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FIBREWISE INJECTIVITY AND KOCK-ZÖBERLEIN MONADS

FRANCESCA CAGLIARI, MARIA MANUEL CLEMENTINO AND SANDRA MANTOVANI

Abstract: Using Escardó's characterization of injectivity via Kock-Zöberlein monads, we introduce suitable monads in comma categories of topological spaces that yield characterizations of fibrewise injectivity in topological T0-spaces, with respect to the class of embeddings, and of dense, of flat and of completely flat embeddings. Characterizations, in the category of topological spaces, of injective maps with respect to the same classes of embeddings follow easily from the results obtained for T0-spaces. Moreover, it is shown that, together with the corresponding embeddings, injective continuous maps form a weak factorization system in the category of topological (T0-)spaces and continuous maps.

Keywords: fibrewise injectivity, Kock-Zöberlein monad, way-below relation.
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Introduction

Filter convergence has proved to be very useful in the study of several classes of continuous maps, such as exponentiable maps, biquotient and triquotient maps, and effective descent maps (as outlined in [7, 8]). In this paper we show that filter convergence can be useful also to study continuous maps injective with respect to classes of embeddings.

While the injective topological spaces with respect to the class of embeddings are known to be the continuous lattices for about forty years [18], a topological characterization of continuous maps which are injective with respect to the class of embeddings was missing. Indeed, there were several approaches to the problem in the last decade (cf. [20, 3, 4, 6]), but none of them produced an internal characterization of injectivity for continuous maps. In [12] Escardó and Flagg, making use of different filter monads and their common property of being Kock-Zöberlein, produced a list of examples of classes of embeddings and their corresponding injective $T0$-spaces. Exploring common features of these filter monads, recently D. Hofmann [13]

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obtained a common characterization of their algebras and therefore, using Escardó's results, of spaces injective with respect to embeddings, dense embeddings, flat embeddings and completely flat embeddings in the category $\mathbf{Top}_0$ of topological T0-spaces and continuous maps. Each of these filter monads $\mathbb{T}$ induces the so-called $\mathbb{T}$-way-below relation, which is the key ingredient to obtain the characterizations.

In this paper we extend Hofmann's approach to the fibrewise case. Indeed, we carry the filter monads into the comma category $\mathbf{Top}_0 \downarrow Z$ so that they inherit the Kock-Zöberlein property, hence Escardó's result applies. Introducing a fibrewise $\mathbb{T}$-way-below relation, we characterize in $\mathbf{Top}_0$ continuous maps injective with respect to $\mathbb{T}$-embeddings as fibrewise sober, fibrewise $\mathbb{T}$-core-compact and fibrewise $\mathbb{T}$-stable. (In case $\mathbb{T}$ is the prime filter monad, the fibrewise $\mathbb{T}$-way-below relation coincides with the fibrewise way-below relation introduced by Richter [16] in order to characterize exponentiability for maps.) These characterizations, together with the properties of the reflection of $\mathbf{Top}_0$ in the category $\mathbf{Top}$ of topological spaces, give corresponding characterizations of injective continuous maps in $\mathbf{Top}$, with respect to embeddings, dense embeddings, flat embeddings and completely flat embeddings, extending results from [5]. Finally, from the description of injective continuous maps as algebras for a monad, we deduce that $\mathbb{T}$-embeddings and injective continuous maps with respect to $\mathbb{T}$-embeddings form a weak factorization system in $\mathbf{Top}_0$ that can be naturally extended to $\mathbf{Top}$.

1. Filters, Kock-Zöberlein monads and injectivity

Throughout this section we will be working in the category $\mathbf{Top}_0$ of topological $T0$-spaces and continuous maps. In each $T0$-space X we consider the order defined by

$$x \leq y \text{ if } y \in \overline{\{x\}},$$

for any $x, y \in X$, that is, $\leq$ is the dual of the *specialization order*. Continuous maps are monotone, with respect to this order, so that $\mathbf{Top}_0$ becomes an enriched category over the category $\mathbf{PoSet}$ of posets and monotone maps. As in [12, 13] we consider the *filter monad* $\mathbb{F} = (F, \eta, \mu)$ on $\mathbf{Top}_0$, where, for every $T0$-space X, with topology $\mathcal{O}X$, and any continuous map $f : X \to Y$ between $T0$-spaces, FX is the set of filters on $\mathcal{O}X$ endowed with the topology generated by the sets

$$U^\sharp := \{\varphi \in FX \,|\, U \in \varphi\}, \text{ for } U \in \mathcal{O}X, \text{ and}$$

$$Ff : FX \to FY,\ \varphi \mapsto Ff(\varphi) := \{V \in \mathcal{O}Y \mid f^{-1}(V) \in \varphi\}.$$

The natural transformations $\eta : \mathrm{Id}_{\mathbf{Top}_0} \to F$ and $\mu : FF \to F$ are defined, for every space X, by:

$$\eta_X : X \to FX,\ x \mapsto \mathcal{O}(x) = \{U \in \mathcal{O}X \mid x \in U\},$$

$$\mu_X : FFX \to FX,\ \Phi \mapsto \{U \in \mathcal{O}X \mid U^\sharp \in \Phi\}.$$

It is easy to check that the order in the $T0$-space FX is given by

$$\varphi \leq \psi \text{ if } \psi \subseteq \varphi,$$

and that F is a monotone functor, so that $\mathbb{F}$ is a monad on the poset-enriched category $\mathbf{Top}_0$. (For information on enriched categories see [14].)

Together with the filter monad we will consider some of its submonads:

(1) the *monad of proper filters*
(2) the *monad of prime filters*
(3) the *monad of completely prime filters.*

We recall that a filter φ is *proper* if $\emptyset \notin \varphi$, it is *prime* if it is inaccessible by finite joins, and it is *completely prime* if it is inaccessible by arbitrary joins. Since the filter $\mathcal{O}(x)$ is completely prime for every x, and $\mu_X(\Phi)$ is proper (prime, completely prime respectively) whenever Φ is, the unit and the multiplication of these monads are (co)restrictions of the unit and multiplication of the filter monad. As in [13], we will denote these monads by $\mathbb{F}_\alpha$, where $\alpha = 0, 1, \omega, \Omega$(=all ordinals), meaning that it is the monad of *filters φ unreachable by α* (or *α-unreachable*), that is, for any family $(U_i)_{i\in I}$ in $\mathcal{O}X$, if $\sharp I < \alpha$, from $\bigcup_{i\in I} U_i \in \varphi$ it follows that $U_j \in \varphi$ for some $j \in I$. Then $\mathbb{F}_0$ is the filter monad $\mathbb{F}$, $\mathbb{F}_1$ is the proper filter monad, $\mathbb{F}_\omega$ is the prime filter monad, and $\mathbb{F}_\Omega$ is the completely prime filter monad.

Next we will see that these monads are lax idempotent, or of Kock-Zöberlein type, as shown by Escardó-Flagg [12]. First we recall that a monad $\mathbb{T} = (T, \eta, \mu)$ on a poset-enriched category $\mathbf{C}$ is said to be *lax idempotent*, or a *Kock-Zöberlein monad*, or simply a *KZ-monad*, if it satisfies one of the following equivalent conditions (see [15] for details):

(i) *for every object X of $\mathbf{C}$, $T\eta_X \leq \eta_{TX}$;*
(ii) *for every object X of $\mathbf{C}$, $T\eta_X \dashv \mu_X$;*
(iii) *for every object X of $\mathbf{C}$, $\mu_X \dashv \eta_{TX}$;*
(iv) *for every object X of $\mathbf{C}$, a $\mathbf{C}$-morphism $l : TX \to X$ is the structure morphism of a $\mathbb{T}$-algebra if, and only if, $l \dashv \eta_X$ with $l \cdot \eta_X = 1_X$.*

Proposition 1.1. *For $\alpha = 0, 1, \omega, \Omega$, the monad $\mathbb{F}_\alpha$ is of Kock-Zöberlein type.*

Proof: We will prove that condition (i) above holds. Let $\mathbb{T} = \mathbb{F}_\alpha$, let X be a $T0$-space and $\varphi \in TX$. To show that $\eta_{TX}(\varphi) \subseteq T\eta_X(\varphi)$, we only need to check that, for every $U \in \mathcal{O}X$, $U^\sharp \in T\eta_X(\varphi)$ whenever $U^\sharp \in \eta_{TX}(\varphi)$, since $\{U^\sharp \,|\, U \in \mathcal{O}X\}$ is a basis for $\mathcal{O}TX$: If $U^\sharp \in \eta_{TX}(\varphi) = \mathcal{O}(\varphi)$, that is, if $\varphi \in U^\sharp$, then $U \in \varphi$ by definition of $U^\sharp$. Since $\eta_X^{-1}(U^\sharp) = U \in \varphi$, we conclude that $U^\sharp \in T\eta_X(\varphi)$ as claimed. ∎

Escardó [11] established an interesting link between KZ-monads and injectivity, proving that injective objects with respect to a class of embeddings $\mathcal{H}$ may be identified as $\mathbb{T}$-algebras whenever $\mathcal{H}$ can be described as the class of T-embeddings for a given KZ-monad $\mathbb{T}$, as we explain in the sequel.

First we recall that, given a class of embeddings $\mathcal{H}$ on a category $\mathbf{C}$, a $\mathbf{C}$-object Z is said to be *injective with respect to* $\mathcal{H}$ if, for every $h : X \to Y$ in $\mathcal{H}$, and every $\mathbf{C}$-morphism $f : X \to Z$, there is a $\mathbf{C}$-morphism $\overline{f} : Y \to Z$ making the following diagram commutative (no unicity is assumed here):

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "\overline{f}"] \\ & Z & \end{tikzcd}$$

Following [11, 12], given a monad $\mathbb{T}$ on a poset-enriched category $\mathbf{C}$, a morphism $h : X \to Y$ is said to be a *T-embedding* if $Th : TX \to TY$ has a right adjoint left-inverse, that is, if there exists $T^*h : TY \to TX$ such that $Th \cdot T^*h \leq 1_{TY}$ and $T^*h \cdot Th = 1_{TX}$.

Theorem 1.2. [11] *For a KZ-monad $\mathbb{T}$ on $\mathbf{C}$, and an object X of $\mathbf{C}$, the following conditions are equivalent:*

(i) *X is injective with respect to T-embeddings;*
(ii) *X has a $\mathbb{T}$-algebra structure.*

What are T-embeddings when $\mathbb{T}$ is one of the filter monads we consider in this paper? As shown in [12], they are well-known classes of topological embeddings. Indeed, for any continuous map $h : X \to Y$ between $T0$-spaces, consider the frame map

$$h^{-1} : \mathcal{O}Y \to \mathcal{O}X$$

and its right adjoint

$$h_* : \mathcal{O}X \to \mathcal{O}Y,\ U \mapsto \bigvee\{V \in \mathcal{O}Y \,|\, h^{-1}(V) \subseteq U\}.$$

In case h is an embedding,

$$h_*(U) = \bigvee\{V \in \mathcal{O}Y \,|\, V \cap X = U\} = \max\{V \in \mathcal{O}Y \,|\, V \cap X = U\}.$$

Proposition 1.3. [12] *Let $h : X \to Y$ be a continuous map between $T0$-spaces, and $\alpha = 0,\ 1,\ \omega$ or Ω. Then the following conditions are equivalent:*

(i) *h is an $\mathbb{F}_\alpha$-embedding;*
(ii) *h is an embedding and h_* preserves α-joins (that is, joins indexed by sets with cardinality less than α).*

Proof: (i) $\Rightarrow$ (ii): Let $\mathbb{T} = \mathbb{F}_\alpha$, and assume that there exists $T^*h : TY \to TX$ such that $Th \cdot T^*h \geq 1_{TY}$ and $T^*h \cdot Th = 1_{TX}$. Then in the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "h"'] & TX \arrow[d, "Th"] \\ Y \arrow[r, "\eta_Y"] & TY \end{tikzcd}$$

Th, η_X and η_Y are embeddings, and so h is an embedding as well. Further, consider the map $r : \mathcal{O}X \to \mathcal{O}Y$ defined by the composition

$$\mathcal{O}X \xrightarrow{(\ )^\sharp} \mathcal{O}TX \xrightarrow{(T^*h)^{-1}} \mathcal{O}TY \xrightarrow{\eta_Y^{-1}} \mathcal{O}Y.$$

As frame maps, η_Y^{-1} and $(T^*h)^{-1}$ preserve joins, and the map $(\ )^\sharp$ preserves α-joins: for an index set I with $\sharp I < \alpha$, $(U_i)_{i\in I}$ a family of open subsets of X, and $\varphi \in TX$,

$$\begin{array}{rcl} \varphi \in (\bigcup_{i\in I} U_i)^\sharp & \Leftrightarrow & \bigcup_{i\in I} U_i \in \varphi \\ & \Rightarrow & \exists j \in I\ :\ U_j \in \varphi \quad (\varphi \text{ is } \alpha\text{-unreachable}) \\ & \Leftrightarrow & \varphi \in \bigcup_{i\in I} U_i^\sharp. \end{array}$$

Moreover,

$$\begin{array}{rcll} h^{-1} \cdot r(U) & = & h^{-1} \cdot \eta_Y^{-1} \cdot (T^*h)^{-1}(U^\sharp) & \\ & = & \eta_X^{-1} \cdot Th^{-1} \cdot (T^*h)^{-1}(U^\sharp) & (\eta \text{ is a natural transformation}) \\ & = & \eta_X^{-1}(U^\sharp) & (T^*h \cdot Th = 1_{TX}) \\ & = & U, & \end{array}$$

for every $U \in \mathcal{O}X$, and, for every $V \in \mathcal{O}Y$,

$$\begin{array}{rcll} r \cdot h^{-1}(V) & = & \eta_Y^{-1} \cdot (T^*h)^{-1}(h^{-1}(V))^\sharp & \\ & = & \eta_Y^{-1} \cdot (T^*h)^{-1} \cdot (Th)^{-1}(V^\sharp) & \\ & \supseteq & \eta_Y^{-1}(V^\sharp) & (Th \cdot T^*h \leq 1_{TY}) \\ & = & V, & \end{array}$$

therefore $h^{-1} \dashv r$, and so $h_* = r$ preserves α-joins.

(ii) $\Rightarrow$ (i): Assume that h is an embedding and $h_* : \mathcal{O}X \to \mathcal{O}Y$ preserves α-joins. Define, for every $\psi \in TY$,

$$T^*h(\psi) := \{U \in \mathcal{O}X \mid h_*(U) \in \psi\}.$$

First we remark that $T^*h(\psi) \in TX$, that is, it is α-unreachable: for any set I with $\sharp I < \alpha$, and any family $(U_i)_{i\in I}$ in $\mathcal{O}X$,

$$\begin{array}{rcl} \bigcup_{i\in I} U_i \in T^*h(\psi) & \Leftrightarrow & h_*(\bigcup_{i\in I} U_i) = \bigcup_{i\in I} h_*(U_i) \in \psi \\ & \Leftrightarrow & \exists j \in I \ : \ h_*(U_j) \in \psi \\ & \Leftrightarrow & \exists j \in I \ : \ U_j \in T^*h(\psi). \end{array}$$

Now it is easy to check that $Th \cdot T^*h \leq 1_{TY}$ and $T^*h \cdot Th = 1_{TX}$. ∎

Note that preservation of 0-joins is trivially satisfied, while preservation of 1-joins means preservation of the empty set, so that $h : X \to Y$ is a $\mathbb{F}_1$-embedding if, and only if, it is a dense embedding. Embeddings $h : X \to Y$ with h_* preserving ω-joins (i.e. finite joins) are *flat embeddings*, while embeddings h with h_* preserving Ω-joins (i.e. arbitrary joins) are *completely flat embeddings* [12]. From now on we will denote by $\mathcal{H}_\alpha$ the class of $\mathbb{F}_\alpha$-embeddings, for $\alpha = 0, 1, \omega, \Omega$.

Using the known characterizations of $\mathbb{F}_\alpha$-algebras – Day [10] and Wyler [22] for $\mathbb{F}_0$, Wyler [23] for $\mathbb{F}_1$-algebras, Simmons [19] and Wyler [22] for $\mathbb{F}_\omega$-algebras (see [12] for details) – one gets:

Corollary 1.4. [12] *Let X be a $T0$-space. Then:*

(1) *X is injective with respect to embeddings if, and only if, it is a continuous lattice;*
(2) *X is injective with respect to dense embeddings if, and only if, it is a continuous Scott domain;*
(3) *X is injective with respect to flat embeddings if, and only if, it is a stably compact space;*
(4) *X is injective with respect to completely flat embeddings if, and only if, it is a sober space.*

In [13] D. Hofmann unified these characterizations, making use of a *way-below relation* on $\mathbf{Top}_0$ *relative to a filter monad* $\mathbb{T}$: for a $T0$-space X, and $U, V \in \mathcal{O}X$,

$$V \ll^{\mathbb{T}} U \;:\Leftrightarrow\; (\forall \varphi \in TX)\; (V \in \varphi \;\Rightarrow\; \lim \varphi \cap U \neq \emptyset),$$

where $\lim \varphi$ is the set of limit points of the filter φ. (When $\mathbb{T}$ is the monad $\mathbb{F}_\omega$ of prime filters, $\ll^{\mathbb{T}}$ is the well-known way-below relation.) A $T0$-space X is said to be $\mathbb{T}$-*core-compact* if

$$(\forall x \in X)\; (\forall U \in \mathcal{O}(x))\; (\exists V \in \mathcal{O}(x)) \;:\; V \ll^{\mathbb{T}} U.$$

And X is $\mathbb{T}$-*stable* if, for any finite set J and J-indexed families $(U_i)_i$ and $(V_i)_i$ on $\mathcal{O}X$, with $V_i \ll^{\mathbb{T}} U_i$ for every $i \in J$, one has $\cap_{i\in J} V_i \ll^{\mathbb{T}} \cap_{i \in J} U_i$. We remark that X is $\mathbb{F}_\omega$-core-compact if, and only if, it is core-compact, while X is $\mathbb{F}_\omega$-stable if it is compact and stable in the sense of [19].

Theorem 1.5. [13] *For $\alpha = 0, 1, \omega, \Omega$, and a $T0$-space X, the following conditions are equivalent:*

(i) *X is injective with respect to $\mathcal{H}_\alpha$-embeddings;*
(ii) *Every $\varphi \in TX$ has a least limit point in X and X is $\mathbb{F}_\alpha$-core-compact.*
(iii) *X is sober, $\mathbb{F}_\alpha$-core-compact and $\mathbb{F}_\alpha$-stable.*

2. Fibrewise filter monads

Throughout this section $\mathbb{T}$ denotes one of the four filter monads we described previously. In order to study injectivity for continuous maps instead of spaces, that is fibrewise injectivity, we will define fibrewise filter monads in the sliced category $\mathbf{Top}_0 \downarrow Z$, for any $T0$-space Z. First we remark that this category is again poset-enriched, when equipped with the order inherited from the order of $\mathbf{Top}_0$.

For each object $f : X \to Z$ of $\mathbf{Top}_0 \downarrow Z$, let $\hat{T}f : \hat{T}X \to Z$, where $\hat{T}X$ is the subspace of $Z \times TX$ defined by

$$\hat{T}X = \{(z, \varphi) \,|\, Tf(\varphi) \leq \eta_Z(z)\} = \{(z, \varphi) \,|\, (\forall W \in \mathcal{O}(z))\; f^{-1}(W) \in \varphi\}$$

and $\hat{T}f(z, \varphi) = z$. The map $\hat{T}f$ is continuous, as a restriction of a product projection into Z, as well as the map

$$\hat{\eta}_f : X \to \hat{T}X,\; x \mapsto (f(x), \mathcal{O}(x))$$

since its compositions with each of the product projections $\hat{T}f:\hat{T}X\to Z$ and $\pi_f:\hat{T}X\to TX$ are continuous:

$$\begin{tikzcd}
X \arrow[rr, "\eta_X"] \arrow[dr, "\hat{\eta}_f"] \arrow[dd, "f"'] & & TX \arrow[dd, "Tf"] \\
& \hat{T}X \arrow[ur, "\pi_f"'] \arrow[dl, "\hat{T}f"] \arrow[dr, phantom, "\geq"] & \\
Z \arrow[rr, "\eta_Z"] & & TZ
\end{tikzcd}$$

For any morphism $h:f\to g$ in $\mathbf{Top}_0\downarrow Z$, that is, for any continuous map $h:X\to Y$ between $T0$-spaces such that the triangle

$$\begin{tikzcd}
X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\
& Z &
\end{tikzcd}$$

commutes, we define $\hat{T}h:\hat{T}X\to\hat{T}Y$ by

$$\hat{T}h(z,\varphi):=(z,Th(\varphi)),$$

for every $(z,\varphi)\in\hat{T}X$, which is a continuous map. Moreover, $\hat{T}h:\hat{T}f\to\hat{T}g$ since $\hat{T}g\cdot\hat{T}h(z,\varphi)=z=\hat{T}f(z,\varphi)$. This way we have defined an endofunctor

$$\hat{T}:\mathbf{Top}_0\downarrow Z\to\mathbf{Top}_0\downarrow Z.$$

It is straightforward to check that $\hat{\eta}_f:f\to\hat{T}f$ is a morphism in $\mathbf{Top}_0\downarrow Z$, and that

$$\hat{\eta}=(\hat{\eta}_f)_{f\in\mathrm{Ob}\mathbf{Top}_0\downarrow Z}$$

defines a natural transformation $\hat{\eta}:\mathrm{Id}_{\mathbf{Top}_0\downarrow Z}\to\hat{T}$: for any $h:f\to g$ in $\mathbf{Top}_0\downarrow Z$, in the diagram

$$\begin{tikzcd}
X \arrow[r, "\hat{\eta}_f"] \arrow[d, "h"'] & \hat{T}X \arrow[d, "\hat{T}h"] \\
Y \arrow[r, "\hat{\eta}_g"] & \hat{T}Y
\end{tikzcd}$$

$$\begin{aligned}
(\hat{T}h\cdot\hat{\eta}_f)(x) &= \hat{T}h(f(x),\mathcal{O}(x))=(f(x),Th(\mathcal{O}(x))) \\
&= (g(h(x)),\mathcal{O}(h(x)))=\hat{\eta}_g(h(x))=(\hat{\eta}_g\cdot h)(x),
\end{aligned}$$

for every $x \in X$. Moreover, the multiplication μ of the monad $\mathbb{T}$ defines a multiplication $\hat{\mu} = (\hat{\mu}_f)_f$ for $\hat{\mathbb{T}}$:

$$\begin{tikzcd} \hat{T}\hat{T}X \arrow[rr, "\hat{\mu}_f"] \arrow[dr, "\hat{T}\hat{T}f"'] & & \hat{T}X \arrow[dl, "\hat{T}f"] \\ & Y & \end{tikzcd}, \quad (z, \Phi) \mapsto (z, \mu_X \cdot T\pi_f(\Phi)).$$

Again, $\hat{\mu}_f$ is continuous because it is continuous componentwise, and both $\hat{T}\hat{T}X$ and $\hat{T}X$ have product topologies. For each $h : f \to g$, the diagram

$$\begin{tikzcd} \hat{T}\hat{T}X \arrow[r, "\hat{\mu}_f"] \arrow[d, "\hat{T}\hat{T}h"'] & \hat{T}X \arrow[d, "\hat{T}h"] \\ \hat{T}\hat{T}Y \arrow[r, "\hat{\mu}_g"] & \hat{T}Y \end{tikzcd}$$

commutes: for every $(z, \Phi) \in \hat{T}\hat{T}X$, to show that the following equality holds

$$(\hat{T}h \cdot \hat{\mu}_f)(z, \Phi) = (\hat{\mu}_g \cdot \hat{T}\hat{T}h)(z, \Phi),$$

that is, $(z, Th \cdot \mu_X \cdot T\pi_f(\Phi)) = (z, \mu_Y \cdot T\pi_g \cdot T\hat{T}h(\Phi))$, we proceed as follows:

$$\begin{array}{rcll} Th \cdot \mu_X \cdot T\pi_f & = & \mu_Y \cdot TTh \cdot T\pi_f & (\mu \text{ is a natural transformation}) \\ & = & \mu_Y \cdot T(Th \cdot \pi_f) & \\ & = & \mu_Y \cdot T(\pi_g \cdot \hat{T}h) & (Th \cdot \pi_f = \pi_g \cdot \hat{T}h) \\ & = & \mu_Y \cdot T\pi_g \cdot T\hat{T}h. & \end{array}$$

Theorem 2.1. *For $\mathbb{T} = \mathbb{F}_0$, $\mathbb{F}_1$, $\mathbb{F}_\omega$ or $\mathbb{F}_\Omega$, and for each $T0$-space Z, the triple $\hat{\mathbb{T}} = (\hat{T}, \hat{\eta}, \hat{\mu})$ is a Kock-Zöberlein monad on $\mathbf{Top}_0 \downarrow Z$.*

Proof: We start by proving that $\hat{\mathbb{T}}$ is a monad. For each object $f : X \to Z$ in $\mathbf{Top}_0 \downarrow Z$, the proof of the commutativity of the diagram

$$\begin{tikzcd} \hat{T}X \arrow[r, "\hat{\eta}_{\hat{T}f}"] \arrow[dr, "1_{\hat{T}X}"'] & \hat{T}\hat{T}X \arrow[d, "\hat{\mu}_f"'] & \hat{T}X \arrow[l, "T\hat{\eta}_f"'] \arrow[dl, "1_{\hat{T}X}"] \\ & \hat{T}X & \end{tikzcd}$$

needs some calculations:

$$\begin{array}{rcll}
\hat{\mu}_f\cdot\hat{\eta}_{\hat{T}f}(z,\varphi) & = & \hat{\mu}_f(z,\eta_{\hat{T}X}(z,\varphi)) & (\text{because } \hat{T}f(z,\Phi)=z)\\
& = & (z,\mu_X\cdot T\pi_f\cdot\eta_{\hat{T}X}(z,\varphi)) & (\text{by definition of } \hat{\mu})\\
& = & (z,\mu_X\cdot\eta_{TX}\cdot\pi_f(z,\varphi)) & (\eta \text{ is a natural transformation})\\
& = & (z,\pi_f(z,\varphi))=(z,\varphi),\ \text{and} &
\end{array}$$

$$\begin{array}{rcll}
\hat{\mu}_f\cdot\hat{T}\hat{\eta}_f(z,\varphi) & = & \hat{\mu}_f(z,T\hat{\eta}_f(\varphi)) & (\text{by definition of } \hat{T})\\
& = & (z,\mu_X\cdot T\pi_f\cdot T\hat{\eta}_f(\varphi)) & \\
& = & (z,\mu_X\cdot T\eta_X(\varphi))=(z,\varphi). &
\end{array}$$

Next, we need to show that the following diagram

$$\begin{tikzcd}
\hat{T}\hat{T}\hat{T}X \arrow[r, "\hat{\mu}_{\hat{T}f}"] \arrow[d, "\hat{T}\hat{\mu}_f"'] & \hat{T}\hat{T}X \arrow[d, "\hat{\mu}_f"] \\
\hat{T}\hat{T}X \arrow[r, "\hat{\mu}_f"] & \hat{T}X
\end{tikzcd}$$

is also commutative, that is,

$$\begin{array}{rcl}
\hat{\mu}_f\cdot\hat{\mu}_{\hat{T}f}(z,\Theta) & = & \hat{\mu}_f(z,\mu_{\hat{T}X}\cdot T\pi_{\hat{T}f}(\Theta))\\
& = & (z,\mu_X\cdot T\pi_f\cdot\mu_{\hat{T}X}\cdot T\pi_{\hat{T}f}(\Theta))\\
& = & (z,\mu_X\cdot T\pi_f\cdot T\hat{\mu}_f(\Theta))=\hat{\mu}_f\cdot\hat{T}\hat{\mu}_f(z,\Theta),
\end{array}$$

for every $(z,\Theta)\in\hat{T}\hat{T}\hat{T}X$, which follows from the commutativity of the following diagram

$$\begin{tikzcd}
T\hat{T}\hat{T}X \arrow[rr, "T\hat{\mu}_f"] \arrow[d, "T\pi_{\hat{T}f}"'] & & T\hat{T}X \arrow[d, "T\pi_f"] \\
TT\hat{T}X \arrow[r, "TT\pi_f"] \arrow[d, "\mu_{eTX}"'] & TTTX \arrow[r, "T\mu_X"] \arrow[d, "\mu_{TX}"] & TTX \arrow[d, "\mu_X"] \\
T\hat{T}X \arrow[r, "T\pi_f"] & TTX \arrow[r, "\mu_X"] & TX;
\end{tikzcd}$$

the bottom squares are naturality diagrams for μ, while commutativity of the upper rectangle follows from an easy calculation showing that $\mu_X\cdot T\pi_f\cdot\pi_{\hat{T}f}=\pi_f\cdot\hat{\mu}_f$.

It remains to be shown that $\mathbb{T}$ is a KZ-monad, that is, for each $f:X\to Z$,

$$\hat{T}\hat{\eta}_f\leq\hat{\eta}_{\hat{T}f}.$$

Recall that

$$\hat{\eta}_{\hat{T}f}:\hat{T}X\to\hat{T}\hat{T}X,\ (z,\varphi)\mapsto(z,\eta_{\hat{T}X}(z,\varphi))=(z,\{\mathcal{U}\in\mathcal{O}\hat{T}X\,|\,(z,\varphi)\in\mathcal{U}\})$$

and

$$\hat{T}\hat{\eta}_f : \hat{T}X \to \hat{T}\hat{T}X,\ (z,\varphi) \mapsto (z, T\hat{\eta}_f(\varphi)) = (z, \{\mathcal{U} \in \mathcal{O}\hat{T}X \mid \hat{\eta}_f^{-1}(\mathcal{U}) \in \varphi\},$$

and we want to show that

$$\eta_{\hat{T}X}(z,\varphi) \subseteq T\hat{\eta}_f(\varphi).$$

Let $\mathcal{U} \in \eta_{\hat{T}X}(z,\varphi)$. Since $\hat{T}X$ is a subspace of the product $Z \times TX$, $\mathcal{O}\hat{T}X$ is generated by

$$W \times V^\sharp := \{(w,\psi) \in \hat{T}X \mid w \in W \text{ and } V \in \psi\},$$

for $W \in \mathcal{O}Z$ and $V \in \mathcal{O}X$. Therefore there exist $W \in \mathcal{O}Z$ and $V \in \mathcal{O}X$ such that $(z,\varphi) \in W \times V^\sharp \subseteq \mathcal{U}$. Now

$$\hat{\eta}_f^{-1}(W \times V^\sharp) = \{x \in X \mid f(x) \in W \text{ and } x \in V\} = f^{-1}(W) \cap V.$$

The proof is finished once we show that $f^{-1}(W) \cap V \in \varphi$, since then also $\mathcal{U} \in \varphi$. That $V \in \varphi$ follows from $(z,\varphi) \in W \times V^\sharp$, while $f^{-1}(W) \in \varphi$ follows from $(z,\varphi) \in \hat{T}X \ \Leftrightarrow\ \mathcal{O}(z) \subseteq Tf(\varphi)$. ∎

3. Fibrewise injectivity

In this section we will use Escardó's result and the fibrewise filter monads in $\mathbf{Top}_0$ to characterize injective continuous maps with respect to some classes of embeddings. We recall that a continuous map $f : X \to Z$ is *injective with respect to a class $\mathcal{H}$ of embeddings*, or *$\mathcal{H}$-injective*, if, for every $h : A \to B$, $u : A \to X$ and $v : B \to Y$ such that $f \cdot u = v \cdot h$, there exists a lifting $d : B \to X$ making the following diagram commutative:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "h"'] & X \arrow[d, "f"] \\ B \arrow[r, "v"'] \arrow[ur, "d"] & Z \end{tikzcd} \tag{$*$}$$

The first step in our study of fibrewise injectivity concerns $\hat{T}$-embeddings.

Theorem 3.1. *Let $\alpha = 0, 1, \omega, \Omega$. For a continuous map $h : f \to g$ in $\mathbf{Top}_0 \downarrow Z$, the following conditions are equivalent:*

(i) *h is a F_α-embedding;*
(ii) *h is a $\hat{F}_\alpha$-embedding;*
(iii) *h is an embedding and h_* preserves α-joins.*

Proof: In Proposition 1.3 we have showed that (i) $\Leftrightarrow$ (iii).

Let $\mathbb{T} = \mathbb{F}_\alpha$. To prove that (ii) $\Rightarrow$ (iii) we use the same arguments as in the proof of (i) $\Rightarrow$ (ii) of Proposition 1.3, replacing η by $\hat{\eta}$. Indeed, observing that in the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\hat{\eta}_X"] \arrow[d, "h"'] & \hat{T}X \arrow[d, "\hat{T}h"] \\ Y \arrow[r, "\hat{\eta}_Y"] & \hat{T}Y, \end{tikzcd}$$

$\hat{\eta}_X$, $\hat{T}h$, $\hat{\eta}_T$ are embeddings, we conclude that h is an embedding as well; and we define the map $r : \mathcal{O}X \to \mathcal{O}Y$ as the composition

$$\mathcal{O}X \xrightarrow{Z\times(\ )^\sharp} \mathcal{O}\hat{T}X \xrightarrow{(\hat{T}^*h)^{-1}} \mathcal{O}\hat{T}Y \xrightarrow{\hat{\eta}_Y^{-1}} \mathcal{O}Y,$$

which preserves α-joins because each of the composites does. An easy calculation shows that $h^{-1} \dashv r$, and so $h_* = r$ preserves α-joins.

To prove that (iii) $\Rightarrow$ (ii) we just observe that the continuous map $T^*h : TY \to TX$ defined in the proof of (ii) $\Rightarrow$ (i) in Proposition 1.3 induces a continuous map

$$\hat{T}^*h : \hat{T}Y \to \hat{T}X,\ (z,\psi) \mapsto (z, T^*h(\psi)),$$

which clearly satisfies $\hat{T}h \cdot \hat{T}^*h \leq 1_{\hat{T}Y}$ and $\hat{T}^*h \cdot \hat{T}h = 1_{\hat{T}X}$. ■

Now we recall, from the equivalent conditions characterizing KZ-monads, that a continuous map $f : X \to Z$ has a $\hat{T}$-algebra structure if, and only if, there exists a continuous map $l : \hat{T}X \to X$ such that $f \cdot l = \hat{T}f$, $l \cdot \hat{\eta}_f = 1_X$ and $\hat{\eta}_f \cdot l \geq 1_{\hat{T}X}$. We introduce the *fibrewise way-below relation with respect to a monad* $\mathbb{T}$, where $\mathbb{T}$ is one of the four filter monads we defined: for each $U, V \in \mathcal{O}X$ and $W \in \mathcal{O}Z$,

$$V \ll_W^{\mathbb{T}} U :\Leftrightarrow\ (\forall (z,\varphi) \in W \times V^\sharp \subseteq \hat{T}X)\ \lim \varphi \cap f^{-1}(z) \cap U \neq \emptyset.$$

In case $\mathbb{T}$ is the prime filter monad, this relation coincides with Richter's fibrewise way-below relation, stated in terms of tied filters (see [16]).

Theorem 3.2. *For $\alpha = 0, 1, \omega, \Omega$, the following conditions are equivalent, for a continuous map $f : X \to Z$:*

(i) *$f : X \to Z$ is $\mathcal{H}_\alpha$-injective,*

(ii) *$f : X \to Z$ has an $\hat{F}_\alpha$-algebra structure,*

(iii) (a) *for every* $(z,\varphi)\in\hat{F}_\alpha X$*, there is a least limit point of* φ *in* $f^{-1}(z)$*,*
(b) $(\forall x\in X)\,(\forall U\in\mathcal{O}(x))\,(\exists W\in\mathcal{O}(f(x)))\,(\exists V\in\mathcal{O}(x))\,:\,V\ll^{\mathbb{F}_\alpha}_W U.$

Proof: Let $\mathbb{T}=\mathbb{F}_\alpha$. (i) $\Leftrightarrow$ (ii) follows from Escardó's Theorem (see Theorem 1.2) since T-embeddings and $\hat{T}$-embeddings coincide, and $\hat{T}$ is a KZ-monad.

(ii) $\Rightarrow$ (iii): Assume the existence of the continuous map $l:\hat{T}X\to X$ as above. By adjointness,

$$(\forall x\in X)\ (\forall (z,\varphi)\in\hat{T}X)\ l(z,\varphi)\le x\ \Leftrightarrow\ (z,\varphi)\le (f(x),\mathcal{O}(x)).$$

Hence $l(z,\varphi)$ is the least limit point of φ in $f^{-1}(z)$, and so condition (a) holds. To prove (b) we use continuity of l: let $x\in X$ and $U\in\mathcal{O}(x)$. Since $(f(x),\mathcal{O}(x))\in l^{-1}(U)$, which is open, $(f(x),\mathcal{O}(x))$ belongs to a basic open subset $W\times V^\sharp$ of $\hat{T}X$ contained in $l^{-1}(U)$. But this means exactly that $V\ll^{\mathbb{T}}_W U$ and (b) follows.

(iii) $\Rightarrow$ (ii): Conversely, assume that (a) and (b) hold. Define, for each $(z,\varphi)\in\hat{T}X$, $l(z,\varphi)$ as the least limit point of φ in $f^{-1}(z)$, whose existence is guaranteed by (a). To check continuity of $l:\hat{T}X\to X$, let $(z,\varphi)\in\hat{T}X$, $x=l(z,\varphi)$ and $U\in\mathcal{O}(x)$. Let W and V be as in (b). Then $(z,\varphi)\in W\times V^\sharp$, and, for any $(w,\psi)\in W\times V^\sharp$, by (b) there exists $x'\in f^{-1}(w)\cap U$ such that ψ converges to x'. Then $l(w,\psi)=x''\le x'$ since both are limit points of ψ contained in $f^{-1}(w)$, and so $x''\in f^{-1}(w)\cap U$ since open subsets are down-sets. Therefore $l(W\times V^\sharp)\subseteq U$ and so the map l is continuous. ∎

As pointed out in the proof above, condition (a) guarantees the existence of a monotone map $l:TX\to X$ left adjoint to η_X. Its continuity is guaranteed by condition (b), which is both a generalization of $\mathbb{T}$-core-compactness in the sense of Hofmann [13] and fibrewise core-compactness in the sense of Richter [16]. Therefore we call a continuous map $f:X\to Z$ satisfying the condition

$$(\forall x\in X)\ (\forall U\in\mathcal{O}(x))\ (\exists W\in\mathcal{O}(f(x)))\ (\exists V\in\mathcal{O}(x))\ :\ V\ll^{\mathbb{T}}_W U$$

fibrewise $\mathbb{T}$*-core-compact*. We observe that, for the prime filter monad, our notion of fibrewise $\mathbb{F}_\omega$-core-compactness coincides with Richter's notion of fibrewise core-compactness [16], and so this notion captures exponentiability exactly as in the object case.

Lemma 3.3. *For a continuous map* $f:X\to Z$*, the following conditions are equivalent:*

(i) f *is fibrewise* $\mathbb{T}$*-core-compact;*

(ii) *for each open subset* U *of* X*,*

$$U = \bigcup \{V \cap f^{-1}(W) \,|\, V \in \mathcal{O}X,\, W \in \mathcal{O}Z,\, V \cap f^{-1}(W) \ll_W^{\mathbb{T}} U \cap f^{-1}(W)\}.$$

Proof: (i) $\Rightarrow$ (ii): Let $U \in \mathcal{O}X$ and $x \in U$. By (i) there is $V \in \mathcal{O}(x)$ and $W \in \mathcal{O}(f(x))$ such that $V \ll_W^{\mathbb{T}} U$. If $(z, \varphi) \in W \times (V \cap f^{-1}(W))^\sharp \subseteq W \times V^\sharp$, then

$$\lim \varphi \cap f^{-1}(z) \cap (U \cap f^{-1}(W)) = \lim \varphi \cap f^{-1}(z) \cap U \neq \emptyset,$$

and therefore $V \cap f^{-1}(W) \ll_W^{\mathbb{T}} U \cap f^{-1}(W)$.

(ii) $\Rightarrow$ (i): Let $x \in X$ and $U \in \mathcal{O}(x)$. By (ii) there exist $V \in \mathcal{O}X$ and $W \in \mathcal{O}Z$ such that $x \in V \cap f^{-1}(W)$ and $V \cap f^{-1}(W) \ll_W^{\mathbb{T}} U \cap f^{-1}(W)$. Then, for $\widetilde{V} = V \cap f^{-1}(W)$, one has $x \in \widetilde{V}$, $f(x) \in W$ and $\widetilde{V} \ll_W^{\mathbb{T}} U \cap f^{-1}(W) \subseteq U$, hence $\widetilde{V} \ll_W^{\mathbb{T}} U$. ∎

In case $\mathbb{T}$ is the filter monad $\mathbb{F}_0$, one can conclude in addition that:

Proposition 3.4. *Every fibrewise* $\mathbb{F}_0$*-core compact continuous map* $f : X \to Z$ *is open.*

Proof: Let $U \in \mathcal{O}X$; $\mathbb{F}_0$-core compactness of f gives that:

$$(\forall x \in U)\ (\exists W_x \in \mathcal{O}(f(x)))\ (\exists V_x \in \mathcal{O}(x))\ :\ V_x \ll_{W_x}^{\mathbb{F}_0} U.$$

In order to conclude that $f(U)$ is open, now we will show that, for each $x \in U$, $W_x \subseteq f(U)$. Let $z \in W_x$. Consider the filter φ on $\mathcal{O}X$ generated by

$$\{f^{-1}(A) \cap V_x \ :\ A \in \mathcal{O}(z)\}.$$

Then $V_x \in \varphi$ and, for each $A \in \mathcal{O}(z)$, $f^{-1}(A) \in \varphi$, that is $F_0 f(\varphi) \to z$. Hence $(z, \varphi) \in W_x \times V^\sharp \subseteq \hat{F}X$ and so, by definition of the fibrewise way-below relation, we have

$$\lim \varphi \cap f^{-1}(z) \cap U \neq \emptyset;$$

this implies that $z \in f(U)$ and the conclusion follows. ∎

Now, in order to characterize fibrewise injectivity, we need to consider a fibrewise version of $\mathbb{T}$-stability. We say that $f : X \to Z$ is *fibrewise* $\mathbb{T}$*-stable* if, for every finite index set J and families $(U_i)_{i \in J}$, $(V_i)_{i \in J}$ of open subsets of X, and $(W_i)_{i \in J}$ of open subsets of Z, with $W := \bigcap_{i \in J} W_i$,

$$(\forall i \in J)\ V_i \ll_{W_i}^{\mathbb{T}} U_i \ \Rightarrow\ \bigcap_{i \in J} V_i \ll_W^{\mathbb{T}} \bigcap_{i \in J} U_i. \qquad (\dagger)$$

We remark that, for $f : X \to Z$, in case the finite set J is empty, condition (†) reads as $X \ll_Z^{\mathbb{T}} X$, which means that f is fibrewise compact, that is f is proper à la Bourbaki [2]. The condition of fibrewise $\mathbb{T}$-stability for non-empty families reduces to:

$$\begin{array}{l}(\forall U_1, U_2, V_1, V_2 \in \mathcal{O}X)\ (\forall W \in \mathcal{O}Z)\\ V_1 \ll_W^{\mathbb{T}} U_1 \ \&\ V_2 \ll_W^{\mathbb{T}} U_2 \ \Rightarrow V_1 \cap V_2 \ll_W^{\mathbb{T}} U_1 \cap U_2.\end{array}$$

Lemma 3.5. *For $\alpha = 0, 1, \omega, \Omega$, if $f : X \to Z$ is fibrewise $\mathbb{F}_\alpha$-core-compact, then the following conditions are equivalent:*

(i) *f is fibrewise $\mathbb{F}_\alpha$-stable;*

(ii) $(\forall (z, \varphi) \in \hat{F}_\alpha X)$ $\lim \varphi \cap f^{-1}(z)$ *is an irreducible set;*

(iii) $(\forall (z, \varphi) \in \hat{F}_\alpha X)$ $\overline{\lim \varphi \cap f^{-1}(z)}$ *is an irreducible set.*

Proof: (ii) $\Leftrightarrow$ (iii) follows from the fact that $A \subseteq X$ is irreducible if, and only if, $\overline{A}$ is irreducible.

(i) $\Rightarrow$ (ii): Let $\mathbb{T} = \mathbb{F}_\alpha$ and $(z, \varphi) \in \hat{T}X$. Then, from $X \ll_Z^{\mathbb{T}} X$ we conclude that $A := \lim \varphi \cap f^{-1}(z) \neq \emptyset$. Let U_1, U_2 be open subsets of X such that $U_1 \cap A \neq \emptyset \neq U_2 \cap A$. Let $x_1 \in U_1 \cap A$ and $x_2 \in U_2 \cap A$. Since f is fibrewise $\mathbb{T}$-core-compact, there exist $W_1 \in \mathcal{O}(f(x_1)) = \mathcal{O}(z)$, $V_1 \in \mathcal{O}(x_1)$, $W_2 \in \mathcal{O}(z)$ and $V_2 \in \mathcal{O}(x_2)$ such that

$$V_1 \ll_{W_1}^{\mathbb{T}} U_1 \text{ and } V_2 \ll_{W_2}^{\mathbb{T}} U_2.$$

By (i) it follows that $V_1 \cap V_2 \ll_{W_1 \cap W_2}^{\mathbb{T}} U_1 \cap U_2$. Since $(z, \varphi) \in (W_1 \cap W_2) \times (V_1 \times V_2)^\sharp$, there is a limit point of φ in $f^{-1}(z) \cap (U_1 \cap U_2)$, that is $U_1 \cap U_2 \cap A \neq \emptyset$. Hence A is irreducible as claimed.

(ii) $\Rightarrow$ (i): If $(z, \varphi) \in \hat{T}X$, irreducibility of $\lim \varphi \cap f^{-1}(z)$ implies that it is non-empty, hence $X \ll_Z^{\mathbb{T}} X$. Now let $V_1 \ll_W^{\mathbb{T}} U_1$ and $V_2 \ll_W^{\mathbb{T}} U_2$, with $U_1, U_2, V_1, V_2 \in \mathcal{O}X$ and $W \in \mathcal{O}Z$. Whenever $(z, \varphi) \in W \times (V_1 \times V_2)^\sharp$, $(z, \varphi) \in W \times V_1^\sharp$ and $(z, \varphi) \in W \times V_2^\sharp$, and therefore $\lim \varphi \cap f^{-1}(z) \cap U_1 \neq \emptyset \neq \lim \varphi \cap f^{-1}(z) \cap U_2$. Irreducibility of $\lim \varphi \cap f^{-1}(z)$ gives then $\lim \varphi \cap f^{-1}(z) \cap (U_1 \cap U_2) \neq \emptyset$, i.e. $V_1 \cap V_2 \ll_W^{\mathbb{T}} U_1 \cap U_2$. ∎

As in [17], we say that a continuous map $f : X \to Z$ is *fibrewise sober* if, for every irreducible closed subset A of X and $z \in Z$,

$$\overline{f(A)} = \overline{\{z\}} \ \Rightarrow\ (\exists!\, x \in f^{-1}(z)) \ :\ A = \overline{\{x\}}.$$

We point out that in [17] Richter and Vauth proved that this notion inherits fibrewisely several properties of sober spaces. Here we add another interesting property of fibrewise sober continuous maps, namely they are the algebras for the fibrewise completely prime filter monad, or, equivalently, the injective continuous maps with respect to completely flat embeddings. This result follows from the characterization theorem below and the fact that, as for spaces, for the completely prime filter monad $\mathbb{F}_\Omega$, fibrewise $\mathbb{F}_\Omega$-core-compactness and fibrewise $\mathbb{F}_\Omega$-stability trivialize.

We are now able to prove our characterization theorem:

Theorem 3.6. *For $\alpha = 0, 1, \omega, \Omega$, and for a continuous map $f : X \to Z$, the following conditions are equivalent:*

(i) *f is $\mathcal{H}_\alpha$-injective;*
(ii) *f is fibrewise sober, fibrewise $\mathbb{F}_\alpha$-core-compact and fibrewise $\mathbb{F}_\alpha$-stable.*

Proof: Let $\mathbb{T} = \mathbb{F}_\alpha$. (ii) $\Rightarrow$ (i): First we want to define $l : \hat{T}X \to X$. Let $(z, \varphi) \in \hat{T}X$. Since $A := \overline{\lim\varphi \cap f^{-1}(z)}$ is irreducible, and $\overline{f(A)} = \overline{\{z\}}$, by fibrewise sobriety of f we obtain a unique $x \in f^{-1}(z)$ such that $A = \overline{\{x\}}$. Therefore $x \in \overline{\lim\varphi \cap f^{-1}(z)} \subseteq \overline{\lim\varphi} = \lim\varphi$. Define $l(z, \varphi) := x$. By construction, it is the least limit point of φ in $f^{-1}(z)$, and it is straightforward to check that such l satisfies the (in)equalities needed. As shown in Theorem 3.2, continuity of l follows from fibrewise $\mathbb{T}$-core-compactness of f.

(i) $\Rightarrow$ (ii): Existence of l guarantees that $X \ll^{\mathbb{T}}_Z X$, while continuity of l implies that f is fibrewise $\mathbb{T}$-core-compact, as it was shown in 3.2. To show that, for $(z, \varphi) \in \hat{T}X$, $A := \lim\varphi \cap f^{-1}(z)$ is irreducible, let U_1, U_2 be open subsets of X such that $U_1 \cap A \neq \emptyset \neq U_2 \cap A$. Then $l(z, \varphi) \leq x_1 \in U_1 \cap A$ and $l(z, \varphi) \leq x_2 \in U_2 \cap A$. Hence $l(z, \varphi) \in U_1 \cap U_2 \cap A$ and so A is irreducible. To show that f is fibrewise sober, let B be an irreducible closed subset of X such that $\overline{f(B)} = \overline{\{z\}}$. Let $\varphi := \{U \in \mathcal{O}X \,|\, U \cap B \neq \emptyset\}$. Then φ is a completely prime filter on $\mathcal{O}X$, hence it belongs to TX, and, moreover, $f(\varphi)$ converges to z, that is $(z, \varphi) \in \hat{T}X$. Then $l(z, \varphi) \in \lim\varphi \cap f^{-1}(z) \subseteq \lim\varphi = B$, and so $\overline{\{l(z, \varphi)\}} \subseteq B$. Let us check that every $x' \in B$ belongs to $\overline{\{l(z, \varphi)\}}$, so that we may conclude that $B = \overline{\{l(z, \varphi)\}}$. If $x' \in B$, then $(f(x'), \varphi) \in \hat{T}X$ and $f(x') \in f(B) \subseteq \overline{\{z\}}$, that is $z \leq f(x')$. Therefore $l(z, \varphi) \leq l(f(x'), \varphi) \leq x'$ and the conclusion follows, since this argument also shows unicity of $l(z, \varphi)$, as the least point in B. ∎

For $\mathbb{T} = \mathbb{F}_0$, this Theorem, together with Proposition 3.4, gives:

Theorem 3.7. *For a continuous map $f : X \to Z$, the following conditions are equivalent:*

(i) *f is injective with respect to embeddings;*

(ii) (a) *f is open;*

(b) *f is fibrewise sober;*

(c) $(\forall x \in X)\ (\forall U \in \mathcal{O}(x))\ (\exists V \in \mathcal{O}(x))\ :\ V \ll^{\mathbb{F}_0}_{f(V)} U$*;*

(d) $(\forall \varphi \in FX)\ (\forall z \in Z)\ \lim \varphi \cap f^{-1}(z)$ *is either empty or irreducible.*

Final Remarks.

(1) The fibrewise construction of the monad $\hat{\mathbb{T}}$ based on a KZ-monad $\mathbb{T}$ can be realized in a more general **PoSet**-enriched category **C**. The proof that, in our setting, the KZ property and the class of T-embeddings are preserved is quite particular, and does not foresee a general treatment of the problem. The setting described in [9] seems to be suitable for this study, since it encompasses quite natural KZ-monads that can be lifted fibrewisely, preserving the class of embeddings. We do not know however under which conditions the KZ-property is preserved.

(2) Theorem 3.6 presents possible candidates for fibrewise notions of continuous lattice and Scott continuous domain, that seem to deserve further study.

4. Fibrewise injectivity in Top

In this section we show that the results on fibrewise injectivity obtained in $\mathbf{Top}_0$ can be extended to the category of topological spaces. First we recall that $\mathbf{Top}_0$ is an epireflective subcategory of **Top**, and we list the properties of the unit $r = (r_X : X \to RX)_{X \in \mathbf{Top}}$ of the adjunction $(\mathbf{Top} \xrightarrow{R} \mathbf{Top}_0) \dashv (\mathbf{Top}_0 \longrightarrow \mathbf{Top})$:

Lemma 4.1. *For a topological space X, let $r_X : X \to RX$ be its $T0$-reflection. Then:*

(1) *for $x, x' \in X$, $r_X(x) = r_X(x')$ if and only if $\overline{\{x\}} = \overline{\{x'\}}$;*

(2) *r_X is an open surjection;*

(3) *r_X is an initial map, that is, for $U \subseteq X$, $U \in \mathcal{O}X$ if and only if $U = r_X^{-1}(V)$ for some $V \in \mathcal{O}(RX)$;*

(4) *r_X is injective with respect to embeddings;*

(5) *for each $U \in \mathcal{O}X$, $V \in \mathcal{O}(RX)$, $r_X^{-1}(r_X(U)) = U$ and $r_X(r_X^{-1}(V)) = V$;*

(6) $(r_X)_* = r_X : \mathcal{O}X \to \mathcal{O}(RX)$, $U \mapsto r_X(U)$.

Proof: (1), (2) and (3) are well-known properties of the $T0$-reflection.

(4) Given a commutative diagram in **Top**

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "h"'] & X \arrow[d, "r_X"] \\ B \arrow[r, "v"] & RX \end{tikzcd}$$

if h is an embedding then there is a map $d : B \to X$ such that $r_X \cdot d = v$ and $d \cdot h = u$, since in **Set** every surjection is injective with respect to any injective map. Due to the initiality of r_X and the fact that $r_X \cdot d = v$ is a continuous map, we can conclude that d is continuous.

(5) Since r_X is surjective, $r_X(r_X^{-1}(V)) = V$ for every open subset V of RX. If $U \in \mathcal{O}X$, then $U = r_X^{-1}(V)$ for some $V \in \mathcal{O}(RX)$, and therefore $r_X^{-1}(r_X(U)) = r_X^{-1}(V) = U$.

(6) From (5) it follows easily that $r_X : \mathcal{O}X \to \mathcal{O}(RX)$, $U \mapsto r_X(U)$, is the right adjoint to r_X^{-1}, that is $(r_X)_* = r_X$. ■

Remark 4.2. The argument used in the proof of (4) shows that, in **Top**, every initial surjective continuous map is injective with respect to embeddings; in particular every surjective continuous map with indiscrete domain is injective with respect to embeddings.

Now it is easy to prove that:

Proposition 4.3. *For a continuous map $h : A \to B$ in* **Top**, *and for $\alpha = 0, 1, \omega, \Omega$, the following conditions are equivalent:*

(i) *h is an embedding and h_* preserves α-joins;*
(ii) *Rh is an F_α-embedding.*

Proof: The proof that h is an embedding if and only if Rh is an embedding can be found in [21]. Moreover, since the diagram

$$\begin{tikzcd} A \arrow[r, "r_A"] \arrow[d, "h"'] & RA \arrow[d, "Rh"] \\ B \arrow[r, "r_B"] & RB \end{tikzcd}$$

is commutative, $(r_B)_* \cdot h_* = (r_B \cdot h)_* = (Rh \cdot r_A)_* = (Rh)_* \cdot (r_A)_*$, and so

$$h_* = r_B^{-1} \cdot (r_B)_* \cdot h_* = r_B^{-1} \cdot (Rh)_* \cdot (r_A)_* = r_B^{-1} \cdot (Rh)_* \cdot r_A, \text{ and}$$

$$(Rh)_* = (Rh)_* \cdot (r_A)_* \cdot r_A^{-1} = (r_B)_* \cdot h_* \cdot r_A^{-1} = r_B \cdot h_* \cdot r_A^{-1}.$$

Since, for every topological space X, $r_X^{-1} : \mathcal{O}(RX) \to X$ and $r_X : \mathcal{O}X \to \mathcal{O}(RX)$ preserve joins, we conclude that h_* preserves α-joins if and only if $(Rh)_*$ preserves α-joins. ∎

From now on $\widetilde{\mathcal{H}}_\alpha$ is the class of embeddings h in **Top** such that h_* preserves α-joins, for $\alpha = 0, 1, \omega, \Omega$, and $\mathcal{H}_\alpha = \widetilde{\mathcal{H}}_\alpha \cap \mathbf{Top}_0$ the class of F_α-embeddings.

Corollary 4.4. *If $f : X \to Z$ is a continuous map between $T0$-spaces, then the following conditions are equivalent:*

(i) *f is injective with respect to $\widetilde{\mathcal{H}}_\alpha$ in* **Top**,
(ii) *f is injective with respect to $\mathcal{H}_\alpha$ in $\mathbf{Top}_0$.*

Proof: (i) ⇒ (ii) is immediate. (ii) ⇒ (i): If the diagram

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "h"'] & X \arrow[d, "f"] \\ B \arrow[r, "v"] & Z \end{tikzcd}$$

is commutative with $h \in \widetilde{\mathcal{H}}_\alpha$, then in the following diagram Rh belongs to $\mathcal{H}_\alpha$

$$\begin{tikzcd} A \arrow[rr, "u"] \arrow[dr, "r_A"] \arrow[dd, "h"'] & & X \arrow[dd, "f"] \\ & RA \arrow[ur, "u'"'] \arrow[dd, "Rh"] & \\ B \arrow[rr, "v"] \arrow[dr, "r_B"'] & & Z \\ & RB \arrow[ur, "v'"'] & \end{tikzcd}$$

and so there is $d' : RB \to X$ such that $f \cdot d' = v'$ and $d' \cdot Rh = u'$. The continuous map $d = d' \cdot r_B$ is the required lifting. ∎

In [5] Cagliari and Mantovani showed that:

Theorem 4.5. *For a continuous map $f : X \to Z$ in* **Top**, *the following assertions are equivalent:*

(i) *f is injective with respect to embeddings;*
(ii) *Rf is injective with respect to embeddings in $\mathbf{Top}_0$ and, for every indiscrete component C in X, $f(C)$ is an indiscrete component in Z.*

(Here by indiscrete component it is meant a maximal indiscrete subspace.)

The techniques of [5] can be used to characterize $\widetilde{\mathcal{H}}_\alpha$-injective maps, in **Top**, using our characterizations of $\mathcal{H}_\alpha$-injective maps, in $\mathbf{Top}_0$. For that we need the following well-known properties of $\mathcal{H}$-injective maps, for a class $\mathcal{H}$ of embeddings.

Lemma 4.6. *For $f : X \to Z$ and $g : Z \to Y$,*

(1) *If $f, g \in \mathrm{Inj}(\mathcal{H})$, then $g \cdot f \in \mathrm{Inj}(\mathcal{H})$.*
(2) *If $g \cdot f \in \mathrm{Inj}(\mathcal{H})$, then $g \in \mathrm{Inj}(\mathcal{H})$.*

Theorem 4.7. *Let $\alpha = 0, 1, \omega, \Omega$. For a continuous map $f : X \to Z$ in* **Top**, *the following conditions are equivalent:*

(i) *f is $\widetilde{\mathcal{H}}_\alpha$-injective;*
(ii) *Rf is $\mathcal{H}_\alpha$-injective and, for every indiscrete component C in X, $f(C)$ is an indiscrete component in Z.*

Proof: (i) $\Rightarrow$ (ii): To conclude that Rf is $\widetilde{\mathcal{H}}_\alpha$-injective, hence $\mathcal{H}_\alpha$-injective by Corollary 4.4, we observe that from $Rf \cdot r_X = r_Z \cdot f$ it follows that $Rf \cdot r_X$ is $\widetilde{\mathcal{H}}_\alpha$-injective, as a composition of the $\widetilde{\mathcal{H}}_\alpha$-injective maps r_Z and f, and then also Rf is $\widetilde{\mathcal{H}}_\alpha$-injective, by Lemma 4.6. To check the condition on images of indiscrete components we note that, for any indiscrete subspace C of X, denoting by D the largest indiscrete subspace of Z containing $f(C)$, the embedding $C \to f^{-1}(D)$ belongs to $\widetilde{\mathcal{H}}_\alpha$, and the existence of a lifting for the diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ f^{-1}(D) \arrow[r, "f|"'] & Z \end{tikzcd}$$

implies that $C = f^{-1}(D)$, and therefore $D = f(C)$ because f is surjective.

(ii) $\Rightarrow$ (i): Given a commutative diagram

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "h"'] & X \arrow[d, "f"] \\ B \arrow[r, "v"] & Z \end{tikzcd}$$

with $h \in \widetilde{\mathcal{H}}_\alpha$, there is a lifting $d : B \to RX$ for the diagram

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "h"'] & X \arrow[r, "r_X"] & RX \arrow[d, "Rf"] \\ B \arrow[r, "v"] & Z \arrow[r, "r_Z"] & RZ \end{tikzcd}$$

If we build now the required lifting $t : B \to X$ in each fibre of r_X so that $r_X \cdot t = d$ we know immediately that t is continuous, by initiality of r_X. In order to do that we first observe that, for each $x \in RX$, $C_x = r_X^{-1}(x)$ is a maximal indiscrete subspace of X, hence $f(C_x)$ is also a maximal indiscrete subspace of Z, and so $f(C_x) = r_Z^{-1}(Rf(x))$. The (co)restriction $f| : C_x \to f(C_x)$ of f is surjective with indiscrete domain, hence it is injective with respect to embeddings. Therefore the commutative diagram

$$\begin{tikzcd} u^{-1}(C_x) \arrow[r, "u|"] \arrow[d, "h|"'] & C_x \arrow[d, "f|"] \\ d^{-1}(x) \arrow[r, "v|"] & f(C_x) \end{tikzcd}$$

has a lifting $t_x : d^{-1}(x) \to C_x$. The map

$$\begin{array}{rcl} t : B = \dot{\bigcup}\, d^{-1}(x) & \to & X = \dot{\bigcup}\, C_x \\ z & \mapsto & t_{d(z)}(z) \end{array}$$

satisfies the equalities $f \cdot t = v$ and $t \cdot h = u$. It is continuous because $r_X \cdot t = d$ and r_X is initial. ■

Since a topological (T0-)space X is $\widetilde{\mathcal{H}}_\alpha$-injective in **Top** ($\mathbf{Top}_0$) if and only if the continuous map $f : X \to \{*\}$ is $\widetilde{\mathcal{H}}_\alpha$-injective in **Top** ($\mathbf{Top}_0$), and $f : X \to \{*\}$ sends trivially indiscrete components to indiscrete components, from the Theorem it follows that:

Corollary 4.8. *Let $\alpha = 0, 1, \omega, \Omega$. For a topological space X, the following conditions are equivalent:*

(i) *X is $\widetilde{\mathcal{H}}_\alpha$-injective;*
(ii) *RX is $\mathcal{H}_\alpha$-injective.*

5. Fibrewise injectivity and weak factorization systems

In this final section we show that these classes of embeddings and respective injective continuous maps form weak factorization systems. First we remark that Diagram $(*)$ in the definition of $\mathcal{H}$-injective morphism states that every morphism in $\mathcal{H}$ has the *left lifting property* with respect to every $\mathcal{H}$-injective morphism (see [1] for details). We recall that a *weak factorization system* in a category is a pair $(\mathcal{L}, \mathcal{R})$ of morphism classes such that:

(1) every morphism has a factorization as an $\mathcal{L}$-morphism followed by an $\mathcal{R}$-morphism, and

(2) every morphism in $\mathcal{L}$ has the left lifting property with respect to every morphism in $\mathcal{R}$.

When $\mathcal{L} = \mathcal{H}$ is a class of embeddings, and $\mathcal{R} = \mathrm{Inj}(\mathcal{H})$ is the class of $\mathcal{H}$-injective morphisms, property (2) follows directly from the definition of injectivity. Moreover, in case $\mathcal{H} = \mathcal{H}_\alpha$, for every continuous map $f : X \to Z$ in $\mathbf{Top}_0$, condition (1) follows from the factorization:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\hat{\eta}_f"'] & & Z \\ & \hat{F}_\alpha X \arrow[ur, "\hat{F}_\alpha f"'] & \end{tikzcd}$$

- $\hat{\eta}_f$ is a $\hat{F}_\alpha$-embedding ($=F_\alpha$-embedding), with $\hat{F}^*_\alpha(\hat{\eta}_f) = \hat{\mu}_f$;
- $\hat{F}_\alpha f$ is injective with respect to F_α-embeddings, since it has a (free) $\hat{F}_\alpha$-algebra structure.

Therefore we can conclude that:

Theorem 5.1. *For $\alpha = 0, 1, \omega, \Omega$, the pair $(\mathcal{H}_\alpha, \mathrm{Inj}(\mathcal{H}_\alpha))$ is a weak factorization system in $\mathbf{Top}_0$.*

Finally we remark that an analogous argument shows that from Escardó's Theorem (cf. Theorem 1.2) it follows that:

Theorem 5.2. $\mathbf{Top}_0$ *has enough $\mathcal{H}_\alpha$-injective objects, for $\alpha = 0, 1, \omega, \Omega$.*

To prove that $(\widetilde{\mathcal{H}}_\alpha, \mathrm{Inj}(\widetilde{\mathcal{H}}_\alpha))$, $\alpha = 0, 1, \omega, \Omega$, is a weak factorization system in $\mathbf{Top}$, we first recall from [1] that:

Proposition 5.3. *Let $\mathbf{C}$ be a category with finite products and $\mathcal{H}$ a left cancellable class of morphisms, containing all isomorphisms. Then the following conditions are equivalent:*

(i) $(\mathcal{H}, \mathrm{Inj}(\mathcal{H}))$ *is a weak factorization system;*
(ii) $\mathbf{C}$ *has enough $\mathcal{H}$-injective objects.*

It is well-known that, in $\mathbf{Top}$, the class $\widetilde{\mathcal{H}}_0$ of embeddings is left cancellable, that is, if $g \cdot f \in \widetilde{\mathcal{H}}_0$, then $f \in \widetilde{\mathcal{H}}_0$. It remains to be shown that:

Lemma 5.4. *For $\alpha = 1, \omega, \Omega$, the class $\widetilde{\mathcal{H}}_\alpha$ is left cancellable.*

Proof: We have to show that, given embeddings $f : X \to Z$ and $g : Z \to Y$, if $(g \cdot f)_*$ preserves α-joins, then also f_* preserves α-joins. Assume that $(g \cdot f)_*$ preserves α-joins. Since, for every $W \in \mathcal{O}Z$,

$$g^{-1}(g_*(W)) = g^{-1}(\bigvee\{V \in \mathcal{O}Y\,;\, g^{-1}(V) = W\}) = W,$$

one has, for every family $(U_i)_{i\in I}$ of open subsets of X indexed by a set I with cardinality less than α,

$$\begin{aligned} f_*(\textstyle\bigcup_i U_i) &= g^{-1}(g_*(f_*(\textstyle\bigcup_i U_i))) = g^{-1}(\textstyle\bigcup_i g_*(f_*(U_i))) \\ &= \textstyle\bigcup_i g^{-1}(g_*(f_*(U_i))) = \textstyle\bigcup_i f_*(U_i), \end{aligned}$$

that is, f_* preserves α-joins. ■

Proposition 5.5. *For $\alpha = 0, 1, \omega, \Omega$,* **Top** *has enough $\widetilde{\mathcal{H}}_\alpha$-injective objects.*

Proof: Given a topological space X and its T0-reflection $r_X : X \to RX$, any section of r_X is continuous, due to its initiality. Let $s : RX \to X$ be such a section. Since $\mathbf{Top}_0$ has enough $\mathcal{H}_\alpha$-injective objects, let $h : RX \to B$ be an $\mathcal{H}_\alpha$-embedding in an $\mathcal{H}_\alpha$-injective object B in $\mathbf{Top}_0$. The image, under the functor $R : \mathbf{Top} \to \mathbf{Top}_0$, of the pushout in **Top** of h along s

$$\begin{tikzcd} RX \arrow[r, "h"] \arrow[d, "s"'] & B \arrow[d, "s'"] \\ X \arrow[r, "k"] & Y \end{tikzcd}$$

is a pushout in $\mathbf{Top}_0$:

$$\begin{tikzcd} RX \arrow[r, "h"] \arrow[d, "Rs"'] & B \arrow[d, "Rs'"] \\ RX \arrow[r, "Rk"] & RY \end{tikzcd}$$

From $Rs = 1_{RX}$ we can conclude that Rs' is an isomorphism, hence Y is $\widetilde{\mathcal{H}}_\alpha$-injective; moreover, $Rk \cong h$ belongs to $\mathcal{H}_\alpha$, and so k belongs to $\widetilde{\mathcal{H}}_\alpha$. ■

Theorem 5.6. *For $\alpha = 0, 1, \omega, \Omega$, the pair $(\widetilde{\mathcal{H}}_\alpha, \mathrm{Inj}(\widetilde{\mathcal{H}}_\alpha))$ is a weak factorization system in* **Top**.

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Francesca Cagliari
Dipartimento di Matematica, Università degli Studi di Bologna, Piazza Porta San Donato, 5, 40127 Bologna, Italia
E-mail address: `cagliari@dm.unibo.it`

Maria Manuel Clementino
CMUC, Department of Mathematics, University of Coimbra, 3001-454 Coimbra, Portugal
E-mail address: `mmc@mat.uc.pt`

Sandra Mantovani
Dipartimento di Matematica, Università degli Studi di Milano, Via C. Saldini, 50, 20133 Milano, Italia
E-mail address: `sandra.mantovani@unimi.it`