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INSERTION AND EXTENSION RESULTS FOR POINT-FREE COMPLETE REGULARITY

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ABSTRACT: There are insertion-type characterizations in point-free topology that extend well known insertion theorems in point-set topology for all relevant higher separation axioms with one notable exception: complete regularity. In this paper we fill this gap. The situation reveals to be an interesting and peculiar one: contrarily to what happens with all the other higher separation axioms, the extension to the point-free setting of the classical insertion result for completely regular spaces characterizes a formally weaker class of frames introduced in this paper (called *completely c-regular frames*). The fact that any compact sublocale (quotient) of a completely regular frame is a C^*-sublocale (C^*-quotient) is obtained as a corollary.

KEYWORDS: Frame, sublocale, compact-like real function, complete regularity, completely separated sublocales, insertion theorem.

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Introduction

Let $\mathfrak{L}(\mathbb{R})$ denote the frame of reals ([2]) and let $\mathcal{S}(L)$ denote the lattice of sublocales of a frame L seen as a frame (that is, turned upside down; see Section 1 below for the details). Among the important examples of sublocales are, for each $a \in L$, the *closed sublocales*

$$\mathfrak{c}(a) = \uparrow a = \{b \in L \mid a \leq b\}$$

and the *open sublocales*

$$\mathfrak{o}(a) = \{a \to b \mid b \in L\}.$$

The class of closed sublocales is usually denoted by $\mathfrak{c}L$ and it is a subframe of $\mathcal{S}(L)$ isomorphic to the given frame L via the mapping $\mathfrak{c}\colon L \to \mathfrak{c}L$ given by the correspondence $a \mapsto \mathfrak{c}(a)$.

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The ring $\mathsf{F}(L)$ of real functions on L ([8, 10]) is the class of all frame homomorphisms

$$\mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$$

partially ordered by

$$\begin{aligned} f \leq g \;\equiv\; & f(r,\text{—}) \leq g(r,\text{—}) \quad \text{for all } r \in \mathbb{Q} \\ \Leftrightarrow\; & g(\text{—},r) \leq f(\text{—},r) \quad \text{for all } r \in \mathbb{Q}. \end{aligned}$$

An $f \in \mathsf{F}(L)$ is *lower* (resp. *upper*) *semicontinuous* if $f(r,\text{—}) \in \mathfrak{c}L$ (resp. $f(\text{—},r) \in \mathfrak{c}L$) for every $r \in \mathbb{Q}$ and it is *continuous* if $f(p,q) \in \mathfrak{c}L$ for every $p,q \in \mathbb{Q}$. We shall denote by $\mathsf{C}(L)$, $\mathsf{LSC}(L)$ and $\mathsf{USC}(L)$ the classes of continuous, lower semicontinuous, and upper semicontinuous members of $\mathsf{F}(L)$, respectively.

An insertion-type theorem in point-free topology has the following structure. Let $\mathcal{F},\mathcal{G},\mathcal{H} \subseteq \mathsf{F}(L)$. Assume $f \in \mathcal{F}, g \in \mathcal{G}$ and $f \leq g$. Then an insertion-type assertion states that

there exists an $h \in \mathcal{H}$ such that $f \leq h \leq g$.

The particular case $L = \mathcal{O}X$ for the topology $\mathcal{O}X$ of a space X gives the corresponding classical insertion theorem.

A fundamental example is the case

$$\mathcal{F} = \mathsf{USC}(L), \quad \mathcal{G} = \mathsf{LSC}(L), \quad \mathcal{H} = \mathsf{C}(L)$$

that characterizes normal frames and extends the celebrated Katětov-Tong insertion theorem for normal spaces (see [12], [9] and [8]; recall that a frame L is normal if $a \vee b = 1$ implies the existence of $x,y \in L$ such that $a \vee x = 1 = b \vee y$ and $x \wedge y = 0$):

Theorem. *The following are equivalent for a frame L:*

(i) *L is normal.*
(ii) *If $f \in \mathsf{USC}(L)$, $g \in \mathsf{LSC}(L)$ and $f \leq g$, then there exists an $h \in \mathsf{C}(L)$ such that $f \leq h \leq g$.*

The corresponding extension theorem asserts that any closed sublocale (quotient) of a normal frame is a C^*-sublocale (C^*-quotient).

For more examples, characterizing monotonically normal, completely normal, perfectly normal, countably paracompact or extremally disconnected frames, consult [8] and [5].

Comparing this with the literature in point-set topology there is one important case missing: complete regularity. Indeed, we know from [6] that

> *A space is completely regular if and only if given $f, g\colon X \to [0,1]$, f compact-like and g lower semicontinuous such that $f \leq g$, then there exists a continuous $h\colon X \to [0,1]$ such that $f \leq h \leq g$.*

This characterization of complete regularity holds for a very simple peculiar reason: every open U in X is the union of the compact subsets $\{x\}$, $x \in U$.

Of course, when dealing with general frames one cannot imitate that: we do not have (enough) points to construct that basic compact subsets. The question naturally arises as to whether this insertion result continues to hold true in general frames. In this paper we address this question. We show that this insertion result extends to completely regular frames but no longer characterizes complete regularity; among fit frames L, it characterizes a formally wider class of frames that we introduce as *completely c-regular frames*. For that we need to revisit completely separated sublocales of a frame L (Section 2) and to introduce compact-like real functions on L (Section 4). The corresponding (Urysohn) separation-type lemma and (Tietze) extension-type theorem are also obtained (sections 4 and 5 respectively).

1. Preliminaries

Useful references for frames and locales are [11] or the more recent [13]. Here we fix some notation and terminology and recall the relevant facts needed later on.

A *frame* (or *locale*) L is a complete lattice with the distributive property

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$$

for all $a \in L$ and $S \subseteq L$; equivalently, it is a complete Heyting algebra with Heyting operation $\to$ satisfying the standard equivalence $a \wedge b \leq c$ if and only if $a \leq b \to c$. The *pseudocomplement* of an $a \in L$ is the element $a^* = a \to 0 = \bigvee\{b \in L \mid a \wedge b = 0\}$.

For any frame L, $k \in L$ is compact if $k \leq \bigvee \mathcal{X}$ implies $k \leq \mathcal{F}$ for some finite $\mathcal{F} \subseteq \mathcal{X}$, and L is called *compact* if its unit 1 is compact. L is *algebraic* if each $a \in L$ is a join of compact elements. Further, a frame L is called *completely regular* if

$$a = \bigvee \{b \in L \mid b \prec\!\prec a\}$$

for each $a \in L$ where $b \prec\!\prec a$ the "completely below" relation) means that there is a sequence $\{c_r \mid r \in \mathbb{Q}\} \subseteq L$ such that $c_r = 1$ if $r < 0$, $c_r = 0$ if $r > 1$, $b \le c_r \le a$ for all $r \in \mathbb{Q} \cap [0,1]$, and $c_r^* \vee c_s = 1$ whenever $r < s$.

Sublocales. A *sublocale set* (briefly, a *sublocale*) S of a frame L is a subset $S \subseteq L$ such that

(S1) for every $A \subseteq S$, $\bigwedge A$ is in S, and
(S2) for every $s \in S$ and every $x \in L$, $x \to s$ is in S.

Each sublocale $S \subseteq L$ is also determined by the frame surjection (quotient map) $c_S \colon L \to S$ given by $c_S(x) = \bigwedge\{s \in S \mid s \ge x\}$ for all $x \in L$. E.g. the quotient maps $c_{\mathfrak{c}(a)}$ and $c_{\mathfrak{o}(a)}$ are given by $c_{\mathfrak{c}(a)}(x) = a \vee x$ and $c_{\mathfrak{o}(a)}(x) = a \to x$, respectively.

Further, each sublocale S of L is itself a frame with the same meets as in L, and since the Heyting operation $\to$ depends on the meet structure only, with the same Heyting operation. However the joins in S and L will not necessarily coincide:

$$\mathop{\bigvee^{S}}_{i \in I} a_i = c_S\Big(\bigvee_{i \in I} a_i\Big) \ge \bigvee_{i \in I} a_i.$$

It follows that $1_S = 1$ but in general $0_S \ne 0$. In particular

$$0_{\mathfrak{c}(a)} = a, \quad x \overset{\mathfrak{c}(a)}{\vee} y = x \vee y, \quad 0_{\mathfrak{o}(a)} = a^* \quad \text{and} \quad x \overset{\mathfrak{o}(a)}{\vee} y = a \to (x \vee y).$$

For notational reasons, we make the co-frame of all sublocales of L into a frame $\mathcal{S}(L)$ by considering the opposite ordering

$$S_1 \le S_2 \quad \Leftrightarrow \quad S_2 \subseteq S_1.$$

Thus, given $\{S_i \in \mathcal{S}(L) \mid i \in I\}$, we have

$$\bigvee_{i \in I} S_i = \bigcap_{i \in I} S_i \quad \text{and} \quad \bigwedge_{i \in I} S_i = \Big\{ \bigwedge A : A \subseteq \bigcup_{i \in I} S_i \Big\}.$$

Further, $\{1\}$ is the top and L is the bottom in $\mathcal{S}(L)$ that we simply denote by 1 and 0.

The *closure* $\overline{S}$ of a sublocale $S \in \mathcal{S}(L)$ is the largest closed sublocale smaller than S, and is given by the formula $\overline{S} = {\uparrow}(\bigwedge S)$. We shall denote the closed sublocales of a sublocale S of L by $\mathfrak{c}^S(a)$.

Facts 1.1. *For every $a, b \in L$, $A \subseteq L$ and $S, T \in \mathcal{S}(L)$, we have:*
(1) $\mathfrak{c}(a) \wedge \mathfrak{c}(b) = \mathfrak{c}(a \wedge b)$ *and* $\mathfrak{o}(a) \vee \mathfrak{o}(b) = \mathfrak{o}(a \wedge b)$.

(2) $\bigvee_{a\in A}\mathfrak{c}(a)=\mathfrak{c}(\bigvee A)$ *and* $\bigwedge_{a\in A}\mathfrak{o}(a)=\mathfrak{o}(\bigvee A)$.
(3) $\mathfrak{c}(a)\vee\mathfrak{o}(a)=1$ *and* $\mathfrak{c}(a)\wedge\mathfrak{o}(a)=0$.
(4) $\mathfrak{o}(a)\geq\mathfrak{c}(b)$ *if and only if* $a\wedge b=0$.
(5) $\mathfrak{o}(a)\leq\mathfrak{c}(b)$ *if and only if* $a\vee b=1$.
(6) $\overline{1}=1$, $\overline{S}\leq S$, $\overline{\overline{S}}=\overline{S}$, *and* $\overline{S\wedge T}=\overline{S}\wedge\overline{T}$.
(7) $\overline{\mathfrak{o}(a)}=\mathfrak{c}(a^*)$.
(8) $\mathfrak{c}(a)\vee S$ *is the closed sublocale* $\mathfrak{c}^S(c_S(a))$ *of* S.
(9) *If* T *is a closed sublocale of* S *then* $T=\mathfrak{c}(x)\vee S$ *for some* $x\in S$.

A sublocale S is said to be *compact* if it is compact as a frame. Equivalently:

Fact 1.2. *A sublocale* S *of* L *is compact if and only if for each* $\{a_i\}_{i\in I}\subseteq L$ *such that* $\bigwedge_{i\in I}\mathfrak{o}(a_i)\leq S$, *there exists a finite* $J\subseteq I$ *such that* $\bigwedge_{i\in J}\mathfrak{o}(a_i)\leq S$.

Proof: Just notice that

$$\begin{aligned}\bigwedge_{i\in I}\mathfrak{o}(a_i)=\mathfrak{o}\Big(\bigvee_{i\in I}a_i\Big)\leq S\quad&\iff\quad 1=\mathfrak{c}\Big(\bigvee_{i\in I}a_i\Big)\vee S=\mathfrak{c}^S\Big(c_S\Big(\bigvee_{i\in I}a_i\Big)\Big)\\&\iff\quad 1=c_S\Big(\bigvee_{i\in I}a_i\Big)=\overset{S}{\bigvee}_{i\in I}c_S(a_i).\quad\blacksquare\end{aligned}$$

Note that in the co-frame of sublocales this just says that *a sublocale is compact iff every open cover has a finite subcover.*

Corollary 1.3. *An element* $k\in L$ *is compact iff the sublocale* $\mathfrak{o}(k)$ *is compact.* $\blacksquare$

Fact 1.4. *If* S *is a compact sublocale of a frame* L *and* T *is a closed sublocale of* S *then* T *is a compact sublocale of* L.

Real functions. The frame $\mathfrak{L}(\mathbb{R})$ of reals is the frame specified by generators $(p,\text{—})$ and $(\text{—},q)$ for $p,q\in\mathbb{Q}$, and defining relations

(R1) $(r,\text{—})\wedge(\text{—},s)=0$ whenever $r\geq s$,
(R2) $(r,\text{—})\vee(\text{—},s)=1$ whenever $r<s$,
(R3) $(r,\text{—})=\bigvee_{s>r}(s,\text{—})$, for every $r\in\mathbb{Q}$,
(R4) $(\text{—},r)=\bigvee_{s<r}(\text{—},s)$, for every $r\in\mathbb{Q}$,
(R5) $\bigvee_{r\in\mathbb{Q}}(r,\text{—})=1$,
(R6) $\bigvee_{r\in\mathbb{Q}}(\text{—},r)=1$.

In order to define a real function $f \in \mathsf{F}(L)$ it suffices to consider two maps from $\mathbb{Q}$ to $\mathcal{S}(L)$ that turn the defining relations (R1)–(R6) of $\mathfrak{L}(\mathbb{R})$ into identities in $\mathcal{S}(L)$. An $f \in \mathsf{F}(L)$ is *lower* (resp. *upper*) *semicontinuous* if $f(r,\text{—}) \in \mathfrak{c}L$ (resp. $f(\text{—},r) \in \mathfrak{c}L$) for every $r \in \mathbb{Q}$. We denote by $\mathsf{LSC}(L)$ (resp. $\mathsf{USC}(L)$) the class of all lower (resp. upper) semicontinuous real functions.

Continuous real functions are usually defined (see [4]) as frame homomorphisms $\varphi\colon \mathfrak{L}(\mathbb{R}) \to L$. As proved in [8], after the isomorphism $\mathfrak{c}\colon L \to \mathfrak{c}L$, they can be identified with the elements of $\mathsf{C}(L) = \mathsf{LSC}(L) \cap \mathsf{USC}(L)$. In what follows, we will freely refer to continuous real function as both the real function $f \in \mathsf{C}(L)$ and the unique frame homomorphism $\varphi\colon \mathfrak{L}(\mathbb{R}) \to L$ such that $\mathfrak{c} \cdot \varphi = f$.

Real functions can be easily defined via scales: a *scale* in $\mathcal{S}(L)$ (see [8]) is a family $(S_r)_{r\in\mathbb{Q}}$ of sublocales of L satisfying

(1) $S_p \vee {S_q}^* = 1$ whenever $p < q$, and
(2) $\bigvee_{p\in\mathbb{Q}} S_p = 1 = \bigvee_{p\in\mathbb{Q}} {S_p}^*$.

In fact, for each scale $(S_r)_{r\in\mathbb{Q}}$ the formulas

$$f(p,\text{—}) = \bigvee_{r>p} S_r \quad \text{and} \quad f(\text{—},q) = \bigvee_{r<q} {S_r}^* \quad (p,q \in \mathbb{Q})$$

determine an $f \in \mathsf{F}(L)$. Moreover, if every S_r is closed (resp. open, resp. clopen) then $f \in \mathsf{LSC}(L)$ (resp. $f \in \mathsf{USC}(L)$, resp. $f \in \mathsf{C}(L)$).

Example 1.5 (Characteristic functions). *Let S be a complemented sublocale of L. We denote by χ_S the real function defined for each $p, q \in \mathbb{Q}$ by*

$$\chi_S(p,\text{—}) = \begin{cases} 1 & \text{if } p < 0 \\ S^* & \text{if } 0 \le p < 1 \\ 0 & \text{if } p \ge 1 \end{cases} \quad \text{and} \quad \chi_S(\text{—},q) = \begin{cases} 0 & \text{if } q \le 0 \\ S & \text{if } 0 < q \le 1 \\ 1 & \text{if } q > 1. \end{cases}$$

2. Complete regularity and completely separated sublocales

The notion of complete separation in pointfree topology was first introduced in [1] in terms of quotient maps and cozero elements and equivalently reformulated in [7] in terms of sublocales and continuous real functions.

Let S and T be sublocales of L. They are said to be *completely separated* if there exists an $f \in \mathsf{C}(L)$ such that

$$f(0, \text{—}) \leq S \quad \text{and} \quad f(\text{—}, 1) \leq T.$$

Equivalently, this means that the corresponding quotient maps c_S and c_T are completely separated, i.e. if there exists a frame homomorphism $\varphi \colon \mathfrak{L}(\mathbb{R}) \to L$ such that $c_S(\varphi(0, \text{—})) = 0_S$ and $c_T(\varphi(\text{—}, 1)) = 0_T$.

Remarks 2.1. (a) 0 and 1 are completely separated by χ_0.
(b) If S and T are completely separated, then for every sublocales $U \geq S$ and $V \geq T$, U and V are also completely separated.
(c) Sublocales S and T are completely separated iff $\overline{S}$ and $\overline{T}$ are completely separated.
(d) If S and T_i $(i = 1, 2)$ are completely separated, then S and $T_1 \wedge T_2$ are also completely separated.

The following theorem from [7, Thm. 4.2] is crucial in our approach.

Theorem 2.2. *Let L be a frame and let $f, g \in \mathsf{F}(L)$. Then the following are equivalent:*

(i) *There exists $h \in \mathsf{C}(L)$ such that $f \leq h \leq g$.*
(ii) *The sublocales $f(\text{—}, q)$ and $g(p, \text{—})$ are completely separated for every $p < q$ in $\mathbb{Q}$.* ■

Proposition 2.3. *Let S and T be sublocales of L. Then S and T are completely separated if and only if there exists an $f \in \mathsf{C}(L)$ such that*

$$\chi_{\overline{S}} \leq f \leq \chi_{\overline{T}^*}.$$

Proof: Let S and T be sublocales. Then

$$\chi_{\overline{S}}(\text{—}, q) = \begin{cases} 0 & \text{if } q \leq 0, \\ \overline{S} & \text{if } 0 < q \leq 1, \\ 1 & \text{if } q > 1, \end{cases} \quad \text{and} \quad \chi_{\overline{T}^*}(p, \text{—}) = \begin{cases} 1 & \text{if } p < 0, \\ \overline{T} & \text{if } 0 \leq p < 1, \\ 0 & \text{if } p \geq 1, \end{cases}$$

and the result follows immediately by Remarks 2.1 (i) and (iii) and Theorem 2.2. ■

The following is also included in [7] (Remark 3.5).

Remark 2.4. Let $a, b \in L$. Then $b \prec\!\!\prec a$ if and only if $\mathfrak{o}(b)$ and $\mathfrak{c}(a)$ are completely separated.

From Proposition 2.3 and Remark 2.4 it follows immediately that

Corollary 2.5. *Let $a, b \in L$. Then $b \prec\!\!\prec a$ if and only there exists an $f \in \mathsf{C}(L)$ such that $\chi_{\overline{\mathfrak{o}(b)}} \le f \le \chi_{\mathfrak{o}(a)}$.* ■

3. Variants of [complete] regularity in frames

Recall that a topological space $(X, \mathcal{O}X)$ is *regular* if for each $U \in \mathcal{O}X$ and $x \in U$ there exists $V \in \mathcal{O}X$ such that $x \in V \subseteq \overline{V} \subseteq U$. The following characterizations are easy to get:

$$\begin{aligned}
(X,\mathcal{O}X) \text{ is regular} \;&\Leftrightarrow\; U = \bigcup\big\{V \in \mathcal{O}X \mid \overline{V} \subseteq U\big\} \text{ for every } U \in \mathcal{O}X \qquad (*)\\
&\Leftrightarrow\; \text{For every compact } K \subseteq X \text{ and } U \in \mathcal{O}X \text{ such}\\
&\qquad \text{that } K \subseteq U, \text{ there exists } V \in \mathcal{O}X \text{ such that}\\
&\qquad K \subseteq V \text{ and } \overline{V} \subseteq U. \qquad (**)
\end{aligned}$$

It is the mimic of condition $(*)$ in frames (more precisely, in the dual lattice of $\mathcal{S}(L)$) that is taken as the definition of a regular frame: a frame L is *regular* if $a = \bigvee\{b \in L \mid b \prec a\}$ for each $a \in L$, or, equivalently, if

$$\mathfrak{o}(a) = \bigwedge\{\mathfrak{o}(b) \mid b \prec a\} \quad \text{for each } a \in L.$$

What about condition $(**)$ in frames? We first note the following:

Proposition 3.1. *Let $(X, \mathcal{O}X)$ be a topological space. Then X is regular if and only if for each compact sublocale S and each $U \in \mathcal{O}X$ satisfying $\mathfrak{o}(U) \le S$, there exists $V_S \in \mathcal{O}X$ such that $\mathfrak{o}(V_S) \le S$ and $V_S \prec U$.*

Proof: Let S be a compact sublocale and $U \in \mathcal{O}X$ such that $\mathfrak{o}(U) \le S$. Then

$$\bigwedge\{\mathfrak{o}(V) \mid V \prec U\} = \mathfrak{o}(U) \le S$$

and a use of Fact 1.2 gives $\{V_i\}_{i=1}^n \subseteq L$ such that $V_i \prec U$ for every $i \in \{1, \ldots, n\}$ and $\bigwedge_{i=1}^n \mathfrak{o}(V_i) \le S$. Take $V_S = \bigcup_{i=1}^n V_i$. Then $\mathfrak{o}(V_S) = \bigwedge_{i=1}^n \mathfrak{o}(V_i) \le S$ and $\overline{V_S} = \bigcup_{i=1}^n \overline{V_i} \subseteq U$.

For the converse: First note that since $X \setminus \overline{\{x\}}$ is prime for each $x \in X$, it follows from [13, III.10.1] that $S_x = \{X \setminus \overline{\{x\}}, X\}$ is a compact sublocale of $\mathcal{O}X$ (a *one-point* sublocale). Moreover, given $U \in \mathcal{O}X$, we have that

$$\begin{aligned}
\mathfrak{o}(U) \le S_x \;&\Leftrightarrow\; X \setminus \overline{\{x\}} \in \mathfrak{o}(U) \;\Leftrightarrow\; U \to X \setminus \overline{\{x\}} = X \setminus \overline{\{x\}}\\
&\Leftrightarrow\; U \not\le X \setminus \overline{\{x\}} \;\Leftrightarrow\; U \cap \overline{\{x\}} \ne \varnothing \;\Leftrightarrow\; x \in U.
\end{aligned}$$

Let $U \in \mathcal{O}X$ and $x \in U$. Then S_x is a compact sublocale such that $\mathfrak{o}(U) \leq S_x$ and hence there exists $V_x \in \mathcal{O}X$ such that $V_x \prec U$ and $\mathfrak{o}(V_x) \leq S_x$. It follows that $x \in V_x$ and $V_x \subseteq U$. ■

In the same vein of the previous ideas, just replacing $\prec$ by $\prec\!\prec$, we have now the following:

Proposition 3.2. *Let $(X, \mathcal{O}X)$ be a topological space. Then X is completely regular if and only if for each compact sublocale S and each $U \in \mathcal{O}X$ satisfying $\mathfrak{o}(U) \leq S$, there exists $V_S \in \mathcal{O}X$ such that $\mathfrak{o}(V_S) \leq S$ and $V_S \prec\!\prec U$.* ■

This suggests the following variants of [complete] regularity in frames:

Definition 3.3. Let L be a frame. L is said to be *c-regular* (resp. *completely c-regular*) if for each compact sublocale S of L and each $a \in L$ such that $\mathfrak{o}(a) \leq S$, there exists $b_S \in L$ such that $\mathfrak{o}(b_S) \leq S$ and $b_S \prec a$ (resp. $b_S \prec\!\prec a$).

It is natural then to study the relationship between these variants and the original notions. One implication is almost obvious:

Proposition 3.4. *Let L be a frame. If L is [completely] regular, then it is [completely] c-regular.*

Proof: The proof follows the lines of the first implication in Proposition 3.1. ■

As it follows from Propositions 3.1 and 3.2, the converse implication in Proposition 3.4 holds for spatial frames. More generally, [complete] regularity coincides with [complete] c-regularity whenever any $a \in L$ satisfies

$$\mathfrak{o}(a) = \bigwedge\{S \in \mathcal{S}(L) \mid \mathfrak{o}(a) \leq S \text{ and } (S \text{ clopen or compact})\}. \qquad (***)$$

Indeed, let L be c-regular and $a \in L$. If $\mathfrak{o}(a) \leq S$ and S is clopen (i.e. $S = \mathfrak{c}(x) = \mathfrak{o}(x^*)$ for some complemented $x \in L$), then $x^* \prec a$ and so by (***), $S \geq \bigwedge \{\mathfrak{o}(b) \mid b \prec a\}$. On the other hand, for each compact sublocale S satisfying $\mathfrak{o}(a) \leq S$, by c-regularity there exists $b_S \in L$ such that $\mathfrak{o}(b_S) \leq S$ and $b_S \prec a$. It follows that $S \geq \mathfrak{o}(b_S) \geq \bigwedge \{\mathfrak{o}(b) \mid b \prec a\}$. Hence

$$\begin{aligned}\mathfrak{o}(a) &= \bigwedge\{S \in \mathcal{S}(L) \mid \mathfrak{o}(a) \leq S \text{ and } (S \text{ clopen or compact})\} \\ &\geq \bigwedge \{\mathfrak{o}(b) \mid \mathfrak{o}(a) \leq b \prec a\}\end{aligned}$$

and since the converse inequality is always true we conclude that a is regular. A similar argument applies in the case of complete regularity.

Note that any complemented $a \in L$ satisfies (***) since in that case $\mathfrak{o}(a)$ is clopen, and the join of any set of elements satisfying (***) also satisfies (***). Therefore, each zero-dimensional frame (in particular, each Boolean frame) satisfies (***). On the other hand, by Corollary 1.3, any compact element satisfies (***). Consequently algebraic frames also satisfy (***).

Question 3.5. The question is left open whether every [completely] c-regular frame is [completely] regular. As shown above, the two notions coincide for a wide class of frames, namely the ones satisfying condition (***) for any $a \in L$ — as all completely regular frames (in particular, zero-dimensional or Boolean frames), spatial frames and algebraic frames—, but we believe this not to be the case in general. However a proof of this has eluded us so far.

4. Insertion theorem: not quite like the classical case

Given a frame L, we say that an $f \in \mathsf{F}(L)$ is *upper compact-like* (resp. *lower compact-like*) if $f(—, p)$ (resp. $f(p, —)$) is a compact sublocale of L for every $p \in \mathbb{Q}$. As a first example we note the following obvious proposition:

Proposition 4.1. *Let S be a complemented sublocale of a frame L. Then:*

(a) *χ_S is upper compact-like if and only if S is compact.*
(b) *χ_S is lower compact-like if and only if S^* is compact.* ∎

Proposition 4.2.

(a) *If L is compact, then all upper semicontinuous functions on L are upper compact-like.*
(b) *If L is Hausdorff or fit, then all upper (resp. lower) compact-like functions on L are upper (resp. lower) semicontinuous.*

Proof: (a) is a consequence of Fact 1.4 and (b) follows immediately from the fact that in any Hausdorff (or fit) frame, compact sublocales are closed [14]. ∎

In order to obtain our insertion result we need first the following Urysohn-type separation result.

Lemma 4.3. *The following statements are equivalent for any frame L:*

(i) *L is completely c-regular.*
(ii) *Every two sublocales S and T of L such that $S \vee T = 1$, one of which is compact and the other closed, are completely separated.*

Proof: (i)$\Rightarrow$(ii): Let S and T be sublocales such that $S \vee T = 1$, with S being compact and $T = \mathfrak{c}(a)$. Then we have $S \vee \mathfrak{c}(a) = 1$ iff $\mathfrak{o}(a) \leq S$ and therefore, by hypothesis, there exists $b_S \in L$ such that $\mathfrak{o}(b_S) \leq S$ and $b_S \prec\!\prec a$. By Proposition 2.4, $\mathfrak{o}(b_S)$ and $T = \mathfrak{c}(a)$ are completely separated and, finally, by Remark 2.1 (2) so are S and T.
(ii)$\Rightarrow$(i): Let $a \in L$ and let S be a compact sublocale such that $\mathfrak{o}(a) \leq S$. Since $\mathfrak{c}(a) \vee S = 1$, it follows by hypothesis that $\mathfrak{c}(a)$ and S are completely separated. Hence there exists $h \in \mathsf{C}(L)$ such that $h(0,\text{—}) \leq \mathfrak{c}(a)$ and $h(\text{—},1) \leq S$ and thus $h\left(0,\frac{1}{2}\right) \leq h\left(\frac{1}{2},\text{—}\right)^* \leq h(\text{—},1) \leq S$. Let $b_S \in L$ such that $h(\frac{1}{2},\text{—})^* = \mathfrak{o}(b_S)$. Then $\mathfrak{o}(b_S)$ and $\mathfrak{c}(a)$ are completely separated, i.e. $b_S \prec\!\prec a$. ■

Remark 4.4. We point out that a proof that complete regularity implies the statement (ii) above appears in [3, Lemma 2.1].

We can now prove our insertion-type result for completely c-regular frames:

Theorem 4.5. *Let L be a completely c-regular frame. If $f, g \in \mathsf{F}(L)$, f is upper compact-like, g is lower semicontinuous and $f \leq g$, then there exists $h \in \mathsf{C}(L)$ such that $f \leq h \leq g$.*

The converse holds for frames in which every compact sublocale is complemented (in particular, Hausdorff or fit frames).

Proof: Let $f, g \in \mathsf{F}(L)$ such that f is upper compact-like, g is lower semicontinuous and $f \leq g$. Then $f(\text{—}, q)$ is compact, $g(p,\text{—})$ is closed and $f(\text{—}, q) \vee g(p,\text{—}) = 1$ for each $p < q$ in $\mathbb{Q}$. It follows from Lemma 4.3 that $f(\text{—}, q)$ and $g(p,\text{—})$ are completely separated. We conclude from Theorem 2.2 that there exists $h \in \mathsf{C}(L)$ such that $f \leq h \leq g$.

Conversely, let S and T be sublocales such that $S \vee T = 1$, with S being compact (hence complemented) and $T = \mathfrak{c}(a)$ closed. Then $S \geq \mathfrak{o}(a)$ and so $\chi_S \leq \chi_{\mathfrak{o}(a)}$ with χ_S being upper compact-like and $\chi_{\mathfrak{o}(a)}$ lower semicontinuous. It follows that there exists $h \in \mathsf{C}(L)$ such that $\chi_S \leq h \leq \chi_{\mathfrak{o}(a)}$, that is $h(\text{—},1) \leq S$ and $h(0,\text{—}) \leq \mathfrak{c}(a)$. ■

Remark 4.6. Recall from [10] that there is an order-isomorphism

$$-(\cdot)\colon \mathsf{F}(L) \to \mathsf{F}(L)$$

defined by

$$(-f)(\text{—}, r) = f(-r, \text{—}) \quad \text{for every } r \in \mathbb{Q},$$

and that $f \in \mathsf{F}(L)$ is upper semicontinuous (resp. compact-like) if and only if $-f$ is lower semicontinuous (resp. compact-like). Consequently, we have the dual result: if L is a completely c-regular frame and $f, g \in \mathsf{F}(L)$ are such that f is upper semicontinuous, g is lower compact-like and $f \leq g$, then there exists $h \in \mathsf{C}(L)$ such that $f \leq h \leq g$.

Remark 4.7. It should be noted that the statement (ii) of the (Urysohn)-type lemma is just the particularization of the insertion statement above to characteristic functions.

5. Extension theorem

Let S be a sublocale of L. Recall from [1] that a frame homomorphism $\varphi\colon \mathfrak{L}(\mathbb{R}) \to S$ is said to have an *extension* to L if there exists a frame homomorphism $\widetilde{\varphi}\colon \mathfrak{L}(\mathbb{R}) \to L$ such that $c_S \cdot \widetilde{\varphi} = \varphi$.

An $f \in \mathsf{C}(S)$ has a *continuous extension* to L if the associated frame homomorphism $\varphi\colon \mathfrak{L}(\mathbb{R}) \to L$ (such that $f = \mathfrak{c} \cdot \varphi$) has an extension to L. The sublocale S is then said to be *C-embedded* if every $f \in \mathsf{C}(S)$ has a continuous extension to L. Denoting by $\mathsf{C}^*(S)$ the functions of $\mathsf{C}(S)$ such that $f((\text{—},0) \vee (1,\text{—})) = 0$, S is said to be *C^*-embedded* if every $f \in \mathsf{C}^*(S)$ has a continuous extension to L.

Theorem 5.1. *Every compact sublocale of a completely regular frame L is C^*-embedded in L.*

Proof: Let S be a compact sublocale and let $\varphi\colon \mathfrak{L}(\mathbb{R}) \to S$ be a frame homomorphism such that $\varphi((\text{—},0) \vee (1,\text{—})) = 0$. Define $\mathcal{S} = (S_r \mid r \in \mathbb{Q}) \subseteq \mathcal{S}(L)$ as follows:

$$S_r = \begin{cases} 0, & \text{if } r \geq 0; \\ \mathfrak{c}(\varphi(\text{—},-r)), & \text{if } -1 \leq r < 0; \\ 1, & \text{if } r < -1. \end{cases}$$

Since $\mathcal{S}$ is antitone, S_r is complemented in $\mathcal{S}(L)$ for every $r \in \mathbb{Q}$ and $\bigvee_{r\in\mathbb{Q}} S_r = 1 = \bigvee_{r\in\mathbb{Q}} {S_r}^*$, it follows that it is a scale that generates an $f_1 \in \mathsf{F}(L)$. Let $f = -f_1$. Then

$$f(p,\text{—}) = f_1(\text{—},-p) = \bigvee_{s<-p} {S_s}^* = \begin{cases} 0 & \text{if } p \geq 1 \\ \displaystyle\bigvee_{r>p} \mathfrak{o}(\varphi(\text{—},r)), & \text{if } 0 \leq p < 1 \\ 1 & \text{if } p < 0 \end{cases}$$

and

$$f(—,q) = f_1(-q,—) = \bigvee_{s>-q} S_s = \begin{cases} 1 & \text{if } q > 1 \\ \mathfrak{c}(\varphi(—,q)) & \text{if } 0 < q \leq 1 \\ 0 & \text{if } q \leq 0. \end{cases}$$

Similarly, $\mathcal{T} = (T_r \mid r \in \mathbb{Q}) \subseteq \mathcal{S}(L)$ defined by

$$T_r = \begin{cases} 0 & \text{if } r \geq 1 \\ \mathfrak{c}(\varphi(r,—)) & \text{if } 0 \leq r < 1 \\ 1 & \text{if } r < 0 \end{cases}$$

is also a scale. The corresponding $g \in \mathsf{F}(L)$ is now given by

$$g(p,—) = T_p \quad \text{and} \quad g(—,q) = \begin{cases} 1 & \text{if } q > 1 \\ \bigvee_{r<q} \mathfrak{o}(\varphi(r,—)), & \text{if } 0 < q \leq 1 \\ 0 & \text{if } q \leq 0. \end{cases}$$

Since L is regular (hence Hausdorff), S is a closed sublocale and so, for each $0 < q \leq 1$,

$$f(—,q) = \mathfrak{c}(\varphi(—,q)) = \mathfrak{c}(\varphi(—,q)) \vee S$$

is a closed sublocale of S and since S is compact, by Fact 1.4 we may conclude that $f(—,q)$ is compact. Hence f is upper compact-like. On the other hand, $g(p,—)$ is a closed sublocale for every p, hence g is lower semicontinuous. Finally, $\varphi(—,r) \vee \varphi(p,—) = 1$ for each $0 \leq p < r < 1$ and thus $\mathfrak{o}(\varphi(—,r)) \leq \mathfrak{c}(\varphi(—,p))$. Hence

$$f(p,—) = \bigvee_{r>p} \mathfrak{o}(\varphi(—,r)) \leq \mathfrak{c}(\varphi(—,p)) = g(p,—),$$

that is, $f \leq g$.

It then follows from Theorem 4.5 that there exists $h \in \mathsf{C}(L)$ such that $f \leq h \leq g$. Consequently, $h(p,—) = 0 = \mathfrak{c}(\varphi(p,—))$ for every $p \geq 1$, $h(p,—) = 1 = \mathfrak{c}(\varphi(p,—))$ for every $p < 0$ and, for each $0 \leq p < 1$, we have

$$\begin{aligned} \mathfrak{c}(\varphi(p,—)) = \bigvee_{r>p} \mathfrak{c}(\varphi(r,—)) \leq \bigvee_{r>p} \mathfrak{o}(\varphi(—,r)) = \\ = f(p,—) \leq h(p,—) \leq g(p,—) = \mathfrak{c}(\varphi(p,—)). \end{aligned}$$

Similarly, $h(\text{—}, q) = 1 = \mathfrak{c}(\varphi(\text{—}, q))$ for every $q > 1$, $h(\text{—}, q) = 0 = \mathfrak{c}(\varphi(\text{—}, q))$ for every $q \leq 0$ and, for each $0 < q \leq 1$,

$$\begin{aligned}\mathfrak{c}(\varphi(\text{—}, q)) = \bigvee_{r<q} \mathfrak{c}(\varphi(\text{—}, r)) \leq \bigvee_{r<q} \mathfrak{o}(\varphi(r, \text{—})) = \\ = g(\text{—}, q) \leq h(\text{—}, q) \leq f(\text{—}, q) = \mathfrak{c}(\varphi(\text{—}, q)).\end{aligned}$$

We then conclude that $h = \mathfrak{c} \cdot \varphi$ and thus

$$\widetilde{\varphi} = \mathfrak{c}^{-1} \cdot h \colon \mathfrak{L}(\mathbb{R}) \to L$$

is the desired extension of φ. ■

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