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SOME OPERATORS THAT PRESERVE THE LOCALITY OF A PSEUDOVARIETY OF SEMIGROUPS

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Abstract: It is shown that if V is a local monoidal pseudovariety of semigroups, then $\mathsf{K}Ⓜ\mathsf{V}$, $\mathsf{D}Ⓜ\mathsf{V}$ and $\mathcal{L}\mathsf{I}Ⓜ\mathsf{V}$ are local. Other operators of the form $\mathsf{Z}Ⓜ(\_)$ are considered. In the process, results about the interplay between the operators $\mathsf{Z}Ⓜ(\_)$ and $(\_) * \mathsf{D}_k$ are obtained.

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1. Introduction

A pseudovariety of monoids V is *local* if the pseudovariety of categories generated by V coincides with the pseudovariety of categories whose local monoids belong to V. Replacing *monoid* by *semigroup*, and *category* by *semigroupoid*, we get the notion of *local pseudovariety of semigroups*. These are important notions due to the key role finite categories/semigroupoids often play in operations between monoid/semigroup pseudovarieties [32, 25].

In [13, 14] one finds some unary operators preserving the locality of sub-pseudovarieties of the pseudovariety DS of monoids whose regular $\mathcal{D}$-classes are semigroups. Further examples related to the operators studied in this paper appear in [28]. For a pseudovariety V of monoids/semigroups, the pseudovariety of semigroups whose local monoids belong to V is denoted by $\mathcal{L}\mathsf{V}$. In [28], Steinberg proves that when H is a pseudovariety of groups and V is a local pseudovariety of monoids, then the Mal'cev product $\mathcal{L}\mathsf{H}Ⓜ\mathsf{V}$ is local if V contains the six element Brandt monoid B_2^1 and H is Fitting, or if H is extension-closed and nontrivial. Also in [28], Steinberg posed the following question concerning the trivial pseudovariety I: is $\mathcal{L}\mathsf{I}Ⓜ\mathsf{V}$ local for *every* local pseudovariety V of monoids? Since I is a Fitting pseudovariety of groups, it suffices to consider pseudovarieties V such that $B_2^1 \notin \mathsf{V}$, that is, such that

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$\mathsf{V} \subseteq \mathsf{DS}$. On the other hand, Almeida had already proved the locality of all subpseudovarieties of DO (the pseudovariety of monoids whose regular $\mathcal{D}$-classes are orthodox semigroups) in the image of the operator $\mathcal{L}\mathsf{I}\circledcirc(\_)$ [2]. An example which is not covered by the previous cases, is that of the pseudovariety $\mathcal{L}\mathsf{I}\circledcirc\mathsf{CR}$ (the locality of CR was proved in [11]).

We answer Steinberg's question affirmatively. First we prove the operator $\mathsf{K}\circledcirc(\_)$ preserves locality, where K is the pseudovariety of semigroups whose idempotents are left zeros. This implies $\mathsf{D}\circledcirc(\_)$ preserves locality, where D is the dual of K. As the operator $\mathcal{L}\mathsf{I}\circledcirc(\_)$ is equal to the infinite alternate composition of $\mathsf{K}\circledcirc(\_)$ and $\mathsf{D}\circledcirc(\_)$ [7], we easily deduce that $\mathcal{L}\mathsf{I}\circledcirc(\_)$ preserves locality. We do not take advantage of those cases where Steinberg and Almeida had already shown that $\mathcal{L}\mathsf{I}\circledcirc(\_)$ preserves locality, but when dealing with the restriction of $\mathsf{K}\circledcirc(\_)$ to subpseudovarieties of DS, we use left basic factorizations of implicit operations, a key idea in Almeida's paper [2].

Results about operators of the form $\mathsf{Z}\circledcirc(\_)$ for pseudovarieties Z other than $\mathcal{L}\mathsf{I}$, K or D, are obtained. Many concern the interplay between the operators $\mathsf{Z}\circledcirc(\_)$ and $(\_) * \mathsf{D}_k$, where k is a positive integer and D_k is the pseudovariety of semigroups satisfying the identity $yx_1 \cdots x_k = x_1 \cdots x_k$ (recall that $\mathsf{D} = \bigcup_{k\geq 1} \mathsf{D}_k$). No matter V is a monoid or semigroup pseudovariety, the semidirect product $\mathsf{V} * \mathsf{D}_k$ is always a semigroup pseudovariety. This leads us to consider $\mathsf{Z}\circledcirc(\_)$ as an operator on the lattice of semigroup pseudovarieties. The translation to an operator on monoid pseudovarieties is easy. As is common in the literature, we often use the same notation for a pseudovariety of monoids and for its generated pseudovariety of semigroups. For example, Sl can denote the pseudovariety of monoid semilattices or the pseudovariety of semigroup semilattices, depending on the context.

As usual, the pseudovarieties of groups and of nilpotent semigroups are denoted by G and N, respectively. The set of semigroup pseudovarieties

$$\mathbb{V} = \{\mathcal{L}\mathsf{I}, \mathsf{K}, \mathsf{D}, \mathsf{N}, \mathcal{L}\mathsf{G}, \mathsf{K} \vee \mathsf{G}, \mathsf{D} \vee \mathsf{G}, \mathsf{N} \vee \mathsf{G}\}$$

plays an important role along this paper, due to the following proposition.

Proposition 1.1. *Let* $\mathsf{Z} \in \mathbb{V}$. *Then* $\mathcal{L}(\mathsf{Z}\circledcirc\mathsf{V}) = \mathsf{Z}\circledcirc\mathcal{L}\mathsf{V}$.

The case where $\mathsf{Z} = \mathcal{L}\mathsf{I}$ is Exercise 4.6.58 in [25]. The general case $\mathsf{Z} \in \mathbb{V}$ offers no additional difficulty, but given its importance to our paper, we give an explicit proof of Proposition 1.1 in the appendix at the end of this paper.

Proposition 1.1 motivates the search for conditions under which the inclusion

$$\mathsf{Z}ⓜ(\mathsf{V} * \mathsf{D}_k) \subseteq (\mathsf{Z}ⓜ\mathsf{V}) * \mathsf{D}_k. \tag{1.1}$$

holds for all $k \geq 1$, since if $\mathcal{L}(\mathsf{Z}ⓜ\mathsf{V}) = \mathsf{Z}ⓜ\mathcal{L}\mathsf{V}$ and (1.1) hold, then the locality of V implies the locality of $\mathsf{Z}ⓜ\mathsf{V}$, if V is monoidal (cf. Lemma 4.5). With this motivation, we introduce in Section 3 *permanent* pseudovarieties. These pseudovarieties have basis of pseudoidentities satisfying certain conditions allowing an application of known results about pseudoidentities basis of Mal'cev products [23] and of semidirect products with D_k [3]. We prove that if Z is permanent then (1.1) holds if $\mathsf{Z}ⓜ\mathsf{V}$ is not contained in DS. All elements of $\mathbb{V}$ are permanent. The pseudovariety A of finite aperiodic semigroups is also permanent.

If $\mathsf{Z} \in \{\mathsf{K}, \mathsf{D}, \mathcal{L}\mathsf{I}\}$, we show that (1.1) also holds when $\mathsf{Sl} \subseteq \mathsf{V} \subseteq \mathsf{DS}$. The proof depends strongly on the locality of DS [14]. The aforementioned preservation of locality by operator $\mathcal{L}\mathsf{G}ⓜ(\_)$, proved by Steinberg [28], gives another proof of the locality of DS, since $\mathsf{DS} = \mathcal{L}\mathsf{G}ⓜ\mathsf{Sl}$ [24, 26] and Sl is local [8, 10]. Other examples are discussed in Section 4, where the main results and applications appear. Among the applications, we deduce from cases where (1.1) holds, a result (Theorem 4.11) about varieties of formal languages, indicating the interest of (1.1) is not limited to locality questions.

The proof of some of the main results in Section 4 is deferred to the three final sections. One of them, Section 6, is dedicated to implicit operations on $\mathsf{DS} * \mathsf{D}_k$, and seems to be of independent interest.

2. Preliminaries

This paper concerns the theory of pseudovarieties of semigroups/monoids (classes of finite semigroups/monoids closed under taking homomorphic images, subsemigroups/submonoids and finitary direct products) to which we assume the reader is familiar with. As supporting references, we indicate books [1, 25].

By an *alphabet* we mean a finite nonempty set. Let A be an alphabet. For each pseudovariety of semigroups we denote by $\overline{\Omega}_A\mathsf{V}$ the A-generated free pro-V semigroup. The elements of $\overline{\Omega}_A\mathsf{V}$ can be interpreted as the *implicit operations* on V, in the sense developed in [1, Chapter 3] and [25, Chapter 3]. The subsemigroup of $\overline{\Omega}_A\mathsf{V}$ generated by A, which is dense in $\overline{\Omega}_A\mathsf{V}$, is denoted by $\Omega_A\mathsf{V}$. If V contains N, then $\Omega_A\mathsf{V}$ is isomorphic to the free semigroup A^+, and every element of $\Omega_A\mathsf{V}$ is isolated in $\overline{\Omega}_A\mathsf{V}$; in this case we identify $\Omega_A\mathsf{V}$ with the semigroup A^+. The elements of $\overline{\Omega}_A\mathsf{V}$ can be viewed as generalizations of words on the alphabet A, and so sometimes they are also called *pseudowords* (of $\overline{\Omega}_A\mathsf{V}$).

Since the cardinal of an alphabet determines $\overline{\Omega}_A\mathsf{V}$ and $\Omega_A\mathsf{V}$, we may write $\overline{\Omega}_{|A|}\mathsf{V}$ and $\Omega_{|A|}\mathsf{V}$ instead of $\overline{\Omega}_A\mathsf{V}$ and $\Omega_A\mathsf{V}$, respectively. Let S be the pseudovariety of all finite semigroups. The semigroup $\overline{\Omega}_2\mathsf{S}$ will play an important role in this paper. The elements of the generating set of $\overline{\Omega}_2\mathsf{S}$ are the implicit operations x_1 and x_2, where x_i is the binary implicit operation "projection on the i-th component".

An ordered pair (u, v) of implicit operations of $\overline{\Omega}_A\mathsf{V}$ can be interpreted as a *pseudoidentity* over the alphabet A, and as such is denoted by the formal equality $u = v$. We will also use the standard notation $S \models u = v$ and $\mathsf{V} \models u = v$ to indicate that the profinite semigroup S and the pseudovariety V satisfy the pseudoidentity $u = v$, and we write $\mathsf{V} = [\![\Sigma]\!]$ to indicate that V is defined by the basis of pseudoidentities Σ. Again, the reader looking for details may consult [1, Chapter 3] or [25, Chapter 3].

The *content* of a pseudoword $u \in \overline{\Omega}_A\mathsf{V}$, with $\mathsf{Sl} \subseteq \mathsf{V}$, is the image of u by the canonical homomorphism $c : \overline{\Omega}_A\mathsf{V} \to \overline{\Omega}_A\mathsf{Sl}$. It amounts to the set of letters on which u depends [1, Section 8.1]. One defines $c(1) = \emptyset$.

For a semigroup S with operation $\star$, its *dual*, denoted S^{op}, is the semigroup with the same underlying set and operation $\star_{op}$ such that $s \star_{op} t = t \star s$ for all $s, t \in S$. If S is a topological semigroup, then S^{op} is a topological semigroup for the same topology. The *dual of a pseudovariety of semigroups* V is the pseudovariety $\mathsf{V}^{op} = \{S^{op} \mid S \in \mathsf{V}\}$. The pseudovariety V is *self-dual* if $\mathsf{V} = \mathsf{V}^{op}$.

Given an alphabet A, the semigroup $(\overline{\Omega}_A\mathsf{S})^{op}$ is profinite, and so there is a unique continuous homomorphism χ_A from $\overline{\Omega}_A\mathsf{S}$ to $(\overline{\Omega}_A\mathsf{S})^{op}$ such that $\chi_A(a) = a$ for every $a \in A$. Applying a forgetful functor, we may view χ_A as a continuous self-mapping of $\overline{\Omega}_A\mathsf{S}$, and thus we may consider the composition $\chi_A \circ \chi_A$. Clearly, one has $\chi_A \circ \chi_A(A^+) = A^+$. Since A^+ is dense in $\overline{\Omega}_A\mathsf{S}$, we have $\chi_A \circ \chi_A = id_{\overline{\Omega}_A\mathsf{S}}$. Therefore χ_A is a continuous isomorphism from $\overline{\Omega}_A\mathsf{S}$ onto $(\overline{\Omega}_A\mathsf{S})^{op}$ (cf. [1, Exercise 5.2.1]).

Lemma 2.1. *Let $u, v \in \overline{\Omega}_A\mathsf{S}$. For a finite semigroup S, one has $S \models u = v$, if and only if $S^{op} \models \chi_A(u) = \chi_A(v)$.*

Proof: In the case where $u, v \in A^+$, the lemma is justified by a simple induction on $|u| + |v|$. For the general case, note that, since S is finite, χ_A is continuous and A^+ is dense in $\overline{\Omega}_A\mathsf{S}$, there are $x, y \in A^+$ such that $S \models u = x$, $S \models v = y$, $S^{op} \models \chi_A(u) = \chi_A(x)$ and $S^{op} \models \chi_A(v) = \chi_A(y)$. Hence, $S \models u = v$ if and only if $S \models x = y$, if and only if $S^{op} \models \chi_A(x) = \chi_A(y)$ (by the already proved case), if and only if $S^{op} \models \chi_A(u) = \chi_A(v)$. ■

Given a semigroup S which is not a monoid, one denotes by S^1 the monoid obtained by S by adjoining an identity; if the semigroup S is a monoid, then one defines $S^1 = S$. Occasionally it will be convenient to consider the profinite monoid $(\overline{\Omega}_A\mathsf{V})^1$, obtained from $\overline{\Omega}_A\mathsf{V}$ by adjoining the empty word 1 as an identity and an isolated point. If S is a profinite semigroup and V is a semigroup pseudovariety, then $S \models 1 = 1$ and $\mathsf{V} \models 1 = 1$.

For a pseudovariety of monoids V, the pseudovariety of semigroups generated by the elements of V viewed as semigroups is denoted by V_S. It is well known that $S \in \mathsf{V}_S$ if and only if $S^1 \in \mathsf{V}$ ([1, Section 7.1]). A pseudovariety of semigroups is *monoidal* if it is of the form V_S for some pseudovariety of monoids V.

Let W be a pseudovariety of semigroups. As is usual in the literature, regardless V is a pseudovariety of semigroups or of monoids, we shall denote by $\mathsf{W} \circledm \mathsf{V}$ the Mal'cev product of W with V and by $\mathsf{W} * \mathsf{V}$ the semidirect product of W by V.

Recall that, if V is a pseudovariety of semigroups, then $\mathsf{W} \circledm \mathsf{V}$ is the pseudovariety of semigroups generated by finite semigroups S for which there is a semigroup homomorphism φ from S into some element T of W such that $\varphi^{-1}(e) \in \mathsf{W}$ for every idempotent e in T. Replacing *monoid* by *semigroup* in the previous sentence, we obtain the characterization of $\mathsf{W} \circledm \mathsf{V}$ when V is a pseudovariety of monoids. We remark the following difference between the Mal'cev product of two semigroup pseudovarieties and the Mal'cev product of a semigroup pseudovariety with a monoid pseudovariety: in the former case, $\mathsf{W} \circledm \mathsf{V}$ contains W, while in the latter case, that may not occur. In both cases, $\mathsf{W} \circledm \mathsf{V}$ contains V. The following proposition seems folklore.

Proposition 2.2. *Let* W *be a pseudovariety of semigroups, and let* V *be a pseudovariety of monoids. If* V *contains* Sl *then* $(\mathsf{W} \circledm \mathsf{V})_\mathsf{S} = \mathsf{W} \circledm \mathsf{V}_\mathsf{S}$.

Proof: Clearly, $\mathsf{W} \circledm \mathsf{V} \subseteq \mathsf{W} \circledm \mathsf{V}_\mathsf{S}$, thus $(\mathsf{W} \circledm \mathsf{V})_\mathsf{S} \subseteq \mathsf{W} \circledm \mathsf{V}_\mathsf{S}$.

Conversely, let S be a finite semigroup for which there is a homomorphism $\varphi : S \to T$ such that $T \in \mathsf{V}_S$ and $\varphi^{-1}(e)$ for every idempotent e of S. For a semigroup R, denote by R^I the monoid obtained from R by adjoining an identity I not in R (regardless if R is a monoid or not). Extend $\varphi : S \to T$

to a monoid homomorphism $\bar{\varphi} : S^I \to T^I$. Suppose that T is not a monoid. Then $T^I \in \mathsf{V}$, since $T \in \mathsf{V}_S$, and so, as $\bar{\varphi}^{-1}(I) = \{I\}$, we conclude that $S^I \in \mathsf{W}ⓜ\mathsf{V}$, thus $S \in (\mathsf{W}ⓜ\mathsf{V})_{\mathsf{S}}$. Suppose now that T is a monoid with identity 1_T. Consider the monoid semilattice $U_1 = \{0,1\}$, with the usual multiplication. Then $T \times U_1 \in \mathsf{V}$. Note that $T \times \{0\} \cup \{(1_T, 1)\}$ is a submonoid of $T \times U_1$ isomorphic to T^I. Hence, $T^I \in \mathsf{V}$. Again, from the extension of φ to a monoid homomorphism $\bar{\varphi} : S^I \to T^I$ we deduce that $S^I \in \mathsf{W}ⓜ\mathsf{V}$ and so $S \in (\mathsf{W}ⓜ\mathsf{V})_{\mathsf{S}}$. This concludes the proof that $\mathsf{W}ⓜ\mathsf{V}_{\mathsf{S}} \subseteq (\mathsf{W}ⓜ\mathsf{V})_{\mathsf{S}}$. ■

Proposition 2.2 fails if V does not contain the monoid semillatices. For example, viewing the pseudovariety G of groups as a semigroup pseudovariety, we have $\mathsf{K}ⓜ\mathsf{G} = \mathsf{K} \vee \mathsf{G}$, while interpreting G as a monoid pseudovariety we have $\mathsf{K}ⓜ\mathsf{G} = \mathsf{G}$ (the deduction of these equalities are easy exercises, cf. [25, Exercise 4.6.9]).

Proposition 2.3. *We have* $(\mathsf{W}ⓜ\mathsf{V})^{op} = \mathsf{W}^{op}ⓜ\mathsf{V}^{op}$ *for every pair* V, W *of pseudovarieties of semigroups.*

Proof: Note that if V is generated by a class $\mathcal{K}$ of semigroups then V^{op} is generated by the class $\{S^{op} \mid S \in \mathcal{K}\}$. So, let S be a generating element of $\mathsf{W}ⓜ\mathsf{V}$ for which there is a homomorphism $\varphi : S \to T$ such that T belongs to V and $\varphi^{-1}(e) \in \mathsf{W}$ for every idempotent e from T. We may consider the homomorphism $\varphi_{op} : S^{op} \to T^{op}$ which as a set-theoretic mapping coincides with φ. Note that $\varphi_{op}^{-1}(e) = \varphi^{-1}(e)^{op}$. Therefore, $S^{op} \in \mathsf{W}^{op}ⓜ\mathsf{V}^{op}$, thus proving $(\mathsf{W}ⓜ\mathsf{V})^{op} \subseteq \mathsf{W}^{op}ⓜ\mathsf{V}^{op}$. Since the pseudovarieties are arbitrary, we also have $\mathsf{W}^{op}ⓜ\mathsf{V}^{op} = ((\mathsf{W}^{op}ⓜ\mathsf{V}^{op})^{op})^{op} \subseteq (\mathsf{W}ⓜ\mathsf{V})^{op}$. ■

Proposition 2.4 ([31]). *If* V *is a pseudovariety of semigroups containing some nontrivial monoid, then* $(\mathsf{V} * \mathsf{D}_k)^{op} = \mathsf{V}^{op} * \mathsf{D}_k$ *for every* $k \geq 1$*, and thus* $(\mathsf{V} * \mathsf{D})^{op} = \mathsf{V}^{op} * \mathsf{D}$.

We remark that in the statement of Proposition 2.4 we choose the semigroup varietal version stated in [1, Exercise 10.6.9], instead of the original monoid version from [31].

3. Permanent pseudovarieties

Definition 3.1. Let $u = v$ be a pseudoidentity between implicit operations u, v of $\overline{\Omega}_2\mathsf{S}$ such that $u^2 = u$, $u(u,v) = u$ and $v(u,v) = v$. If additionally we have $v = uv$ (respectively $v = vu$) then we say that $u = v$ is a *left-permanent* (respectively *right-permanent*) pseudoidentity.

A pseudovariety of semigroups is *left-permanent* (respectively *right-permanent*) if it has a basis of left-permanent (respectively *right-permanent*) pseudoidentities.

A pseudovariety which is the intersection of a family of left-permanent and right-permanent pseudovarieties is called a *permanent* pseudovariety. Alternatively, a permanent pseudovariety is a pseudovariety having a basis consisting of permanent pseudoidentities, where by *permanent pseudoidentity* we mean a pseudoidentity which is either left-permanent or right-permanent.

Proposition 3.2. *A pseudovariety of semigroups* Z *is left-permanent if and only if the pseudovariety* Z^{op} *is right-permanent.*

Proof: Since a left-permanent (respectively right-permanent) pseudovariety is the intersection of pseudovarieties defined by only one left-permanent (respectively right-permanent) pseudoidentity, it suffices to consider the case where $\mathsf{Z} = [\![u = v]\!]$ for some left-permanent pseudoidentity $u = v$.

Then, by Lemma 2.1, we have $\mathsf{Z}^{op} = [\![\chi_2(u) = \chi_2(v)]\!]$. Now,

$$\chi_2(v) = \chi_2(uv) = \chi_2(u) \cdot_{op} \chi_2(v) = \chi_2(v) \cdot \chi_2(u).$$

It is also clear that $\chi_2(u)$ is an idempotent.

Finally, let $\varphi : \overline{\Omega}_2\mathsf{S} \to \overline{\Omega}_2\mathsf{S}$ be the unique continuous endomorphism $\varphi : \overline{\Omega}_2\mathsf{S} \to \overline{\Omega}_2\mathsf{S}$ such that $\varphi(x_1) = u$ and $\varphi(x_2) = v$, and consider the unique continuous endomorphism $\psi : \overline{\Omega}_2\mathsf{S} \to \overline{\Omega}_2\mathsf{S}$ such that $\psi(x_i) = \chi_2(\varphi(x_i))$, for $i \in \{1, 2\}$. By induction on the length of u, it is routine to check that $\psi(u) = \chi_2 \circ \varphi \circ \chi_2(u)$ for every $u \in \Omega_2\mathsf{S}$ (recall that the composition $\chi_2 \circ \varphi \circ \chi_2$ is well defined via the application of a forgetful functor). Hence, since $\Omega_2\mathsf{S}$ is dense in $\overline{\Omega}_2\mathsf{S}$ and ψ, φ and χ_2 are continuous mappings, it follows that the function ψ equals the composite $\chi_2 \circ \varphi \circ \chi_2$. Then

$$\chi_2(u)(\chi_2(u), \chi_2(v)) = \psi(\chi_2(u)) = \chi_2 \circ \varphi \circ \chi_2 \circ \chi_2(u).$$

Since $\chi_2 \circ \chi_2$ is the identity and $\varphi(u) = u(u, v) = u$, we conclude that $\chi_2(u)(\chi_2(u), \chi_2(v)) = \chi_2(u)$. Similarly, $\chi_2(v)(\chi_2(u), \chi_2(v)) = \chi_2(v)$. Therefore, $\chi_2(u) = \chi_2(v)$ is a right-permanent pseudoidentity and Z^{op} is a right-permanent pseudovariety. ∎

We now give some examples. The five pseudoidentities

$$x_1^\omega = x_1^\omega x_2, \quad x_1^\omega = x_1^\omega x_2^\omega, \quad x_1^\omega = x_1^\omega x_2 x_1^\omega, \quad x_1^\omega = (x_1^\omega x_2 x_1^\omega)^\omega, \quad x_2^\omega = x_2^{\omega+1},$$

are left-permanent. The corresponding left-permanent pseudovarieties defined by each of them are K, $\mathsf{K} \vee \mathsf{G}$, $\mathcal{L}\mathsf{I}$, $\mathcal{L}\mathsf{G}$ and A, respectively. The pseudovarieties D, $\mathsf{D} \vee \mathsf{G}$, $\mathcal{L}\mathsf{I}$, $\mathcal{L}\mathsf{G}$ and A are right-left permanent pseudovarieties, a fact easy to verify directly, but that also follows immediately from Proposition 3.2 because $\mathsf{D} = \mathsf{K}^{op}$, $\mathsf{D} \vee \mathsf{G} = (\mathsf{K} \vee \mathsf{G})^{op}$ and $\mathcal{L}\mathsf{I}$, $\mathcal{L}\mathsf{G}$, A are self-dual. Therefore, $\mathsf{N} = \mathsf{K} \cap \mathsf{D}$ and $\mathsf{N} \vee \mathsf{G} = (\mathsf{K} \vee \mathsf{G}) \cap (\mathsf{D} \vee \mathsf{G})$ are permanent pseudovarieties.

In all the preceeding examples, the permanent pseudoidentities are formed by ω-words. That is not the case of the following example. For a prime p, the pseudovariety G_p of p-groups is defined by the pseudoidentity $1 = x^{p^\omega}$, where $x^{p^\omega} = \lim_{n\to\infty} x^{p^{n!}}$ [25], so $\mathcal{L}\mathsf{G}_p$ is defined by the left-permanent pseudoidentity $x_1^\omega = (x_1^\omega x_2 x_1^\omega)^{p^\omega}$.

Lemma 3.3. *If Z is a left-permanent semigroup pseudovariety then $\mathsf{K} \subseteq \mathsf{Z}$. Dually, if Z is a right-permanent semigroup pseudovariety then $\mathsf{D} \subseteq \mathsf{Z}$.*

Proof: Let $u = v$ be a left-permanent pseudoidentity. Then $u = u^2$ and $v = uv$. The equality $u = u^2$ implies $u \notin \Omega_2\mathsf{S}$, thus $\mathsf{K} \models u = uv$. That is, $\mathsf{K} \subseteq [\![u = v]\!]$, since $v = uv$. If Z is left-permanent, then it is the intersection of pseudovarieties of the form $[\![u = v]\!]$, with $u = v$ a left-permanent pseudoidentity, and so $\mathsf{K} \subseteq \mathsf{Z}$. If Z is right-permanent, we reduce to the left-permanent case by application of Proposition 3.2. $\blacksquare$

Definition 3.4. Let Σ be a set of pseudoidentities between elements of $\overline{\Omega}_2\mathsf{S}$. For each $(u = v) \in \Sigma$, let $\mathcal{F}_{(u=v)}$ be a set of continuous homomorphisms of the form $\varphi : \overline{\Omega}_2\mathsf{S} \to \overline{\Omega}_{n(\varphi)}\mathsf{S}$, where $n(\varphi)$ runs over the set of positive integers. Denote by $\mathcal{F}$ the family $(\mathcal{F}_{(u=v)})_{(u=v)\in\Sigma}$. Consider the set of pseudoidentities

$$\Sigma_{\mathcal{F}} = \{\varphi(u) = \varphi(v) \mid (u = v) \in \Sigma,\ \varphi \in \mathcal{F}_{(u=v)}\}.$$

Let V be the pseudovariety $[\![\Sigma_{\mathcal{F}}]\!]$. We say that $\mathcal{F}$ is a Σ*-collection of substitutions defining* V.

We use the language introduced in Definition 3.4 to make the following statement of the Pin-Weil basis theorem for Mal'cev products, for the special case where the first pseudovariety in the Mal'cev product is defined by a basis of pseudoidentities between elements of $\overline{\Omega}_2\mathsf{S}$.

Theorem 3.5 (cf. [23, Theorem 4.1])**.** *Let Z and V be two pseudovarieties of semigroups. Suppose that Z has a basis Σ of pseudoidentities between elements of $\overline{\Omega}_2\mathsf{S}$. Consider the set $\mathcal{H}$ of continuous homomorphisms of the* *form* $\varphi : \overline{\Omega}_2\mathsf{S} \to \overline{\Omega}_{n(\varphi)}\mathsf{S}$*, where* $n(\varphi)$ *runs over the set of positive integers, such that* $\mathsf{V} \models \varphi(x_1) = \varphi(x_2) = \varphi(x_2)^2$*. For each* $(u = v) \in \Sigma$*, take* $\mathcal{F}_{(u=v)} = \mathcal{H}$*. Then* $\mathsf{Z}ⓜ\mathsf{V} = [\![\Sigma_{\mathcal{F}}]\!]$*.*

The "only if" part of the following lemma is crucial to obtain many of our main results, because of its application in the proof of Proposition 5.6.

Lemma 3.6. *Let* Z *be a pseudovariety of semigroups with a basis* Σ *consisting of permanent pseudoidentities. Let* V *be a pseudovariety of semigroups. Then there is a* Σ*-collection of substitutions defining* V *if and only if* $\mathsf{Z}ⓜ\mathsf{V} = \mathsf{V}$*.*

Proof: Let V be a pseudovariety of semigroups for which there is a Σ-collection $\mathcal{F}$ of substitutions defining V. Let $(u = v) \in \Sigma$ and $\varphi \in \mathcal{F}_{(u=v)}$. As $u = u^2$, we have $\varphi(u) = \varphi(u)^2$. Hence $\mathsf{V} \models \varphi(u) = \varphi(v) = \varphi(v)^2$. Since $\mathsf{Z} \models u = v$, from Theorem 3.5 we obtain $\mathsf{Z}ⓜ\mathsf{V} \models u(\varphi(u), \varphi(v)) = v(\varphi(u), \varphi(v))$. Note that, for each $w \in \{u, v\}$, we have $w(\varphi(u), \varphi(v)) = \varphi(w(u, v)) = \varphi(w)$. Hence $\mathsf{Z}ⓜ\mathsf{V} \models \varphi(u) = \varphi(v)$. Therefore $\mathsf{Z}ⓜ\mathsf{V} \models \Sigma_{\mathcal{F}}$, that is $\mathsf{Z}ⓜ\mathsf{V} \subseteq \mathsf{V}$. The reverse inclusion is trivial.

Conversely, if $\mathsf{Z}ⓜ\mathsf{V} = \mathsf{V}$, then it follows immediately from Theorem 3.5 that there is a Σ-collection of substitutions defining V. ■

Corollary 3.7. *Let* Z *be a permanent pseudovariety of semigroups. For every pseudovariety of semigroups* V *we have* $\mathsf{Z}ⓜ(\mathsf{Z}ⓜ\mathsf{V}) = \mathsf{Z}ⓜ\mathsf{V}$*. In particular,* $\mathsf{Z} = \mathsf{Z}ⓜ\mathsf{Z}$*.*

Proof: By Theorem 3.5, $\mathsf{Z}ⓜ\mathsf{V}$ is defined by a Σ-collection of substitutions, so the result follows immediately from Lemma 3.6. ■

4. Main results and applications

4.1. Interplay between operators $\mathsf{Z}ⓜ(\_)$ and $(\_) * \mathsf{D}_k$. The proof of the following theorem is deferred to Section 5. Recall that B_2 denotes the five-element Brandt semigroup, and that $B_2 \notin \mathsf{V}$ if and only if $\mathsf{V} \subseteq \mathsf{DS}$, for every semigroup pseudovariety V [25, Lemma 2.2.3].

Theorem 4.1. *Let* Z *be a permanent pseudovariety of semigroups. Let* V *be a pseudovariety of semigroups such that* $\mathsf{Z}ⓜ\mathsf{V} = \mathsf{V}$*. Suppose moreover that* $B_2 \in \mathsf{V}$*, or that* V *is local and contains some nontrivial monoid. Then*

$$\mathsf{Z}ⓜ(\mathsf{V} * \mathsf{D}_k) = \mathsf{V} * \mathsf{D}_k$$

for every $k \geq 1$*.*

As a corollary of Theorem 4.1, we get the following result.

Theorem 4.2. *Let* Z *be a permanent pseudovariety of semigroups. Let* V *be a pseudovariety of semigroups such that* $B_2 \in \mathsf{Z}ⓜ\mathsf{V}$. *Then*

$$\mathsf{Z}ⓜ(\mathsf{V} * \mathsf{D}_k) \subseteq (\mathsf{Z}ⓜ\mathsf{V}) * \mathsf{D}_k$$

for every $k \geq 1$.

Proof: By Corollary 3.7, we have $\mathsf{Z}ⓜ(\mathsf{Z}ⓜ\mathsf{V}) = \mathsf{Z}ⓜ\mathsf{V}$. Hence, from Theorem 4.1 we obtain the equality

$$\mathsf{Z}ⓜ((\mathsf{Z}ⓜ\mathsf{V}) * \mathsf{D}_k) = (\mathsf{Z}ⓜ\mathsf{V}) * \mathsf{D}_k.$$

On the other hand, clearly $\mathsf{Z}ⓜ(\mathsf{V} * \mathsf{D}_k) \subseteq \mathsf{Z}ⓜ((\mathsf{Z}ⓜ\mathsf{V}) * \mathsf{D}_k)$. ■

Recall that all pseudovarieties of $\mathbb{V}$ are permanent. Let $\mathsf{Z} \in \mathbb{V}$. Then $\mathsf{DS} = \mathsf{Z}ⓜ\mathsf{DS}$. This can be easily deduced directly, using the fact that $B_2 \notin \mathsf{V}$ if and only if $\mathsf{V} \subseteq \mathsf{DS}$, for every semigroup pseudovariety V. Alternatively, one can deduce it from equality $\mathsf{DS} = \mathcal{L}\mathsf{G}ⓜ\mathsf{Sl}$ [24, 26] and inclusion $\mathsf{Z} \subseteq \mathcal{L}\mathsf{G}$. Hence, if $\mathsf{V} \subseteq \mathsf{DS}$, we have $B_2 \in \mathsf{Z}ⓜ\mathsf{V}$ if and only if $B_2 \in \mathsf{V}$.

Theorem 4.3. *Let* $\mathsf{Z} \in \{\mathsf{K}, \mathsf{D}, \mathcal{L}\mathsf{I}\}$. *If* V *is a pseudovariety of semigroups containing* Sl *then*

$$\mathsf{Z}ⓜ(\mathsf{V} * \mathsf{D}_k) \subseteq (\mathsf{Z}ⓜ\mathsf{V}) * \mathsf{D}_k \tag{4.1}$$

for all $k \geq 1$.

Note that if $B_2 \in \mathsf{V}$ then (4.1) holds by Theorem 4.2. We shall prove Theorem 4.3 in Section 7, after some preparations in Section 6 for dealing with the case in which $B_2 \notin \mathsf{V}$.

Theorem 4.3 has the following immediate corollary, which improves Theorem 4.1 in some cases.

Corollary 4.4. *Let* $\mathsf{Z} \in \{\mathsf{K}, \mathsf{D}, \mathcal{L}\mathsf{I}\}$. *If* V *is a pseudovariety of semigroups containing* Sl *and such that* $\mathsf{Z}ⓜ\mathsf{V} = \mathsf{V}$, *then*

$$\mathsf{Z}ⓜ(\mathsf{V} * \mathsf{D}_k) = \mathsf{V} * \mathsf{D}_k$$

for all $k \geq 1$.

4.2. Preservation of locality. By Tilson's Delay Theorem [32], a monoid pseudovariety V is local if and only if $\mathcal{L}\mathsf{V} = \mathsf{V} * \mathsf{D}$. In contrast, there are non-local semigroup pseudovarieties that are solutions of equation $\mathcal{L}\mathsf{X} = \mathsf{X} * \mathsf{D}$. That is the case of the nontrivial pseudovarieties of groups viewed as semigroup pseudovarieties (cf. [1, Theorem 10.6.14] and the discussion in [6, page 14]); on the other hand, nontrivial pseudovarieties of groups are local as monoid pseudovarieties [31]. But if V is a monoid pseudovariety not contained in G, then V is local if and only if V_S is local, from which it follows that for monoidal pseudovarieties of semigroups not contained in G, there is equivalence between being local and being solution of $\mathcal{L}\mathsf{X} = \mathsf{X} * \mathsf{D}$ [6].

Lemma 4.5. *Let* Z *and* V *be pseudovarieties of semigroups for which we have* $\mathcal{L}(\mathsf{Z}\circledm\mathsf{V}) = \mathsf{Z}\circledm\mathcal{L}\mathsf{V}$ *and* $\mathsf{Z}\circledm(\mathsf{V} * \mathsf{D}) \subseteq (\mathsf{Z}\circledm\mathsf{V}) * \mathsf{D}$*. If* V *is a solution of the equation* $\mathcal{L}\mathsf{X} = \mathsf{X} * \mathsf{D}$*, then so is* $\mathsf{Z}\circledm\mathsf{V}$*. In particular, if* V *is monoidal and local, then* $\mathsf{Z}\circledm\mathsf{V}$ *is local.*

Proof: By hypothesis, $\mathcal{L}(\mathsf{Z}\circledm\mathsf{V}) = \mathsf{Z}\circledm\mathcal{L}\mathsf{V} = \mathsf{Z}\circledm(\mathsf{V} * \mathsf{D}) \subseteq (\mathsf{Z}\circledm\mathsf{V}) * \mathsf{D}$, and so $\mathsf{Z}\circledm\mathsf{V}$ is a solution of equation $\mathcal{L}\mathsf{X} = \mathsf{X} * \mathsf{D}$ (recall that $\mathsf{W} * \mathsf{D} \subseteq \mathcal{L}\mathsf{W}$ for every pseudovariety of semigroups W [1, Proposition 10.6.13]). The remaining part of the lemma is justified by the remarks in the paragraph preceding its statement (note that Proposition 2.2 guarantees $\mathsf{Z}\circledm\mathsf{V}$ is monoidal). ∎

Theorem 4.6. *Let* $\mathsf{Z} \in \mathbb{V}$ *and let* V *be a monoidal pseudovariety of semigroups such that* $B_2 \in \mathsf{V}$*. If* V *is local then* $\mathsf{Z}\circledm\mathsf{V}$ *is local.*

Proof: Thanks to Proposition 1.1 and Theorem 4.2, we are in the conditions of Lemma 4.5, and so one gets immediately the theorem. ∎

Theorem 4.7. *Let* V *be a monoidal pseudovariety of semigroups. If* V *is local then the pseudovarieties of semigroups* $\mathsf{K}\circledm\mathsf{V}$*,* $\mathsf{D}\circledm\mathsf{V}$ *and* $\mathcal{L}\mathsf{I}\circledm\mathsf{V}$ *are local.*

Proof: Since V is local, we do not have $\mathsf{V} \subseteq \mathsf{G}$. Therefore, V contains Sl, since it is monoidal. The theorem now follows immediately from Proposition 1.1, Theorem 4.3 and Lemma 4.5. ∎

Next, we translate Theorem 4.7 to the context of monoid pseudovarieties.

Theorem 4.8. *Let* V *be a pseudovariety of monoids. If* V *is local then the pseudovarieties of monoids* $\mathsf{K}\circledm\mathsf{V}$*,* $\mathsf{D}\circledm\mathsf{V}$ *and* $\mathcal{L}\mathsf{I}\circledm\mathsf{V}$ *are local.*

Proof: Let $\mathsf{Z} \in \{\mathsf{K}, \mathsf{D}, \mathcal{L}\mathsf{I}\}$. If V consists only of groups, then $\mathsf{Z}\circledm\mathsf{V} = \mathsf{V}$. If V contains a monoid which is not a group, then V_S contains Sl. Since V_S is local, so is $\mathsf{Z}ⓜ\mathsf{V}_S$, by Theorem 4.7. But $\mathsf{Z}ⓜ\mathsf{V}_S = (\mathsf{Z}ⓜ\mathsf{V})_S$, by Proposition 2.2. Therefore, $\mathsf{Z}ⓜ\mathsf{V}$ is local. ■

Theorems 4.6 and 4.7 unify several known results concerning locality. For example, it is well known that the pseudovarieties R of $\mathcal{R}$-trivial semigroups, and DA of semigroups whose regular $\mathcal{D}$-classes are aperiodic semigroups, are local [29, 2]. The equalities $\mathsf{R} = \mathsf{K}ⓜ\mathsf{Sl}$ and $\mathsf{DA} = \mathcal{L}\mathsf{I}ⓜ\mathsf{Sl}$ are also well known, and were used in Almeida's proof of the locality of R and DA [2]. Their locality may be seen as an application of Theorem 4.7, since Sl is local [8, 10].

More generally, let $(\mathsf{R}_m)_{m\geq 1}$ and $(\mathsf{L}_m)_{m\geq 1}$ be the families of pseudovarieties recursively defined by $\mathsf{R}_1 = \mathsf{L}_1 = \mathsf{Sl}$ and $\mathsf{R}_{m+1} = \mathsf{K}ⓜ\mathsf{L}_m$, $\mathsf{L}_{m+1} = \mathsf{K}ⓜ\mathsf{R}_m$ for $m \geq 1$. These two families, each one with union equal to DA, have received some attention [33, 17, 18]*. As Sl is local, Theorem 4.7 provides the first proof, as far as we know, that all pseudovarieties in this pair of families are local.

Next, we use Theorem 4.6 to obtain a variant proof of a result of Steinberg, which states in particular that $\mathsf{G}*\mathsf{A}$ is local. If H is a pseudovariety of groups, then $\bar{\mathsf{H}}$ denotes the pseudovariety of semigroups whose maximal subgroups belong to H.

Theorem 4.9 ([27]). *If* H *is a pseudovariety of groups closed under semidirect product, then* $\mathsf{G} * \bar{\mathsf{H}}$ *is local.*

Proof: The pseudovariety $\bar{\mathsf{H}}$ is also closed under semidirect product. Moreover, it is closed under adjoining identities. Therefore, we are in the conditions of Theorem 4.11.4 from [25], and so $\mathsf{G} * \bar{\mathsf{H}} = (\mathsf{K} \vee \mathsf{G})ⓜ\bar{\mathsf{H}}$. As $\bar{\mathsf{H}}$ is local and contains B_2, and $\mathsf{K} \vee \mathsf{G} \in \mathbb{V}$, the result follows from Theorem 4.7. ■

The operator $\mathsf{N}ⓜ(\_)$ does not preserve locality of subpseudovarieties of DS. Indeed, the pseudovariety J of $\mathcal{J}$-trivial semigroups is equal to $\mathsf{N}ⓜ\mathsf{Sl}$ [20], and it is not local [16], while Sl is local.

We do not know whether in the cases $\mathsf{Z} \in \{\mathsf{K} \vee \mathsf{G}, \mathsf{D} \vee \mathsf{G}, \mathsf{N} \vee \mathsf{G}\}$ the operator $\mathsf{Z}ⓜ(\_)$ preserves locality when restricted to the interval $[\mathsf{Sl}, \mathsf{DS}]$. Since $\mathsf{DG} = (\mathsf{N} \vee \mathsf{G})ⓜ\mathsf{Sl}$ (cf. proof of Proposition 5.12 in [23]), a research on this subject could lead to a new proof that DG (the pseudovariety of semigroups whose regular $\mathcal{D}$-classes are groups) is local, a difficult result proved by Kad'ourek [15].

*In [17, 18] one has $\mathsf{R}_1 = \mathsf{L}_1 = \mathsf{J}$, where J is the pseudovariety of $\mathcal{J}$-trivial semigroups. But the remaining pseudovarieties are the same, since $\mathsf{K}ⓜ\mathsf{Sl} = \mathsf{K}ⓜ\mathsf{J}$ and $\mathsf{D}ⓜ\mathsf{Sl} = \mathsf{D}ⓜ\mathsf{J}$. We prefer to define $\mathsf{R}_1 = \mathsf{L}_1 = \mathsf{Sl}$ because J is not local [16].

4.3. An application to varieties of formal languages. Eilenberg's theorem relating pseudovarieties of semigroups (or monoids) with varieties of formal languages gave a strong motivating framework to study the former. The following theorem summarizes a series of results which are good examples of the adequacy of that framework.

Theorem 4.10 ([19, 20, 22, 30], see also survey [21]†). *Given a $+$-variety $\mathcal{V}$ of languages, let V be the pseudovariety of semigroups generated by the syntactic semigroups of elements of $\mathcal{V}$. Then, we have:*

(1) $\mathcal{V}$ is closed under concatenation product if and only if $\mathsf{V} = \mathsf{A}ⓜ\mathsf{V}$;
(2) $\mathcal{V}$ is closed under unambiguous product if and only if $\mathsf{V} = \mathcal{L}\mathsf{I}ⓜ\mathsf{V}$;
(3) $\mathcal{V}$ is closed under left deterministic product if and only if $\mathsf{V} = \mathsf{K}ⓜ\mathsf{V}$, and dually, $\mathcal{V}$ is closed under right deterministic product if and only if $\mathsf{V} = \mathsf{D}ⓜ\mathsf{V}$.

We denote by $(\_) * \mathcal{D}_k$ the operator in the lattice of $+$-varieties of languages corresponding to the operator $(\_) * \mathsf{D}_k$, and by $\mathcal{S}l$ the $+$-variety of languages recognized by semigroups of Sl.

Theorem 4.11. *Let $\mathcal{V}$ be a variety of $+$-languages.*

(1) If $\mathcal{V}$ is closed under concatenation product, then so is $\mathcal{V} * \mathcal{D}_k$.
(2) If $\mathcal{V}$ contains $\mathcal{S}l$ and is closed under unambiguous product (respectively, left deterministic product, right deterministic product), then $\mathcal{V} * \mathcal{D}_k$ is also closed under unambiguous product (respectively, left deterministic product, right deterministic product).

Proof: Thanks to Theorem 4.10, this is an immediate application of Theorem 4.1 (recall that A is permanent and $B_2 \in \mathsf{A}$) and Corollary 4.4. ∎

5. Proof of Theorem 4.1

We assume familiarity with the fundamentals of profinite semigroupoids, namely the equational theory of semigroupoid pseudovarieties. See [12, 6, 25]‡. Recall in particular that for a pseudovariety of semigroups V, the pseudovariety of semigroupoids generated by V, the *global of* V, is denoted by $g\mathsf{V}$. The pseudovariety of semigroupoids whose local semigroups belong to V, the

†In [21] one only finds the characterization of $*$-varieties of languages closed under concatenation product. Nevertheless, with the appropriated translation, that characterization holds for $+$-varieties of languages, thanks to [9, Theorem 4.1], a more general result.

‡For delicate questions related with infinite-vertex semigroupoids, see also [4]. However, all semigroupoids in this paper are finite-vertex.

local of V, is denoted by $\ell\mathsf{V}$. Hence V is local if and only if $g\mathsf{V} = \ell\mathsf{V}$. The free profinite semigroupoid, over the pseudovariety Sd of all finite semigroupoids, and generated by a finite graph A, is denoted by $\overline{\Omega}_A\mathsf{Sd}$.

Definition 5.1. Let Z be a pseudovariety of semigroups with a basis Σ of left-permanent pseudoidentities. Let W be a pseudovariety of semigroupoids such that W has a basis

$$\Gamma = \bigcup_{(u=v)\in\Sigma} \Gamma_{(u=v)}$$

in which an element $\mathfrak{p}$ of $\Gamma_{(u=v)}$, with $(u = v) \in \Sigma$, is a path pseudoidentity of the form $(u_L(r_1, r_2) = v_L(r_1, r_2); A_{\mathfrak{p}})$, where $A_{\mathfrak{p}}$ is a finite graph, L is the local semigroup at a vertex c of $\overline{\Omega}_{A_{\mathfrak{p}}}\mathsf{Sd}$, and r_1 and r_2 are elements of L. We say that W is *left* Z*-based*.

In [3], given an alphabet X and pseudoword $w \in \overline{\Omega}_X\mathsf{S}$, it is associated a finite graph denoted by A_w. In this paper we shall not recall the somewhat technical definition of A_w, but we will next highlight some of its features which will be used in the proof of Proposition 5.5. The precise definitions and results can be found in [3, Section 5].

The edges of A_w are elements of X, and every element of X^+ that labels a path p in A_w can be interpreted as being the path p. The pseudoword w can also be interpreted uniquely as profinite path of A_w, as follows from the next lemma.

Lemma 5.2 ([3, Lemma 5.7])**.** *If $(w_n)_{n\geq 1}$ is a sequence of elements of X^+ converging to w, then there is $N \geq 1$ such that $(w_n)_{n\geq N}$ is a sequence of elements of $\overline{\Omega}_{A_w}\mathsf{Sd}$ converging to a profinite path L of A_w. The profinite path L depends only of w, not of the choice of the sequence $(w_n)_{n\geq 1}$.*

The limit L in Lemma 5.2 is the interpretation of w in $\overline{\Omega}_{A_w}\mathsf{Sd}$.

It follows also from Lemma 5.2 that every factor u of w can be interpreted as a profinite path of A_w, in the manner which we next explain. Let $x, y \in (\overline{\Omega}_A\mathsf{S})^1$ be such that $w = xuy$. Consider a sequence $(x_n, u_n, y_n)_n$ of elements of $X^* \times X^+ \times X^*$ converging to (x, u, y). Then, for large enough n, the word $x_n u_n y_n$ can be interpreted as an element of $\overline{\Omega}_{A_w}\mathsf{Sd}$ by Lemma 5.2. In particular, u_n can be interpreted as an element of $\overline{\Omega}_{A_w}\mathsf{Sd}$. Therefore, in this manner, an accumulation point of $(u_n)_n$ in $\overline{\Omega}_{A_w}\mathsf{Sd}$ can be seen as an interpretation of u in $\overline{\Omega}_{A_w}\mathsf{Sd}$. From a careful reading of the paragraph preceeding [3, Lemma 5.7], where Lemma 5.2 is proved, one concludes that the interpretation of u is unique, but we shall not need to use this more precise information.

The following property will be used without reference.

Proposition 5.3 (cf. [3, Theorem 5.9]). *For an alphabet X, let $u, v \in \overline{\Omega}_X\mathsf{S}$. If $B_2 \models u = v$ then $A_u = A_v$.*

The next result explains the importance of the graphs of the form A_w.

Theorem 5.4 ([3, Theorem 5.9]). *Let V be a pseudovariety of semigroups containing B_2. If $\mathsf{V} = [\![u_i = v_i \mid i \in I]\!]$ then $g\mathsf{V} = [\![(u_i = v_i; A_{u_i}) \mid i \in I]\!]$.*

Theorem 5.4 is crucial for proving the following proposition.

Proposition 5.5. *Let Z be a left-permanent pseudovariety of semigroups, and let V be a pseudovariety of semigroups such that $\mathsf{V} = \mathsf{Z}\,ⓜ\,\mathsf{V}$. If V is local or if $B_2 \in \mathsf{V}$, then $g\mathsf{V}$ is left Z-based.*

Proof: Let Σ be a basis of Z comprised by left-permanent pseudoidentities. Let $\mathcal{H}$ be the set of continuous homomorphisms and $\mathcal{F}$ the Σ-collection of substitutions as described in Theorem 3.5. Then $\mathsf{V} = [\![\Sigma_{\mathcal{F}}]\!]$, since $\mathsf{V} = \mathsf{Z}\,ⓜ\,\mathsf{V}$.

The pseudovariety $\ell\mathsf{V}$ has as a basis formed by path pseudoidentities of the form $(\varphi(u) = \varphi(v); X_\varphi)$, where $(u = v)$ runs over Σ and $\varphi : \overline{\Omega}_2\mathsf{S} \to \overline{\Omega}_{X_\varphi}\mathsf{S}$ runs over $\mathcal{H}$. Therefore, $\ell\mathsf{V}$ is left Z-based. In particular, if V is local then $g\mathsf{V}$ is left Z-based.

Suppose now that $B_2 \in \mathsf{V}$. The set of path pseudoidentities of the form

$$\Big(\varphi(u) = \varphi(v);\, A_{\varphi(v)}\Big), \tag{5.1}$$

where $(u = v)$ runs over Σ and φ runs over $\mathcal{H}$, defines a basis for $g\mathsf{V}$, by Theorem 5.4.

Fix $(u = v) \in \Sigma$ and $\varphi \in \mathcal{H}$. Suppose first that $|c(v)| = 2$. Then $\varphi(x_1)$ and $\varphi(x_2)$ are factors of $\varphi(v)$ and so they can be interpreted as profinite paths in the graph $A_{\varphi(v)}$. We denote these paths by r_1 and r_2 respectively. By the definition of $\mathcal{H}$, we have $\mathsf{V} \models \varphi(x_1) = \varphi(x_2)$. Since $\mathsf{V} = \mathsf{Z}\,ⓜ\,\mathsf{V}$, we have $\mathsf{Z} \subseteq \mathsf{V}$, thus $\mathsf{K} \models \varphi(x_1) = \varphi(x_2)$ by Lemma 3.3. In particular, $\varphi(x_1)$ and $\varphi(x_2)$ start with same letter, and so r_1 and r_2 have the same origin. It will be convenient to use the notation $\alpha(e)$ and $\omega(e)$ for the origin and the terminus of an edge e, respectively. Hence,

$$\alpha(r_1) = \alpha(r_2). \tag{5.2}$$

Note that $c(u)$ is nonempty and is contained in $c(v)$, the latter because $v = uv$. Without loss of generality, suppose that $x_1 \in c(u)$. Then, as $x_2 \in c(v)$, the word x_1x_2 is a factor of v, thus $\varphi(x_1)\varphi(x_2)$ is a factor of $\varphi(v)$. Therefore,

$$\omega(r_1) = \alpha(r_2). \tag{5.3}$$

Hence, by (5.2) and (5.3), r_1 is a loop rooted at vertex $c = \alpha(r_2)$.

If $|c(u)| = 2$ or x_2 is not the last letter of v, then x_2x_1 is a factor of v, thus $\varphi(x_2)\varphi(x_1)$ is a factor of $\varphi(v)$ and $\omega(r_2) = \alpha(r_1) = c$. The other case to consider occurs if $x_2 \notin c(u)$ and x_2 is the last letter of v. Then $\omega(r_1) = \omega(\varphi(u))$ and $\omega(r_2) = \omega(\varphi(v))$. But $\omega(\varphi(u)) = \omega(\varphi(v))$, by the definition of path pseudoidentity, and so $\omega(r_2) = \omega(r_1) = c$. In both cases, we conclude that r_2 is a loop rooted at c.

Denote by L the local semigroup of $A_{\varphi(v)}$ at c. We claim that if $w \in \{u, v\}$ then the interpretation of the pseudoword $\varphi(w)$ as a profinite path of $A_{\varphi(v)}$ is precisely the profinite path $w_L(r_1, r_2)$. To prove the claim, let $(w_n)_n$, $(r_{1,n})_n$ and $(r_{2,n})_n$ be sequences of finite words converging respectively to the pseudowords w, $\varphi(x_1)$ and $\varphi(x_2)$, where $r_{i,n}$ as a path of $A_{\varphi(v)}$ converges to r_i, for $i \in \{1, 2\}$. Since the sequence of finite words $(w_n(r_{1,n}, r_{2,n}))_n$ converges to $w_{\overline{\Omega}_{X_\varphi}\mathsf{S}}(\varphi(x_1), \varphi(x_2)) = \varphi(w)$, by Lemma 5.2 it suffices to prove that the sequence of finite paths $(w_n(r_{1,n}, r_{2,n}))_n$ converges to $w_L(r_1, r_2)$ in $\overline{\Omega}_{A_{\varphi(v)}}\mathsf{Sd}$. We may assume that $r_{1,n}$ and $r_{2,n}$ are elements of the local semigroup L for all n, and so $w_n(r_{1,n}, r_{2,n})$ as path of $A_{\varphi(v)}$ equals $(w_n)_L(r_{1,n}, r_{2,n})$. Let ψ be a continuous homomorphism from $\overline{\Omega}_{A_{\varphi(v)}}\mathsf{Sd}$ onto a finite semigroupoid S. Then ψ induces, by restriction, a continuous semigroup homomorphism ψ_c from L into the local semigroup of S at the vertex $\psi(c)$, denoted $S_{\psi(c)}$. Then, for large enough n,

$$\begin{aligned}
\psi_c((w_n)_L(r_{1,n}, r_{2,n})) &= (w_n)_{S_{\psi(c)}}(\psi_c(r_{1,n}), \psi_c(r_{2,n})) \\
&= (w_n)_{S_{\psi(c)}}(\psi_c(r_1), \psi_c(r_2)) \\
&= w_{S_{\psi(c)}}(\psi_c(r_1), \psi_c(r_2)) \\
&= \psi_c(w_L(r_1, r_2)).
\end{aligned}$$

Hence $((w_n)_L(r_{1,n}, r_{2,n}))_n$ converges to $w_L(r_1, r_2)$, proving the claim. Therefore, when $|c(v)| = 2$, the pseudoidentity (5.1) is actually the pseudoidentity

$$\Big(u_L(r_1, r_2) = v_L(r_1, r_2);\ A_{\varphi(v)}\Big). \tag{5.4}$$

Let us now consider the case $|c(v)| = 1$. Then $c(u) = c(v)$ because $v = uv$. If $c(v) = \{x_1\}$, then, as $u = u^2$ and $v = v(u, v)$, we have $u = x_1^\omega = v$. Suppose that $c(v) = \{x_2\}$ and let r be the interpretation of $\varphi(x_2)$ as a profinite path of $A_{\varphi(v)}$. Since x_2^2 is factor of v, we know that r is a loop rooted at a vertex c. For each $w \in \{u, v\}$, we have $w_{\overline{\Omega}_{X_\varphi}\mathsf{S}}(\varphi(x_1), \varphi(x_2)) = w_{\overline{\Omega}_{X_\varphi}\mathsf{S}}(\varphi(x_2), \varphi(x_2))$. Then, with the same kind of arguments used in the case $|c(v)| = 2$, we conclude that the interpretation of $w_{\overline{\Omega}_{X_\varphi}\mathsf{S}}(\varphi(x_1), \varphi(x_2))$ as a profinite path of $A_{\varphi(v)}$ is the profinite path $w_L(r, r)$.

Putting cases $|c(v)| = 1$ and $|c(v)| = 2$ together, we conclude that $g\mathsf{V}$ has a basis of pseudoidentities of the form (5.4), which proves that $g\mathsf{V}$ is left Z-based. ■

For the proof of the next proposition we need to add some more notation and definitions.

For a graph A, we denote by $V(A)$ the set of its vertices, and by $E(A)$ the set of its edges.

The following definitions are taken from [1, Section 5.2]. They will be used also in Section 6. If u is an element of $\overline{\Omega}_A\mathsf{S}$, then there is a unique word $\mathsf{t}_n(u)$ of length at most n such that $\mathsf{D}_n \models u = \mathsf{t}_n(u)$. If u is a word of A^+ of length at most n, then $u = \mathsf{t}_n(u)$, otherwise $\mathsf{t}_n(u)$ is the unique word w of length n such that $u \in (\overline{\Omega}_A\mathsf{S})^1 \cdot w$. Dually, one may consider the unique word $\mathsf{i}_n(u)$ such that $\mathsf{K}_n \models u = \mathsf{i}_n(u)$, where K_n is the dual of D_n. These definitions are extended to the empty word by letting $\mathsf{i}_n(1) = \mathsf{t}_n(1) = 1$.

Proposition 5.6. *Let Z be a left-permanent pseudovariety of semigroups, and let V be a pseudovariety of semigroups such that $g\mathsf{V}$ is left Z-based. Then $\mathsf{Z} \textcircled{m} (\mathsf{V} * \mathsf{D}_n) = \mathsf{V} * \mathsf{D}_n$, for every $n \geq 1$.*

Proof: For $\mathsf{W} = g\mathsf{V}$, we retain the notation from Definition 5.1. By the application of the Almeida-Weil basis theorem[§] to semidirect products with D_n, which is explained in the proof of Theorem 4.1 in [3], the pseudovariety $\mathsf{V} * \mathsf{D}_n$ has a basis

$$\Upsilon = \bigcup \Big\{ \Upsilon_{\mathfrak{p}} \mid (u = v) \in \Sigma,\ \mathfrak{p} \in \Gamma_{(u=v)} \Big\}$$

[§]The Almeida-Weil basis theorem describes a basis of pseudoidentities for semidirect products $\mathsf{V} * \mathsf{W}$ of semigroup pseudovarieties. The proof has a gap found by Rhodes and Steinberg, but it works when $g\mathsf{V}$ has a basis of pseudoidentities with bounded number of vertices or when W is locally finite (cf. [25, Theorem 3.7.15]). Since D_n is locally finite, the result holds in the cases we are interested.

where each $\Upsilon_{\mathfrak{p}}$ has the following property: an element $\mathfrak{e}$ of $\Upsilon_{\mathfrak{p}}$ is a pseudoidentity of the form

$$\pi_c \cdot \delta(u_L(r_1, r_2)) = \pi_c \cdot \delta(v_L(r_1, r_2)), \tag{5.5}$$

where $\delta : \overline{\Omega}_{A_{\mathfrak{p}}}\mathsf{Sd} \to \overline{\Omega}_{B_{\mathfrak{e}}}\mathsf{S}$ is a continuous semigroupoid homomorphism and $(\pi_d)_{d \in V(A_{\mathfrak{p}})}$ is a family of words of $B_{\mathfrak{e}}^+$ with length at most n, such that, among other restrictions, one has

$$\mathsf{D}_n \models \pi_{\alpha s}\delta(s) = \pi_{\omega s}, \quad \text{for every } s \in E(A_{\mathfrak{p}}). \tag{5.6}$$

We shall denote by $\mathcal{D}_{\mathfrak{p}}$ the set of all continuous homomorphisms δ arising in this way, for a fixed $\mathfrak{p}$.

Let us fix $(u = v) \in \Sigma$, $\mathfrak{p} \in \Gamma_{(u=v)}$ and $\mathfrak{e} \in \Upsilon_{\mathfrak{p}}$ with the above definitions and notations.

Since $u_L(r_1, r_2) = u_L(r_1, r_2)^2$, the profinite path $u_L(r_1, r_2)$ is not finite, thus $u_L(r_1, r_2) = ts_1 \cdots s_n$ for some $t \in E(\overline{\Omega}_{A_{\mathfrak{p}}}\mathsf{Sd})$ and $s_1, \ldots, s_n \in E(A_{\mathfrak{p}})$. Note that $\omega s_n = c$. Applying (5.6) consecutively, we then get

$$\mathsf{D}_n \models \pi_{\alpha s_1}\delta(s_1 s_2 s_3 \cdots s_n) = \pi_{\alpha s_2}\delta(s_2 s_3 \cdots s_n) = \cdots = \pi_{\omega s_n} = \pi_c,$$

which means that $\mathsf{t}_n(\delta(u_L(r_1, r_2))) = \mathsf{t}_n(\delta(s_1 \cdots s_n)) = \pi_c$. Therefore, there is a pseudoword w such that $\delta(u_L(r_1, r_2)) = w\pi_c$. Hence, if a semigroup S satisfies (5.5), it also satisfies $\delta(u_L(r_1, r_2))^2 = \delta(u_L(r_1, r_2)v_L(r_1, r_2))$. Since $u = u^2$ and $v = uv$, we conclude that S satisfies (5.5) if and only if it satisfies

$$\delta(u_L(r_1, r_2)) = \delta(v_L(r_1, r_2)). \tag{5.7}$$

Let ψ_δ be the continuous homomorphism $\overline{\Omega}_2\mathsf{S} \to \overline{\Omega}_{B_{\mathfrak{e}}}\mathsf{S}$ such that $\psi_\delta(x_i) = \delta(r_i)$ for each $i \in \{1, 2\}$. Then (5.7), which is equivalent to (5.5), can be rewritten as

$$\psi_\delta(u) = \psi_\delta(v).$$

Therefore, denoting by $\mathcal{F}_{(u=v)}$ the set $\{\psi_\delta \mid \delta \in \mathcal{D}_{\mathfrak{p}},\ \mathfrak{p} \in \Gamma_{(u=v)}\}$, we proved that the family $(\mathcal{F}_{(u=v)})_{(u=v)\in\Sigma}$ is a Σ-collection of substitutions defining $\mathsf{V} * \mathsf{D}_n$. The result now follows from Lemma 3.6. ■

Conclusion of the proof of Theorem 4.1: Let Z be a permanent pseudovariety of semigroups, and let V be a pseudovariety of semigroups such that $\mathsf{Z}ⓜ\mathsf{V} = \mathsf{V}$. Suppose also that $B_2 \in \mathsf{V}$, or that V is local and contains some nontrivial monoid. If Z is left-permanent, it follows immediately from Propositions 5.5 and 5.6 that the equality

$$\mathsf{Z}ⓜ(\mathsf{V} * \mathsf{D}_k) = \mathsf{V} * \mathsf{D}_k \tag{5.8}$$

holds.

If Z is right-permanent, then Z^{op} is left-permanent by Proposition 3.2, and so, from the already proved case, together with Propositions 2.4 and 2.3, we obtain

$$(\mathsf{Z}\,ⓜ(\mathsf{V} * \mathsf{D}_k))^{op} = \mathsf{Z}^{op}ⓜ(\mathsf{V}^{op} * \mathsf{D}_k) = \mathsf{V}^{op} * \mathsf{D}_k = (\mathsf{V} * \mathsf{D}_k)^{op},$$

proving equality (5.8) for the case of right-permanent pseudovarieties.

Finally, suppose that Z is a permanent pseudovariety of semigroups. Then $\mathsf{Z} = \bigcap_{i\in I} \mathsf{Z}_i$ for some family $(\mathsf{Z}_i)_{i\in I}$ consisting of left-permanent or right-permanent pseudovarieties of semigroups. The Mal'cev product of pseudovarieties is right-distributive over intersections of pseudovarieties [23, Corollary 3.2], whence $\mathsf{Z}ⓜ(\mathsf{V} * \mathsf{D}_k) = \bigcap_{i\in I} \mathsf{Z}_iⓜ(\mathsf{V} * \mathsf{D}_k)$. This concludes the proof, since we have shown that $\mathsf{Z}_iⓜ(\mathsf{V} * \mathsf{D}_k) = \mathsf{V} * \mathsf{D}_k$. ■

6. The pseudovarieties $\mathsf{DS} * \mathsf{D}_k$

In this section we obtain some properties of implicit operations on $\mathsf{DS} * \mathsf{D}_k$ similar to fundamental properties of implicit operations on DS presented in [1, Chapter 8]. The former properties are obtained by reduction to the latter via general properties of semidirect products of the form $\mathsf{V} * \mathsf{D}_k$ and the use of iterated left basic factorizations of implicit operations.

6.1. Semidirect products of the form $\mathsf{V}*\mathsf{D}_k$ and the mapping Φ_k. Let k be a positive integer. Consider the mapping $A^+ \to (A^{k+1})^*$ that maps to 1 each word of A^+ with length less than $k+1$, and maps each word u of A^+ with length at least $k+1$ to the word of $(A^{k+1})^+$ formed by the consecutive factors of length $k+1$ of u. This mapping has a unique continuous extension to a mapping $\Phi_k : \overline{\Omega}_A\mathsf{S} \to (\overline{\Omega}_{A^{k+1}}\mathsf{S})^1$ [1, Lemma 10.6.11]. Note that, for $u \in \overline{\Omega}_A\mathsf{S}$, one has $\Phi_k(u) = 1$ if and only if u has length less than $k+1$. The mapping Φ_k has the following property, which we use frequently without explicit reference: for every $u, v \in \overline{\Omega}_A\mathsf{S}$ one has

$$\Phi_k(uv) = \Phi_k(u\,\mathsf{i}_k(v)) \cdot \Phi_k(v) = \Phi_k(u) \cdot \Phi_k(\mathsf{t}_k(u)v).$$

One may informally think that, for $u \in \overline{\Omega}_A\mathsf{S}$, the pseudoword $\Phi_k(u)$ is the result of "reading" the consecutive factors of length $k+1$ of u.

The mapping Φ_k appears frequently in the study of semidirect products of the form $\mathsf{V} * \mathsf{D}_k$. It is in this context that it is introduced in [1, Section 10.6].

Theorem 6.1 ([1, Theorem 10.6.12]). *Let* V *be a pseudovariety of semigroups containing some nontrivial monoid. Let* $u, v \in \overline{\Omega}_A\mathsf{S}$. *Then* $\mathsf{V} * \mathsf{D}_k \models u = v$ *if and only if* $\mathsf{i}_k(u) = \mathsf{i}_k(u)$, $\mathsf{t}_k(u) = \mathsf{t}_k(u)$ *and* $\mathsf{V} \models \Phi_k(u) = \Phi_k(v)$.

For the following corollary of Theorem 6.1, we introduce some notation. For an alphabet A and a pseudovariety of semigroups V, the image of an element u of $\overline{\Omega}_A\mathsf{S}$ by the canonical homomorphism $\overline{\Omega}_A\mathsf{S} \to \overline{\Omega}_A\mathsf{V}$ will be denoted by $[u]_\mathsf{V}$. Recall that $\mathsf{V} \models u = v$ if and only if $[u]_\mathsf{V} = [v]_\mathsf{V}$. More generally, if θ is a binary relation on $\overline{\Omega}_A\mathsf{V}$ and $[u]_\mathsf{V}\ \theta\ [v]_\mathsf{V}$, then we may use the notation $\mathsf{V} \models u\ \theta\ v$.

Corollary 6.2. *Let* V *be a pseudovariety of semigroups containing some nontrivial monoid. Let* $\mathcal{K}$ *be one of the Green's relations* $\mathcal{J}$, $\mathcal{R}$ *or* $\mathcal{L}$. *For every* $u, v \in \overline{\Omega}_A\mathsf{S}$, *the following properties hold:*

(1) if $\mathsf{V} * \mathsf{D}_k \models u \leq_\mathcal{K} v$ *then* $\mathsf{V} \models \Phi_k(u) \leq_\mathcal{K} \Phi_k(v)$;

(2) if $[u]_{\mathsf{V}*\mathsf{D}_k}$ *is regular then* $[\Phi_k(u)]_\mathsf{V}$ *is regular.*

Proof: (1): We write the proof for $\mathcal{K} = \mathcal{R}$ only, since the other cases are similar. If $\mathsf{V} * \mathsf{D}_k \models u \leq_\mathcal{R} v$, then $\mathsf{V} * \mathsf{D}_k \models u = vx$ for some $x \in (\overline{\Omega}_A\mathsf{S})^1$. Hence $\mathsf{V} \models \Phi_k(u) = \Phi_k(vx)$ by Theorem 6.1. As $\Phi_k(vx) = \Phi_k(v) \cdot \Phi_k(\mathsf{t}_k(v)x)$, this establishes $\mathsf{V} \models \Phi_k(u) \leq_\mathcal{R} \Phi_k(v)$.

(2): If $[u]_{\mathsf{V}*\mathsf{D}_k}$ is regular then $\mathsf{V} * \mathsf{D}_k \models u = uxu$, for some $x \in \overline{\Omega}_A\mathsf{S}$. Then $\mathsf{V} \models \Phi_k(u) = \Phi_k(uxu)$ by Theorem 6.1. Since $\Phi_k(uxu) = \Phi_k(u) \cdot \Phi_k(\mathsf{t}_k(u)x\,\mathsf{i}_k(u)) \cdot \Phi_k(u)$, this establishes (2). ∎

As another consequence of Theorem 6.1, we have $c(\Phi_k(u)) = c(\Phi_k(v))$ if $\mathsf{SI} * \mathsf{D}_k \models u = v$. Sometimes we use the notation $c_{k+1}(u)$ for $c(\Phi_k(u))$. Note that $c_{k+1}(u)$ is just the set of factors of length $k+1$ of u.

6.2. Left basic factorizations. Left basic factorizations, whose definition we next recall, are explained in detail and extensively used in [33, Section 3], which we consider as our supporting reference for this particular subject. This tool had already proved to be quite useful in [2, 5].

Let V be a pseudovariety of semigroups containing SI. Let u be an element of $\overline{\Omega}_A\mathsf{V}$. A *left basic factorization* of u is a factorization $u = xay$ such that $x, y \in (\overline{\Omega}_A\mathsf{V})^1$, $a \in A$, $a \notin c(x)$ and $c(u) = c(xa)$. Every element of $\overline{\Omega}_A\mathsf{V}$ has a left basic factorization.

Suppose furthermore that $\mathsf{V} = \mathsf{K} \circledcirc \mathsf{V}$. Then the left basic factorization of u is unique: if $u = xay$ and $u = zbt$ are left basic factorizations of u then $x = z$, $a = b$ and $y = t$.

For $u \in \overline{\Omega}_A \mathsf{V}$ consider the following recursive definition:

(1) the left basic factorization of u is $u = u_1 a_1 r_1$;
(2) if $c(r_n) = c(u)$, then u_{n+1}, a_{n+1} and r_{n+1} are such that the left basic factorization of r_n is $r_n = u_{n+1} a_{n+1} r_{n+1}$;
(3) if $c(r_n) \subsetneq c(u)$ then the recursive definition stops.

If this recursive definition stops after a finite number of steps, that is, if $c(r_n) \subsetneq c(u)$ for some $n \geq 1$, then $u = u_1 a_1 u_2 a_2 \cdots u_{n-1} a_{n-1} u_n a_n r_n$ is the *iterated left basic factorization* of u. Moreover, one says that u has a *finite* iterated left basic factorization of *length* $\ell = n$ and *reminder* $r = r_n$. If the recursive definition does not stop after a finite number of steps then u has an *infinite iterated left basic factorization.* We then write $\ell = \infty$. We also define $r = 1$ if $\ell = \infty$.

The following notation encompasses both cases $\ell \in \mathbb{N}$ and $\ell = \infty$, where $\mathsf{ilbf}(u)$ stands for *iterated left basic factorization of* u:

$$\mathsf{ilbf}(u) = ((u_i, a_i)_{1 \leq i < \ell+1}; r).$$

From the uniqueness of left basic factorizations one sees that $\mathsf{ilbf}(u)$ is well-defined, that is, iterated left basic factorizations are unique.

Remark 6.3. *Let* V *be a pseudovariety of semigroups such that* $\mathsf{Sl} \subseteq \mathsf{V}$ *and* $\mathsf{V} = \mathsf{K} \circledcirc \mathsf{V}$. *For every* $u \in \overline{\Omega}_A \mathsf{S}$,

$$\mathsf{ilbf}(u) = ((u_i, a_i)_{1 \leq i < \ell+1}; r) \implies \mathsf{ilbf}([u]_{\mathsf{V}}) = (([u_i]_{\mathsf{V}}, a_i)_{1 \leq i < \ell+1}; [r]_{\mathsf{V}}),$$

where $[1]_{\mathsf{V}} = 1$.

Proof: The result follows from the fact that $c([w]_{\mathsf{V}}) = c(w)$ for every $w \in \overline{\Omega}_A \mathsf{S}$, and from the uniqueness of iterated left basic factorizations. ∎

The following proposition expresses the significance of a pseudoword having infinite iterated left basic factorization.

Proposition 6.4 ([33, Proposition 3.3])**.** *Let* V *be a pseudovariety of semigroups such that* $\mathsf{Sl} \subseteq \mathsf{V} \subseteq \mathsf{DS}$ *and* $\mathsf{V} = \mathsf{K} \circledcirc \mathsf{V}$. *Let* $x \in \overline{\Omega}_A \mathsf{V}$. *Then* x *is regular if and only if* x *has an infinite iterated left basic factorization.*

Let u be an element of $\overline{\Omega}_A \mathsf{S}$ with length greater than k. Suppose that

$$\mathsf{ilbf}(\Phi_k(u)) = ((w_i, z_i)_{1 \leq i < \ell+1}; r).$$

For each integer i, with $1 \leq i < \ell + 1$, the pseudoword w_i is a factor of $\Phi_k(u)$ so, by [1, Proposition 10.9.2], there is u_i such that $\Phi_k(u_i) = w_i$ and u_i is unique if $w_i \neq 1$. Also, $\Phi_k(q) = r$ for some q, which is unique if $r \neq 1$.

Suppose that $w_i \neq 1$. Since $w_i z_i$ and $z_{i-1} w_i$ (the latter being defined only if $i \neq 1$) are factors of $\Phi_k(u)$, we have $\mathsf{i}_k(z_i) = \mathsf{t}_k(u_i)$ and $\mathsf{t}_k(z_{i-1}) = \mathsf{i}_k(u_i)$ (otherwise $\Phi_k(u)$ would have factors of length 2 which are not in $\Phi_k(A^+)$).

We shall focus on the special case $k = 1$. It is easier to handle with this case because $z_i = \mathsf{t}_1(u_i)\,\mathsf{i}_1(u_{i+1})$, while in the case $k > 1$ we do not have $z_i = \mathsf{t}_k(u_i)\mathsf{i}_k(u_{i+1})$.

Note that, assuming $k = 1$, if some w_i is 1 then all w_i and r are 1, and $c(\Phi_1(u))$ has only one element, say ab, with $a, b \in A$. Then either $u = ab$ or $a = b$. In the former case, we have $\ell = 1$ and $z_1 = ab$, and we choose u_1 to be a; in the latter case we have $z_i = aa$ for all i, and we choose u_i to be a. In any case, if $\ell \in \mathbb{N}$, we choose q to be b.

If $\ell \in \mathbb{N}$ then

$$\Phi_1(u_1 u_2 \cdots u_\ell q) = \Phi_1(u_1) z_1 \Phi_1(u_2) z_2 \Phi_1(u_3) \cdots \Phi_1(u_\ell) z_\ell \Phi_1(q) = \Phi_1(u).$$

Since, for $w = u_1 u_2 \cdots u_\ell q$, we also have $\mathsf{i}_1(w) = \mathsf{i}_1(u)$ and $\mathsf{t}_1(w) = \mathsf{t}_1(u)$, it follows from Theorem 6.1, applied to the pseudovariety S, that $u = w$.

More generally, whether $\ell \in \mathbb{N}$ or $\ell = \infty$, the equality

$$\Phi_1(u_1 u_2 \cdots u_i) = \Phi_1(u_1) z_1 \Phi_1(u_2) z_2 \Phi_1(u_3) \cdots \Phi_1(u_{i-1}) z_{i-1} \Phi_1(u_i)$$

holds for all i such that $1 \leq i < \ell + 1$, and $u_1 u_2 \cdots u_i$ is a prefix of u.

We define

$$\mathsf{ilfb}_2(u) = ((u_i)_{1 \leq i < \ell+1}; q),$$

where $q = 1$ if $\ell = \infty$.

Lemma 6.5. *Let* V *be a pseudovariety of semigroups such that* $\mathsf{Sl} \subseteq \mathsf{V}$ *and* $\mathsf{V} = \mathsf{K} ⓜ \mathsf{V}$. *Let* u *and* v *be elements of* $\overline{\Omega}_A \mathsf{S} \setminus A$ *such that*

$$\mathsf{ilfb}_2(u) = ((u_i)_{1 \leq i < \ell_u+1}; q_u), \quad \mathsf{ilfb}_2(v) = ((v_i)_{1 \leq i < \ell_v+1}; q_v).$$

If $\mathsf{V} * \mathsf{D}_1 \models u = v$ *then* $\ell_u = \ell_v$, $\mathsf{V} * \mathsf{D}_1 \models u_i = v_i$ *for all* i *such that* $1 \leq i < \ell_u + 1$, *and* $\mathsf{V} * \mathsf{D}_1 \models q_u = q_v$.

Proof: If $\mathsf{V} * \mathsf{D}_1 \models u = v$ then $\mathsf{V} \models \Phi_1(u) = \Phi_1(v)$, $\mathsf{i}_1(u) = \mathsf{i}_1(v)$ and $\mathsf{t}_1(u) = \mathsf{t}_1(v)$ by Theorem 6.1. By the definition of ilfb_2, the iterated left basic factorizations of $\Phi_1(u)$ and $\Phi_1(v)$ are of the following form,

$$\mathsf{ilbf}(\Phi_1(u)) = ((\Phi_1(u_i), z_{u,i})_{1 \leq i < \ell_u+1}; q_u),$$
$$\mathsf{ilbf}(\Phi_1(v)) = ((\Phi_1(v_i), z_{v,i})_{1 \leq i < \ell_v+1}; q_v),$$

with $z_{u,i} = \mathsf{t}_1(u_i)\mathsf{i}_1(u_{i+1})$ and $z_{v,i} = \mathsf{t}_1(v_i)\mathsf{i}_1(v_{i+1})$, where $u_{\ell_u+1} = q_u$ and $v_{\ell_v+1} = q_v$.

Since $\mathsf{V} = \mathsf{K} ⓜ \mathsf{V}$, the elements of $\overline{\Omega}_{A^2}\mathsf{V}$ have unique left basic factorizations. Therefore, taking into account Remark 6.3, we conclude that ℓ_u and ℓ_v are equal, say to ℓ, and

$$z_{u,i} = z_{v,i}, \tag{6.1}$$

$$\mathsf{V} \models \Phi_1(u_i) = \Phi_1(v_i), \quad \mathsf{V} \models \Phi_1(q_u) = \Phi_1(q_v), \tag{6.2}$$

for all i such that $1 \leq i < \ell + 1$.

Since $\mathsf{i}_1(u) = \mathsf{i}_1(v)$ and $\mathsf{t}_1(u) = \mathsf{t}_1(v)$, we have $\mathsf{i}_1(u_1) = \mathsf{i}_1(v_1)$ and $\mathsf{t}_1(q_u) = \mathsf{t}_1(q_v)$. By (6.1) we also know that $\mathsf{i}_1(u_{i+1}) = \mathsf{i}_1(v_{i+1})$ and $\mathsf{t}_1(u_i) = \mathsf{t}_1(v_i)$ for every integer i such that $1 \leq i < \ell + 1$ (in particular, $\mathsf{i}_1(q_u) = \mathsf{i}_1(q_v)$). Then, from (6.2) and Theorem 6.1, we obtain $\mathsf{V} * \mathsf{D}_1 \models u_i = v_i$ for every integer i such that $1 \leq i < \ell + 1$, and $\mathsf{V} * \mathsf{D}_1 \models q_u = q_v$. ■

6.3. From DS to $\mathsf{DS} * \mathsf{D}_k$. The following theorem and corollary state known properties of implicit operations on DS from which we obtain similar properties of implicit operations on $\mathsf{DS} * \mathsf{D}_k$, expressed in Theorem 6.10.

Theorem 6.6 ([1, Theorem 8.1.7]). *Let $u, v \in \overline{\Omega}_A\mathsf{S}$ be such that $[u]_{\mathsf{DS}}$ and $[v]_{\mathsf{DS}}$ are regular. Then $[u]_{\mathsf{DS}} \mathcal{J} [v]_{\mathsf{DS}}$ if and only if $c(u) = c(v)$.*

Corollary 6.7. *Let $u, v \in \overline{\Omega}_A\mathsf{S}$ be such that $c(u) = c(v)$. Let $\mathcal{K}$ be one of the Green's relations $\mathcal{J}$, $\mathcal{R}$ or $\mathcal{L}$. Suppose that $[u]_{\mathsf{DS}} \leq_{\mathcal{K}} [v]_{\mathsf{DS}}$ and that $[v]_{\mathsf{DS}}$ is regular. Then $[u]_{\mathsf{DS}} \mathcal{K} [v]_{\mathsf{DS}}$.*

Proof: If $[u]_{\mathsf{DS}} \leq_{\mathcal{K}} [v]_{\mathsf{DS}}$ then $[u^\omega]_{\mathsf{DS}} \leq_{\mathcal{K}} [u]_{\mathsf{DS}} \leq_{\mathcal{K}} [v]_{\mathsf{DS}}$. Since $[u^\omega]_{\mathsf{DS}}$ is regular and $c(u^\omega) = c(u) = c(v)$, it follows immediately from Theorem 6.6 that $[u]_{\mathsf{DS}} \mathcal{J} [v]_{\mathsf{DS}}$, thus $[u]_{\mathsf{DS}} \mathcal{K} [v]_{\mathsf{DS}}$ by stability of $\overline{\Omega}_A\mathsf{DS}$. ■

Lemma 6.8. *Let u be an element of $\overline{\Omega}_A\mathsf{S}$ with a factorization of the form $u = pxqypzq$, with p, q elements of A^+ with length greater than or equal to k. If $[\Phi_k(u)]_{\mathsf{DS}}$ is regular then $[u]_{\mathsf{DS}*\mathsf{D}_k}$ is regular.*

Proof: Clearly, we only need to consider the case where $|p| = |q| = k$. Let $v = (uy)^\omega u$. Note that $v \mathcal{J} (uy)^\omega$, thus v is regular. Therefore, it suffices to prove that $[v]_{\mathsf{DS}*\mathsf{D}_k} = [u]_{\mathsf{DS}*\mathsf{D}_k}$.

Consider the idempotents $e = \Phi_k(uyp)^\omega$ and $f = \Phi_k(qyu)^\omega$. Note that $e = \Phi_k((uy)^\omega p)$ and $f = \Phi_k(q(uy)^\omega)$ because $p = \mathsf{i}_k(u)$ and $q = \mathsf{t}_k(u)$. Moreover, since $v = (uy)^\omega u(yu)^\omega$, we have

$$\Phi_k(v) = e \cdot \Phi_k(u) \cdot f. \tag{6.3}$$

Moreover, the factorization $u = pxqypzq$ and the lengths of p and q assure us that $c_{k+1}(u) = c_{k+1}(uyu) = c_{k+1}(v)$, and $c(\Phi_k(v)) = c(\Phi_k(u)) = c(e) = c(f)$. Since $[\Phi_k(u)]_{\mathsf{DS}}$ is regular, by Theorem 6.6 we know that $[\Phi_k(u)]_{\mathsf{DS}}$, $[e]_{\mathsf{DS}}$ and $[f]_{\mathsf{DS}}$ belong to the same $\mathcal{J}$-class of $\overline{\Omega}_{A^{k+1}}\mathsf{DS}$. As $e \leq_{\mathcal{R}} \Phi_k(u)$ and $f \leq_{\mathcal{L}} \Phi_k(u)$, it follows that $[e]_{\mathsf{DS}}\ \mathcal{R}\ [\Phi_k(u)]_{\mathsf{DS}}$ and $[f]_{\mathsf{DS}}\ \mathcal{L}\ [\Phi_k(u)]_{\mathsf{DS}}$. Therefore, from (6.3) we obtain $[\Phi_k(v)]_{\mathsf{DS}} = [\Phi_k(u)]_{\mathsf{DS}}$. As $\mathsf{i}_k(v) = \mathsf{i}_k(u)$ and $\mathsf{t}_k(v) = \mathsf{t}_k(u)$, applying Theorem 6.1 we get $[v]_{\mathsf{DS}*\mathsf{D}_k} = [u]_{\mathsf{DS}*\mathsf{D}_k}$. ■

Since $\mathsf{DS} = \mathsf{K} \textcircled{m} \mathsf{DS}$, the elements of $\overline{\Omega}_A\mathsf{DS}$ have unique iterated left basic factorizations, for every alphabet A.

Theorem 6.9. *Let u be an element of $\overline{\Omega}_A\mathsf{S}$ with length greater than k. The following conditions are equivalent:*

(1) the length of the iterated left basic factorization of $\Phi_k(u)$ is infinite;
(2) $[u]_{\mathsf{DS}*\mathsf{D}_k}$ is regular;
(3) $[\Phi_k(u)]_{\mathsf{DS}}$ is regular.

Proof: (1)⇔(3): The iterated left basic factorizations of $\Phi_k(u)$ and $[\Phi_k(u)]_{\mathsf{DS}}$ have the same length (cf. Remark 6.3), hence this equivalence follows immediately from Proposition 6.4.

(2)⇒(3): This implication follows from Corollary 6.2.

(1)⇒(2): By hypothesis, there is a factorization

$$\Phi_k(u) = w_1 z_1 w_2 z_2 w_3 z_3 w_4 z_4 w_5,$$

for some $w_i \in \overline{\Omega}_{A^{k+1}}\mathsf{S}$ $(1 \leq i \leq 5)$, $z_j \in A^{k+1}$ $(1 \leq j \leq 4)$, such that $c(w_i z_i) = c(\Phi_k(u))$ for $1 \leq i \leq 4$. Moreover, we may choose w_i to be an infinite pseudoword, for every i.

Let $p = \mathsf{i}_{k+1}(u)$ and $q = \mathsf{t}_{k+1}(u)$. It follows from [1, Proposition 10.9.2] that there is some factorization of the form $u = pxv_1yv_2zq$, with $x, y, z \in (\overline{\Omega}_A\mathsf{S})^1$ and $v_1, v_2 \in \overline{\Omega}_A\mathsf{S}$ such that $\Phi_k(v_1) = w_2 z_2$ and $\Phi_k(v_2) = w_4 z_4$ (the hypothesis that w_3 is infinite guarantees that the factors v_1 and v_2 of u appear in a non-overlapping way when reading factors of u from left to right). Since $q \in c(w_2 z_2) = c(\Phi_k(u))$ and $p \in c(w_4 z_4) = c(\Phi_k(u))$, we conclude that p and q are respectively factors of v_1 and v_2, thus $u = px'qy'pz'q$ for some $x', y', z' \in (\overline{\Omega}_A\mathsf{S})^1$. Note that $[\Phi_k(u)]_{\mathsf{DS}}$ is regular, by the already proved equivalence (1)⇔(3). This shows we are in the conditions of Lemma 6.8, thus concluding the proof of the implication (1)⇒(2). ■

The following result is an analog of Corollary 6.7.

Theorem 6.10. *Let u and v be elements of $\overline{\Omega}_A\mathsf{S}$ with length greater than k. Let $\mathcal{K}$ be one of Green's relations $\mathcal{J}$, $\mathcal{R}$ or $\mathcal{L}$. Suppose that $c_{k+1}(u) = c_{k+1}(v)$, $[u]_{\mathsf{DS}*\mathsf{D}_k} \leq_{\mathcal{K}} [v]_{\mathsf{DS}*\mathsf{D}_k}$ and that $[v]_{\mathsf{DS}*\mathsf{D}_k}$ is regular. Then $[u]_{\mathsf{DS}*\mathsf{D}_k} \mathrel{\mathcal{K}} [v]_{\mathsf{DS}*\mathsf{D}_k}$.*

Proof: Since $\overline{\Omega}_A(\mathsf{DS}*\mathsf{D}_k)$ is stable, it suffices to consider the case $\mathcal{K} = \mathcal{J}$.

By Corollary 6.2, we know that $[\Phi_k(u)]_{\mathsf{DS}} \leq_{\mathcal{J}} [\Phi_k(v)]_{\mathsf{DS}}$ and that $[\Phi_k(v)]_{\mathsf{DS}}$ is regular. Since $c(\Phi_k(u)) = c(\Phi_k(v))$, it follows from Corollary 6.7 that $[\Phi_k(u)]_{\mathsf{DS}} \mathrel{\mathcal{J}} [\Phi_k(v)]_{\mathsf{DS}}$, and so $[\Phi_k(u)]_{\mathsf{DS}}$ is regular. Therefore, $[u]_{\mathsf{DS}*\mathsf{D}_k}$ is regular by Theorem 6.9.

Then there are $x, y \in \overline{\Omega}_A\mathsf{S}$ such that

$$\mathsf{DS}*\mathsf{D}_k \models u = xvy = (xvy)^{\omega+1}.$$

Let $z = (vyx)^{\omega}v$. Then

$$\Phi_k(z) = \Phi_k((vyx)^{\omega}\,\mathsf{i}_k(v)) \cdot \Phi_k(v) = \Phi_k(vyx\,\mathsf{i}_k(v))^{\omega} \cdot \Phi_k(v). \tag{6.4}$$

Moreover, as $\mathsf{SI}*\mathsf{D}_k \models (xvy)^{\omega+1} = u$, we have $c_{k+1}((xvy)^{\omega+1}) = c_{k+1}(u) = c_{k+1}(v)$ and so

$$c(\Phi_k(vyx\,\mathsf{i}_k(v))^{\omega}) = c_{k+1}(vyxv) = c_{k+1}((xvy)^{\omega+1}) = c(\Phi_k(v)).$$

Since $[\Phi_k(v)]_{\mathsf{DS}}$ and $[\Phi_k(vyx\,\mathsf{i}_k(v))^{\omega}]_{\mathsf{DS}}$ are regular elements of $\overline{\Omega}_{A^{k+1}}\mathsf{DS}$, and since $\Phi_k(vyx\,\mathsf{i}_k(v))^{\omega} \leq_{\mathcal{R}} \Phi_k(v)$, it follows from Theorem 6.6 that $[\Phi_k(v)]_{\mathsf{DS}} \mathrel{\mathcal{R}} [\Phi_k(vyx\,\mathsf{i}_k(v))^{\omega}]_{\mathsf{DS}}$.

Therefore, and as $[\Phi_k(vyx\,\mathsf{i}_k(v))^{\omega}]_{\mathsf{DS}}$ is idempotent, we deduce from (6.4) that $\mathsf{DS} \models \Phi_k(z) = \Phi_k(v)$. Clearly, $\mathsf{i}_k(z) = \mathsf{i}_k(v)$ and $\mathsf{t}_k(z) = \mathsf{t}_k(v)$. Hence, $\mathsf{DS}*\mathsf{D}_k \models z = v$. Since $\mathsf{DS}*\mathsf{D}_k \models z \leq_{\mathcal{J}} xvy \leq_{\mathcal{J}} v$, we conclude that $\mathsf{DS}*\mathsf{D}_k \models u \mathrel{\mathcal{J}} v$. ∎

7. Proof of Theorem 4.3

The following lemma is implicitly shown in the last paragraph of the proof of Theorem 3.6 in [33]. For the reader's convenience, we write down its statement and proof explicitely.

Lemma 7.1. *Let V be a pseudovariety of semigroups. Given an alphabet A, consider the canonical homomorphism $\rho : \overline{\Omega}_A(\mathsf{K} \textcircled{m} \mathsf{V}) \to \overline{\Omega}_A\mathsf{V}$. If x and y are $\mathcal{R}$-equivalent regular elements of $\overline{\Omega}_A(\mathsf{K} \textcircled{m} \mathsf{V})$ such that $\rho(x) = \rho(y)$ then $x = y$.*

Proof: From $\rho(x) = \rho(y)$, we get $\rho(x^{\omega}) = \rho(y^{\omega})$. By the Pin-Weil basis theorem for Mal'cev products [23], given an idempotent e of $\overline{\Omega}_A\mathsf{V}$, the semigroup $\rho^{-1}(e)$ is a pro-K semigroup. Therefore, $x^{\omega}y^{\omega} = x^{\omega}$ and $y^{\omega}x^{\omega} = y^{\omega}$, whence

$x^\omega \mathrel{\mathcal{L}} y^\omega$. Since x and y are regular, we have $x \mathrel{\mathcal{H}} x^\omega$ and $y \mathrel{\mathcal{H}} y^\omega$, thus $x \mathrel{\mathcal{L}} y$. By the hypothesis $x \mathrel{\mathcal{R}} y$, we obtain $x \mathrel{\mathcal{H}} y$ and $x^\omega = y^\omega$. The equality $\rho(x) = \rho(y)$ also implies $\rho(xy^{\omega-1}) = \rho(x^\omega)$, and so $x^\omega \cdot xy^{\omega-1} = x^\omega$, that is, $xy^{\omega-1} = x^\omega$. Therefore, $y = y^{\omega+1} = x^\omega y = xy^{\omega-1}y = xy^\omega = xx^\omega = x$. ∎

The following auxiliary result was inspired by Theorem 3.6 from [33], with which it has similarities. The main difference is that the pseudovarieties considered in Theorem 3.6 from [33] are contained in DS, while in the following result we need to go out from that realm and consider pseudovarieties contained in $\mathsf{DS} * \mathsf{D}_1$. The locality of DS is crucial in the proof, as it guarantees pseudovarieties appearing there are indeed contained in $\mathsf{DS} * \mathsf{D}_1$.

Proposition 7.2. *Let* V *be a pseudovariety of semigroups such that* $\mathsf{Sl} \subseteq \mathsf{V} \subseteq \mathsf{DS}$. *Let* $u, v \in \overline{\Omega}_A\mathsf{S} \setminus A$, *with*

$$\mathsf{ilbf}_2(u) = ((u_i)_{1 \leq i < \ell_u + 1}; q_u) \quad \textit{and} \quad \mathsf{ilbf}_2(v) = ((v_i)_{1 \leq i < \ell_v + 1}; q_v). \tag{7.1}$$

Then $\mathsf{K} ⓜ (\mathsf{V} * \mathsf{D}_1) \models u = v$ *if the following conditions hold:*

(1) $\mathsf{V} * \mathsf{D}_1 \models u = v$;
(2) $\ell_u = \ell_v$;
(3) $\mathsf{K} ⓜ (\mathsf{V} * \mathsf{D}_1) \models u_i = v_i$ *for all* $i \geq 1$ *and* $\mathsf{K} ⓜ (\mathsf{V} * \mathsf{D}_1) \models q_u = q_v$.

Proof: Let $\ell = \ell_u = \ell_v$. If ℓ is finite, then $u = u_1u_2u_3 \cdots u_\ell q_u$ and $v = v_1v_2v_3 \cdots v_\ell q_v$. Therefore, thanks to condition (3), the proof of the theorem for this case is immediate.

Suppose that $\ell = \infty$. Denote by W the pseudovariety $\mathsf{K} ⓜ (\mathsf{V} * \mathsf{D}_1)$. Then $\mathsf{W} \subseteq \mathsf{K} ⓜ (\mathsf{DS} * \mathsf{D}_1)$. The pseudovariety DS is local [14] and one has $\mathsf{K} ⓜ \mathsf{DS} = \mathsf{DS}$. Therefore, by Theorem 4.1, we have $\mathsf{K} ⓜ (\mathsf{DS} * \mathsf{D}_1) = \mathsf{DS} * \mathsf{D}_1$, whence $\mathsf{W} \subseteq \mathsf{DS} * \mathsf{D}_1$. Since $\ell = \infty$, the implicit operations $[u]_{\mathsf{DS}*\mathsf{D}_1}$ and $[v]_{\mathsf{DS}*\mathsf{D}_1}$ are regular by Theorem 6.9. Hence, since $\mathsf{W} \subseteq \mathsf{DS} * \mathsf{D}_1$, the implicit operations $[u]_\mathsf{W}$ and $[v]_\mathsf{W}$ are also regular.

For each $n \geq 1$, let $w_n = u_1u_2 \cdots u_n$. Note that $u \leq_\mathcal{R} w_n$. Let w be an accumulation point of the sequence $(w_n)_n$. Then $\mathsf{ilbf}(\Phi_1(w))$ is infinite and $u \leq_\mathcal{R} w$. In particular, $[w]_\mathsf{W}$ is regular by Theorem 6.9. Moreover, $c_2(w) = c_2(u)$, and so applying Theorem 6.10 we obtain $[u]_\mathsf{W} \mathrel{\mathcal{R}} [w]_\mathsf{W}$. On the other hand, since $v_1 \cdots v_n$ is a prefix of v, by Condition (3) in the statement, we have $[v]_\mathsf{W} \leq_\mathcal{R} [w_n]_\mathsf{W}$ for all n. Hence $[v]_\mathsf{W} \mathrel{\mathcal{R}} [w]_\mathsf{W}$, again by Theorem 6.9. Note also that $c_2(v) = c_2(u) = c_2(w)$, since $\mathsf{Sl} * \mathsf{D}_1 \models u = v$. Therefore $[u]_\mathsf{W}$ and $[v]_\mathsf{W}$ are $\mathcal{R}$-equivalent regular elements of $\overline{\Omega}_A\mathsf{W}$. Since $[u]_{\mathsf{V}*\mathsf{D}_1} = [v]_{\mathsf{V}*\mathsf{D}_1}$, applying Lemma 7.1 we get $[u]_\mathsf{W} = [v]_\mathsf{W}$. ∎

Proposition 7.3. *Let* V *be a pseudovariety of semigroups such that* $\mathsf{SI} \subseteq \mathsf{V} \subseteq \mathsf{DS}$*. Then* $(\mathsf{K} \textcircled{m} \mathsf{V}) * \mathsf{D}_1 \models u = v$ *implies* $\mathsf{K} \textcircled{m} (\mathsf{V} * \mathsf{D}_1) \models u = v$*, for every* $u, v \in \overline{\Omega}_A \mathsf{S}$*.*

Proof: Since $(\mathsf{K} \textcircled{m} \mathsf{V}) * \mathsf{D}_1 \models u = v$, in particular we have $\mathsf{K} \models u = v$. Therefore, $u \in A^+$ if and only if $v \in A^+$, and if $u, v \in A^+$ then $u = v$.

Next, we prove the proposition by induction on $|c_2(u)|$. If $|c_2(u)| = 0$, then $u \in A$, and so $u = v$.

Suppose that $|c_2(u)| > 0$, and that the proposition holds for pseudoidentities between elements with less factors of length two than u. Let

$$\mathsf{ilfb}_2(u) = ((u_i)_{1 \le i < \ell_u + 1}; q_u), \quad \mathsf{ilfb}_2(v) = ((v_i)_{1 \le i < \ell_v + 1}; q_v).$$

Since $(\mathsf{K} \textcircled{m} \mathsf{V}) * \mathsf{D}_1 \models u = v$, from Lemma 6.5 we deduce that ℓ_u and ℓ_v are equal, say to ℓ, and that

$$(\mathsf{K} \textcircled{m} \mathsf{V}) * \mathsf{D}_1 \models u_i = v_i \quad \text{and} \quad (\mathsf{K} \textcircled{m} \mathsf{V}) * \mathsf{D}_1 \models q_u = q_v, \tag{7.2}$$

for all i such that $1 \le i < \ell + 1$.

By the definition of ilfb_2, we know that $c_2(u_i)$ and $c_2(q_u)$ have less elements than $c_2(u)$. Applying the induction hypothesis to (7.2), we obtain

$$\mathsf{K} \textcircled{m} (\mathsf{V} * \mathsf{D}_1) \models u_i = v_i \quad \text{and} \quad \mathsf{K} \textcircled{m} (\mathsf{V} * \mathsf{D}_1) \models q_u = q_v, \tag{7.3}$$

for all i such that $1 \le i < \ell + 1$. Therefore $\mathsf{K} \textcircled{m} (\mathsf{V} * \mathsf{D}_1) \models u = v$ by Proposition 7.2. ■

We are now ready to conclude the proof of Theorem 4.3. We do it in three steps, one for each of the pseudovarieties K, D and $\mathcal{L}\mathsf{I}$.

The case of pseudovariety K: We want to show that the following inclusion holds for every pseudovariety of semigroups V containing SI, and every $k \ge 1$:

$$\mathsf{K} \textcircled{m} (\mathsf{V} * \mathsf{D}_k) \subseteq (\mathsf{K} \textcircled{m} \mathsf{V}) * \mathsf{D}_k. \tag{7.4}$$

If $B_2 \in \mathsf{V}$, then there is nothing to prove thanks to Theorem 4.2. Suppose that $B_2 \notin \mathsf{V}$, that is, $\mathsf{V} \subseteq \mathsf{DS}$.

Since every pseudovariety is defined by a basis of pseudoidentities, Proposition 7.3 states that

$$\mathsf{K} \textcircled{m} (\mathsf{V} * \mathsf{D}_1) \subseteq (\mathsf{K} \textcircled{m} \mathsf{V}) * \mathsf{D}_1. \tag{7.5}$$

It is well known that $B_2 \in \mathsf{V} * \mathsf{D}_1$ (cf. [31, Theorem 5.7], for example). Let k be an integer greater than 1. Since $\mathsf{V} * \mathsf{D}_k = \mathsf{V} * \mathsf{D}_1 * \mathsf{D}_{k-1}$ (recall that

$\mathsf{D}_m * \mathsf{D}_n = \mathsf{D}_{m+n}$ for all $m, n \geq 1$ [1, Lemma 10.4.1]) and $B_2 \in \mathsf{V} * \mathsf{D}_1$, we may apply Theorem 4.2 to get

$$\mathsf{K} \circledcirc (\mathsf{V} * \mathsf{D}_k) \subseteq (\mathsf{K} \circledcirc (\mathsf{V} * \mathsf{D}_1)) * \mathsf{D}_{k-1}.$$

By (7.5), we then obtain

$$\mathsf{K} \circledcirc (\mathsf{V} * \mathsf{D}_k) \subseteq (\mathsf{K} \circledcirc \mathsf{V}) * \mathsf{D}_1 * \mathsf{D}_{k-1},$$

which is precisely inclusion (7.4). ■

The case of pseudovariety D: The inclusion

$$\mathsf{D} \circledcirc (\mathsf{V} * \mathsf{D}_k) \subseteq (\mathsf{D} \circledcirc \mathsf{V}) * \mathsf{D}_k. \tag{7.6}$$

is obtained by applying Propositions 2.4 and 2.3 to the inclusion (7.4). ■

The case of pseudovariety $\mathcal{L}\mathsf{I}$: Let α and β be the operators on the lattice of semigroup pseudovarieties defined by $\alpha(\mathsf{W}) = \mathsf{K} \circledcirc \mathsf{W}$ and $\beta(\mathsf{W}) = \mathsf{D} \circledcirc \mathsf{W}$, where W is a pseudovariety of semigroups. It is proved in [7] (more precisely, see Theorems 6.4 and 8.2 from [7]) that

$$\mathcal{L}\mathsf{I} \circledcirc \mathsf{W} = \bigcup_{n \geq 0} (\beta \circ \alpha)^n(\mathsf{W}) = \bigcup_{n \geq 0} (\alpha \circ \beta)^n(\mathsf{W}) \tag{7.7}$$

for every pseudovariety of semigroups W. If V is a pseudovariety of semigroups containing Sl, then it follows easily from (7.4) and (7.6), and from induction on n, that $(\beta \circ \alpha)^n(\mathsf{V} * \mathsf{D}_k) \subseteq (\beta \circ \alpha)^n(\mathsf{V}) * \mathsf{D}_k$ for every $n \geq 0$ and $k \geq 1$. Thanks to (7.7), this shows $\mathcal{L}\mathsf{I} \circledcirc (\mathsf{V} * \mathsf{D}_k) \subseteq (\mathcal{L}\mathsf{I} \circledcirc \mathsf{V}) * \mathsf{D}_k$. ■

All cases were considered, whence Theorem 4.3 is proved. ■

Appendix: Proof of Proposition 1.1

To write an explicit proof of Proposition 1.1 (which, as remarked in the introduction, is a straightforward generalization of the result presented in [25, Exercise 4.6.58]) we need the language and results of the "Semilocal Theory" of finite semigroups, the subject of Section 4.6 in [25]. We follow some definitions and notations from [25, Subsections 4.6.1 and 4.6.2]. However, we also introduce definitions and notations of our own, in order to make a uniform proof of Proposition 1.1.

We use the expression K-*semigroup* for a right mapping semigroup and the expression $(\mathsf{K} \vee \mathsf{G})$-*semigroup* for a right letter mapping semigroup. The expressions D-*semigroup* and $(\mathsf{D} \vee \mathsf{G})$-*semigroup* have dual meanings. Finally, the expression $\mathcal{L}\mathsf{I}$-*semigroup* is used for generalized group mapping semigroups, and $\mathcal{L}\mathsf{G}$-*semigroup* for generalized group mapping semigroups with an aperiodic distinguished $\mathcal{J}$-class.

The following result is also from [25] (cf. [25, Proposition 4.6.56 and the comment following its proof]).

Proposition 7.4. *Let* $\mathsf{Z} \in \mathbb{V} \setminus \{\mathsf{N}, \mathsf{N} \vee \mathsf{G}\}$*. If S is a* Z*-semigroup then every local monoid of S is a* Z*-semigroup.*

Next, we uniformize some of the definitions appearing in [25, Subsection 4.6.2]. Let S be a finite semigroup and let J be a regular $\mathcal{J}$-class of S. The canonical homomorphisms

$$S \to \mathsf{AGGM}_J(S),\ S \to \mathsf{GGM}_J(S),\ S \to \mathsf{RLM}_J(S) \text{ and } S \to \mathsf{RM}_J(S)$$

will be denoted respectively by $\mu^S_{\mathcal{L}\mathsf{G},J}$, $\mu^S_{\mathcal{L}\mathsf{I},J}$, $\mu^S_{\mathsf{K}\vee\mathsf{G},J}$ and $\mu^S_{\mathsf{K},J}$.

Similarly, the canonical homomorphisms

$$S \to S/\mathsf{AGGM},\ S \to S/\mathsf{GGM},\ S \to S/\mathsf{RLM}, \text{ and } S \to S/\mathsf{RM}$$

will be denoted respectively by $\mu^S_{\mathcal{L}\mathsf{G}}$, $\mu^S_{\mathcal{L}\mathsf{I}}$, $\mu^S_{\mathsf{K}\vee\mathsf{G}}$ and μ^S_{K}.

The homomorphisms $\mu^S_{\mathsf{D}\vee\mathsf{G},J}$, $\mu^S_{\mathsf{D},J}$, $\mu^S_{\mathsf{D}\vee\mathsf{G}}$ and μ^S_{D} have the obvious dual definitions. Let $Jr(S)$ be the set of regular $\mathcal{J}$-classes of S. Recall from [25, Definition 4.6.44] that $\operatorname{Ker}\mu^S_{\mathsf{Z}} = \bigcap_{J\in Jr(S)} \operatorname{Ker}\mu^S_{\mathsf{Z},J}$ for every $\mathsf{Z} \in \mathbb{V}\setminus\{\mathsf{N}\vee\mathsf{G}, \mathsf{N}\}$, whence $\mu^S_{\mathsf{Z}}(S)$ is a subdirect product of $\prod_{J\in Jr(S)} \mu^S_{\mathsf{Z},J}(S)$.

We divide the proof of Proposition 1.1 in two parts.

Proof of Proposition 1.1: We treat first the case $\mathsf{Z} \in \mathbb{V} \setminus \{\mathsf{N}, \mathsf{N} \vee \mathsf{G}\}$.

Suppose that $S \in \mathsf{Z}ⓜ\mathcal{L}\mathsf{V}$. Then $\mu^S_{\mathsf{Z}}(S) \in \mathcal{L}\mathsf{V}$, by [25, Theorem 4.6.50]. If e is an idempotent of S, then $\mu^S_{\mathsf{Z}}(eSe)$ is a local monoid of $\mu^S_{\mathsf{Z}}(S)$, thus $\mu^S_{\mathsf{Z}}(eSe) \in \mathsf{V}$. Therefore $eSe \in \mathsf{Z}ⓜ\mathsf{V}$ by [25, Theorem 4.6.50]. This proves $\mathsf{Z}ⓜ\mathcal{L}\mathsf{V} \subseteq \mathcal{L}(\mathsf{Z}ⓜ\mathsf{V})$.

Conversely, suppose that $S \in \mathcal{L}(\mathsf{Z}ⓜ\mathsf{V})$. Consider a regular $\mathcal{J}$-class K of S. Let e be an idempotent of S. Then $M = \mu^S_{\mathsf{Z},K}(eSe)$ is a local monoid of $\mu^S_{\mathsf{Z},K}(S)$, and every local monoid of $\mu^S_{\mathsf{Z},K}(S)$ is of this form. The semigroup $\mu^S_{\mathsf{Z},K}(S)$ is a Z-semigroup by Proposition 4.6.29, 4.6.31, 4.6.35 or 4.6.37 from [25], depending on which Z is considered. Hence M is also a Z-semigroup by Proposition 7.4. Then, also by Proposition 4.6.29, 4.6.31, 4.6.35 or 4.6.37 from [25], if J is the distinguished $\mathcal{J}$-class of M, then the identity on M factors through $\mu^M_{\mathsf{Z},J}$, thus $\mu^M_{\mathsf{Z},J}$ and μ^M_{Z} are the identity on M. Since $S \in \mathcal{L}(\mathsf{Z}ⓜ\mathsf{V})$ and the monoid M is a homomorphic image of eSe, we know that

$M \in \mathsf{Z}ⓜ\mathsf{V}$. And since $\mu_{\mathsf{Z}}^{M}(M) = M$, we actually have $M \in \mathsf{V}$ by [25, Theorem 4.6.50]. This proves that $\mu_{\mathsf{Z},K}^{S}(S) \in \mathcal{L}\mathsf{V}$. As $\mu_{\mathsf{Z}}^{S}(S)$ divides $\prod_{K \in Jr(S)} \mu_{\mathsf{Z},K}^{S}(S)$, we deduce that $\mu_{\mathsf{Z}}^{S}(S) \in \mathcal{L}\mathsf{V}$, and so $S \in \mathsf{Z}ⓜ\mathcal{L}\mathsf{V}$, again by [25, Theorem 4.6.50]. This concludes the proof of the inclusion $\mathcal{L}(\mathsf{Z}ⓜ\mathsf{V}) \subseteq \mathsf{Z}ⓜ\mathcal{L}\mathsf{V}$ in the case $\mathsf{Z} \in \mathbb{V} \setminus \{\mathsf{N}, \mathsf{N} \vee \mathsf{G}\}$.

Finally, let us proof the proposition for $\mathsf{Z} \in \{\mathsf{N}, \mathsf{N} \vee \mathsf{G}\}$. In this case $\mathsf{Z} = \mathsf{Z}_1 \cap \mathsf{Z}_2$ for some $\mathsf{Z}_1, \mathsf{Z}_2 \in \mathbb{V}$. Then by [23, Corollary 3.2] and the already proved cases, we have

$$\mathsf{Z}ⓜ\mathcal{L}\mathsf{V} = \mathsf{Z}_1ⓜ\mathcal{L}\mathsf{V} \cap \mathsf{Z}_2ⓜ\mathcal{L}\mathsf{V} = \mathcal{L}(\mathsf{Z}_1ⓜ\mathsf{V}) \cap \mathcal{L}(\mathsf{Z}_2ⓜ\mathsf{V}).$$

Since $\mathcal{L}(\mathsf{W}_1 \cap \mathsf{W}_2) = \mathcal{L}\mathsf{W}_1 \cap \mathcal{L}\mathsf{W}_2$ for every pair of pseudovarieties W_1, W_2, and again taking into account [23, Corollary 3.2], we therefore obtain $\mathsf{Z}ⓜ\mathcal{L}\mathsf{V} = \mathcal{L}(\mathsf{Z}ⓜ\mathsf{V})$. ■

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