Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 11–33

ON THE CHARACTERIZATION OF JÓNSSON-TARSKI AND OF SUBTRACTIVE VARIETIES

MARINO GRAN AND DIANA RODELO

Abstract: We investigate unital, subtractive and strongly unital regular categories with enough projectives and give characterizations of their projective covers. The categorical equation *strongly unital = unital + subtractive* is explored: this leads to proofs of their varietal characterizations in terms of the categorical properties of the corresponding algebraic theories.

Keywords: regular category, projective cover, unital category, subtractive category, Jónnson-Tarski variety.
AMS Subject Classification (2000): 08C05, 18A35, 18B99, 18E10.

Introduction

Over the past years, there have been several investigations devoted to the construction of "free" categories of a specific type, a classical instance of this being the free abelian category over a (pre)additive one [13, 2].

Abelian categories are, in particular, exact categories [3]: this naturally led to the study of the free exact category over a category with finite limits, that was first described in [7]. In particular, this result was useful to obtain a conceptual construction in two steps of the free abelian category mentioned above. It was later observed that, for the free exact completion, the assumption that the original category had finite *limits* was never fully used, and only the existence of finite *weak limits* was actually needed (recall that for a weak limit the uniqueness requirement in the definition of a limit is simply dropped). The exact completion over a category with weak finite limits [10], called *weakly lex* in the following, is general enough to cover many further examples, such as any algebraic category.

An interesting feature of free exact categories over weakly lex ones is that they always possess enough regular projectives, a property which is quite useful in homological algebra. One actually knows even more: an exact category is the exact completion of a weakly lex category if and only if it

Received November 11, 2011.
Research supported by the F.N.R.S. grant *Crédit aux chercheurs* 1.5.016.10F, and by FCT/Centro de Matemática da Universidade de Coimbra.

has enough regular projectives and, in this case, it is the completion of its full subcategory of regular projectives. More generally, any exact category with enough projectives $\mathbb{E}$ is the exact completion of any of its *projective covers*, that is of any full subcategory $\mathbb{C}$ of $\mathbb{E}$ whose objects are regular projectives and that has, moreover, the property that for any object A in $\mathbb{E}$ there is a regular epimorphism $\tilde{A}\twoheadrightarrow A$ with $\tilde{A}$ in $\mathbb{C}$. For example, any variety of universal algebras turns out to be the exact completion of its full subcategory of free algebras. Similar considerations also apply to the *free regular completion* over a weakly lex category (see [21] and references therein). Many results on abelian categories can be recovered by using these completions, and the proofs are general and simple (see [19]).

An important aspect in the study of these completions concerns the possibility of characterizing those categories that appear as *projective covers* of a free exact category. For instance, already in [13] it was shown that a category is equivalent to a projective cover of an abelian category if and only if it is preadditive with weak finite products and weak kernels. Similar characterizations have been later established for the projective covers of categories that are Mal'tsev [19], protomodular, semi-abelian [11], extensive [12], (locally) cartesian closed categories [9, 18] and toposes [17].

The aim of this work is to extend the above list, by giving characterizations of the projective covers of regular categories $\mathbb{E}$ which are unital, subtractive [14] and strongly unital [5]. The investigation of these specific properties is inspired from the "categorical equation"

$$\textit{strongly unital} = \textit{unital} + \textit{subtractive}.$$

When $\mathbb{E}$ is a variety of universal algebras, our results isolate those properties of the categories of free algebras that "force" $\mathbb{E}$ to be a unital, subtractive or strongly unital variety, respectively. As a consequence we obtain some new proofs of their varietal characterizations in terms of the categorical properties of the corresponding algebraic theories (in the sense of [16]).

1. Projective covers

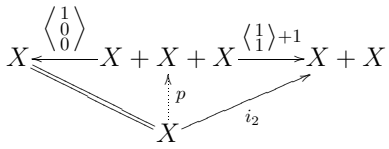
When $\mathbb{C}$ is a full subcategory of a category $\mathbb{E}$ one says that $\mathbb{C}$ is a *projective cover* of $\mathbb{E}$ if two conditions are satisfied: 1) any object in $\mathbb{C}$ is regular projective in $\mathbb{E}$; 2) for any object A in $\mathbb{E}$ there exists a $\mathbb{C}$-cover $\tilde{A}$ of A, that is an object $\tilde{A}$ in $\mathbb{C}$ and a regular epimorphism $\tilde{A}\twoheadrightarrow A$.

[15] B. Jónsson and A. Tarski, *Direct Decompositions of Finite Algebraic Systems*, Notre Dame Mathematical Lectures, Notre Dame, Indiana (1947)
[16] F. W. Lawvere, *Functorial Semantics of Algebraic Theories*, Ph.D. thesis, Columbia University (1963), Reprint in Theory and Applications of Categories **5** (2004) 1-121.
[17] M. Menni, *A characterization of the left exact categories whose exact completions are toposes*, J. Pure Appl. Algebra **177** (2003) 287-301.
[18] J. Rosicky, *Cartesian closed exact completions*, J. Pure Appl. Algebra **142** (1999) 261-270.
[19] J. Rosicky and E.M. Vitale, *Exact completions and representations in abelian categories*, Homology, Hom. Appl. **3** (2001) 453-466.
[20] A. Ursini, *On subtractive varieties*, I. Algebra Universalis **31** (1994) 204-222.
[21] E. Vitale, *Left covering functors*, PhD thesis, Université catholique de Louvain (1994).

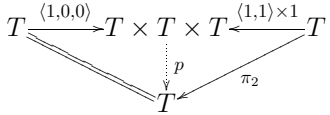
Marino Gran
Insitut de Mathématique Pure et Appliquées, Université Catholique de Louvain, B-1348 Louvain-la-Neuve, Belgium
E-mail address: Marino.Gran@uclouvain.be

Diana Rodelo
Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade do Algarve, Campus de Gambelas, 8005-139 Faro, Portugal;
Centre for Mathematics of the University of Coimbra, 3001-454 Coimbra, Portugal
E-mail address: drodelo@ualg.pt

3. $\Leftrightarrow$ 4. Via the duality between varieties of algebras and algebraic theories, the existence of a morphism $p : X \to X + X + X$ making the diagram



commutative in $\mathbb{C}$ (where $X = F(1)$) corresponds to the existence of a morphism $p : T \times T \times T \to T$ making the diagram



commutative in the algebraic theory $\mathbb{T}$ of $\mathbb{E}$. □

References

[1] J. Adámek and H. Porst, *Algebraic theories of quasivarieties*, J. Algebra, **208** (1998) 379-398.

[2] M. Adelman, *Abelian categories over additive ones*, J. Pure Appl. Algebra **3** (1973) 103-117.

[3] M. Barr, Exact Categories, in: *Lecture Notes in Math.* 236, Springer (1971) 1-120.

[4] F. Borceux and D. Bourn, *Mal'cev, protomodular, homological and semi-abelian categories*, Kluwer, 2004.

[5] D. Bourn, *Mal'cev categories and fibration of pointed objects*, Appl. Categ. Structures **4** (1996) 307-327.

[6] A. Carboni, J. Lambek, and M.C. Pedicchio, *Diagram chasing in Malcev categories*, J. Pure Appl. Algebra **69** (1991) 271-284.

[7] A. Carboni and R. Celia Magno, *The free exact category on a left exact one*, J. Austr. Math. Soc. Ser., A 33 (1982) 295-301.

[8] A. Carboni and M.C. Pedicchio, *A new proof of the Mal'cev theorem*, Rend. Circolo Mat. Palermo, S. II, Suppl. **64** (2000) 13-16.

[9] A. Carboni and G. Rosolini, *Locally cartesian closed exact completions*, J. Pure Appl. Algebra **154** (2000) 103-116.

[10] A. Carboni and E. Vitale, *Regular and exact completions*, J. Pure Appl. Algebra **125** (1998) 79-116.

[11] M. Gran, *Semi-abelian exact completions*, Homology, Hom. Appl. **4** (2002) 175-189.

[12] M. Gran and E.M. Vitale, *On the exact completion of the homotopy category*, Cah. Top. Geom. Diff. Categoriques **XXXIX-4** (1998) 287-297.

[13] P. Freyd, *Representations in abelian categories*, in: Proc. Conference on Categorical Algebra, La Jolla, 1965 (Springer, Berlin, 1966).

[14] Z. Janelidze, *Subtractive categories*, Appl. Categ. Structures **13** (2005) 343-350.

For example, if $\mathbb{E}$ is a (quasi)variety of universal algebras and $\mathbb{C}$ its full subcategory of free algebras, then $\mathbb{C}$ is a projective cover of the (regular) exact category $\mathbb{E}$.

Recall that a finitely complete category is called *regular* [3] when it has a pullback-stable (regular epimorphism, monomorphism) factorization system.

Remark 1.1. Let $\mathbb{C}$ be a projective cover of a category $\mathbb{E}$. As a consequence of Proposition 4 in [10], we have that $\mathbb{C}$ is *weakly lex* (i.e. it has weak finite limits) whenever $\mathbb{E}$ is weakly lex. For instance, if X, Y are objects in $\mathbb{C}$, $X \leftarrow E \to Y$ is a weak product in $\mathbb{E}$ and $\tilde{E} \twoheadrightarrow E$ a $\mathbb{C}$-cover of E, then $X \leftarrow \tilde{E} \to Y$ is a weak product in $\mathbb{C}$. On the other hand, if $\mathbb{E}$ is a regular category and $X \leftarrow C \to Y$ is a weak product in $\mathbb{C}$, then the factorization $C \twoheadrightarrow X \times Y$ is a regular epimorphism. Similar remarks apply also to all (weak) finite limits.

Note that, if $\mathbb{C}$ is a pointed projective cover of a category $\mathbb{E}$ with kernel pairs, then $\mathbb{E}$ is also pointed. Let us recall the argument: if P is the zero object of $\mathbb{C}$, then it must be the initial object of $\mathbb{E}$, since, for any object A of $\mathbb{E}$, we can consider the $\mathbb{C}$-cover $a : \tilde{A} \twoheadrightarrow A$ and the unique morphism $i : P \to \tilde{A}$ in $\mathbb{C}$; this gives a morphism $P \to A$. For the uniqueness, suppose that there exists another morphism $x : P \to A$. The fact that P is projective gives a morphism $j : P \to \tilde{A}$ such that $x = a \cdot j$, and $i = j$ since P is an initial object in $\mathbb{C}$. As for being a terminal object in $\mathbb{E}$, first we notice that there exists a unique morphism $t : \tilde{A} \to P$ in $\mathbb{C}$. Next, we consider the $\mathbb{C}$-cover $r : \tilde{R} \to R[a]$ of the kernel pair $(R[a], a_1, a_2)$ of a. Then, we have the equality $t \cdot a_1 \cdot r = t \cdot a_2 \cdot r$ of two morphisms onto the terminal object of $\mathbb{C}$. This gives a (unique) morphism $\pi : A \to P$ such that $\pi \cdot a = t$ since a is the coequalizer of its kernel pair in $\mathbb{E}$. The uniqueness of the morphism $A \to P$ comes from the fact that precomposition with the epimorphism a is the unique morphism t in $\mathbb{C}$.

In the next sections we shall give characterizations of the projective covers of regular categories which are unital, subtractive and strongly unital.

2. Unital categories

We begin this section by recalling the main properties of unital categories [5]. We then give a "weak" version of such categories in order to characterize the projective covers of regular unital categories. In the presence of finite

coproducts, we shall obtain a simple proof of the characterization of algebraic (quasi)varieties which are unital.

In a category with pullbacks, strong and extremal epimorphisms coincide. If, moreover, the category is regular, then the notions of regular, strong and extremal epimorphisms coincide.

To simplify notation, we write 1 (instead of 1_X) to denote the identity morphism on any object X and just 0 to denote the null morphism from an object X to an object Y, when the category is pointed.

Definition 2.1. ([5]) A category $\mathbb{E}$ is called *unital* when it is pointed, has finite limits and, for every pair of objects X, Y in $\mathbb{E}$, the pair of morphisms $X \xrightarrow{\langle 1,0\rangle} X \times Y \xleftarrow{\langle 0,1\rangle} Y$ is jointly strongly epimorphic.

Let $\mathbb{E}$ be a pointed category. A span $X \xleftarrow{f} Z \xrightarrow{g} Y$ in $\mathbb{E}$ is called *punctual* if there exist morphisms $s : X \to Z$ and $t : Y \to Z$ such that $f \cdot s = 1, g \cdot t = 1, f \cdot t = 0$ and $g \cdot s = 0$. For example, for any pair of objects X, Y in $\mathbb{E}$, we have a punctual span $X \underset{\langle 1,0\rangle}{\overset{\pi_1}{\leftrightarrows}} X \times Y \underset{\langle 0,1\rangle}{\overset{\pi_2}{\leftrightarrows}} Y$. Unital categories can be characterized through such spans as follows (see Theorem 1.2.12 of [4]):

Theorem 2.2. *A pointed category with finite limits $\mathbb{E}$ is unital if and only if for every punctual span $X \underset{s}{\overset{f}{\leftrightarrows}} Z \underset{t}{\overset{g}{\leftrightarrows}} Y$ in $\mathbb{E}$, the induced factorisation $\langle f, g\rangle : Z \twoheadrightarrow X \times Y$ is a strong epimorphism.*

We introduce the following definition, which is suggested by the observations made in Remark 1.1.

Definition 2.3. A category $\mathbb{C}$ is called *w-unital* when it is pointed, has weak finite limits and for every punctual span $X \underset{s}{\overset{f}{\leftrightarrows}} Z \underset{t}{\overset{g}{\leftrightarrows}} Y$, the span $X \xleftarrow{f} Z \xrightarrow{g} Y$ is a weak product.

It is easy to see that any finitely complete w-unital category is necessarily unital, since the factorization to the "true" product $\langle f, g\rangle : Z \twoheadrightarrow X \times Y$ is a split epimorphism, thus a strong epimorphism. Similar observations can be made also for subtractive and strongly unital categories, which shall be analysed in the next sections.

Recall that a pointed variety is strongly unital if and only if its corresponding theory contains a unique constant 0 and a ternary operation p such that $p(x, 0, 0) = x$ and $p(x, x, y) = y$ (Theorem 1.8.16 of [4]). This result is illustrated in the next theorem, that uses the w-strongly unital property. An object X equipped with a morphism $p : X \to X + X + X$ such that $\left\langle \begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix} \right\rangle \cdot p = 1$ and $(\langle \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \rangle + 1) \cdot p = i_2$ is called a *strongly unital coalgebra*.

Theorem 4.6. *Let $\mathbb{E}$ be an algebraic variety and $\mathbb{C}$ its full subcategory of free algebras. Then following statements are equivalent:*

1. *$\mathbb{E}$ is a strongly unital category;*
2. *$\mathbb{C}$ is a strongly w-unital category;*
3. *the free algebra $X = F(1)$ on one generator is a strongly unital coalgebra;*
4. *$\mathbb{E}$ is a pointed strongly unital variety.*

Proof: 1. $\Leftrightarrow$ 2. is given by Corollary 4.4.
2. $\Rightarrow$ 3. If $\mathbb{C}$ is w-strongly unital, then for the free algebra $X = F(1)$, the split right punctual span $X \underset{i_1}{\overset{\left\langle \begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix} \right\rangle}{\leftrightarrows}} X + X + X \underset{i_{2,3}}{\overset{\langle \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \rangle + 1}{\leftrightarrows}} X + X$ gives a weak product. Therefore, there exists a morphism $p : X \to X + X + X$ such that $\left\langle \begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix} \right\rangle \cdot p = 1$ and $(\langle \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \rangle + 1) \cdot p = i_2$; X is a strongly unital coalgebra.
3. $\Rightarrow$ 2. By using the same argument as in the implication 3. $\Rightarrow$ 2. of Theorem 2.6 we know that any free algebra A has a strongly unital coalgebra structure since the free algebra $X = F(1)$ has such a structure. Consider then free algebra morphisms $b, c : A \to Y$. There exists a morphism

$$A \xrightarrow{\ p\ } A + A + A \xrightarrow{\ b + \langle \begin{smallmatrix} b \\ c \end{smallmatrix} \rangle\ } Y + Y$$

such that $\langle \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \rangle \cdot (b + \langle \begin{smallmatrix} b \\ c \end{smallmatrix} \rangle) \cdot p = b$ and $\langle \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \rangle \cdot (b + \langle \begin{smallmatrix} b \\ c \end{smallmatrix} \rangle) \cdot p = c$. This proves that the span $Y \xleftarrow{\langle \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \rangle} Y + Y \xrightarrow{\langle \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \rangle} Y$ is a weak product for any $Y \in \mathbb{C}$, and $\mathbb{C}$ is w-strongly unital by Proposition 4.5.

Proof: Let $\mathbb{C}$ be a w-strongly unital category. Since any punctual span is a split right punctual span, it then determines a weak product, and the category $\mathbb{C}$ is w-unital. To prove that $\mathbb{C}$ is w-subtractive, consider a right punctual reflexive graph $X \xleftarrow{d} G \underset{t}{\overset{c}{\rightleftarrows}} X$. It is a split right punctual span, thus a weak product. Then, there exists a morphism $s : X \rightarrowtail G$ such that $d \cdot s = 1$ and $c \cdot s = 0$, which makes the reflexive graph also left punctual.

Conversely, suppose that $\mathbb{C}$ is w-unital and w-subtractive. Let $X \underset{s}{\overset{f}{\leftrightarrows}} Z \underset{t}{\overset{g}{\leftrightarrows}} Y$ be a split right punctual span and consider arbitrary morphisms $x : A \rightarrowtail X$ and $y : A \rightarrowtail Y$. Any weak pullback of $\langle f, g\rangle$ along $f \times g$

$$\begin{array}{ccc} G & \xrightarrow{\quad h \quad} & Z \\ {\scriptstyle \langle d,c\rangle}\downarrow & & \downarrow{\scriptstyle \langle f,g\rangle} \\ Z \times Z & \xrightarrow[f\times g]{} & X \times Y \end{array}$$

produces a right punctual *reflexive* graph $Z \xleftarrow{d} G \underset{\tau}{\overset{c}{\rightleftarrows}} Z$, where the reflexive morphism is $e = (\langle 1, 1\rangle, 1)$ and $\tau = (\langle 0, t\rangle, t)$. Then, it must also be left punctual and, consequently, it is a weak product. This gives a morphism $w : A \rightarrowtail G$ such that $d \cdot w = s \cdot x$ and $c \cdot w = t \cdot y$. Then, there exists a morphism $h \cdot w : A \rightarrowtail Z$ such that $f \cdot h \cdot w = x$ and $g \cdot h \cdot w = y$, proving that (f, g) is a weak product. □

Corollary 4.4. *Let $\mathbb{C}$ be a pointed projective cover of a regular category $\mathbb{E}$. Then $\mathbb{E}$ is strongly unital if and only if $\mathbb{C}$ is w-strongly unital.*

Proof: The proof follows from Proposition 4.3 and Proposition 3 in [14]. □

In a context that admits binary coproducts, (w-)strongly unital categories can also be characterized through a special kind of punctual span. For strongly unital categories, the characterization is similar to that of Proposition 1.2.18 in [4]. We omit the proof of the following result, which is similar to the one of Proposition 2.5.

Proposition 4.5. *A pointed weakly lex category with binary coproducts $\mathbb{C}$ is w-strongly unital if and only the span $X \xleftarrow{\langle 1 \atop 0\rangle} X + X \xrightarrow{\langle 0 \atop 1\rangle} X$ is a weak product, for every object X in $\mathbb{C}$.*

Proposition 2.4. *Let $\mathbb{C}$ be a pointed projective cover of a regular category $\mathbb{E}$. Then $\mathbb{E}$ is unital if and only if $\mathbb{C}$ is w-unital.*

Proof: Suppose that $\mathbb{E}$ is unital. Since $\mathbb{C}$ is a projective cover of a regular category $\mathbb{E}$, which has finite limits, then $\mathbb{C}$ has weak finite limits (Remark 1.1). Now, let $\tilde{X} \underset{\tilde{s}}{\overset{\tilde{f}}{\leftrightarrows}} \tilde{Z} \underset{\tilde{t}}{\overset{\tilde{g}}{\leftrightarrows}} \tilde{Y}$ be a punctual span in $\mathbb{C}$. Since it is also a punctual span in $\mathbb{E}$, which is unital, then $\langle \tilde{f}, \tilde{g}\rangle : \tilde{Z} \twoheadrightarrow \tilde{X} \times \tilde{Y}$ is a regular epimorphism in $\mathbb{E}$. In other words, it represents a $\mathbb{C}$-cover of the true product, so it is a weak product in $\mathbb{C}$ (Remark 1.1).
Conversely, $\mathbb{E}$ is pointed, because $\mathbb{C}$ is pointed (see Section 1), and it has finite limits. Let $X \underset{s}{\overset{f}{\leftrightarrows}} Z \underset{t}{\overset{g}{\leftrightarrows}} Y$ be a punctual span in $\mathbb{E}$. We consider the $\mathbb{C}$-covers $x : \tilde{X} \twoheadrightarrow X$ and $y : \tilde{Y} \twoheadrightarrow Y$, form the pullback U of $x \times y$ and $\langle f, g\rangle$ and take its $\mathbb{C}$-cover $\alpha : \tilde{Z} \twoheadrightarrow U$

$$\begin{array}{ccccc} \tilde{Z} & \xrightarrow{\quad \alpha \quad} & U & \xrightarrow{\quad u \quad} & Z \\ & {\scriptstyle \langle \tilde{f}=f'\cdot\alpha,\, \tilde{g}=g'\cdot\alpha\rangle}\searrow & {\scriptstyle \langle f', g'\rangle}\downarrow & \lrcorner & \downarrow{\scriptstyle \langle f,g\rangle} \\ & & \tilde{X} \times \tilde{Y} & \xrightarrow[x\times y]{} & X \times Y. \end{array}$$

Note that, $\tilde{X} \underset{s'}{\overset{f'}{\leftrightarrows}} U \underset{t'}{\overset{g'}{\leftrightarrows}} \tilde{Y}$ is a punctual span in $\mathbb{E}$, for $s' = (\langle 1, 0\rangle, s \cdot x)$ and $t' = (\langle 0, 1\rangle, t \cdot y)$. Since $\tilde{X}$ and $\tilde{Y}$ are projective and α is a regular epimorphism, then there exists morphisms $\tilde{s}$ and $\tilde{t}$ such that $s' = \alpha \cdot \tilde{s}$ and $t' = \alpha \cdot \tilde{t}$. We obtain a punctual span $\tilde{X} \underset{\tilde{s}}{\overset{\tilde{f}}{\leftrightarrows}} \tilde{Z} \underset{\tilde{t}}{\overset{\tilde{g}}{\leftrightarrows}} \tilde{Y}$ in $\mathbb{C}$. Since $\mathbb{C}$ is w-unital, then $(\tilde{f}, \tilde{g})$ is a weak product and, consequently, $\langle \tilde{f}, \tilde{g}\rangle : \tilde{Z} \twoheadrightarrow \tilde{X} \times \tilde{Y}$ is a regular epimorphism (Remark 1.1). Finally, the equality $\langle f, g\rangle \cdot u \cdot \alpha = x \times y \cdot \langle \tilde{f}, \tilde{g}\rangle$ allows us to conclude that $\langle f, g\rangle$ is a regular epimorphism, thus a strong epimorphism and, therefore, $\mathbb{E}$ is unital. □

When working in a context that admits binary coproducts, (w-)unital categories can be characterized through a special type of punctual span (see Proposition 1.2.18 in [4] for unital categories):

Proposition 2.5. *A pointed weakly lex category with binary coproducts $\mathbb{C}$ is w-unital if and only, for every pair of objects X, Y in $\mathbb{C}$, the canonical span $X \xleftarrow{\langle 1 \rangle} X + Y \xrightarrow{\langle 0 \rangle} Y$ is a weak product.*

Proof: When $\mathbb{C}$ is w-unital, the span $X \xleftarrow{\langle 1 \rangle} X + Y \xrightarrow{\langle 0 \rangle} Y$ is a weak product, since the span $X \underset{i_1}{\overset{\langle 1 \rangle}{\rightleftarrows}} X + Y \underset{i_2}{\overset{\langle 0 \rangle}{\rightleftarrows}} Y$ is punctual.

Conversely, consider a punctual span $X \underset{s}{\overset{f}{\rightleftarrows}} Z \underset{t}{\overset{g}{\rightleftarrows}} Y$ and arbitrary morphisms $x : A \to X, y : A \to Y$. By assumption, there is a morphism $w : A \to X + Y$ such that $\langle {}^1_0 \rangle \cdot w = x$ and $\langle {}^0_1 \rangle \cdot w = y$. So, there exists a morphism $\langle {}^s_t \rangle \cdot w : A \to Z$ such that $f \cdot \langle {}^s_t \rangle \cdot w = x$ and $g \cdot \langle {}^s_t \rangle \cdot w = y$, proving that (f, g) is a weak product. □

Recall that a variety of universal algebras is called a *Jónsson-Tarski* variety when its theory contains a unique constant 0 and a binary operation $+$ such that $x + 0 = x = 0 + x$ [15]. An *(internal) unital coalgebra* in any category $\mathbb{E}$ with binary coproducts is defined as an object X equipped with a morphism $+ : X \to X + X$ such that $\langle {}^1_0 \rangle \cdot + = 1 = \langle {}^0_1 \rangle \cdot +$.

It is known that a variety is Jónsson-Tarski if and only if it is a unital category (Theorem 1.2.15 of [4]). This result is illustrated in the next theorem, where we give a new proof based on the structural property of the category of free algebras (given in Proposition 2.5).

Given a pointed algebraic variety $\mathbb{E}$ and its full subcategory $\mathbb{C}$ of free algebras, we are in the situation where $\mathbb{C}$ is a pointed projective cover with binary coproducts of the regular category $\mathbb{E}$.

Theorem 2.6. *Let $\mathbb{E}$ be an algebraic variety and $\mathbb{C}$ its full subcategory of free algebras. Then the following statements are equivalent:*

1. *$\mathbb{E}$ is a unital category;*
2. *$\mathbb{C}$ is a w-unital category;*
3. *the free algebra $X = F(1)$ on one generator is a unital coalgebra;*
4. *$\mathbb{E}$ is a Jónsson-Tarski variety.*

Proof: 1. $\Leftrightarrow$ 2. is exactly given by Proposition 2.4.

is punctual in $\mathbb{C}$ if and only if the corresponding span

$$T \underset{\langle 1,0 \rangle}{\overset{s}{\rightleftarrows}} T \times T \underset{\langle 1,1 \rangle}{\overset{\pi_2}{\rightleftarrows}} T$$

is punctual in the algebraic theory $\mathbb{T}$ of $\mathbb{E}$. □

4. Strongly unital categories

In this last section, we are going to examine the characterization for projective covers of regular strongly unital categories. The link with unital and subtractive categories is given by the categorical equation (see Proposition 3 of [14]):

$$\text{unital} + \text{subtractive} = \text{strongly unital}.$$

Definition 4.1. ([5]) A category $\mathbb{E}$ is called *strongly unital* when it is pointed, has finite limits and for every object X in $\mathbb{E}$, the pair of morphisms $X \xrightarrow{\langle 1,1 \rangle} X \times X \xleftarrow{\langle 0,1 \rangle} X$ is jointly strongly epimorphic.

Let $\mathbb{E}$ be a pointed category. A span $X \xleftarrow{f} Z \xrightarrow{g} Y$ in $\mathbb{E}$ is called *split right punctual* if there exist morphisms $s : X \to Z$ and $t : Y \to Z$ such that $f \cdot s = 1, g \cdot t = 1$ and $f \cdot t = 0$, i.e. it is right punctual and f is split by s. For example, for any object X in $\mathbb{E}$, we have a split right punctual span $X \underset{\langle 1,1 \rangle}{\overset{\pi_1}{\rightleftarrows}} X \times X \underset{\langle 0,1 \rangle}{\overset{\pi_2}{\rightleftarrows}} X$. It is actually possible to characterize strongly unital categories through such spans (see Proposition 1.8.14 [4]). This fact led us to introduce the following definition:

Definition 4.2. A category $\mathbb{C}$ is called *w-strongly unital* when it is pointed, has weak finite limits and, for every split right punctual span $X \underset{s}{\overset{f}{\rightleftarrows}} Z \underset{t}{\overset{g}{\rightleftarrows}} Y$, the span $X \xleftarrow{f} Z \xrightarrow{g} Y$ is a weak product.

The "equation" relating the notions of (strongly) unital and of subtractive categories still holds in the weakened context:

Proposition 4.3. *A pointed weakly lex category $\mathbb{C}$ is w-strongly unital if and only if it is w-unital and w-subtractive.*

Proposition 3.4. *A pointed weakly lex category with binary coproducts $\mathbb{C}$ is w-subtractive if and only, for any object X in $\mathbb{C}$, the right punctual reflexive graph $X \xleftarrow{\langle 1 \atop 0\rangle} X + X \underset{i_2}{\overset{\langle 1 \atop 1\rangle}{\rightleftarrows}} X$ is left punctual.*

Proof: The statement is obvious when $\mathbb{C}$ is w-subtractive. Conversely, consider a right punctual reflexive graph $X \xleftarrow{d} G \underset{t}{\overset{c}{\rightleftarrows}} X$. By assumption, there exists a morphism $s : X \to X + X$ such that $\langle {1 \atop 0} \rangle \cdot s = 1$ and $\langle {1 \atop 1} \rangle \cdot s = 0$. Consequently, there exists a morphism $\langle {e \atop t} \rangle \cdot s : X \to G$, such that $d \cdot \langle {e \atop t} \rangle \cdot s = 1$ and $c \cdot \langle {e \atop t} \rangle \cdot s = 0$, proving that the reflexive graph is also left punctual. □

Recall that a subtractive variety $\mathbb{E}$, in the sense of Ursini [20], is such that its theory contains a constant 0 and a binary operation s, called *subtraction*, such that $s(x, 0) = x$ and $s(x, x) = 0$. When in the theory of a variety there is a *unique* constant, then $\mathbb{E}$ is a pointed variety. In this case it is known that $\mathbb{E}$ is a subtractive variety if and only if $\mathbb{E}$ is a subtractive category (Theorem 2 of [14]).

The morphism $s : X \to X + X$ arising in the proof of Proposition 3.4 then equips X with a *subtractive coalgebra* structure, i.e. such that the equalities $\langle {1 \atop 0} \rangle \cdot s = 1$ and $\langle {1 \atop 1} \rangle \cdot s = 0$ hold.

Corollary 3.5. *A pointed weakly lex category with binary coproducts $\mathbb{C}$ is w-subtractive if and only every object X in $\mathbb{C}$ is a subtractive coalgebra.*

We are then ready to prove the main result in this section (see the proof of Theorem 2.6 for the omitted details which we avoid repeating):

Theorem 3.6. *Let $\mathbb{E}$ be an algebraic variety and $\mathbb{C}$ its full subcategory of free algebras. Then following statements are equivalent:*

1. *$\mathbb{E}$ is a subtractive category;*
2. *$\mathbb{C}$ is a w-subtractive category;*
3. *the free algebra $X = F(1)$ on one generator is a subtractive coalgebra;*
4. *$\mathbb{E}$ is a pointed subtractive variety.*

Proof: The conditions 1. and 2. are equivalent by Proposition 3.3.
The conditions 2. and 3. are equivalent by Corollary 3.5.
3. ⇔ 4. By duality, the span

$$X \underset{s}{\overset{\langle {1 \atop 0} \rangle}{\rightleftarrows}} X + X \underset{i_2}{\overset{\langle {1 \atop 1} \rangle}{\rightleftarrows}} X$$

2. ⇒ 3. If $\mathbb{C}$ is w-unital, we can consider the free algebra $X = F(1)$ on one generator, and then the diagram

$$X \xleftarrow{\langle {1 \atop 0} \rangle} X + X \xrightarrow{\langle {0 \atop 1} \rangle} X$$

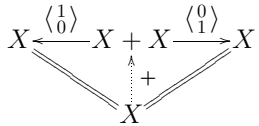
is a weak product in $\mathbb{C}$ by Proposition 2.5. It follows that there exists a morphism $+ : X \to X + X$ with the property that $\langle {1 \atop 0} \rangle \cdot + = 1 = \langle {0 \atop 1} \rangle \cdot +$, and X is a unital coalgebra.
3. ⇒ 2. If $X = F(1)$ is a unital coalgebra, then *every* free algebra A is a unital coalgebra (since A is a copower $A = \coprod_{i \in I} X$ for some set I). Consider then two morphisms $b : A \to B$ and $c : A \to C$ in $\mathbb{C}$. Then the unital coalgebra structure on A allows one to define a morphism

$$A \xrightarrow{+} A + A \xrightarrow{(b+c)} B + C$$

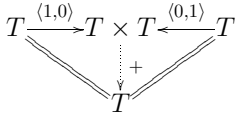
such that $\langle {1 \atop 0} \rangle \cdot (b + c) \cdot + = b$ and $\langle {0 \atop 1} \rangle \cdot (b + c) \cdot + = c$. This proves that the span

$$B \xleftarrow{\langle {1 \atop 0} \rangle} B + C \xrightarrow{\langle {0 \atop 1} \rangle} C$$

is a weak product, and $\mathbb{C}$ is w-unital.
3. ⇔ 4. Let $\mathbb{T}$ be an algebraic theory of $\mathbb{E}$ (in the sense of Lawvere [16]), namely a small category with finite products with the property that $\mathbb{E}$ is the category of product preserving functors from $\mathbb{C}$ to the category of sets. Via the duality between varieties of algebras and algebraic theories, the existence of a morphism $+ : X \to X + X$ making the diagram



commute in $\mathbb{C}$ corresponds exactly to the existence of a morphism $+ : T \times T \to T$ making the diagram



commute in the algebraic theory $\mathbb{T}$ (here T is the "generic object" of $\mathbb{T}$). □

The proof above of the equivalence between conditions 3. and 4. follows the technique used in [8] in the case of Mal'tsev varieties [6].

Remark 2.7. The arguments used in Theorem 2.6 can be adapted also to the context of *quasivarieties* [1] (we never used the fact that $\mathbb{E}$ is an exact category, but only the fact that $\mathbb{E}$ is regular). An interesting example of a unital quasivariety is provided by the category of *torsion-free monoids*, which are the monoids M satisfying the implications

$$a^n = b^n \Rightarrow a = b, \qquad \forall a, b \in M, \quad \forall n \in \mathbb{N}^*.$$

3. Subtractive categories

In this section we shall characterize the projective covers of regular subtractive categories [14]. In the presence of finite coproducts, this will lead to a simple proof of the characterization of algebraic pointed subtractive (quasi)varieties, in the sense of Ursini [20].

Given a pointed category $\mathbb{E}$, a span $X \xleftarrow{f} Z \xrightarrow{g} Y$ in $\mathbb{E}$ is called *right (resp. left) punctual* if there exists a morphism $t : Y \to Z$ (resp. $s : X \to Z$) such that $g \cdot t = 1$ and $f \cdot t = 0$ (resp. $f \cdot s = 1$ and $g \cdot s = 0$). Recall that a reflexive graph is a span $X \underset{e}{\overset{d}{\rightleftarrows}} G \underset{e}{\overset{c}{\rightleftarrows}} X$ such that $d \cdot e = c \cdot e = 1$; a reflexive relation is a reflexive graph such that d and c are jointly monomorphic. In the text, we shall omit representing the reflexivity morphism e for reflexive graphs and relations.

Definition 3.1. ([14]) A category $\mathbb{E}$ is called *subtractive* when it is pointed, has finite limits and every right punctual reflexive relation is also left punctual.

Equivalently, a subtractive category could be defined by demanding that every left punctual reflexive relation is also right punctual. We now introduce a "weak version" of this definition.

Definition 3.2. A category $\mathbb{C}$ is called *w-subtractive* when it is pointed, has weak finite limits and every right punctual reflexive graph is also left punctual.

Proposition 3.3. *Let $\mathbb{C}$ be a pointed projective cover of a regular category $\mathbb{E}$. Then $\mathbb{E}$ is subtractive if and only if $\mathbb{C}$ is w-subtractive.*

Proof: Suppose that $\mathbb{E}$ is subtractive. Since $\mathbb{C}$ is a projective cover of a regular category $\mathbb{E}$, then $\mathbb{C}$ has weak finite limits (Remark 1.1). Now, let $\tilde{X} \xleftarrow{\tilde{d}} \tilde{G} \underset{\tilde{t}}{\overset{\tilde{c}}{\rightleftarrows}} \tilde{X}$ be a right punctual reflexive graph in $\mathbb{C}$. Then, the (regular epimorphism, monomorphism)-factorization $(\tilde{d}, \tilde{c}) = (r_1, r_2) \cdot p$ in $\mathbb{E}$, gives a right punctual reflexive relation $\langle r_1, r_2 \rangle : R \rightarrowtail \tilde{X} \times \tilde{X}$ in $\mathbb{E}$. So, it is left punctual, i.e. there exists a morphism $s : \tilde{X} \to R$ such that $r_1 \cdot s = 1$ and $r_2 \cdot s = 0$. By the projectivity of $\tilde{X}$, there exists a morphism $\tilde{s} : \tilde{X} \to \tilde{G}$ such that $p \cdot \tilde{s} = s$. This morphism $\tilde{s}$ makes the original reflexive graph also left punctual.

Conversely, $\mathbb{E}$ is pointed, since $\mathbb{C}$ is (see Section 1). Let $X \xleftarrow{r_1} R \underset{t}{\overset{r_2}{\rightleftarrows}} X$ be a right punctual reflexive relation in $\mathbb{E}$. Given $\mathbb{C}$-covers $x : \tilde{X} \twoheadrightarrow X$ and $\alpha : \tilde{R} \twoheadrightarrow x^{-1}(R)$,

$$\begin{array}{ccccc} \tilde{R} & \xrightarrow{\alpha} & x^{-1}(R) & \xrightarrow{u} & R \\ & \searrow^{\langle \tilde{r}_1, \tilde{r}_2 \rangle} & \downarrow{\scriptstyle \langle r_1', r_2' \rangle} & & \downarrow{\scriptstyle \langle r_1, r_2 \rangle} \\ & & \tilde{X} \times \tilde{X} & \xrightarrow[x \times x]{} & X \times X \end{array}$$

the inverse image $\tilde{X} \xleftarrow{r_1'} x^{-1}(R) \underset{t'}{\overset{r_2'}{\rightleftarrows}} \tilde{X}$ is necessarily a reflexive relation in $\mathbb{E}$ which is also right punctual for $t' = (\langle 0, 1 \rangle, t \cdot x)$. Using the projectivity of $\tilde{X}$ and the fact that α is a regular epimorphism, there exists a morphism $\tilde{t} : \tilde{X} \to \tilde{R}$ such that $\tilde{X} \xleftarrow{\tilde{r}_1} \tilde{R} \underset{\tilde{t}}{\overset{\tilde{r}_2}{\rightleftarrows}} \tilde{X}$ is a right punctual reflexive graph in $\mathbb{C}$. Therefore, it is also left punctual, i.e. there exists a morphism $\tilde{s} : \tilde{X} \to \tilde{R}$ such that $\tilde{r}_1 \cdot \tilde{s} = 1$ and $\tilde{r}_2 \cdot \tilde{s} = 0$. Finally, the fact that x is a strong (=regular) epimorphism, gives a unique factorization s in

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{x} & X \\ {\scriptstyle u \cdot \alpha \cdot \tilde{s}} \downarrow & \overset{s}{\swarrow} & \downarrow{\scriptstyle \langle 1,0 \rangle} \\ R & \xrightarrow[\langle r_1, r_2 \rangle]{} & X \times X \end{array}$$

which makes the original reflexive relation R in $\mathbb{E}$ also left punctual. □

In the presence of binary coproducts, w-subtractive categories can be characterized through a special kind of right punctual reflexive graph as follows: