we consider the endomorphism J^* defined by $\langle u, J^* v \rangle = \langle Ju, v \rangle$, then J is skew-symmetric if and only if $J + J^* = 0$. Vector bundle endomorphisms of E will be seen as $(1,1)$-tensors on E.

Let A be a vector bundle. When $E = A \oplus A^*$ and $\langle ., . \rangle$ is the usual symmetric bilinear form, a skew-symmetric $(1,1)$-tensor $J : E \to E$ is of the type

$$J = \begin{pmatrix} N & \pi^\sharp \\ \omega^\flat & -N^* \end{pmatrix}, \tag{3}$$

with $N : A \to A, \pi \in \Gamma(\bigwedge^2 A)$ and $\omega \in \Gamma(\bigwedge^2 A^*)$.

The deformation of the Dorfman bracket $[., .]$ by a $(1,1)$-tensor $J : E \to E$ is defined, for all $X, Y \in \Gamma(E)$, by

$$[X, Y]_J = [JX, Y] + [X, JY] - J[X, Y].$$

If $J + J^* = \lambda id_E$, for some $\lambda \in \mathbb{R}$, then $[., .]_J$ satisfies (1) and (2) [3], so that $(\rho \circ J, [., .]_J)$ is a pre-Courant structure on $(E, \langle ., . \rangle)$.

When the $(1,1)$-tensor $J : E \to E$ is skew-symmetric, the deformed pre-Courant structure $(\rho \circ J, [., .]_J)$ is given, in supergeometric terms, by $\Theta_J := \{J, \Theta\} \in \mathcal{F}^3_E$. In the case where $E = A \oplus A^*$ and J is skew-symmetric so that it is of type (3), and we consider the supergeometric framework, J corresponds to the function $N + \pi + \omega$, which we also denote by J. Therefore, we have $\Theta_J = \{N + \pi + \omega, \Theta\}$. The deformation of Θ_J by the skew-symmetric $(1,1)$-tensor I is denoted by $\Theta_{J,I}$, i.e. $\Theta_{J,I} = \{I, \{J, \Theta\}\}$, while the deformed Dorfman bracket $([., .]_J)_I$ is denoted by $[., .]_{J,I}$.

Recall that a vector bundle endomorphism $I : E \to E$ is a *Nijenhuis* tensor on the Courant algebroid (E, Θ) if its torsion vanishes. The torsion $\mathcal{T}_\Theta I$ is given, for all $X, Y \in \Gamma(E)$, by

$$\mathcal{T}_\Theta I(X,Y) = [IX,IY] - I([X,Y]_I)$$

or, equivalently, by

$$\mathcal{T}_\Theta I(X,Y) = \frac{1}{2}([X,Y]_{I,I} - [X,Y]_{I^2}), \tag{4}$$

where $I^2 = I \circ I$. When $I^2 = \lambda\, id_E$, for some $\lambda \in \mathbb{R}$, (4) is given [7], in supergeometric terms, by

$$\mathcal{T}_\Theta I = \frac{1}{2}(\Theta_{I,I} - \lambda\Theta). \tag{5}$$

When (E,Θ) is a pre-Courant algebroid, the definition of Nijenhuis tensor is the same as in the case of a Courant algebroid.

Example 1.3. For every Lie algebra $\mathcal{G}$, any linear operator I valued in the center and such that the kernel of I^2 contains the commutator $[\mathcal{G},\mathcal{G}]$ is a Nijenhuis operator.

The notion of *deforming* tensor for a Courant structure Θ on E was introduced in [13]. The definition holds in the case of a pre-Courant algebroid and it will play an important role in this article.

Definition 1.4. The skew-symmetric $(1,1)$-tensor J on (E,Θ) is said to be *deforming for* Θ if $\Theta_{J,J} = \lambda\Theta$, for some $\lambda \in \mathbb{R}$.

Remark 1.5. If I is Nijenhuis for Θ and satisfies $I^2 = \lambda id_E$, for some $\lambda \in \mathbb{R}$, then, from (5), we have $\mathcal{T}_\Theta I = 0 \Rightarrow \Theta_{I,I} = \lambda\Theta$, i.e. I is deforming for Θ. This was also noticed in [13].

Now, we present several examples of skew-symmetric tensors which are deforming or/and Nijenhuis, in the case where $(E = A \oplus A^*, \Theta)$ is a Courant algebroid with $\Theta = \mu$ and μ a Lie algebroid on A (see the end of section 1.1). As we have already remarked, a skew-symmetric $(1,1)$-tensors on E is of type (3).

Example 1.6.

a) Let π be a bivector on A. Then, $J_\pi = \begin{pmatrix} 0 & \pi \\ 0 & 0 \end{pmatrix}$ is deforming for $\Theta = \mu$ if and only if π is a Poisson bivector on (A,μ).

We have $\mu_{J_\pi,J_\pi} = \{\pi, \{\pi, \mu\}\}$. Since μ and μ_{J_π,J_π} do not have the same bidegree, we get

$$\mu_{J_\pi,J_\pi} = \lambda\,\mu \Leftrightarrow \lambda = 0 \quad \text{and} \quad \{\pi, \{\pi, \mu\}\} = 0,$$

which in turn is equivalent to

$$[\pi, \pi]_\mu = 0,$$

where $[.,.]_\mu$ is the Gerstenhaber bracket on $\Gamma(\wedge^\bullet A)$ associated to the Lie algebroid (A, μ).

Now, we remark that $J_\pi \circ J_\pi = 0$ so that, using (5) with $\lambda = 0$, we deduce that the torsion of J_π is given by $\mathcal{T}_\mu J_\pi = \frac{1}{2}\{\pi, \{\pi, \mu\}\}$. Therefore,

$$\mathcal{T}_\mu J_\pi = 0 \Leftrightarrow [\pi, \pi]_\mu = 0$$

which means that J_π is Nijenhuis for $\Theta = \mu$ if and only if π is a Poisson bivector on (A, μ).

b) Let ω be a 2-form on A. Then, $J_\omega = \begin{pmatrix} 0 & 0 \\ \omega & 0 \end{pmatrix}$ is a deforming and a Nijenhuis tensor for the Courant algebroid $(A \oplus A^*, \mu)$. This is an immediate consequence of $J_\omega \circ J_\omega = 0$ and $\{\omega, \{\omega, \mu\}\} = 0$.

c) Let $N : A \to A$ be a $(1,1)$-tensor on A, such that $N^2 = \lambda\, id_A$, for some $\lambda \in \mathbb{R}$. Then, $I_N = \begin{pmatrix} N & 0 \\ 0 & -N^* \end{pmatrix}$ is a Nijenhuis tensor for the Courant algebroid $(A \oplus A^*, \mu)$ if and only if N is Nijenhuis tensor for the Lie algebroid (A, μ) [13]. Just noticed that $\mathcal{T}_\mu N = \mathcal{T}_\mu I_N$.

d) Let π be a bivector on A and $N : A \to A$ a $(1,1)$-tensor on A. Then, $J = \begin{pmatrix} N & \pi \\ 0 & -N^* \end{pmatrix}$ is deforming for μ if and only if $\begin{cases} N \text{ is deforming for } \mu \\ \mu_{N,\pi} + \mu_{\pi,N} = 0 \\ \pi \text{ is Poisson for } \mu. \end{cases}$

We have,

$$\begin{aligned} \mu_{J,J} &= \{N + \pi, \{N + \pi, \mu\}\} \\ &= \{N, \{N, \mu\}\} + \{\pi, \{N, \mu\}\} + \{N, \{\pi, \mu\}\} + \{\pi, \{\pi, \mu\}\} \\ &= \mu_{N,N} + \mu_{N,\pi} + \mu_{\pi,N} + \mu_{\pi,\pi} \end{aligned}$$

and, by counting the bidegrees, we deduce that $\mu_{J,J} = \lambda\,\mu$ if and only if

$$\mu_{N,N} = \lambda\,\mu, \quad \mu_{N,\pi} + \mu_{\pi,N} = 0, \quad [\pi,\pi]_\mu = 0.$$

$\diamondsuit$

Let us consider the Courant algebroid $(A \oplus A^*, \mu+\gamma)$, which is the double of a Lie bialgebroid $((A,\mu),(A^*,\gamma))$ and the skew-symmetric $(1,1)$-tensor $J : A\oplus A^* \to A \oplus A^*$, given by

$$J = \begin{pmatrix} \frac{1}{2}\,id_A & \pi \\ 0 & -\frac{1}{2}\,id_{A^*} \end{pmatrix}. \tag{6}$$

Proposition 1.7. *Let $((A,\mu),(A^*,\gamma))$ be a Lie bialgebroid. Then, the $(1,1)$-tensor J given by (6) is a deforming tensor for the Courant structure $\mu+\gamma$ if and only if π is a solution of the Maurer-Cartan equation*

$$\mathrm{d}_\gamma \pi = \frac{1}{2}[\pi,\pi]_\mu.$$

Proof: The $(1,1)$-tensor $J = \frac{1}{2}\,id_A + \pi$ is a deforming tensor for $\mu+\gamma$ if there exists $\lambda \in \mathbb{R}$ such that

$$\{\frac{1}{2}\,id_A + \pi, \{\frac{1}{2}\,id_A + \pi, \mu+\gamma\}\} = \lambda(\mu+\gamma).$$

We have,

$$\begin{aligned}
\{\frac{1}{2}\,id_A + \pi, \{\frac{1}{2}\,id_A + \pi, \mu+\gamma\}\} &= \frac{1}{4}\{\,id_A, \{\,id_A,\mu\} + \{\,id_A,\gamma\}\} \\
&\quad + \frac{1}{2}\{\,id_A, \{\pi,\mu\} + \{\pi,\gamma\}\} + \frac{1}{2}\{\pi, \{\,id_A, \mu+\gamma\} + \{\pi,\{\pi,\mu\} + \{\pi,\gamma\}\} \\
&= \frac{1}{4}(\mu+\gamma) - 2\{\pi,\gamma\} - \{\{\pi,\mu\},\pi\} \\
&= \frac{1}{4}(\mu+\gamma) + 2\mathrm{d}_\gamma\pi - [\pi,\pi]_\mu,
\end{aligned}$$

where we used $\{id_A, u\} = (q-p)u$, for all u of bidegree (p,q). So,

$$\frac{1}{4}(\mu+\gamma) + 2\mathrm{d}_\gamma\pi - [\pi,\pi]_\mu = \lambda(\mu+\gamma)$$

if and only if

$$\lambda = \frac{1}{4} \quad \text{and} \quad \mathrm{d}_\gamma\pi = \frac{1}{2}[\pi,\pi]_\mu.$$

$\blacksquare$

1.3. Anti-commuting skew-symmetric tensors. Let E be a vector bundle. In general, the composition of two skew-symmetric endomorphisms of E is not a skew-symmetric endomorphism.

Lemma 1.8. *Let I and J be two skew-symmetric $(1,1)$-tensors on E that anti-commute, i.e, $I \circ J = -J \circ I$. Then,*

$$I \circ J = \frac{1}{2}\{J, I\} \tag{7}$$

and, for any $n \in \mathbb{N}$,

- $I^n \circ J = (-1)^n J \circ I^n$;
- $I^n \circ J$ *is skew-symmetric;*
- $I^n \circ J$ *anti-commutes with* I,

where $I^n = \underbrace{I \circ \ldots \circ I}_{n}$, *for* $n \geq 1$, *and* $I^0 = Id_E$.

Proof: It is a straightforward computation. ■

The notion of concomitant of two $(1,1)$-tensors on a manifold was introduced in [17] and then extended to Lie algebroids in [14]. For pre-Courant algebroids it can be defined as follows:

Definition 1.9. The concomitant of two skew-symmetric $(1,1)$-tensors I and J on a pre-Courant algebroid (E, Θ) is given by

$$C_\Theta(I, J) = \{J, \{I, \Theta\}\} + \{I, \{J, \Theta\}\} = \Theta_{I,J} + \Theta_{J,I}. \tag{8}$$

If $(\rho, [.,.])$ is the pre-Courant structure on E corresponding to Θ, (8) reads as follows:

$$\{\{X, C_\Theta(I, J)\}, Y\} = [X, Y]_{I,J} + [X, Y]_{J,I} \tag{9}$$

and

$$\{\{X, C_\Theta(I, J)\}, f\} = (\rho \circ (I \circ J + J \circ I))(X).f, \tag{10}$$

for all $X, Y \in \Gamma(E)$ and $f \in C^\infty(M)$.

In the sequel, we denote the left hand side of (9) by $C_\Theta(I, J)(X, Y)$, i.e., we write $C_\Theta(I, J)(X, Y) = [X, Y]_{I,J} + [X, Y]_{J,I}$. When I and J anti-commute, we have

$$\{\{X, C_\Theta(I, J)\}, f\} = 0,$$

for all $X \in \Gamma(E)$ and $f \in C^\infty(M)$. Therefore, in this case,

$$C_\Theta(I, J) = 0 \Leftrightarrow C_\Theta(I, J)(X, Y) = 0, \ \forall X, Y \in \Gamma(E). \tag{11}$$

Using the Jacobi identity, we easily check that (8) is equivalent to

$$C_\Theta(I,J) = \Theta_{\{J,I\}} + 2\Theta_{J,I}. \tag{12}$$

Lemma 1.10. *Let I and J be two skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E,Θ) that anti-commute. Then,*

$$C_\Theta(I,J) = 2(\Theta_{I,J} - \Theta_{I\circ J}).$$

Proof: Since I and J anti-commute, we know from (7) in Lemma 1.8, that $\{I,J\} = -2\, I\circ J$. Using the Jacobi identity, we have

$$\Theta_{J,I} = -2\Theta_{I\circ J} + \Theta_{I,J}.$$

Therefore,

$$C_\Theta(I,J) = \Theta_{I,J} + \Theta_{J,I} = 2(\Theta_{I,J} - \Theta_{I\circ J}).$$

■

2. Hierarchies of compatible tensors and structures

Let (E,Θ) be a pre-Courant algebroid. We introduce the following notation, where $I, J, \ldots, K$ are skew-symmetric $(1,1)$-tensors on (E,Θ):

- $\Theta_{I,J,\ldots,K} = (((\Theta_I)_J)_{\ldots})_K$;
- $\Theta_n = \underbrace{(((\Theta_I)_I)_{\ldots})_I}_{n} = \Theta_{\underbrace{I,\ldots,I}_{n}}$, $n\in\mathbb{N}$; $\Theta_0 = \Theta$.

2.1. Hierarchy of compatible Courant structures. We construct a hierarchy of compatible Courant structures on $(E, \langle .,.\rangle)$.

The next proposition generalizes a result in [14].

Proposition 2.1. *Let I be a skew-symmetric $(1,1)$-tensor on a pre-Courant algebroid (E,Θ) and let X, Y be any sections of E. Then,*

$$\mathcal{T}_{\Theta_n} I(X,Y) = \mathcal{T}_{\Theta_{n-1}} I(IX,Y) + \mathcal{T}_{\Theta_{n-1}} I(X,IY) - I(\mathcal{T}_{\Theta_{n-1}} I(X,Y)),\ n\in\mathbb{N}. \tag{13}$$

Proof: Let us denote by $[-,-]_n$ the Dorfman bracket associated to Θ_n. It is obvious that

$$[X,Y]_n = [IX,Y]_{n-1} + [X,IY]_{n-1} - I[X,Y]_{n-1},$$

and therefore we have,

$$\begin{aligned}
\mathcal{T}_{\Theta_n} I(X,Y) &= [IX,IY]_n - I[IX,Y]_n - I[X,IY]_n + I^2[X,Y]_n \\
&= [I^2X,IY]_{n-1} - I[I^2X,Y]_{n-1} - I[IX,IY]_{n-1} + I^2[IX,Y]_{n-1} \\
&\quad + [IX,I^2Y]_{n-1} - I[IX,IY]_{n-1} - I[X,I^2Y]_{n-1} + I^2[X,IY]_{n-1} \\
&\quad - I([IX,IY]_{n-1} - I[IX,Y]_{n-1} - I[X,IY]_{n-1} + I^2[X,Y]_{n-1}) \\
&= \mathcal{T}_{\Theta_{n-1}} I(IX,Y) + \mathcal{T}_{\Theta_{n-1}} I(X,IY) - I(\mathcal{T}_{\Theta_{n-1}} I(X,Y)).
\end{aligned}$$

■

Corollary 2.2. *If I is Nijenhuis for Θ, then I is Nijenhuis for Θ_n, $\forall n \in \mathbb{N}$.*

When (E,Θ) is a Courant algebroid, it is well known [2] that if I is a skew-symmetric Nijenhuis tensor for Θ, then (E,Θ_I) is a Courant algebroid.

Proposition 2.3. *Let (E,Θ) be a Courant algebroid and I a skew-symmetric Nijenhuis tensor for Θ. Then, (E,Θ_n) is a Courant algebroid, for all $n \in \mathbb{N}$.*

Proof: Let (E,Θ) be a Courant algebroid and I a skew-symmetric Nijenhuis tensor for Θ. As we already noticed, (E,Θ_I) is a Courant algebroid [2]. From (13), we know that if I is Nijenhuis for Θ, then I is Nijenhuis for Θ_I. Therefore, $(E,\Theta_{I,I})$ is a Courant algebroid.

By recursion, we get that (E,Θ_n) is a Courant algebroid for all $n \in \mathbb{N}$. ■

Let us compute the torsion $\mathcal{T}_\Theta I^n$, for all $n \in \mathbb{N}$.

Proposition 2.4. *Let I be a $(1,1)$-tensor on a pre-Courant algebroid (E,Θ). Then, for all sections X and Y on E,*

$$\begin{aligned}
\mathcal{T}_\Theta I^n(X,Y) =& \mathcal{T}_\Theta I(I^{n-1}X, I^{n-1}Y) + I(\mathcal{T}_\Theta I^{n-1}(IX,Y) + \mathcal{T}_\Theta I^{n-1}(X,IY)) \\
&- I^2(\mathcal{T}_\Theta I^{n-2}(IX,IY)) + I^{2n-2}(\mathcal{T}_\Theta I(X,Y)),\ n \geq 2.
\end{aligned}$$

Proof: It is enough to use the definition of Nijenhuis torsion. In fact, we have, for all sections X and Y of E,

$$\begin{aligned}
\mathcal{T}_\Theta I(I^{n-1}X, I^{n-1}Y) =& [I^nX, I^nY] - I([I^nX, I^{n-1}Y] + [I^{n-1}X, I^nY]) \\
&+ I^2[I^{n-1}X, I^{n-1}Y]; \qquad (14)
\end{aligned}$$

$$\begin{aligned} I(\mathcal{T}_\Theta I^{n-1}(IX,Y)+\mathcal{T}_\Theta I^{n-1}(X,IY)) = & I([I^nX,I^{n-1}Y]+[I^{n-1}X,I^nY]) \\ & - I^n([I^nX,Y]+[IX,I^{n-1}Y]+[I^{n-1}X,IY]+[X,I^nY]) \\ & + I^{2n-1}([IX,Y]+[X,IY]); \end{aligned} \tag{15}$$

$$\begin{aligned} -I^2(\mathcal{T}_\Theta I^{n-2}(IX,IY)) = & - I^2[I^{n-1}X,I^{n-1}Y]+I^n([I^{n-1}X,IY]+[IX,I^{n-1}Y]) \\ & - I^{2n-2}[IX,IY]; \end{aligned} \tag{16}$$

and

$$\begin{aligned} I^{2n-2}(\mathcal{T}_\Theta I(X,Y)) = & I^{2n-2}[IX,IY] - I^{2n-1}([IX,Y]+[X,IY]) \\ & + I^{2n}[X,Y]. \end{aligned} \tag{17}$$

The sum of the right hand sides of equations (14), (15), (16) and (17) gives

$$[I^nX,I^nY]-I^n([I^nX,Y]+[X,I^nY])+I^{2n}[X,Y]=\mathcal{T}_\Theta I^n(X,Y).$$

■

As an immediate consequence of the previous proposition, we have the following:

Corollary 2.5. *If I is a Nijenhuis tensor for Θ, then I^n is a Nijenhuis tensor for Θ, for all $n \in \mathbb{N}$.*

Proposition 2.6. *Let I be a skew-symmetric $(1,1)$-tensor on a pre-Courant algebroid (E,Θ). If I is Nijenhuis for Θ, then I^n is Nijenhuis for Θ_m, for all $m,n \in \mathbb{N}$.*

Proof: Let I be a Nijenhuis tensor for Θ. Then, according to Corollary 2.2, I is Nijenhuis for Θ_m, for all $m \in \mathbb{N}$. Applying Corollary 2.5, the result follows. ■

Recall that two Courant structures Θ_1 and Θ_2 on the vector bundle $(E,\langle\,,\,\rangle)$ are said to be *compatible* if their sum $\Theta_1+\Theta_2$ is a Courant structure on $(E,\langle\,,\,\rangle)$. As an immediate consequence, we have that Θ_1 and Θ_2 are compatible if and only if

$$\{\Theta_1,\Theta_2\}=0.$$

Two arbitrary pre-Courant structures Θ_1 and Θ_2 on $(E,\langle\,,\,\rangle)$ are compatible, in the sense that the sum $\Theta_1+\Theta_2$ is always a pre-Courant structure.

Theorem 2.7. *Let I be a skew-symmetric $(1,1)$-tensor on a Courant algebroid (E,Θ). If I is Nijenhuis for Θ, then the Courant structures Θ_m and Θ_n on $(E,\langle\,,\,\rangle)$ are compatible, for all $m,n\in\mathbb{N}$.*

In particular, Θ is compatible with Θ_n, for all $n\in\mathbb{N}$.

Proof: We start by remarking that if $m=n$, then we have $\{\Theta_m,\Theta_m\}=0$ by Proposition 2.3. Also, for any Courant structure Θ and any skew-symmetric $(1,1)$-tensor I, the relation $\{\Theta,\Theta_I\}=0$ follows from the Jacobi identity and the graded symmetry of the Poisson bracket. We use induction on $m+n$ to finish the proof.

Case $m+n=2$:

- i) $m=n=1$,
$$\{\Theta_I,\Theta_I\}=0;$$
- ii) $m=2$, $n=0$,
$$\{\Theta_{I,I},\Theta\}=\{I,\{\Theta,\Theta_I\}\}-\{\Theta_I,\Theta_I\}=0.$$

Now, suppose that $\{\Theta_m,\Theta_n\}=0$ holds with $m+n=k-1$ and take m and n such that $m+n=k$.

- i) if $m=n$, we already noticed that $\{\Theta_m,\Theta_m\}=0$;
- ii) if $m\neq n$, suppose that $m>n$. Then,

$$\begin{aligned}
\{\Theta_m,\Theta_n\} &= \{\{I,\Theta_{m-1}\},\Theta_n\}=\{I,\{\Theta_n,\Theta_{m-1}\}\}-\{\Theta_{n+1},\Theta_{m-1}\}\\
&= -\{\Theta_{m-1},\Theta_{n+1}\}\\
&= -\{I,\{\Theta_{m-2},\Theta_{n+1}\}\}+\{\Theta_{m-2},\Theta_{n+2}\}\\
&= \{\Theta_{m-2},\Theta_{n+2}\}\\
&= \dots\\
&= \begin{cases} (-1)^{m-l}\{\Theta_l,\Theta_l\}, & \text{if } m+n=2l\\ (-1)^{m-(l+1)}\{\Theta_{l+1},\Theta_l\}, & \text{if } m+n=2l+1,\end{cases}\\
&= \begin{cases} 0, & \text{if } m+n=2l\\ (-1)^{m-(l+1)}\frac{1}{2}\{I,\{\Theta_l,\Theta_l\}\}=0, & \text{if } m+n=2l+1.\end{cases}
\end{aligned}$$

■

Remark 2.8. If I is a deforming tensor for Θ, i.e., $\Theta_{I,I}=\lambda\Theta$, for some $\lambda\in\mathbb{R}$, then, a straightforward computation provides
$$\Theta_{2k}=\lambda^k\Theta,\quad \Theta_{2k+1}=\lambda^k\Theta_I,\quad \text{for all } k\in\mathbb{N}.$$

In this case, Theorem 2.7 is trivially satisfied.

We investigated so far the Courant structure Θ_n obtained by deforming n times Θ by a skew-symmetric tensor I. It is logical to ask what happens when one deforms Θ by I^n, and the answer is that we get precisely the same pre-Courant structure Θ_n. But this result can not be written as $\Theta_n = \Theta_{I^n}$ for even n, since I^n is then not a skew-symmetric $(1,1)$-tensor anymore. We bypass this difficulty by considering directly the Dorfman brackets, rather than the functions of degree 3 associated with.

Proposition 2.9. *Let I be a skew-symmetric $(1,1)$-tensor on a pre-Courant algebroid (E,Θ). Then, for all sections X and Y of E,*

a) $[X,Y]_{I^{2n+1}} = [X,Y]_{I^{2n},I} - \sum\limits_{\substack{0\leq i,j\leq 2n-1\\ i+j=2n-1}} I^j(\mathcal{T}_\Theta I(I^iX,Y) + \mathcal{T}_\Theta I(X,I^iY))$;

b) *If I is Nijenhuis for Θ then, for any $n\in\mathbb{N}$,*
$[X,Y]_{I^n} = [X,Y]_{\underbrace{I,\dots,I}_{n}}$;

c) *If I is Nijenhuis for Θ then, for any $m,n\in\mathbb{N}$,*
$[X,Y]_{I^m,I^n} = [X,Y]_{I^{m+n}}$.

Proof: a) It is an easy but cumbersome computation that uses the definition of $\mathcal{T}_\Theta I$.

b) First, we observe that, if two skew-symmetric $(1,1)$-tensors I and J commute, then $[X,Y]_{I,J} = [X,Y]_{J,I}$, for all sections X and Y of E. In particular we have, for all $m,n\in\mathbb{N}$,

$$[X,Y]_{I^m,I^n} = [X,Y]_{I^n,I^m}. \tag{18}$$

i) If n is odd, $n=2k+1$, we use a):

$$[X,Y]_{I^n} = [X,Y]_{I^{2k+1}} = [X,Y]_{I^{2k},I},$$

and it is case ii).

ii) If n is even, $n=2l$, since I^l is Nijenhuis for Θ, using (4) we may write

$$[X,Y]_{I^n} = [X,Y]_{I^l\circ I^l} = [X,Y]_{I^l,I^l}.$$

If l is even, we repeat the procedure. If l is odd, we are back to case i).

Repeating the procedure, and taking into account (18), we end up with

$$[X,Y]_{I^n} = [X,Y]_{\underbrace{I,\dots,I}_{n}},\ \forall n \in \mathbb{N}.$$

c) We use b) and (18):

$$[X,Y]_{I^n,I^m} = [X,Y]_{\underbrace{I,\dots,I}_{n},I^m} = [X,Y]_{I^m,\underbrace{I,\dots,I}_{n}} = [X,Y]_{\underbrace{I,\dots,I}_{m+n}} = [X,Y]_{I^{m+n}}.$$

■

Given a Courant structure $(\rho, [.,.])$ on $(E, \langle\, ,\,\rangle)$, we denote by $(\rho, [.,.])_I$ the pre-Courant structure on $(E, \langle\, ,\,\rangle)$ defined by

$$(\rho, [.,.])_I = (\rho \circ I, [.,.]_I),$$

where I is a skew-symmetric $(1,1)$-tensor on E. If I is Nijenhuis for $(\rho, [.,.])$, then

$$(\rho, [.,.])_{I^{k_1},\cdots,I^{k_n}} = (\rho, [.,.])_{\underbrace{I,\cdots,I}_{k_1+\cdots+k_n}} = (\rho, [.,.])_{I^{k_1+\cdots+k_n}} \tag{19}$$

is a Courant structure on $(E, \langle\, ,\,\rangle)$, for all $k_1, \cdots, k_n \in \mathbb{N}$. This result follows directly from Proposition 2.9 b) and c). In supergeometric terms, (19) means that the deformation of Θ, either by $I^{k_1+\cdots+k_n}$ or successively by $I^{k_1}, I^{k_2}, \cdots, I^{k_n}$, is the pre-Courant structure $\Theta_{\underbrace{I,\cdots,I}_{k_1+\cdots+k_n}} = \Theta_{k_1+\cdots+k_n}$.

2.2. Hierarchy of compatible tensors w.r.t. Θ. In this section, we introduce the notion of compatible pair of $(1,1)$-tensors with respect to a pre-Courant structure Θ on E and construct a hierarchy of pairs of tensors satisfying this type of compatibility.

Definition 2.10. We say that two skew-symmetric $(1,1)$-tensors I and J on a pre-Courant algebroid (E, Θ) *anti-commute with respect to* Θ, if $\Theta_{I,J} = -\Theta_{J,I}$ or, equivalently, if $C_\Theta(I,J) = 0$.

In terms of the pre-Courant structure $(\rho, [.,.])$ on $(E, \langle .,. \rangle)$, the condition $C_\Theta(I,J) = 0$ is equivalent to

$$\begin{cases} \rho \circ I \circ J + \rho \circ J \circ I = 0 \\ [X,Y]_{I,J} + [X,Y]_{J,I} = 0, \end{cases}$$

for all sections X and Y of E.

Proposition 2.11. *If I and J are skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E,Θ) that anti-commute with respect to Θ, then $\Theta_{\{I,J\}} = -2\Theta_{I,J} = 2\Theta_{J,I}$.*

Proof: We have, by the Jacobi identity,

$$\Theta_{\{I,J\}} = \{\{I,J\},\Theta\} = \{I,\{J,\Theta\}\} + \{\{I,\Theta\},J\} = \Theta_{J,I} - \Theta_{I,J}.$$

Since I and J anti-commute with respect to Θ,

$$\Theta_{\{I,J\}} = 2\Theta_{J,I} = -2\Theta_{I,J}.$$

■

Proposition 2.12. *Let I and J be two skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E,Θ) that anti-commute. Then, for any sections X and Y of E,*

$$C_\Theta(I, I\circ J)(X,Y) = I(C_\Theta(I,J)(X,Y)) + 2\,\mathcal{T}_\Theta I(JX,Y) + 2\,\mathcal{T}_\Theta I(X,JY). \tag{20}$$

Proof: Using the equality $C_\Theta(I,J)(X,Y) = [X,Y]_{I,J} + [X,Y]_{J,I}$ and the definition of $\mathcal{T}_\Theta I$, and taking into account that I and J anti-commute as well as I and $I\circ J$, we have

$$\begin{aligned} I(C_\Theta(I,J)(X,Y)) &= 2I([JX,IY] - I[JX,Y] - J[X,IY] + [IX,JY] \\ &\quad -I[X,JY] - J[IX,Y]), \end{aligned}$$

$$2\,\mathcal{T}_\Theta I(JX,Y) = 2([IJX,IY] - I[IJX,Y] - I[JX,IY] + I^2[JX,Y]),$$

and

$$2\,\mathcal{T}_\Theta I(X,JY) = 2([IX,IJY] - I[IX,JY] - I[X,IJY] + I^2[X,JY]).$$

The sum of the right-hand sides of the three last equations gives:

$$\begin{aligned} &2([IX,IJY] - IJ[IX,Y] + [IJX,IY] - IJ[X,IY] - I[IJX,Y] - I[X,IJY]) \\ &= C_\Theta(I, I\circ J)(X,Y). \end{aligned}$$

■

For the various classes of pairs of skew-symmetric $(1,1)$-tensors that will be introduced in the sequel, we shall require that the skew-symmetric $(1,1)$-tensors are compatible in the following sense:

Definition 2.13. A pair (I, J) of skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E, Θ) is said to be a *compatible pair w.r.t.* Θ, if I and J anti-commute and anti-commute w.r.t. Θ.

Let I and J be two $(1,1)$-tensors on a pre-Courant algebroid (E, Θ). Recall that the *Nijenhuis concomitant* of I and J is defined, for all sections X and Y of E, as follows [10]:

$$\begin{aligned}\mathcal{N}_\Theta(I,J)(X,Y) = [IX, JY] - I[X, JY] - J[IX, Y] + IJ[X, Y] \\ + [JX, IY] - J[X, IY] - I[JX, Y] + JI[X, Y]. \quad (21)\end{aligned}$$

Notice that, if $I = J$, then $\mathcal{N}_\Theta(I, I) = 2\mathcal{T}_\Theta I$ and, if I and J anti-commute, then $\mathcal{N}_\Theta(I, J) = \frac{1}{2}C_\Theta(I, J)$.

Lemma 2.14. *Let I and J be two skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E, Θ). Then, $\mathcal{T}_\Theta(I + J) = \mathcal{T}_\Theta I + \mathcal{T}_\Theta J + \mathcal{N}_\Theta(I, J)$.*

Proof: Let I and J be two skew-symmetric $(1,1)$-tensors on (E, Θ) and X, Y any sections of E. Then, using the definition of Nijenhuis torsion, we get:

$$\begin{aligned}\mathcal{T}_\Theta(I+J)(X,Y) =& \mathcal{T}_\Theta I(X,Y) + \mathcal{T}_\Theta J(X,Y) + [IX, JY] + [JX, IY] - I[X, JY] \\ & - J[X, IY] - I[JX, Y] - J[IX, Y] + IJ[X, Y] + JI[X, Y] \\ =& \mathcal{T}_\Theta I(X,Y) + \mathcal{T}_\Theta J(X,Y) + \mathcal{N}_\Theta(I,J)(X,Y).\end{aligned}$$

■

The next proposition gives a characterization of compatible pairs.

Proposition 2.15. *Let I and J be two skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E, Θ) that anti-commute. Then, (I, J) is a compatible pair w.r.t. Θ if and only if $\mathcal{T}_\Theta(I + J) = \mathcal{T}_\Theta I + \mathcal{T}_\Theta J$.*

Proof: Let I and J be two skew-symmetric $(1,1)$-tensors on (E, Θ) that anti-commute and X, Y any sections of E. From Lemma 2.14, and taking into account that I and J anti-commute, we get

$$\mathcal{T}_\Theta(I+J)(X,Y) = \mathcal{T}_\Theta I(X,Y) + \mathcal{T}_\Theta J(X,Y) + \frac{1}{2}C_\Theta(I,J)(X,Y)$$

and, according to Definition 2.13, the proof is complete. ■

Theorem 2.16. *Let I and J be two skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E, Θ) such that $\mathcal{T}_\Theta I(JX, Y) = \mathcal{T}_\Theta I(X, JY) = 0$, for all* *sections X and Y of E. If (I, J) is a compatible pair w.r.t. Θ, then $(I, I^n \circ J)$ is a compatible pair w.r.t. Θ, for all $n \in \mathbb{N}$, in particular,*

$$C_\Theta(I, I^n \circ J) = 0, \ \forall n \in \mathbb{N}. \tag{22}$$

Proof: For $n = 0$, (22) reduces to $C_\Theta(I, J) = 0$, which is one of the assumptions, since (I, J) is a compatible pair w.r.t. Θ. Using (20) and (11), we get

$$C_\Theta(I, I \circ J) = 0,$$

and (22) holds with $n = 1$.

From Lemma 1.8, we know that $I^n \circ J$ is skew-symmetric and anti-commutes with I. Therefore, we may apply Proposition 2.12 to I and $I^n \circ J$, to yield

$$\begin{aligned} C_\Theta(I, I^n \circ J)(X,Y) =& I(C_\Theta(I, I^{n-1} \circ J)(X,Y)) \\ &+ 2\,\mathcal{T}_\Theta I((I^{n-1} \circ J)X, Y) + 2\,\mathcal{T}_\Theta I(X, (I^{n-1} \circ J)Y) \\ =& I(C_\Theta(I, I^{n-1} \circ J)(X,Y)), \end{aligned}$$

where we have used Lemma 1.8 to obtain

$$\mathcal{T}_\Theta I((I^{n-1} \circ J)X, Y) = (-1)^{n-1}\mathcal{T}_\Theta I(J(I^{n-1}X), Y) = 0$$

and

$$\mathcal{T}_\Theta I(X, (I^{n-1} \circ J)Y) = 0.$$

Now, it is obvious that if $C_\Theta(I, I^{n-1} \circ J) = 0$, then $C_\Theta(I, I^n \circ J) = 0$ and the result follows by recursion. ■

2.3. Compatible tensors w.r.t. Θ_n, $n \in \mathbb{N}$.

In this section, we address the general case of hierarchies of tensors which are compatible w.r.t. each term of the family $(\Theta_n)_{n \in \mathbb{N}}$ of pre-Courant structures on E.

Proposition 2.17. *Let I and J be two skew-symmetric $(1,1)$-tensors on (E, Θ). Then,*

$$C_{\Theta_I}(I, J) = C_\Theta(I, \{J, I\}) + \{I, C_\Theta(I, J)\}. \tag{23}$$

In particular, if I and J anti-commute, then,

$$C_{\Theta_I}(I, J) = 2C_\Theta(I, I \circ J) + \{I, C_\Theta(I, J)\}. \tag{24}$$

Proof: Applying twice the Jacobi identity, we get

$$\begin{aligned} \Theta_{I,I,J} &= \Theta_{I,\{J,I\}} + \Theta_{I,J,I} \\ &= \Theta_{I,\{J,I\}} + \Theta_{\{J,I\},I} + \Theta_{J,I,I}, \end{aligned}$$

which can be written as

$$C_\Theta(I,\{J,I\}) = \Theta_{I,I,J} - \Theta_{J,I,I}.$$

From the definition of $C_\Theta(I,J)$, we have $\Theta_{J,I,I} = \{I, C_\Theta(I,J)\} - \Theta_{I,J,I}$. Substituting this result in the last equality, we get

$$\begin{aligned} C_\Theta(I,\{J,I\}) &= \Theta_{I,I,J} - \{I, C_\Theta(I,J)\} + \Theta_{I,J,I} \\ &= C_{\Theta_I}(I,J) - \{I, C_\Theta(I,J)\}, \end{aligned}$$

proving the first statement. If I and J anti-commute, using (7) in Lemma 1.8, we can replace $\{J,I\}$ by $2\, I\circ J$ and the second statement follows. ∎

The next theorem extends the result of Theorem 2.16.

Theorem 2.18. *Let I and J be two skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E,Θ) such that $\mathcal{T}_\Theta I(JX,Y) = \mathcal{T}_\Theta I(X,JY) = 0$, for all sections X and Y of E. If (I,J) is a compatible pair w.r.t. Θ, then $(I, I^t\circ J)$ is a compatible pair w.r.t. Θ_n, for all $n,t\in\mathbb{N}$, in particular,*

$$C_{\Theta_n}(I, I^t\circ J) = 0, \quad \forall n,t\in\mathbb{N}.$$

Proof: Let I and J be two skew-symmetric $(1,1)$-tensors which are compatible w.r.t Θ. Suppose that $\mathcal{T}_\Theta I(JX,Y) = \mathcal{T}_\Theta I(X,JY) = 0$, for all sections X and Y on E. We will prove, by induction on n, that

$$C_{\Theta_n}(I, I^t\circ J) = 0, \quad \forall n,t\in\mathbb{N}.$$

For $n=0$, this is the content of Theorem 2.16.

Suppose now that, for some $n\in\mathbb{N}$, $C_{\Theta_n}(I, I^t\circ J) = 0$, for all $t\in\mathbb{N}$. Then, by Proposition 2.17 we have, for all $t\in\mathbb{N}$,

$$\begin{aligned} C_{\Theta_{n+1}}(I, I^t\circ J) &= 2C_{\Theta_n}(I, I^{t+1}\circ J) + \{I, C_{\Theta_n}(I, I^t\circ J)\} \\ &= 0, \end{aligned}$$

where we used the induction hypothesis in the last equality. Since the skew-symmetric tensor $I^t\circ J$ anti-commutes with I, for all $t\in\mathbb{N}$, the proof is complete. ∎

In order to establish the main results of this section, we need the following lemma.

Lemma 2.19. *Let I and J be two skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E,Θ) such that I is Nijenhuis for Θ. If (I,J) is a compatible pair w.r.t. Θ, then, for all sections X and Y of E,*

$$[X,Y]_{\underbrace{I,\dots,I}_{n},J} = [X,Y]_{I^n\circ J}.$$

Proof: Theorem 2.18 ensures that, for all $n\in\mathbb{N}$, $C_{\Theta_n}(I,J)=0$ and, applying Lemma 1.10 to the pre-Courant structure Θ_{n-1}, we get

$$\begin{aligned}[X,Y]_{\underbrace{I,\dots,I}_{n},J} &= [X,Y]_{\underbrace{I,\dots,I}_{n-1},I,J} = \{\{X,(\Theta_{n-1})_{I,J}\},Y\}\\ &= \{\{X,(\Theta_{n-1})_{I\circ J}\},Y\} = [X,Y]_{\underbrace{I,\dots,I}_{n-1},I\circ J}.\end{aligned}$$

Since, for every $k\in\mathbb{N}$, I anti-commutes with $I^k\circ J$, we may repeat $n-1$ times this procedure, and we end up with

$$[X,Y]_{\underbrace{I,\dots,I}_{n},J} = [X,Y]_{I^n\circ J}.$$

■

Remark 2.20. In Lemma 2.19, we may replace the assumption that I is Nijenhuis for Θ by $\mathcal{T}_\Theta I(JX,Y)=\mathcal{T}_\Theta I(X,JY)=0$, for all sections X and Y on E.

Theorem 2.21. *Let I and J be two skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E,Θ), such that I is Nijenhuis and (I,J) is a compatible pair w.r.t. Θ. Then, $(I^{2s+1}, I^t\circ J)$ is a compatible pair w.r.t. Θ_n, for all $n,s,t\in\mathbb{N}$, in particular,*

$$C_{\Theta_n}(I^{2s+1}, I^t\circ J)=0,\quad \forall n,s,t\in\mathbb{N}. \tag{25}$$

Moreover, if J is Nijenhuis tensor, then $(I^{2s+1}, I^t\circ J^{2k+1})$ is a compatible pair w.r.t. Θ_n, for all $n,s,t,k\in\mathbb{N}$, in particular,

$$C_{\Theta_n}(I^{2s+1}, I^t\circ J^{2k+1})=0,\quad \forall n,s,t,k\in\mathbb{N}. \tag{26}$$

Proof: Let I and J be two skew-symmetric $(1,1)$-tensors which are compatible w.r.t Θ and such that such that $\mathcal{T}_\Theta I=0$. Let us first prove that

$$C_\Theta(I^{2s+1}, I^t\circ J)=0,\quad \forall s,t\in\mathbb{N}.$$

Since I^{2s+1} anti-commutes with $I^t \circ J$, we may apply Lemma 1.10:

$$C_\Theta(I^{2s+1}, I^t \circ J)(X,Y) = 2([X,Y]_{I^{2s+1}, I^t \circ J} - [X,Y]_{I^{2s+1} \circ (I^t \circ J)}).$$

Lemma 2.19 gives

$$\begin{aligned} C_\Theta(I^{2s+1}, I^t \circ J)(X,Y) &= 2([X,Y]_{I^{2s+1}, I^t \circ J} - [X,Y]_{\underbrace{I, \dots, I}_{2s+1}, I^t \circ J}) \\ &= 2([X,Y]_{I^{2s+1}} - [X,Y]_{I^{2s+1}})_{I^t \circ J} = 0, \end{aligned}$$

where we have used Lemma 2.9 b) in the second equality. From (11), we get $C_\Theta(I^{2s+1}, I^t \circ J) = 0$.

In order to prove the result for a general Θ_n, notice that, due to Corollary 2.2 and Theorem 2.18, the hypothesis originally satisfied for Θ, are also satisfied for any of the pre-Courant structures $\Theta_n, n \in \mathbb{N}$. Therefore, we can replace in the above arguments Θ by any $\Theta_n, n \in \mathbb{N}$.

Now, suppose that I and J are both Nijenhuis for Θ. Since they play symmetric roles, we may intertwine them in (25). Specializing (25) to the case $n = 0$, $t = 0$ and $s = k$, we obtain $C_\Theta(I, J^{2k+1}) = 0$ and, because I and J^{2k+1} anti-commute, we conclude that (I, J^{2k+1}) is a compatible pair w.r.t. Θ. Thus, we may apply again (25), taking J^{2k+1} instead of J, to get $C_{\Theta_n}(I^{2s+1}, I^t \circ J^{2k+1}) = 0$. ■

Summarizing, we have started with two skew-symmetric $(1,1)$-tensors I and J on a pre-Courant algebroid (E, Θ) which form a compatible pair w.r.t. Θ and we have considered two families of $(1,1)$-tensors on E:

$$(I^{2s+1})_{s \in \mathbb{N}} \quad \text{and} \quad (I^t \circ J^{2k+1})_{t,k \in \mathbb{N}}.$$

We showed that, if I and J are Nijenhuis for Θ, then any element of the first hierarchy together with any element of the second hierarchy form a pair of tensors which is a compatible pair with respect to any pre-Courant structure Θ_n on $(E, \langle.,.\rangle)$, for all $n \in \mathbb{N}$.

The case where both I and J are Nijenhuis for Θ will be discussed in more detail in the last section.

3. Hierarchies of deforming-Nijenhuis pairs

We introduce the notion of deforming-Nijenhuis pair as well as the definition of Poisson tensor on a pre-Courant algebroid and construct several hierarchies of deforming-Nijenhuis and Poisson-Nijenhuis pairs.

3.1. Hierarchy of deforming-Nijenhuis pairs for Θ_m, $m \in \mathbb{N}$. Starting with a deforming-Nijenhuis pair (J, I) for Θ, we prove, in a first step, that it is also a deforming-Nijenhuis pair for Θ_n, for all $n \in \mathbb{N}$. Then, we construct a hierarchy $(J, I^{2n+1})_{n\in\mathbb{N}}$ of deforming-Nijenhuis pairs for Θ_m, for all $m \in \mathbb{N}$.

Definition 3.1. Let I and J be two skew-symmetric $(1,1)$-tensors on the pre-Courant algebroid (E, Θ). The pair (J, I) is said to be a *deforming-Nijenhuis pair* for Θ if

- (I, J) is a compatible pair w.r.t. Θ;
- J is deforming for Θ;
- I is Nijenhuis for Θ.

We need the following lemmas.

Lemma 3.2. *Let I and J be two skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E, Θ) that anti-commute. We have*

$$(\Theta_n)_{\{J,\{I,J\}\}} = (\Theta_{\{J,\{I,J\}\}})_{\underbrace{I,\ldots,I}_{n}}, \ \forall n \in \mathbb{N}. \tag{27}$$

In particular,

i) *if $\Theta_{\{J,\{I,J\}\}} = \lambda_0 \Theta_{J,J,I}$, for some $\lambda_0 \in \mathbb{R}$, then*
$$(\Theta_n)_{\{J,\{I,J\}\}} = \lambda_0 (\Theta_{J,J})_{\underbrace{I,\ldots,I}_{n+1}}, \ \forall n \in \mathbb{N};$$

ii) *if $\{J, \{I, J\}\}$ is a Θ-cocycle, then it is a Θ_n-cocycle, for all $n \in \mathbb{N}$.*

Proof: Let I and J be two skew-symmetric $(1,1)$-tensors on (E, Θ) that anti-commute. Then, we have

$$I \circ (I \circ J^2) = (I \circ J^2) \circ I \Leftrightarrow \{I, I \circ J^2\} = 0 \Leftrightarrow \{I, \{J, \{J, I\}\}\} = 0. \tag{28}$$

Using the Jacobi identity, (28) implies

$$\Theta_{I,\{J,\{J,I\}\}} = \Theta_{\{J,\{J,I\}\},I}. \tag{29}$$

Since (29) holds for any pre-Courant structure on E, we may write

$$\begin{aligned}(\Theta_{\underbrace{I,\ldots,I}_{n}})_{\{J,\{I,J\}\}} &= (\Theta_{\underbrace{I,\ldots,I}_{n-1}})_{\{J,\{I,J\}\},I} = (\Theta_{\underbrace{I,\ldots,I}_{n-2}})_{\{J,\{I,J\}\},I,I} \\ &= \cdots = (\Theta_{\{J,\{I,J\}\}})_{\underbrace{I,\ldots,I}_{n}},\end{aligned}$$

and (27) is proved. The particular cases follow immediately. ■

Lemma 3.3. *Let I and J be two skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E,Θ). Then,*

$$\Theta_{J,I,J} = \frac{1}{3}\left(\Theta_{J,J,I} + \Theta_{\{J,\{I,J\}\}} + \{J, C_\Theta(I,J)\}\right); \tag{30}$$

$$\Theta_{I,J,J} = -\frac{1}{3}\left(\Theta_{J,J,I} + \Theta_{\{J,\{I,J\}\}} - 2\{J, C_\Theta(I,J)\}\right). \tag{31}$$

Proof: By application of the Jacobi identity, we have

$$\begin{aligned}\Theta_{J,I,J} &= \Theta_{J,\{J,I\}} + \Theta_{J,J,I}\\ &= \Theta_{\{\{J,I\},J\}} + \Theta_{\{J,I\},J} + \Theta_{J,J,I}\\ &= \Theta_{\{\{J,I\},J\}} + \{J, C_\Theta(I,J) - 2\Theta_{J,I}\} + \Theta_{J,J,I},\end{aligned}$$

where we used (12) in the last equality. The last equation reads:

$$3\,\Theta_{J,I,J} = \Theta_{\{J,\{I,J\}\}} + \{J, C_\Theta(I,J)\} + \Theta_{J,J,I},$$

proving (30).

The equality (31) is a consequence of (30), taking into account the relation $\Theta_{I,J,J} = -\Theta_{J,I,J} + \{J, C_\Theta(I,J)\}$. ■

As a particular case of the previous lemma, we have the following:

Corollary 3.4. *If I and J anti-commute w.r.t. Θ, and if $\Theta_{\{J,\{I,J\}\}} = \lambda_0\Theta_{J,J,I}$, $\lambda_0 \in \mathbb{R}$, then*

$$\Theta_{I,J,J} = \alpha\Theta_{J,J,I}, \tag{32}$$

with $\alpha = -\frac{\lambda_0+1}{3}$. Moreover, if J is deforming for Θ, i.e., $\Theta_{J,J} = k\Theta$, with $k \in \mathbb{R}$, then J is deforming for Θ_I. More precisely,

$$\Theta_{I,J,J} = \beta\Theta_I,$$

with $\beta = k\alpha$.

Lemma 3.5. *Let I and J be two skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E,Θ) such that (I,J) is a compatible pair w.r.t. Θ and $\mathcal{T}_\Theta I(JX,Y) = \mathcal{T}_\Theta I(X,JY) = 0$, for all sections X and Y of E. If $\Theta_{\{J,\{I,J\}\}} = \lambda_0\Theta_{J,J,I}$, for some $\lambda_0 \in \mathbb{R}$, such that $\lambda_0 \neq \frac{4}{(-3)^m-1}$ for all $m \in \mathbb{N}$, then, for all $n \in \mathbb{N}$,* *(a)* $(\Theta_n)_{\{J,\{I,J\}\}} = \lambda_n(\Theta_n)_{J,J,I}$, *where* λ_n *is defined recurrently*[†] *by* $\lambda_n = \frac{-3\lambda_{n-1}}{1+\lambda_{n-1}}$, $n \geq 1$,

(b) $\lambda_n(\Theta_n)_{J,J,I} = \lambda_0\Theta_{J,J,\underbrace{I,\dots,I}_{n+1}}$.

(c) If, in particular, $\lambda_0 = 0$, *then* $(\Theta_n)_{J,J} = (-\frac{1}{3})^n\Theta_{J,J,\underbrace{I,\dots,I}_{n}}$, *for all* $n \in \mathbb{N}$.

Proof: (a) We will prove this statement by induction. Suppose that, for some $n \geq 1$, $(\Theta_{n-1})_{\{J,\{I,J\}\}} = \lambda_{n-1}(\Theta_{n-1})_{J,J,I}$. Using Lemma 3.2 and the induction hypothesis, we have

$$\begin{aligned}(\Theta_n)_{\{J,\{I,J\}\}} &= (\Theta_{n-1})_{\{J,\{I,J\}\},I}\\ &= \lambda_{n-1}(\Theta_{n-1})_{J,J,I,I}.\end{aligned}$$

Applying formula (32) for Θ_{n-1}, we obtain

$$(\Theta_n)_{\{J,\{I,J\}\}} = \frac{-3\lambda_{n-1}}{1+\lambda_{n-1}}(\Theta_{n-1})_{I,J,J,I} = (\Theta_n)_{\{J,\{I,J\}\}} = \lambda_n(\Theta_n)_{J,J,I},$$

with $\lambda_n = \frac{-3\lambda_{n-1}}{1+\lambda_{n-1}}$.

(b) Starting from the previous statement, then using the Lemma 3.2 and the hypothesis, we have,

$$\lambda_n(\Theta_n)_{J,J,I} = (\Theta_n)_{\{J,\{I,J\}\}} = \Theta_{\{J,\{I,J\}\},\underbrace{I,\dots,I}_{n}} = \lambda_0\Theta_{J,J,\underbrace{I,\dots,I}_{n+1}}.$$

(c) From Lemma 3.2 i), we get

$$(\Theta_n)_{\{J,\{I,J\}\}} = 0,\ \ \forall n \in \mathbb{N},$$

while Theorem 2.18 gives

$$C_{\Theta_n}(I,J) = 0,\ \ \forall n \in \mathbb{N}.$$

Thus, applying successively the formula (31), yields

$$(\Theta_n)_{J,J} = -\frac{1}{3}(\Theta_{n-1})_{J,J,I} = \dots = (-\frac{1}{3})^n\Theta_{J,J,\underbrace{I,\dots,I}_{n}}.$$

∎

[†]Explicitly, $\lambda_n = \frac{(-3)^n\lambda_0}{1+\frac{1-(-3)^n}{4}\lambda_0}$, for all $n \in \mathbb{N}$.

Proposition 3.6. *Let I and J be two skew-symmetric* $(1,1)$*-tensors on a pre-Courant algebroid* (E,Θ) *such that* (I,J) *is a compatible pair w.r.t.* Θ, $\Theta_{\{J,\{I,J\}\}} = \lambda_0 \Theta_{J,J,I}$, *for some* $\lambda_0 \in \mathbb{R}$, *such that* $\lambda_0 \neq \frac{4}{(-3)^m-1}$ *for all* $m \in \mathbb{N}$, *and* $\mathcal{T}_\Theta I(JX,Y) = \mathcal{T}_\Theta I(X,JY) = 0$, *for all sections X and Y of E. If J is a deforming tensor for Θ, then J is also a deforming tensor for Θ_n, for all $n \in \mathbb{N}$.*

Proof: We consider two cases, depending on the value of λ_0.

i) Case $\lambda_0 \neq 0$.

From Theorem 2.18, we have that $C_{\Theta_n}(I,J) = 0$, for all $n \in \mathbb{N}$. We compute[‡], using Lemma 3.3 and both statements of Lemma 3.5:

$$\begin{aligned}
(\Theta_n)_{J,J} = (\Theta_{n-1})_{I,J,J} &= -\frac{1}{3}((\Theta_{n-1})_{J,J,I} + (\Theta_{n-1})_{\{J,\{I,J\}\}}) \\
&= -\frac{1}{3}((\Theta_{n-1})_{J,J,I} + \lambda_{n-1}(\Theta_{n-1})_{J,J,I}) \\
&= -\frac{1+\lambda_{n-1}}{3}(\Theta_{n-1})_{J,J,I} \\
&= -\frac{(1+\lambda_{n-1})\lambda_0}{3\lambda_{n-1}}\Theta_{J,J,\underbrace{I,\ldots,I}_{n}} \\
&= \frac{\lambda_0}{\lambda_n}\Theta_{J,J,\underbrace{I,\ldots,I}_{n}}.
\end{aligned}$$

The tensor J being deforming for Θ, we have $\Theta_{J,J} = k\Theta$, for some $k \in \mathbb{R}$, and the last equality becomes

$$(\Theta_n)_{J,J} = \frac{\lambda_0}{\lambda_n}\, k\, \Theta_n,$$

which means that J is a deforming tensor for Θ_n.

(ii) Case $\lambda_0 = 0$.

If J is deforming for Θ, i.e., $\Theta_{J,J} = k\Theta$, with $k \in \mathbb{R}$, then, from Lemma 3.5 c) we immediately get

$$(\Theta_n)_{J,J} = (-\frac{1}{3})^n k\Theta_n, \quad \forall n \in \mathbb{N},$$

which means that J is deforming for Θ_n.

■

[‡]Notice that if $\lambda_0 \neq 0$ then $\lambda_n \neq 0, \forall n \in \mathbb{N}$.

Combining Corollary 2.2, Theorem 2.18 and Proposition 3.6, we deduce the following:

Theorem 3.7. *Let I and J be two skew-symmetric $(1,1)$-tensors on a pre-Courant (respectively, Courant) algebroid (E,Θ). If (J,I) is a deforming-Nijenhuis pair for Θ, then (J,I) is a deforming-Nijenhuis pair for the pre-Courant (respectively, Courant) structures Θ_n, for all $n \in \mathbb{N}$.*

Now, we establish the main result of this section.

Theorem 3.8. *Let I and J be two skew-symmetric $(1,1)$-tensors on a pre-Courant (respectively, Courant) algebroid (E,Θ). If (J,I) is a deforming-Nijenhuis pair for Θ, then (J,I^{2n+1}) is a deforming-Nijenhuis pair for the pre-Courant (respectively, Courant) structures Θ_m, for all $m,n \in \mathbb{N}$.*

Proof: Let (J,I) be a deforming-Nijenhuis pair for Θ. Applying Theorem 3.7, we have that (J,I) is a deforming-Nijenhuis pair for Θ_m, for all $m \in \mathbb{N}$. From Proposition 2.6, we get that I^{2n+1} is Nijenhuis for Θ_m, for all $m,n \in \mathbb{N}$. According to Lemma 1.8, I^{2n+1} and J anti-commute and, from Theorem 2.21, we have that $C_{\Theta_m}(I^{2n+1},J)=0$, for all $m,n \in \mathbb{N}$. Thus, (J,I^{2n+1}) is a deforming-Nijenhuis pair for Θ_m, for all $m,n \in \mathbb{N}$. ■

3.2. Hierarchy of Poisson tensors for Θ_k, $k \in \mathbb{N}$. We introduce the notions of Poisson tensor and of compatible Poisson tensors for a pre-Courant algebroid (E,Θ). We construct a hierarchy $(I^n \circ J)_{n\in\mathbb{N}}$ of Poisson tensors which are pairwise compatible for each element of the hierarchy $(\Theta_k)_{k\in\mathbb{N}}$ of pre-Courant structures.

We start by introducing the main notion of this section.

Definition 3.9. A skew-symmetric $(1,1)$-tensor J on a pre-Courant algebroid (E,Θ) satisfying $\Theta_{J,J}=0$ is said to be a *Poisson* tensor for Θ.

In the next example, we show that the previous definition extends the usual definition of a Poisson bivector on a Lie algebroid.

Example 3.10. Let (A,μ) be a Lie algebroid. Consider the Courant algebroid $(E=A\oplus A^*,\Theta=\mu)$ and the $(1,1)$-tensor and J_π of Example 1.6 a). Then, J_π is a Poisson tensor for $\Theta=\mu$ if and only if π is a Poisson tensor on (A,μ):

$$\Theta_{J_\pi,J_\pi}=0 \Leftrightarrow \{\pi,\{\pi,\mu\}\}=0 \Leftrightarrow [\pi,\pi]_\mu=0.$$

$\diamondsuit$

Example 3.11. The operators introduced in example 1.3 are Poisson operators on Lie algebras.

The next theorem follows directly from Lemma 3.5 c).

Theorem 3.12. *Let I and J be two skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E,Θ) such that (I,J) is a compatible pair w.r.t. Θ, $\Theta_{\{J,\{I,J\}\}}=0$ and $\mathcal{T}_\Theta I(JX,Y)=\mathcal{T}_\Theta I(X,JY)=0$, for all sections X and Y of E. If J is Poisson for Θ, then J is Poisson for Θ_n, for all $n\in\mathbb{N}$.*

Now, we introduce the notion of compatible Poisson tensors.

Definition 3.13. Let J and J' be two Poisson tensors for the pre-Courant structure Θ on the vector bundle $(E,\langle .,.\rangle)$. The tensors J and J' are said to be *compatible* Poisson tensors for Θ if $J+J'$ is a Poisson tensor for Θ, i.e, $\Theta_{J+J',J+J'}=0$.

An immediate consequence of this definition is the following:

Lemma 3.14. *Let J and J' be two Poisson tensors for Θ. Then, J and J' are compatible Poisson tensors for Θ if and only if $\Theta_{J,J'}+\Theta_{J',J}=0$. In other words, J and J' are compatible Poisson tensors for Θ if and only if J and J' anti-commute w.r.t. Θ.*

Example 3.15. Let (A,μ) be a Lie algebroid, consider the Courant algebroid $(E=A\oplus A^*,\Theta=\mu)$ and take two Poisson tensors for $\Theta=\mu$, J_π and $J_{\pi'}$, of the type considered in Example 1.6 a). Then,

$$\begin{aligned}\Theta_{J_\pi,J_{\pi'}}+\Theta_{J_{\pi'},J_\pi}=0 &\Leftrightarrow \{\pi',\{\pi,\mu\}\}+\{\pi,\{\pi',\mu\}\}=0\\ &\Leftrightarrow 2\{\pi',\{\pi,\mu\}\}=0\\ &\Leftrightarrow [\pi,\pi']_\mu=0,\end{aligned}$$

and we recover the notion of compatible Poisson tensors on a Lie algebroid.

In order to construct a hierarchy of pairwise compatible Poisson tensors, we need the next proposition.

Proposition 3.16. *Let I and J be two skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E,Θ), that anti-commute. Then, for all sections X and Y of E,*

$$\mathcal{T}_{\Theta_I}J(X,Y)=-J(C_\Theta(I,J)(X,Y))-\mathcal{T}_\Theta J(IX,Y)-\mathcal{T}_\Theta J(X,IY)-I(\mathcal{T}_\Theta J(X,Y)). \tag{33}$$

Proof: We compute $\mathcal{T}_{\Theta_I} J$ and $C_\Theta(I, J)$. For any sections X, Y of E, we have

$$\begin{aligned}\mathcal{T}_{\Theta_I} J(X,Y) &= [JX, JY]_I - J[JX,Y]_I - J[X,JY]_I + J^2[X,Y]_I \\ &= [IJX, JY] + [JX, IJY] - I[JX, JY] \\ &- J[IJX, Y] - J[JX, IY] + JI[JX, Y] \\ &- J[IX, JY] - J[X, IJY] + JI[X, JY] \\ &+ J^2[IX, Y] + J^2[X, IY] - J^2I[X, Y]\end{aligned}$$

and

$$\begin{aligned}C_\Theta(I,J)(X,Y) &= 2([JX, IY] + [IX, JY] - I([JX, Y] + [X, JY]) \\ & \quad -J([IX, Y] + [X, IY])).\end{aligned}$$

Thus,

$$\begin{aligned}&\mathcal{T}_{\Theta_I} J(X,Y) + J(C_\Theta(I,J)(X,Y)) = -[JIX, JY] - [JX, JIY] - I[JX, JY] \\ &\qquad +J[JIX, Y] + J[JX, IY] + IJ[JX, Y] + J[IX, JY] + J[X, JIY] \\ &\qquad +IJ[X, JY] - J^2[IX, Y] - J^2[X, IY] - IJ^2[X, Y] \\ &= -\mathcal{T}_\Theta J(IX, Y) - \mathcal{T}_\Theta J(X, IY) - I(\mathcal{T}_\Theta J(X,Y)).\end{aligned}$$

■

Remark 3.17. The roles of I and J can be reversed in the previous proposition, so that the following result also holds:

$$\mathcal{T}_{\Theta_J} I(X,Y) = -I(C_\Theta(I,J)(X,Y)) - \mathcal{T}_\Theta I(JX, Y) - \mathcal{T}_\Theta I(X, JY) - J(\mathcal{T}_\Theta I(X,Y)). \tag{34}$$

Theorem 3.18. *Let I and J be two skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E, Θ), such that (I, J) is a compatible pair w.r.t. Θ, I is Nijenhuis for Θ, J is Poisson for Θ and $\Theta_{\{J,\{I,J\}\}} = 0$. Then, for all $m, n, k \in \mathbb{N}$,*

$$(\Theta_k)_{I^m \circ J, I^n \circ J} = 0.$$

In particular,

1. *$I^n \circ J$ is a Poisson tensor for Θ_k, for all $n, k \in \mathbb{N}$;*
2. *$(I^n \circ J)_{n \in \mathbb{N}}$ is a hierarchy of pairwise compatible Poisson tensors for Θ_k, for all $k \in \mathbb{N}$.*

Requiring $\Theta_{\{J,\{I,J\}\}} = 0$ might seem a bit arbitrary, but it is not. Indeed, for many Poisson $1(1,1)$ tensors, $J^2 = 0$, so that the condition is automatically satisfied. Indeed, this condition may be interpreted as meaning that the strong condition $J^2 = 0$ can be weakened and turned into the condition $I \circ J^2$ is a Θ-cocycle.

The proof of the above theorem needs two auxiliary lemmas.

Lemma 3.19. *Let I and J be two skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E, Θ), such that (I, J) is a compatible pair w.r.t. Θ. If I is Nijenhuis for Θ, then I is Nijenhuis for $(\Theta_m)_J$, for all $m \in \mathbb{N}$.*

Proof: Fix $m \in \mathbb{N}$. From Corollary 2.2, I is Nijenhuis for Θ_m. Also, applying Theorem 2.18, we get $C_{\Theta_m}(I, J) = 0$. Finally, using (34) for the pre-Courant structure Θ_m, we conclude that I is Nijenhuis for $(\Theta_m)_J$. ■

Lemma 3.20. *Let I and J be two skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E, Θ) such that J is Poisson for Θ, $\Theta_{\{J,\{I,J\}\}} = 0$ and $\mathcal{T}_\Theta I(JX, Y) = \mathcal{T}_\Theta I(X, JY) = 0$, for all sections X and Y of E. If (I, J) is a compatible pair w.r.t. Θ, then (I, J) is a compatible pair w.r.t. $(\Theta_m)_J$, for all $m \in \mathbb{N}$.*

Proof: Fix $m \in \mathbb{N}$. By definition, $C_{(\Theta_m)_J}(I, J) = (\Theta_m)_{J,I,J} + (\Theta_m)_{J,J,I}$. In order to compute $(\Theta_m)_{J,I,J}$, remember formula (30) for the pre-Courant structure Θ_m:

$$(\Theta_m)_{J,I,J} = \frac{1}{3}\left((\Theta_m)_{J,J,I} + (\Theta_m)_{\{J,\{I,J\}\}} + \{J, C_{\Theta_m}(I, J)\}\right).$$

Since (I, J) is a compatible pair w.r.t. Θ, applying Theorem 2.18, we get $C_{\Theta_m}(I, J) = 0$. Furthermore, from Lemma 3.2(ii), we have $(\Theta_m)_{\{J,\{I,J\}\}} = 0$. Then, the formula above turns into

$$(\Theta_m)_{J,I,J} = \frac{1}{3}(\Theta_m)_{J,J,I},$$

so that

$$C_{(\Theta_m)_J}(I, J) = \frac{4}{3}(\Theta_m)_{J,J,I}.$$

Using Theorem 3.12, we get $(\Theta_m)_{J,J,I} = 0$ and, therefore, (I, J) is a compatible pair w.r.t. $(\Theta_m)_J$. ■

We address now the proof of the above theorem.

Proof of Theorem 3.18: Let (I, J) be a compatible pair w.r.t. Θ such that I is Nijenhuis for Θ, J is Poisson for Θ and $\Theta_{\{J,\{I,J\}\}} = 0$. From the above auxiliary lemmas, (I, J) is a compatible pair w.r.t. $(\Theta_{k+m})_J$ and I is Nijenhuis for $(\Theta_{k+m})_J$. Then, using Lemma 2.19 for the pre-Courant structure $(\Theta_{k+m})_J$, we obtain

$$(\Theta_k)_{I^m \circ J, I^n \circ J} = ((\Theta_{k+m})_J)_{I^n \circ J} = ((\Theta_{k+m})_J)_{\underbrace{I, \dots, I}_{n}, J} = \Theta_{\underbrace{I, \dots, I}_{k+m}, J, \underbrace{I, \dots, I}_{n}, J}$$

and, applying successively Theorem 2.18, we get

$$(\Theta_k)_{I^m \circ J, I^n \circ J} = (-1)^n \, \Theta_{\underbrace{I, \dots, I}_{k+m+n}, J, J}. \tag{35}$$

Using Theorem 3.12, we deduce that

$$(\Theta_k)_{I^m \circ J, I^n \circ J} = 0.$$

■

3.3. Hierarchy of Poisson-Nijenhuis pairs for Θ_k, $k \in \mathbb{N}$. We introduce the notion of Poisson-Nijenhuis pair for a pre-Courant algebroid (E, Θ) and construct a hierarchy of Poisson-Nijenhuis pairs.

Definition 3.21. Let I and J be two skew-symmetric $(1, 1)$-tensors on a pre-Courant algebroid (E, Θ). The pair (J, I) is said to be a *Poisson-Nijenhuis pair* for Θ if

(1) (J, I) is a compatible pair w.r.t. Θ;
(2) J is Poisson for Θ;
(3) I is Nijenhuis for Θ.

Remark 3.22. If (J, I) is a Poisson-Nijenhuis pair for Θ, then it is a deforming-Nijenhuis pair for Θ.

Recall that a Poisson-Nijenhuis structure on a Lie algebroid (A, μ) is a pair (π, N), where π is a Poisson tensor and $N : A \to A$ is a Nijenhuis tensor such that $N\pi^\# = \pi^\# N^*$ and $C_\mu(\pi, N) = 0$.

The next example shows the relation between Definition 3.21 and the notion of Poisson-Nijenhuis structure on a Lie algebroid.

Example 3.23. Let (π, N) be a Poisson-Nijenhuis structure on a Lie algebroid (A, μ), with $N^2 = \lambda id_A$, $\lambda \in \mathbb{R}$. Consider the Courant algebroid (E, Θ), with $E = A \oplus A^*$ and $\Theta = \mu$, J_π and I_N as in Example 1.6 a) and c), respectively. Then, (J_π, I_N) is a Poisson-Nijenhuis pair for Θ. In fact, $N\pi^\# = \pi^\# N^* \Leftrightarrow I_N \circ J_\pi = -J_\pi \circ I_N$ and $C_\mu(\pi, N) = C_\mu(J_\pi, I_N) = 0$, so that

(J_π, I_N) is a compatible pair w.r.t. μ. Moreover, π is a Poisson bivector on (A, μ) if and only if J_π is Poisson for $\Theta = \mu$ (see Example 3.10) and I_N is Nijenhuis for $\Theta = \mu$ (see Example 1.6 c)). Conversely, if (J_π, I_N) is a Poisson-Nijenhuis pair for $\Theta = \mu$ with $N^2 = \lambda id_A$, then (π, N) is a Poisson-Nijenhuis structure on (A, μ).

$\diamondsuit$

The next theorem defines a hierarchy of Poisson-Nijenhuis pairs.

Theorem 3.24. *Let I and J be two skew-symmetric $(1,1)$-tensors on the pre-Courant algebroid (E, Θ), such that (J, I) is a Poisson-Nijenhuis pair for Θ and $\Theta_{\{J,J_1\}} = 0$. Then, $(I^n \circ J, I^{2m+1})$ is a Poisson-Nijenhuis pair for Θ_k, for all $m, n, k \in \mathbb{N}$.*

Proof: Let (J, I) be a Poisson-Nijenhuis pair for Θ such that $\Theta_{\{J,J_1\}} = 0$. We have to prove that:

(1) $(I^n \circ J, I^{2m+1})$ is a compatible pair w.r.t. Θ_k,
(2) $I^n \circ J$ is Poisson for Θ_k,
(3) I^{2m+1} is Nijenhuis for Θ_k,

for all $m, n, k \in \mathbb{N}$.

Applying Theorem 2.21, statement (1) follows. From Theorem 3.18, we get statement (2), while statement (3) is a particular case of Proposition 2.6. ∎

Using the Poisson-Nijenhuis pair arising from a Poisson-Nijenhuis structure as in Example 3.23, we recover most of the hierarchy already studied by [14], up to a minor difference. In this general setting it is not possible to consider I^{2n} since it is not a skew-symmetric $(1,1)$-tensor.

To conclude this section, we come back to the deforming-Nijenhuis pairs to discuss a particular case.

Proposition 3.25. *Let I and J be two skew-symmetric $(1,1)$-tensors on the pre-Courant algebroid (E, Θ), such that I is Nijenhuis, $I^2 = \alpha\, id_E$, $\alpha \in \mathbb{R}$, and (I, J) is a compatible pair w.r.t Θ. If J is deforming for Θ, then $I^n \circ J$ is also deforming for Θ.*

Proof: First, we recall from (5), that if I is Nijenhuis and satisfies $I^2 = \alpha\, id_E$, then $\Theta_{I,I} = \alpha\Theta$ and, by recursion, we get $\Theta_{\underbrace{I, \dots, I}_{2n}} = \alpha^n \Theta$. From the equality

(35), with $k = 0$ and $m = n$, we get

$$\begin{aligned}\Theta_{I^n \circ J, I^n \circ J} &= (-1)^n \Theta_{\underbrace{I, \dots, I}_{2n}, J, J} \\ &= (-\alpha)^n \Theta_{J,J} \\ &= (-\alpha)^n \lambda \Theta,\end{aligned}$$

where, in the last equality, we use the fact that J is deforming for Θ, ie, $\Theta_{J,J} = \lambda\Theta$, for some $\lambda \in \mathbb{R}$. ■

From the previous proposition, we conclude that if (J, I) is a deforming-Nijenhuis pair for Θ such that $I^2 = \alpha Id_E$, then, for all $n \in \mathbb{N}$, $(I^n \circ J, I)$ is still a deforming-Nijenhuis pair for Θ. But $(I^n \circ J, I)_{n\in\mathbb{N}}$ is a very poor hierarchy of deforming-Nijenhuis pairs since all the pairs are proportional either to (J, I) or to $(I \circ J, I)$. In fact we have, for all $n \in \mathbb{N}$,

$$I^{2n} \circ J = \alpha^n J, \quad I^{2n+1} \circ J = \alpha^n I \circ J.$$

4. Hierarchies of Nijenhuis pairs

The last part of this article is devoted to the study of pairs of Nijenhuis tensors on pre-Courant algebroids.

4.1. Nijenhuis pair for a hierarchy of pre-Courant structures. We start by introducing the notion of Nijenhuis pair for a pre-Courant algebroid.

Definition 4.1. Let I and J be two skew-symmetric tensors on a pre-Courant algebroid (E, Θ). The pair (I, J) is called a *Nijenhuis pair* for Θ, if it is a compatible pair w.r.t. Θ and I and J are both Nijenhuis for Θ.

Example 4.2. Let J be a deforming tensor on (E, Θ), i.e. $\Theta_{J,J} = \lambda\Theta$, with $\lambda \in \mathbb{R}$. If (J, I) is a deforming-Nijenhuis pair, with $J^2 = \lambda id_E$, then (J, I) is a Nijenhuis pair. In particular, if (J, I) is Poisson-Nijenhuis pair, and $J^2 = 0$, then (J, I) is a Nijenhuis pair.

◇

In the next proposition we compute the torsion of the composition $I \circ J$.

Proposition 4.3. *Let I and J be two skew-symmetric tensors on a pre-Courant algebroid (E, Θ) that anti-commute. Then, for all sections X and*

Y of E,

$$2\mathcal{T}_\Theta(I\circ J)(X,Y)=\Big(\mathcal{T}_\Theta I(JX,JY)-J\left(\mathcal{T}_\Theta I(JX,Y)+\mathcal{T}_\Theta I(X,JY)\right)-\\ -J^2(\mathcal{T}_\Theta I(X,Y))\Big)+\underset{I,J}{\circlearrowleft}, \quad (36)$$

where $\underset{I,J}{\circlearrowleft}$ stands for permutation of I and J.

Proof: Let us compute the first four terms of the right hand side of equation (36):

$$\begin{array}{rlllll}
\mathcal{T}_\Theta I(JX,JY) & = & [IJX,IJY] & -I[IJX,JY] & -I[JX,IJY] & +I^2[JX,JY]\\
-J\left(\mathcal{T}_\Theta I(JX,Y)\right) & = & -J[IJX,IY] & +JI[IJX,Y] & +JI[JX,IY] & -JI^2[JX,Y]\\
-J\left(\mathcal{T}_\Theta I(X,JY)\right) & = & -J[IX,IJY] & +JI[IX,JY] & +JI[X,IJY] & -JI^2[X,JY]\\
-J^2\left(\mathcal{T}_\Theta I(JX,Y)\right) & = & -J^2[IX,IY] & +J^2I[IX,Y] & +J^2I[JX,IY] & -J^2I^2[X,Y].
\end{array}$$

The terms appearing on the right hand sides of the above equalities can be addressed in a matrix form:

$$M(I,J)(X,Y)=\begin{bmatrix}
[IJX,IJY] & -I[IJX,JY] & -I[JX,IJY] & I^2[JX,JY]\\
-J[IJX,IY] & JI[IJX,Y] & JI[JX,IY] & -JI^2[JX,Y]\\
-J[IX,IJY] & JI[IX,JY] & JI[X,IJY] & -JI^2[X,JY]\\
-J^2[IX,IY] & J^2I[IX,Y] & J^2I[JX,IY] & -J^2I^2[X,Y]
\end{bmatrix}.$$

Because I and J anti-commute, intertwining the tensors I and J, we obtain the matrix $M(J,I)$ with entries given by

$$M(J,I)_{m,n}=\begin{cases}-M(I,J)_{n,m}, & \text{if } m\neq n\\ M(I,J)_{m,n}, & \text{if } m=n\end{cases}$$

for all $m,n=1,\dots,4$.

Note that the right hand side of equation (36) is the sum of all the entries of both matrices $M(I,J)(X,Y)$ and $M(J,I)(X,Y)$. Thus,

$$\begin{aligned}
\mathcal{T}_\Theta I(JX,JY)-J\left(\mathcal{T}_\Theta I(JX,Y)+\mathcal{T}_\Theta I(X,JY)\right)-J^2(\mathcal{T}_\Theta I(X,Y))+\underset{I,J}{\circlearrowleft}&=\\
&=2\left([IJX,IJY]+JI[IJX,Y]+JI[X,IJY]-J^2I^2[X,Y]\right)\\
&=2\left([IJX,IJY]-IJ[IJX,Y]-IJ[X,IJY]+(IJ)^2[X,Y]\right)\\
&=2\mathcal{T}_\Theta(I\circ J)(X,Y),
\end{aligned}$$

and the proof is complete. ■

Proposition 4.4. *Let I and J be two skew-symmetric tensors on a pre-Courant algebroid (E, Θ). If (I, J) is a Nijenhuis pair for Θ, then $(I, I \circ J)$ and $(J, I \circ J)$ are also Nijenhuis pairs for Θ.*

Proof: It is obvious that I and $I \circ J$ anti-commute, as well as J and $I \circ J$. From (36) we conclude that $I \circ J$ is a Nijenhuis tensor and from (20) we get $C_\Theta(I, I \circ J) = C_\Theta(J, I \circ J) = 0$. ■

Proposition 4.4 allows us to establish a relationship between Nijenhuis pairs and hypercomplex triples, which will be defined in the sequel.

The triple (I, J, K) of skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E, Θ) is called a *hypercomplex triple* if $I^2 = J^2 = K^2 = I \circ J \circ K = -id_E$ and all the six Nijenhuis concomitants $\mathcal{N}_\Theta(I, I)$, $\mathcal{N}_\Theta(J, J)$, $\mathcal{N}_\Theta(K, K)$, $\mathcal{N}_\Theta(I, J)$, $\mathcal{N}_\Theta(J, K)$ and $\mathcal{N}_\Theta(I, K)$ vanish [22]. (See (21) for the definition of $\mathcal{N}_\Theta$).

Example 4.5. Given a Nijenhuis pair (I, J) such that $I^2 = J^2 = -id_E$, the triple $(I, J, I \circ J)$ is a hypercomplex structure. Conversely, for every hypercomplex structure (I, J, K), the pairs (I, J), (J, K) and (K, I) are Nijenhuis pairs.

$\diamondsuit$

The main result of this section is the following.

Theorem 4.6. *Let I and J be two $(1,1)$-tensors on a pre-Courant algebroid (E, Θ). If (I, J) is a Nijenhuis pair for Θ, then (I, J) is a Nijenhuis pair for $\Theta_{k_1, k_2, \ldots, k_n}$, for all $n \in \mathbb{N}$, where k_i stands either for I or for J, for every $i = 1, \ldots, n$.*

Proof: Let (I, J) be a Nijenhuis pair for Θ. Combining formulae (20) and (24), we get

$$C_{\Theta_I}(I, J)(X, Y) = I(C_\Theta(I, J)(X, Y)) + C_\Theta(I, J)(IX, Y) + C_\Theta(I, J)(X, IY) \\ +4\,\mathcal{T}_\Theta I(X, Y) + 4\,\mathcal{T}_\Theta I(X, Y). \tag{37}$$

Now, from (33), (13) applied with $n = 1$ and (37), we conclude that (I, J) is a Nijenhuis pair for Θ_I. Since we may exchange the roles of I and J, we also conclude that (I, J) is a Nijenhuis pair for Θ_J.

Since the formulae (33), (13) and (37) hold for any anti-commuting tensors I and J, and for any pre-Courant structure Θ on E, we can repeat the previous argument iteratively to conclude that (I, J) is a Nijenhuis pair for $\Theta_{k_1,k_2,\ldots,k_n}$, for all $n \in \mathbb{N}$, where k_i stands either for I or for J, for every $i = 1, \ldots, n$. ■

4.2. First hierarchy of Nijenhuis pairs. We start with the construction of a hierarchy $(I^{2m+1}, J)_{m\in\mathbb{N}}$ of Nijenhuis pairs where one of the Nijenhuis tensors keeps unchanged.

Proposition 4.7. *Let I and J be two $(1,1)$-tensors on a pre-Courant algebroid (E, Θ). If (I, J) is a Nijenhuis pair for Θ, then, for all $m \in \mathbb{N}$, (I^{2m+1}, J) is a Nijenhuis pair for Θ.*

Proof: The proof follows from Corollary 2.5 and Theorem 2.21. ■

Combining Theorem 4.6 and Proposition 4.7, we deduce:

Proposition 4.8. *Let I and J be two $(1,1)$-tensors on a pre-Courant algebroid (E, Θ). If (I, J) is a Nijenhuis pair for Θ then, for all $m \in \mathbb{N}$, (I^{2m+1}, J) is a Nijenhuis pair for $\Theta_{k_1,k_2,\ldots,k_s}$, for all $s \in \mathbb{N}$, where k_i stands either for I or for J, for every $i = 1, \ldots, s$.*

Now we consider the hierarchy (I^{2m+1}, J^{2n+1}), $m, n \in \mathbb{N}$. This case follows from the previous one: for every $m \in \mathbb{N}$, (I^{2m+1}, J) is a Nijenhuis pair. Applying Proposition 4.7 to each one of these pairs, we get that $(I^{2m+1}, J^{2n+1})_{m,n\in\mathbb{N}}$ is a hierarchy of Nijenhuis pairs. If, moreover, we take into account Theorem 4.6, we end up with the following.

Theorem 4.9. *Let I and J be two $(1,1)$-tensors on a pre-Courant algebroid (E, Θ). If (I, J) is a Nijenhuis pair for Θ then, for all $m, n \in \mathbb{N}$, (I^{2m+1}, J^{2n+1}) is a Nijenhuis pair for $\Theta_{k_1,k_2,\ldots,k_s}$, for all $s \in \mathbb{N}$, where k_i stands either for I or for J, for every $i = 1, \ldots, s$.*

4.3. General hierarchy of Nijenhuis pairs. Before considering the general case, we will construct a hierarchy $(I^{2m+1} \circ J^n, J)_{m,n\in\mathbb{N}}$ of Nijenhuis pairs.

Let I and J be two skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E, Θ). If I and J are Nijenhuis tensors, we know (see Corollary 2.5) that, for any $m, n \in \mathbb{N}$, I^m and J^n are also Nijenhuis tensors for Θ. The next lemma gives a condition granting that $I^m \circ J^n$ is also Nijenhuis.

Lemma 4.10. *Let I and J be two skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E,Θ). If I and J are anti-commuting Nijenhuis tensors, then $I^m \circ J^n$ is a Nijenhuis tensor provided that one at least of the integers m,n is odd.*

Proof: As the roles of the tensors I and J are symmetric, we can suppose that m is odd (and n is even or odd). If n is also odd then I^m and J^n anti-commute because $I^m \circ J^n = (-1)^{mn} J^n \circ I^m = -J^n \circ I^m$ and the result follows from Proposition 4.3. Suppose now that m is odd and n is even. By the previous case, $I^m \circ J^{n-1}$ is Nijenhuis and anti-commutes with J:

$$(I^m \circ J^{n-1}) \circ J = I^m \circ J^n = -J \circ (I^m \circ J^{n-1}).$$

Then, using again Proposition 4.3, we conclude that $I^m \circ J^n$ is a Nijenhuis tensor. ■

The next proposition generalizes Propositions 4.4 and 4.7.

Proposition 4.11. *Let I and J be two skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E,Θ). If (I,J) is a Nijenhuis pair for Θ, then the pair*
$(I^{2m+1} \circ J^n, J^{2t+1})$ (and analogously $(I^m \circ J^{2n+1}, I^{2t+1})$) is a Nijenhuis pair for Θ, for all $m,n,t \in \mathbb{N}$.

Proof: We already know that $I^{2m+1} \circ J^n$ is Nijenhuis (see Lemma 4.10) and that J^{2t+1} is Nijenhuis (see Corollary 2.5). Moreover, $I^{2m+1} \circ J^n$ anti-commutes with J^{2t+1} and, applying (26), we get $C_\Theta(I^{2m+1} \circ J^n, J^{2t+1}) = 0$. ■

Using Theorem 4.6, the result of Proposition 4.11 can be extended to all pre-Courant structures $\Theta_{k_1,k_2,\ldots,k_s}$, where k_i stands either for I or for J, for every $i = 1,\ldots,s$.

Theorem 4.12. *Let I and J be two skew-symmetric $(1,1)$-tensors on a pre-Courant algebroid (E,Θ). If (I,J) is a Nijenhuis pair for Θ, then for all $m,n,t \in \mathbb{N}$, $(I^{2m+1} \circ J^n, J^{2t+1})$ is a Nijenhuis pair for $\Theta_{k_1,k_2,\ldots,k_s}$, for all $s \in \mathbb{N}$, where k_i stands either for I or for J, for every $i = 1,\ldots,s$.*

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