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SOME REMARKS ON COSYMPLECTIC 3-STRUCTURES

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Abstract: In this note we briefly review some recent results of the authors on the topological and geometrical properties of 3-cosymplectic manifolds.

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1. Introduction

Cosymplectic manifolds were introduced in the frame of quasi-Sasakian manifolds by Blair in [2] as the closest odd-dimensional counterpart of Kähler manifolds. Since then cosymplectic geometry has attracted the interest of many researchers also due to its role in mechanics and physics. Recently, a great deal of work on the topological properties of cosymplectic manifolds was done (see [6, 7, 9] among others). In particular, in [6] Chinea, de León and Marrero proved several important results for the Betti numbers of a compact cosymplectic manifold.

The notion of 3-cosymplectic manifold is the transposition of the notion of cosymplectic manifold to the setting of 3-structures. Namely, a 3-cosymplectic manifold is a smooth manifold endowed with three distinct cosymplectic structures related to each other by means of some relations formally similar to the quaternionic identities (see Section 2 for more details). This note contains a concise review of the main properties of 3-cosymplectic manifolds, recently obtained by the authors in [4, 5]. Especially, we emphasize our results concerning Betti numbers of compact 3-cosymplectic manifolds. Finally, we present a method for constructing non-trivial examples of such compact manifolds.

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2. 3-cosymplectic geometry

An *almost contact manifold* is an odd-dimensional smooth manifold M endowed with a tensor field ϕ of endomorphisms on the tangent spaces, a vector field ξ and a 1-form η satisfying $\phi^2 = -I + \eta \otimes \xi$, where I denotes the identity mapping of TM. It is known that there exists a Riemannian metric g which is compatible with the structure, in the sense that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.1}$$

for any $X, Y \in \Gamma(TM)$. When one fixes one compatible metric, the resulting geometric structure (ϕ, ξ, η, g) is called an *almost contact metric structure* on M. From (2.1) it follows that the bilinear form $\Phi := g(\cdot, \phi\cdot)$ is in fact a 2-form, called the *fundamental* 2*-form* of the almost contact metric manifold. An *almost cosymplectic manifold* is an almost contact metric manifold (M, ϕ, ξ, η, g) such that both the 1-form η and the fundamental 2-form Φ are closed. If in addition the structure is *normal*, that is, if the Nijenhuis tensor field of ϕ vanishes identically, (M, ϕ, ξ, η, g) is said to be a *cosymplectic manifold*. In terms of the covariant derivative of the structure tensor field ϕ, this condition is equivalent to $\nabla\phi = 0$. Now, we come to the main topic of the paper. A triple of almost contact structures (ϕ_1, ξ_1, η_1), (ϕ_2, ξ_2, η_2), (ϕ_3, ξ_3, η_3) on a manifold M, related by the identities

$$\begin{aligned} \phi_\gamma &= \phi_\alpha\phi_\beta - \eta_\beta \otimes \xi_\alpha = -\phi_\beta\phi_\alpha + \eta_\alpha \otimes \xi_\beta, \\ \xi_\gamma &= \phi_\alpha\xi_\beta = -\phi_\beta\xi_\alpha, \quad \eta_\gamma = \eta_\alpha \circ \phi_\beta = -\eta_\beta \circ \phi_\alpha, \end{aligned} \tag{2.2}$$

for any even permutation (α, β, γ) of the set $\{1, 2, 3\}$, is called an *almost contact 3-structure* on M. Then, the dimension of the manifold is necessarily of the form $4n+3$. This notion was introduced independently by Kuo ([8]) and Udriste ([12]). In particular, Kuo proved that one can always find a Riemannian metric g which is compatible with each almost contact structure. If we fix a compatible metric, we speak of *almost contact metric 3-structure*. Any smooth manifold endowed with an almost contact metric 3-structure carries two orthogonal distributions: the *Reeb distribution* $\mathcal{V} := \mathrm{span}\{\xi_1, \xi_2, \xi_3\}$ and the *horizontal distribution* $\mathcal{H} := \ker(\eta_1) \cap \ker(\eta_2) \cap \ker(\eta_3)$.

A remarkable case is when each structure is cosymplectic. In this case we say that M is a *3-cosymplectic manifold*. In any 3-cosymplectic manifold the forms η_α and Φ_α are harmonic. Moreover, the tensors ξ_α, η_α, ϕ_α, Φ_α are all ∇-parallel. In particular, since the Reeb vector fields commute with each other, it follows that the Reeb distribution is integrable and defines a

3-dimensional foliation $\mathcal{F}_3$ of M. As it was proven in [4], $\mathcal{F}_3$ is a Riemannian and transversely hyper-Kähler foliation with totally geodesic leaves. Moreover, since $d\eta_\alpha = 0$, also the horizontal distribution $\mathcal{H}$ is integrable and hence defines a Riemannian, totally geodesic foliation complementary to $\mathcal{F}_3$.

Another important property of 3-cosymplectic manifolds that should be mentioned is that they are Ricci-flat ([4]).

3. The cohomology of a 3-cosymplectic manifold

Let M be a compact 3-cosymplectic manifold of dimension $4n+3$. We will denote by $H^*_{dR}(M)$ the usual de Rham cohomology of M. By the Hodge-de Rham theory each vector space $H^k_{dR}(M)$ can be identified with the vector space $\Omega^k_H(M)$ of harmonic k-forms on M. Recall also that the space of basic k-forms (with respect to $\mathcal{F}_3$) is defined by

$$\Omega^k_B(M) := \left\{\omega \in \Omega^k(M) \;\middle|\; i_{\xi_\alpha}\omega = 0,\ i_{\xi_\alpha}d\omega = 0, \quad \text{for each } \alpha = 1,2,3\right\}.$$

Since the differential d preserves basic forms, it induces a cohomology $H^*_B(M)$ which is called *basic cohomology*.

For each $\alpha \in \{1,2,3\}$ we define two linear operators $l_\alpha : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$, $\omega \mapsto \eta_\alpha \wedge \omega$, and $\lambda_\alpha : \Omega^{k+1}(M) \longrightarrow \Omega^k(M)$, $\omega \mapsto i_{\xi_\alpha}\omega$. Moreover, we define $e_\alpha := l_\alpha \circ \lambda_\alpha$. By [6, Proposition 1] the operators l_α, λ_α, and hence e_α, preserve harmonic forms. Then one can prove the following decomposition

$$\Omega^k_H(M) = \bigoplus_{\epsilon_1,\epsilon_2,\epsilon_3 \in \{0,1\}} \Omega^k_{H,\epsilon_1\epsilon_2\epsilon_3}(M), \tag{3.1}$$

where we have put, for each triple $\epsilon_1, \epsilon_2, \epsilon_3 \in \{0,1\}$,

$$\Omega^k_{H,\epsilon_1\epsilon_2\epsilon_3}(M) := \left\{\omega \in \Omega^k_H(M) \;\middle|\; e_\alpha\omega = \epsilon_\alpha\omega,\ \alpha = 1,2,3\right\}.$$

Moreover, one can prove that the operators l_1, l_2, l_3 induce isomorphisms between the vector spaces $\Omega^*_{H,\epsilon_1\epsilon_2\epsilon_3}$ according to the following diagram

$$\begin{tikzcd}
 & \Omega^{k+1}_{H,100}(M) \arrow[rr, "l_2"] \arrow[dd, "l_3"] & & \Omega^{k+2}_{H,110}(M) \arrow[dd, "l_3"] \\
\Omega^{k}_{H,000}(M) \arrow[ur, "l_1"] \arrow[rr, "l_2"] \arrow[dd, "l_3"] & & \Omega^{k+1}_{H,010}(M) \arrow[ur, "l_1"] \arrow[dd, "l_3"] & \\
 & \Omega^{k+2}_{H,101}(M) \arrow[rr, "l_2"] & & \Omega^{k+3}_{H,111}(M) \\
\Omega^{k+1}_{H,001}(M) \arrow[ur, "l_1"] \arrow[rr, "l_2"] & & \Omega^{k+2}_{H,011}(M) \arrow[ur, "l_1"] &
\end{tikzcd}$$

for each $0 \leq k \leq 4n$. Therefore, the whole information about cohomology groups of M is contained in the vector spaces $\Omega^k_{H,000}(M)$, $0 \leq k \leq 4n$. It is worth to mention that $\Omega^k_{H,000}(M)$ can be identified with the space of basic harmonic k-forms on M (with respect to $\mathcal{F}_3$). In particular, $b^h_k := \dim(\Omega^k_{H,000}(M))$ is the k-th basic Betti number. Now, taking the decomposition (3.1) into account and using the above isomorphisms between the vector spaces $\Omega^*_{H,\epsilon_1\epsilon_2\epsilon_3}(M)$, one gets the following formula for the k-th Betti number of M

$$b_k = b^h_k + 3b^h_{k-1} + 3b^h_{k-2} + b^h_{k-3}. \tag{3.2}$$

On the other hand, one can prove (see [5] for more details) that, for each odd integer k, $\Omega^k_{H,000}(M)$ is a $\mathbb{H}$-module and thus b^h_k is divisible by 4. Then by (3.2) it follows that, for any odd integer k, $b_{k-1}+b_k$ is divisible by 4. Another restriction on the Betti numbers of a compact 3-cosymplectic manifold is the following inequality

$$b_k \geq \binom{k+2}{2} \tag{3.3}$$

for $0 \leq k \leq 2n+1$, which is stronger than the analogous inequality for hyper-Kähler manifolds, due to Wakakuwa ([13]), namely $b_{2k} \geq \binom{k+2}{2}$. We conclude the section by describing an action of the Lie algebra $so(4,1)$ on $\Omega^k_{H,000}(M)$. For every even permutation (α,β,γ) of $\{1,2,3\}$ let us consider the 2-form $\Xi_\alpha := \frac{1}{2}(\Phi_\alpha + 2\eta_\beta \wedge \eta_\gamma)$. Then we define the operators $L_\alpha : \Omega^k(M) \longrightarrow \Omega^{k+2}(M)$ and $\Lambda_\alpha : \Omega^{k+2}(M) \longrightarrow \Omega^k(M)$ by $L_\alpha\omega := \Xi_\alpha \wedge \omega$

and $\Lambda_\alpha := *L_\alpha *$. Since L_α and Λ_α preserve harmonicity, one can consider them as endomorphisms of $\Omega^*_{H,000}(M)$. Then, by [5, Proposition 4.3] one has that, on $\Omega^*_{H,000}$, $[L_\alpha, \Lambda_\alpha] = -H$, where $H : \Omega^k_{H,000}(M) \longrightarrow \Omega^k_{H,000}(M)$ is the operator defined by $H\omega = (2n-k)\omega$. Moreover, for each $\alpha \in \{1,2,3\}$ we define another operator K_α on $\Omega^k_{H,000}(M)$ by $K_\alpha := [L_\beta, \Lambda_\gamma]$, where (α, β, γ) is an even permutation of $\{1,2,3\}$. Then we have the following result.

Theorem 3.1 ([5]). *The linear span $\mathfrak{g}$ of the operators H, L_α, Λ_α, K_α, $\alpha \in \{1,2,3\}$, is a Lie algebra isomorphic to $so(4,1)$. Consequently $\Omega^*_{H,000}(M)$ is an $so(4,1)$-module.*

4. Examples of compact 3-cosymplectic manifolds

The standard example of 3-cosymplectic manifold is $\mathbb{R}^{4n+3}$ with the almost contact metric 3-structure described in [4] in terms of Darboux coordinates. Since this structure is invariant by translations, we get a 3-cosymplectic structure on the flat torus $\mathbb{T}^{4n+3}$ (see also [10]). Both these examples are global products of a hyper-Kähler manifold with an abelian Lie group. In fact, locally this is always true: every 3-cosymplectic manifold is locally a Riemannian product of a hyper-Kähler factor with a 3-dimensional flat abelian Lie group. Thus it makes sense to ask whether there are examples of 3-cosymplectic manifolds which are not global products of a hyper-Kähler manifold with a 3-dimensional Lie group. The answer to this question is affirmative and now we describe a procedure for constructing such examples. Let (M^{4n}, J_α, G) be a compact hyper-Kähler manifold and f a hyper-Kähler isometry on it. We define an action φ of $\mathbb{Z}^3$ on $M^{4n} \times \mathbb{R}^3$ by

$$\varphi((k_1,k_2,k_3),(x,t_1,t_2,t_3)) = (f^{k_1+k_2+k_3}(x), t_1+k_1, t_2+k_2, t_3+k_3).$$

We define a 3-cosymplectic structure on the orbit space $M_f^{4n+3} := (M^{4n} \times \mathbb{R}^3)/\mathbb{Z}^3$ in the following way. Let us consider the vector fields $\xi_\alpha := \frac{\partial}{\partial t_\alpha}$ and the 1-forms $\eta_\alpha := dt_\alpha$ on $M^{4n} \times \mathbb{R}^3$. Next we define, for each $\alpha \in \{1,2,3\}$, a tensor field ϕ_α on $M^{4n} \times \mathbb{R}^3$ by putting $\phi_\alpha X := J_\alpha X$ for any $X \in \Gamma(TM^{4n})$ and $\phi_\alpha \xi_\alpha := 0$, $\phi_\alpha \xi_\beta := \epsilon_{\alpha\beta\gamma}\xi_\gamma$, where $\epsilon_{\alpha\beta\gamma}$ denotes the sign of the permutation (α,β,γ) of $\{1,2,3\}$. Then $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$, where g denotes the product metric, is a 3-cosymplectic structure on $M^{4n} \times \mathbb{R}^3$. Being invariant under the action φ, the structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ descends to a 3-cosymplectic structure on M_f^{4n+3}. By using this general procedure we can construct non-trivial examples of compact 3-cosymplectic manifolds. In fact, let us consider the hyper-Kähler manifold $\mathbb{T}^4 = \mathbb{H}/\mathbb{Z}^4$ and the hyper-Kähler isometry f given by the multiplication by the quaternionic unit $\mathbf{i}$ on the right. Then $M^7_f := (\mathbb{T}^4 \times \mathbb{R}^3)/\mathbb{Z}^3$, endowed with the geometric structure described above, is a compact 3-cosymplectic manifold which is not the global product of a compact 4-dimensional hyper-Kähler manifold K^4 with the flat torus. Indeed, we have only two possibilities for a compact 4-dimensional hyper-Kähler manifold: either $K^4 \cong \mathbb{T}^4$ or it is a complex K3-surface. In the first case $b_2(K^4 \times \mathbb{T}^3) = 21$, in the second $b_2(K^4 \times \mathbb{T}^3) = 25$. However, in [5] it was proven that $b_2(M^7_f) < 21$.

Other examples can be obtained from the previous ones by applying a $\mathcal{D}_a$-*homothetic deformation*, that is a change of the structure tensors of the following type

$$\bar{\phi} := \phi, \quad \bar{\xi} := \frac{1}{a}\xi, \quad \bar{\eta} := a\eta, \quad \bar{g} := ag + a(a-1)\eta \otimes \eta, \tag{4.1}$$

where $a > 0$. This notion was introduced by Tanno ([11]) in the contact metric case, but it can be easily extended to the more general context of almost contact metric structures. In particular, it can be proved that the class of cosymplectic structures is preserved by $\mathcal{D}$-homothetic deformations. Now, let $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$, $\alpha \in \{1, 2, 3\}$, be a 3-cosymplectic manifold. Then by applying the same $\mathcal{D}_a$-homothetic deformation to each cosymplectic structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$, one obtains three new cosymplectic structures $(\bar{\phi}_1, \bar{\xi}_1, \bar{\eta}_1, \bar{g})$, $(\bar{\phi}_2, \bar{\xi}_2, \bar{\eta}_2, \bar{g})$, $(\bar{\phi}_3, \bar{\xi}_3, \bar{\eta}_3, \bar{g})$, which are still related to each other by means of the quaternionic-like relations (2.2). Thus $(\bar{\phi}_\alpha, \bar{\xi}_\alpha, \bar{\eta}_\alpha, \bar{g})$, $\alpha \in \{1, 2, 3\}$, is a new 3-cosymplectic structure on M. In particular, this procedure allows to define other 3-cosymplectic structures on M^{4n+3}_f from the structure described before.

We conclude with the following remark concerning the existence of 3-cosymplectic structures on almost cosymplectic Einstein manifolds. In the context of Sasakian manifolds, Apostolov, Draghici and Moroianu proved the following theorem:

Theorem 4.1 ([1])**.** *Let (M, ϕ, ξ, η, g) be a Sasakian Einstein manifold. Then any contact metric structure (ϕ', ξ', η', g) on M, with the same metric g is Sasakian. Moreover, if $\xi' \neq \pm\xi$, then either (M, g) admits a 3-Sasakian structure or (M, g) is covered by a round sphere.*

It could be interesting to investigate on the cosymplectic counterpart, if any, of Theorem 4.1. In fact, this would permit to construct new examples of 3-cosymplectic manifolds. In this context we mention the following result on compact Einstein cosymplectic manifolds.

Theorem 4.2 ([3]). *Every compact Einstein almost cosymplectic manifold (M, ϕ, ξ, η, g), such that ξ is Killing, is cosymplectic and Ricci-flat. Furthermore, any other almost cosymplectic structure (ϕ', ξ', η', g) on M is necessarily cosymplectic and Ricci-flat.*

Notice that the proof of Theorem 4.1 does not work in the case of cosymplectic manifolds. In fact, it uses a property of the cone metric which holds only for Sasakian manifolds.

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