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SEMIDIRECT PRODUCTS AND CROSSED MODULES IN MONOIDS WITH OPERATIONS

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Abstract: We describe actions, semidirect products and crossed modules in categories of monoids with operations. Moreover we characterize, in this context, the internal categories corresponding to crossed modules. Concrete examples in the cases of monoids, semirings and distributive lattices are given.

Keywords: actions, semidirect products, crossed modules, monoids with operations.
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1. Introduction

In the category of groups, there is a well known equivalence between actions and split extensions, obtained via the semidirect product construction. It is also well known (see, for example, [4]) that internal categories in the category of groups are equivalent to crossed modules. In the paper [9], Porter proved the same equivalence in the case of categories of groups with operations, which includes the examples of rings, associative algebras, Lie algebras, Jordan algebras and many others.

The equivalence between internal categories and crossed modules is not true in weaker algebraic contexts, such as monoids. However, in the paper [8], Patchkoria introduced, in the category of monoids, a particular kind of internal categories, called Schreier internal categories, and he proved the equivalence between them and what he called crossed semimodules.

The aim of the present paper is to generalize Patchkoria's result to a wider class of categories, whose objects are called monoids with operations. This class, which includes monoids, commutative monoids, semirings, semilattices

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with a bottom element and distributive lattices with a bottom element, actually generalizes at the same time Patchkoria's result concerning monoids and Porter's result concerning groups with operations.

The paper is organized as follows. In Section 2 we introduce the notion of monoids with operations and we describe actions and the construction of semidirect products in this context. In Section 3 we define crossed modules in monoids with operations and we prove that they are equivalent to Schreier internal categories. Section 4 is devoted to compare, in the case of monoids, the notion of semidirect product described in Section 2 with the categorical one introduced by Bourn and Janelidze in [3]. In Section 5 the case of semirings, and of distributive lattices as a particular case, is developed with concrete examples.

2. Monoids with operations

The following definition is inspired by the analogous one, given by Porter in [9], of groups with operations.

Definition 2.1. Let Ω be a set of finitary operations such that the following conditions hold: if Ω_i is the set of i-ary operations in Ω, then:

(1) $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$;
(2) There is a binary operation $+ \in \Omega_2$ (not necessarily commutative) and a constant $0 \in \Omega_0$ satisfying the usual axioms for monoids;
(3) $\Omega_0 = \{0\}$;
(4) Let $\Omega_2' = \Omega_2 \backslash \{+\}$; if $* \in \Omega_2'$, then $*^\circ$ defined by $x *^\circ y = y * x$ is also in Ω_2';
(5) Any $* \in \Omega_2'$ is left distributive w.r.t. $+$, i.e.:

$$a * (b + c) = a * b + a * c;$$

(6) For any $* \in \Omega_2'$ we have $b * 0 = 0$;
(7) Any $\omega \in \Omega_1$ satisfies the following conditions:
 - $\omega(x + y) = \omega(x) + \omega(y)$;
 - for any $* \in \Omega_2'$ $\omega(a * b) = \omega(a) * b$.

Let moreover E be a set of axioms including the ones above. We will denote by $\mathbb{C}$ the category of (Ω, E)-algebras. We will call the objects of $\mathbb{C}$ *monoids with operations.*

other operation is given by the concatenation of words: given two languages L and L', a word τ belongs to the product LL' if and only if there exist $\sigma \in L$ and $\sigma' \in L'$ such that $\tau = \sigma\sigma'$. It is immediate to see that this concatenation is associative and distributive with respect to the union.

An important particular instance of a semiring is a distributive lattice: a distributive lattice is a semi-ring $(A, +, 0, \times)$ where, in particular, $(A, +, 0)$ is an idempotent commutative monoid and $(A, \times)$ is an idempotent commutative semigroup.

In the particular case when the operation $\times$ is commutative, $\alpha_\times = \alpha_{\times^\circ}$. As a concrete example, we can study actions of $\mathbb{N}$ on itself, where $\mathbb{N}$ denotes the semiring of natural numbers. There are exactly two actions of $\mathbb{N}$ on itself: $\alpha_\times(n, m) = nm$ and $\alpha_\times(n, m) = 0$. Indeed, from (8), $\alpha_\times$ must be of the form $\alpha_\times(n, m) = knm$, with $k = \alpha_\times(1, 1)$, but in order to satisfy (11), k must be idempotent: $k = k^2$. The only two natural numbers with this property are $k = 0$ and $k = 1$.

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and, if for simplicity we write $\alpha_{(\cdot)}(b,x) = b^{\odot}x$ *and* $\alpha_{(\cdot)^{\circ}}(b,x) = x_{\odot}b$, *the following conditions are satisfied for every* $b, b' \in B$ *and* $x, x' \in X$*:*

$$(b+b')^{\odot}(x+x') = b^{\odot}x + b^{\odot}x' + b'^{\odot}x + b'^{\odot}x', \tag{8}$$

$$(x+x')_{\odot}(b+b') = x_{\odot}b + x_{\odot}b' + x'_{\odot}b + x'_{\odot}b', \tag{9}$$

$$(b^{\odot}x)\cdot x' = b^{\odot}(x\cdot x') \quad , \quad x\cdot(x'_{\odot}b) = (x\cdot x')_{\odot}b, \tag{10}$$

$$(b\cdot b')^{\odot}x = b^{\odot}(b'^{\odot}x) \quad , \quad x_{\odot}(b\cdot b') = (x_{\odot}b)_{\odot}b', \tag{11}$$

$$x\cdot(b^{\odot}x') = (x_{\odot}b)\cdot x' \quad , \quad (b^{\odot}x)_{\odot}b' = b^{\odot}(x_{\odot}b'), \tag{12}$$

$$0^{\odot}x = 0_{\odot}b = b^{\odot}0 = x_{\odot}0 = 0. \tag{13}$$

Proof: Condition (7) is due to the fact that $+$ is commutative, together with the specifications $\alpha_+(0,x) = x$ and $\alpha_+(b,0) = 0$, as it follows from Lemma 2.9. Conditions (8) to (13) are equivalent to the distributivity of $\cdot$ and $\cdot^{\circ}$ with respect to $+$, the fact that 0 is absorvent with respect to $\cdot$, and the associativity of $\cdot$. Indeed, for any element (x,b) in the semidirect product $X \rtimes_{\alpha} B$ with the operations as specified in Definition 2.4, we have

$$(x,b)\cdot(0,0) = (0,0) = (0,0)\cdot(x,b)$$

and hence

$$b^{\odot}0 + x_{\odot}0 = 0 = 0^{\odot}x + 0_{\odot}b,$$

now using the distributivity, as specified in Lemma 2.9, we also have

$$\begin{aligned} 0 = 0^{\odot}x + 0_{\odot}b &= (0+0)^{\odot}x + 0_{\odot}b \\ &= 0^{\odot}x + (0^{\odot}x + 0_{\odot}b) \\ &= 0^{\odot}x + 0 = 0^{\odot}x. \end{aligned}$$

The other identities in (13) are obtained in a similar way. It is now a routine calculation to check that equations (8) and (9) follow from the distributivity of $\cdot$, while the equations in (10), (11) and (12) follow from the associativity of $\cdot$ in $X \rtimes_{\alpha} B$. ■

The example $(\mathbb{N}, +, 0, \times)$ of natural numbers with zero, addition and multiplication is perhaps the paradigmatic example of a semiring. Other examples are $\hom(B,B)$, the set of all endomorphisms on a commutative monoid B, with the zero map, the componentwise addition and the composition of morphisms as multiplication. Moreover, given a set A, the set of languages over the alphabet A (i.e. the set of subsets of the free monoid A^* over A) is a semiring, where the monoid operation is the set-theoretical union, while the

Remark. The definition above does not include the case of groups, or more generally, the one of groups with operations. Indeed, the unary operation given by the group inverses, denoted by $-$, does not satisfy Condition 7. However, in order to recover all these structures, it suffices to add another condition: if the base monoid structure (given by the operations $+$ and 0) is a group, then the operation $-$ should be distinguished from the other unary operations. In other terms, Condition 7 should be satisfied only by operations in $\Omega_1' = \Omega_1 \backslash \{-\}$. In this way, our definition becomes a generalization of the concept of groups with operations.

Example 2.2. Among the known structures covered by Porter's definition, such as groups, rings, associative algebras, Lie algebras and many others, our definition includes the cases of monoids, commutative monoids, semirings (i.e. rings where the additive structure is not necessarily a group, but just a commutative monoid), semilattices with a bottom element, distributive lattices with a bottom element (or a top one).

Remark. Let us observe that requiring left and right distributivity of any $* \in \Omega_2'$ with respect to $+$, as in Definition 2.1 (or, in other terms, left distributivity of $*$ and $*^{\circ}$), implies a partial commutativity of $+$. Indeed, consider the element $(a+b)*(c+d)$; on one hand we have:

$$(a+b)*(c+d) = (a+b)*c + (a+b)*d = a*c + b*c + a*d + b*d,$$

while, on the other hand:

$$(a+b)*(c+d) = a*(c+d) + b*(c+d) = a*c + a*d + b*c + b*d,$$

and hence the two expressions on the right are equal.

From now on, let $\mathbb{C}$ be a category of (Ω, E)-algebras as in the definition above.

Definition 2.3. Let X and B be two objects of $\mathbb{C}$. A *pre-action* of B on X is a set, indexed by the set Ω_2 of binary operations, of set-theoretical maps $\alpha_* \colon B \times X \to X$, $* \in \Omega_2$.

What we call pre-action is what was called set of actions in [7], in the more restricted context of categories of interest (which are particular groups with operations, in the sense of Porter).

Given a pre-action of B on X, we can construct a semidirect product of X and B with respect to this pre-action, following the analogous construction already known for groups with operations.

Definition 2.4. Given a pre-action $\alpha = \{\alpha_* | * \in \Omega_2\}$ of B on X, the *semidirect product* $X \rtimes_\alpha B$ of X and B with respect to α is the Ω-algebra with underlying set $X \times B$ and operations defined by:

$$(x_1, b_1) + (x_2, b_2) = (x_1 + \alpha_+(b_1, x_2), b_1 + b_2),$$
$$(x_1, b_1) * (x_2, b_2) = (x_1 * x_2 + \alpha_*(b_1, x_2) + \alpha_{*^\circ}(b_2, x_1), b_1 * b_2), \text{ for } * \in \Omega_2',$$
$$\omega(x, b) = (\omega(x), \omega(b)), \text{ for } \omega \in \Omega_1.$$

For a generic pre-action α, $X \rtimes_\alpha B$ is not a (Ω, E)-algebra. The main goal of this section is to characterize those pre-actions for which the corresponding semidirect product is a (Ω, E)-algebra.

Let B be an object of $\mathbb{C}$. The category $Pt(B)$ is the category of the points of the comma category $\mathbb{C}$ over B, i.e. the cocomma category 1_B over $\mathbb{C}/B$. This amounts to the category whose objects are the split epimorphisms with codomain B. In fact a morphism from the terminal $1_B \colon B \to B$ to an object $\alpha \colon A \to B$, is precisely an arrow $\beta \colon B \to A$ such that $\alpha\beta = 1_B$. An object of $Pt(B)$ will be called *point over* B. We will consider, in the context of monoids with operations, a particular kind of point. The definition below is inspired by the definition of Schreier internal category given in [8] in the category of monoids:

Definition 2.5. A point

$$\begin{tikzcd} X = Ker\ p \arrow[r, "k"] & A \arrow[r, shift left, "p"] & B \arrow[l, shift left, "s"] \end{tikzcd} \tag{1}$$

is said to be a *Schreier point* if, for any $a \in A$, there exists a unique $x \in X$ such that $a = k(x) + sp(a)$ (where, as in Definition 2.1, we use the symbol $+$ for the monoid operation).

In other terms, a Schreier point is a point of the form (1) equipped with a unique set-theoretical map $q \colon A \to X$ with the property that

$$a = kq(a) + sp(a)$$

for any $a \in A$.

in the same way as it happens in the category of groups (see [3] for a more detailed description of this bijection in the category of groups). Conversely, given an external action α of B on X, we can consider the following commutative diagram:

$$\begin{tikzcd} B\flat X \arrow[r, "k_0"] \arrow[d, dotted, "\xi"'] & X + B \arrow[r, shift right, "{[0,1]}"'] \arrow[d, "{[\langle 1,0\rangle,\langle 0,1\rangle]}"] & B \arrow[l, shift right, "\iota_B"'] \arrow[d, equal] \\ X \arrow[r, "{\langle 1,0\rangle}"'] & X \rtimes_\alpha B \arrow[r, shift right, "\pi_B"'] & B. \arrow[l, shift right, "{\langle 0,1\rangle}"'] \end{tikzcd}$$

Then we can define an internal action ξ just by restriction of the morphism $[\langle 1,0\rangle, \langle 0,1\rangle]$ to $B\flat X$. It is straightforward to prove that in this way we obtain a bijection between $IntAct(B, X)$ and $ExtAct(B, X)$. ■

5. The case of semirings

In this section we explore in more details the example of semirings.

A semiring $(A, +, 0, \cdot)$ is an algebraic structure with one constant and two binary operations, in which $(A, +, 0)$ is a commutative monoid, $(A, \cdot)$ is a semigroup, and the following conditions are satisfied for every $x, y, z \in A$:

$$\begin{aligned} x \cdot (y + z) &= (x \cdot y) + (x \cdot z), \\ (x + y) \cdot z &= (x \cdot z) + (y \cdot z), \\ x \cdot 0 = & \ 0 = 0 \cdot x. \end{aligned}$$

If $X = (X, +, 0, \cdot)$ and $B = (B, +, 0, \cdot)$ are two semirings, a pre-action of B on X consists of three maps

$$\alpha_+, \alpha_{(\cdot)}, \alpha_{(\cdot)^\circ} \colon B \times X \to X.$$

Proposition 5.1. *A pre-action* $\alpha = \{\alpha_+, \alpha_{(\cdot)}, \alpha_{(\cdot)^\circ}\}$ *is an action if and only if, for all* $b \in B$ *and all* $x \in X$,

$$\alpha_+(b, x) = x, \tag{7}$$

of $\mathbb{N}$ over X is just a singleton.

However, the set $ExtAct(\mathbb{N}, X)$ of external actions of $\mathbb{N}$ over X is not a singleton in general. To see that, we can choose also X to be the monoid $\mathbb{N}$ of natural numbers. Consider then, for any natural number n, different from 0, the following pre-action of $\mathbb{N}$ on itself:

$$\alpha_n(b, x) = n^b x.$$

It is straightforward to verify that the semidirect product defined using any of these pre-actions, as in Definition 2.4, is a monoid. Hence α_n is an external action for any n. It is easy to see that these actions do not give rise to semidirect products that are all isomorphic: it suffices to observe that the semidirect product $\mathbb{N} \rtimes_{\alpha_1} \mathbb{N}$ is just the direct product of $\mathbb{N}$ with itself, hence it is a commutative monoid, while the semidirect products $\mathbb{N} \rtimes_{\alpha_n} \mathbb{N}$ are not commutative if $n \neq 1$. Hence $IntAct(\mathbb{N}, \mathbb{N}) \neq ExtAct(\mathbb{N}, \mathbb{N})$.

There are particular cases, however, where internal and external actions coincide. One of them is described in the following

Proposition 4.3. *If B is a group (and X is a generic monoid), then there is a bijection between $IntAct(B, X)$ and $ExtAct(B, X)$.*

Proof: Let us first observe that every point

$$\begin{tikzcd} X \arrow[r, "k"] & A \arrow[r, shift right, "p"'] & B \arrow[l, shift right, "s"'] \end{tikzcd} \tag{6}$$

such that B is a group is actually a Schreier point. Indeed, we can define a pre-action of B on X in the following way:

$$\alpha(b, x) = s(b) + k(x) - s(b),$$

and it is immediate to show that the corresponding semidirect product $X \rtimes_\alpha B$ is a monoid, hence this pre-action is an external action and the point (6) is a Schreier one: in fact we have that, in this case, $q(a) = a - sp(a)$.

Moreover, when B is a group, $B\flat X$ is the submonoid of the free product $X+B$ generated by chains of the form $(b, x, -b)$ for $b \in B$ and $x \in X$. Hence, given an internal action $\xi\colon B\flat X \to X$, we can define a pre-action (which is actually an external action) by:

$$\alpha(b, x) = \xi(b, x, -b),$$

It comes immediately from the definition above that, in a Schreier point, the morphisms k and s are jointly epimorphic. Hence they have the following interesting property:

Proposition 2.6. *In a point of the form*

$$\begin{tikzcd} X \arrow[r, "k"] & A \arrow[r, shift left, "p"] & B, \arrow[l, shift left, "s"] \end{tikzcd}$$

if k and s are jointly epimorphic, then p is the cokernel of k. In other terms the sequence

$$0 \longrightarrow X \xrightarrow{k} A \xrightarrow{p} B \longrightarrow 0$$

is exact and the point is a split extension.

Proof: Given a morphism $f\colon A \to D$ such that $fk = 0$, we have that fs makes the triangle below commutative:

$$\begin{tikzcd} X \arrow[r, "k"] & A \arrow[r, shift left, "p"] \arrow[dr, "f"'] & B \arrow[l, shift left, "s"] \arrow[d, dotted, "fs"] \\ & & D. \end{tikzcd}$$

Indeed:

$$fsps = fs \quad \text{and} \quad fspk = 0 = fk,$$

and since k and s are jointly epimorphic, we have that $fsp = f$. Moreover, given any $g\colon B \to D$ such that $gp = f$, we have that

$$g = gps = fs. \qquad \blacksquare$$

It is known that, in a category $\mathbb{C}$ of monoids with operations, there are points that are not split extensions. For example, in the category Mon of monoids, consider the following point, where $\mathbb{N}$ is the monoid of natural numbers with the usual sum:

$$\begin{tikzcd} 0 \arrow[r, "0"] & \mathbb{N} \times \mathbb{N} \arrow[r, shift right, "+"'] & \mathbb{N}. \arrow[l, shift right, "\langle 0,1 \rangle"'] \end{tikzcd}$$

Now we can introduce the concept of action, which corresponds to the one of set of derived actions, introduced in [7] for categories of interest.

Given a Schreier point over B with kernel X, we can define a pre-action of B on X in the following way:

$$\alpha_+(b,x) = q(s(b)+k(x)),$$
$$\alpha_*(b,x) = q(s(b)*k(x)), \text{ for } * \in \Omega_2'.$$

Definition 2.7. A pre-action defined as above, starting from a Schreier point, will be called an *action* of B on X.

Now we can state the main result of this section:

Theorem 2.8. *A pre-action α of B on X is an action if and only if the semidirect product $X \rtimes_\alpha B$ is an object of $\mathbb{C}$.*

Proof: Let

$$\begin{tikzcd} X \arrow[r, shift right, "k"'] & A \arrow[l, dotted, shift right, "q"'] \arrow[r, shift left, "p"] & B \arrow[l, shift left, "s"] \end{tikzcd}$$

be a Schreier point. First let us observe that $\alpha_+(b,x)$ is the unique element of X such that:

$$s(b)+k(x) = k\alpha_+(b,x)+s(b)$$

(this follows from the Schreier condition applying q to the element $a = s(b)+k(x)$). Now, considering α as in Definition 2.7, we have to show that A is isomorphic to the semidirect product $X \rtimes_\alpha B$ of X and B with respect to the action α. Consider the map $\psi\colon A \to X \rtimes_\alpha B$ sending an element $a \in A$ to the pair $(q(a), p(a))$. It is a bijection, whose inverse is the map $\varphi\colon X \rtimes_\alpha B \to A$ sending a pair (x,b) to the element $k(x)+s(b)$. Indeed:

$$\varphi\psi(a) = \varphi(q(a),p(a)) = kq(a)+sp(a) = a$$

and

$$\psi\varphi(x,b) = \psi(k(x)+s(b)) = (q(k(x)+s(b)), b),$$

so it remains to prove that $q(k(x)+s(b)) = x$. Putting $a = k(x)+s(b)$ and $q(a) = x'$, we have that x' is the unique element of X such that

$$k(x)+s(b) = a = kq(a)+s(b) = k(x')+s(b),$$

and hence $x = x'$. Finally, φ (and hence ψ) is a homomorphism, in fact preservation of unary operations is obvious, and moreover:

$$\varphi((x_1,b_1)+(x_2,b_2)) = \varphi(x_1+\alpha_+(b_1,x_2), b_1+b_2) =$$
$$= k(x_1+\alpha_+(b_1,x_2))+s(b_1+b_2) = k(x_1)+k\alpha_+(b_1,x_2)+s(b_1)+s(b_2) =$$
$$= k(x_1)+s(b_1)+k(x_2)+s(b_2) = \varphi(x_1,b_1)+\varphi(x_2,b_2);$$

Moreover, in the paper [6] the authors proved that, if $\mathbb{C}$ is finitely complete, it has pushouts of split monomorphisms and an initial object, then it is not necessary to consider all morphisms p in $\mathbb{C}$, but it is sufficient to consider only the morphisms $i_B\colon 0 \to B$ with the initial object as domain. Indeed:

Proposition 4.2. *([6], Corollary 3) Let $\mathbb{C}$ be a category with finite limits, pushouts of split monomorphisms and initial object. Then the following statements are equivalent:*

(i) *all pullback functors i_B^* defined by the initial arrows are monadic;*
(ii) *for any morphism p in $\mathbb{C}$, the pullback functor p^* is monadic, i.e. $\mathbb{C}$ admits semidirect products.*

The algebras for the monad T^{i_B} are called *internal actions* in [1]. The monad T^{i_B} is usually denoted by $B\flat(-)$; for any object X, $B\flat X$ is the kernel of the morphism $[0,1]\colon X+B \to B$. Algebras for this monad are hence morphisms $\xi\colon B\flat X \to X$ satisfying the usual conditions for an algebra. Our aim is to compare internal actions with the actions defined by a Schreier split extension, as in Section 2, that will be called *external actions* from now on.

Let now $\mathbb{C}$ be the category Mon of monoids. It is known that this category doesn't have semidirect products in the categorical sense, or, in other terms, that the points are not equivalent to the internal actions. Indeed, the category Mon is not protomodular [2], and it is known that protomodularity is a necessary condition in order to have semidirect products (see, for example, [6], or [5], where the authors give a characterization of pointed categories that admit semidirect products). On the other hand, Theorem 2.8 gives an equivalence between Schreier points and external actions (i.e. pre-actions such that the corresponding semidirect product is an object of $\mathbb{C}$). Hence it is worth comparing internal and external actions in this context.

In general, internal and external actions are not equivalent. To see that, we can consider the monoid $\mathbb{N}$ of natural numbers (with the usual sum as operation) as acting monoid B. In this case, $\mathbb{N}\flat X = X$ for any monoid X. Indeed, it is easy to see that the kernel of the morphism $[0,1]\colon X+\mathbb{N} \to \mathbb{N}$ is just X. Hence an internal action is a morphism $\xi\colon X \to X$ satisfying the usual conditions; in particular, ξ should be a split epimorphism, with section given by the inclusion $\eta\colon X \to B\flat X$. But in this case $\eta = 1_X$, and this forces ξ to be the identity. In other terms, the set $IntAct(\mathbb{N}, X)$ of internal actions

Dually, the same diagram defines a split pushout of (γ, δ) along p when, for any other split commutative square

$$\begin{tikzcd} D \arrow[r, "q'"] \arrow[d, shift left, "\gamma"] & A' \arrow[d, shift left, "\alpha'"] \\ E \arrow[u, shift left, "\delta"] \arrow[r, "p"'] & B, \arrow[u, shift left, "\beta'"] \end{tikzcd}$$

there exists a unique morphism $a\colon A \to A'$ such that

$$\alpha' a = \alpha, \qquad a\beta = \beta', \qquad aq = q'.$$

We say that the category $\mathbb{C}$ has split pullbacks (resp. split pushouts) if it admits split pullbacks (resp. split pushouts) along *any* morphism $p\colon E \to B$.

The existence of split pullbacks defines a contravariant pseudofunctor

$$Pt\colon \mathbb{C}^{op} \to \mathbf{Cat}$$

(the pseudofunctor of *points*) that assigns to a morphism $p\colon E \to B$, the pullback functor

$$p^*\colon Pt(B) \to Pt(E),$$

where the category $Pt(B)$ is the category of points over B, as in Section 2.

Hence the following is a purely categorical definition:

Definition 4.1. ([3], Definition 3.2) A category $\mathbb{C}$ with split pullbacks is said to be a *category with semidirect products* if, for any arrow $p\colon E \to B$ in $\mathbb{C}$, the pullback functor p^* (has a left adjoint and) is monadic.

In this case, denoting by T^p the monad defined by this adjunction, given a T^p-algebra (D, ξ) the semidirect product $(D, \xi) \rtimes (B, p)$ is an object in $Pt(B)$ corresponding to (D, ξ) via the canonical equivalence K:

$$\begin{tikzcd} & {[Pt(E)]^{T^p}} \arrow[d, shift right] \arrow[d, phantom, "\vdash"] \\ Pt(B) \arrow[ur, dashed, "K"] \arrow[r, shift right, "p^*"'] \arrow[r, phantom, "\perp"] & Pt(E) \arrow[l, shift right, "p_!"'] \arrow[u, shift right] \end{tikzcd} \tag{5}$$

Let us observe that, if $\mathbb{C}$ is finitely complete, the pullback functors p^* have left adjoints $p_!$ (for any p in $\mathbb{C}$) if and only if $\mathbb{C}$ has split pushouts.

$$\varphi(0,0) = k(0) + s(0) = 0;$$

and, for any $* \in \Omega_2'$:

$$\begin{aligned} \varphi((x_1, b_1) * (x_2, b_2)) &= \varphi(x_1 * x_2 + \alpha_*(b_1, x_2) + \alpha_{*^\circ}(b_2, x_1), b_1 * b_2) = \\ &= \varphi(x_1 * x_2 + q(s(b_1) * k(x_2)) + q(s(b_2) *^\circ k(x_1)), b_1 * b_2) = \\ &= \varphi(x_1 * x_2 + q(s(b_1) * k(x_2)) + q(k(x_1) * s(b_2)), b_1 * b_2) = \\ &= k(x_1 * x_2 + q(s(b_1) * k(x_2)) + q(k(x_1) * s(b_2))) + s(b_1 * b_2) = \\ &= k(x_1) * k(x_2) + kq(s(b_1) * k(x_2)) + kq(k(x_1) * s(b_2)) + s(b_1) * s(b_2). \end{aligned}$$

But $kq(s(b_1) * k(x_2)) = s(b_1) * k(x_2)$; indeed:

$$s(b_1) * k(x_2) = kq(s(b_1) * k(x_2)) + sp(s(b_1) * k(x_2)),$$

but $sp(s(b_1)*k(x_2)) = 0$, because $s(b_1)*k(x_2) \in \operatorname{Ker} p$; analogously, $kq(k(x_1)* s(b_2)) = k(x_1) * s(b_2)$. Hence:

$$\begin{aligned} \varphi((x_1, b_1)*(x_2, b_2)) &= k(x_1)*k(x_2)+s(b_1)*k(x_2)+k(x_1)*s(b_2)+s(b_1)*s(b_2) = \\ &= (k(x_1) + s(b_1)) * (k(x_2) + s(b_2)) = \varphi(x_1, b_1) * \varphi(x_2, b_2). \end{aligned}$$

Being $X \rtimes_\alpha B$ isomorphic to A, it is an object of $\mathbb{C}$.

Conversely, let α be a pre-action of B on X such that $X \rtimes_\alpha B$ is an object of $\mathbb{C}$. Then we have the following point in $\mathbb{C}$:

$$\begin{tikzcd} X \arrow[r, "\langle 1,0 \rangle"] & X \rtimes_\alpha B \arrow[r, shift left, "\pi_B"] & B. \arrow[l, shift left, "\langle 0,1 \rangle"] \end{tikzcd}$$

This is a Schreier point, where $q = \pi_X$; the uniqueness of q comes from the fact that $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$ are jointly epimorphic. Moreover, it is immediate to see that the action defined by this Schreier point is exactly the pre-action α with which we started. This completes the proof. ∎

We conclude this section with a remark that will be useful in the following:

Lemma 2.9. *Let α be an action of B on X in $\mathbb{C}$. For any $b, b_1, b_2 \in B$, $x, x_1, x_2 \in X$ and $* \in \Omega_2'$ we have:*
(1) $\alpha_+(b, x_1 + x_2) = \alpha_+(b, x_1) + \alpha_+(b, x_2)$;
(2) $\alpha_+(b_1 + b_2, x) = \alpha_+(b_1, \alpha_+(b_2, x))$;
(3) $\alpha_+(0, x) = x$;
(4) $\alpha_+(b, 0) = 0$;
(5) $\alpha_*(b, x_1 + x_2) = \alpha_*(b, x_1) + \alpha_*(b, x_2)$;
(6) $\alpha_*(b_1 + b_2, x) = \alpha_*(b_1, x) + \alpha_*(b_2, x)$.

Proof: The equalities above follow immediately from the fact that $X \rtimes_\alpha B$ is an object of $\mathbb{C}$, and hence $+$ is a monoid operation on it, with identity given by $(0,0)$, and any $* \in \Omega_2'$ is distributive with respect to $+$. ∎

3. Crossed modules and Schreier internal categories

Theorem 2.8 allows us to obtain, in the context of monoids with operations, an equivalence between crossed modules and particular internal categories, that will be called Schreier internal categories (following [8]). This fact is a generalization of the known equivalence for groups with operations, described in [9], and for monoids, as in [8].

We start describing what is a crossed module in a category of monoids with operations. Throughout all the section, $\mathbb{C}$ will be a category of (Ω, E)-algebras as in Definition 2.1.

Definition 3.1. Given two objects X and B of $\mathbb{C}$, an action α of B on X and a morphism $f\colon X \to B$, we say that the pair (α, f) is a *crossed module* if, for any $x, x_1, x_2 \in X$, $b \in B$ and $* \in \Omega_2'$, the following conditions hold:

(i) $f(\alpha_+(b,x)) + b = b + f(x)$;
(ii) $\alpha_+(f(x_1), x_2) + x_1 = x_1 + x_2$;
(iii) $f(\alpha_*(b,x)) = b * f(x)$;
(iv) $\alpha_*(f(x_1), x_2) = \alpha_{*^\circ}(f(x_2), x_1) = x_1 * x_2$.

Given two crossed modules (X, B, α, f) and (X', B', α', f'), a morphism between them is a pair (β, γ) of morphisms in $\mathbb{C}$, where $\beta\colon X \to X'$ and $\gamma\colon B \to B'$, such that the following conditions hold:

(a) $\beta(\alpha_+(b,x)) = \alpha'_+(\gamma(b), \beta(x))$ for any $b \in B$, $x \in X$;
(b) $\beta(\alpha_*(b,x)) = \alpha'_*(\gamma(b), \beta(x))$ for any $b \in B$, $x \in X$ and $* \in \Omega_2'$;
(c) $\gamma f = f' \beta$.

Crossed modules in $\mathbb{C}$ and morphisms between them form a category, which will be denoted by $XMod(\mathbb{C})$. We will show that this category is equivalent to a category whose objects are particular internal categories. Recall that an internal category in $\mathbb{C}$ is a reflexive graph:

$$\begin{tikzcd} A \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & B \arrow[l, "e" description] \end{tikzcd}$$

and X is a group.

Conversely, let (X, B, α, f) a crossed module such that X is a group. Consider the corresponding Schreier internal category

$$\begin{tikzcd} X \arrow[r, "\langle 1,0\rangle"'] & X \rtimes_\alpha B \arrow[r, shift left=2, "\pi_B"] \arrow[r, shift right=2, "c"'] & B, \arrow[l, "\langle 0,1\rangle" description] \end{tikzcd}$$

where $c(x,b) = f(x) + b$ and $m((x', f(x)+b), (x,b)) = (x'+x, b)$. We can define i by:

$$i(x,b) = (-x, f(x)+b).$$

It is immediate to see that i gives inverses for m. ∎

4. The case of monoids

The aim of this section is to compare, in the case of monoids, the semidirect product defined in Section 2 with the categorical one, defined by D. Bourn and G. Janelidze in [3]. We start recalling the categorical definition of semidirect products introduced in [3].

Let $\mathbb{C}$ be a category. A diagram

$$\begin{tikzcd} D \arrow[r, "q"] \arrow[d, shift left, "\gamma"] & A \arrow[d, shift left, "\alpha"] \\ E \arrow[r, "p"'] \arrow[u, shift left, "\delta"] & B \arrow[u, shift left, "\beta"] \end{tikzcd} \tag{4}$$

is called a *split commutative square* if $\alpha\beta = 1_B$, $\gamma\delta = 1_E$ and it commutes both upwards and downwards, i.e. $\alpha q = p\gamma$ and $q\delta = \beta p$.

A *split pullback* is a universal such square. More precisely, the diagram (4) is a split pullback of (α, β) along p if, for any other split commutative square

$$\begin{tikzcd} D' \arrow[r, "q'"] \arrow[d, shift left, "\gamma'"] & A \arrow[d, shift left, "\alpha"] \\ E \arrow[r, "p"'] \arrow[u, shift left, "\delta'"] & B, \arrow[u, shift left, "\beta"] \end{tikzcd}$$

there exists a unique morphism $d\colon D' \to D$ such that

$$\gamma d = \gamma', \qquad d\delta' = \delta, \qquad qd = q'.$$

Definition 3.5. A *Schreier groupoid* is an internal category

$$\begin{tikzcd} X \arrow[r, "k"] & A \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & B \arrow[l, "e" description] \end{tikzcd} \tag{3}$$

in $\mathbb{C}$ endowed with a set-theoretical map $i\colon A \to A$ giving inverses for the composition m, i.e.:

$$di = c,\ ci = d,\ m(i(a), a) = ed(a),\ m(a, i(a)) = ec(a) \text{ for any } a \in A.$$

In the case of monoids, the notion above was already considered in [8], where the author called these groupoids internal. However, we prefer to use a different name, because, with the classical terminology used in category theory, an internal groupoid in $\mathbb{C}$ is an internal category such that the inverse map i is a morphism in $\mathbb{C}$, and not only a set-theoretical map.

Corollary 3.6. *A Schreier internal category in $\mathbb{C}$ is a Schreier groupoid if and only if, in the corresponding crossed modules (X, B, α, f) in $\mathbb{C}$, X is a group.*

Proof: Given a Schreier groupoid of the form (3), for every $y \in X$ we have that

$$m(ik(y), k(y)) = edk(y) = 0, \quad m(k(y), ik(y)) = eck(y).$$

By Schreier condition, there exists a unique $x \in X$ such that

$$ik(y) = k(x) + edik(y) = k(x) + eck(y),$$

hence

$$m(k(x) + eck(y), k(y)) = 0, \quad m(k(y), k(x) + eck(y)) = eck(y).$$

Using Lemma 3.3, we obtain

$$kq(k(x){+}eck(y)){+}k(y) = 0, \quad k(y){+}kq(k(x){+}eck(y)){+}ed(k(x){+}eck(y)) = eck(y),$$

hence

$$kq(k(x) + eck(y)) + k(y) = 0, \quad k(y) + kq(k(x) + eck(y)) + eck(y) = eck(y).$$

By Schreier condition we have that $kq(k(x) + eck(y)) = k(x)$ and so:

$$k(x) + k(y) = 0, \quad k(y) + k(x) + eck(y) = eck(y).$$

Again by Schreier condition, the second equality gives $k(y) + k(x) = 0$; since k is a monomorphism, we have that

$$x + y = y + x = 0,$$

(i.e. $de = ce = 1_B$) with a morphism (giving the composition of arrows) $m\colon A \times_B A \to A$ ($A \times_B A$ is the pullback of d along c) satisfying associativity and identity axioms. A morphism between two internal categories (A, B, d, c, e, m) and (A', B', d', c', e', m'), also called internal functor, is a pair (g_1, g_0), where $g_1\colon A \to A'$ and $g_0\colon B \to B'$, preserving domain, codomain, composition and identities.

Definition 3.2. An internal category (A, B, d, c, e, m) in $\mathbb{C}$ is a *Schreier internal category* if the point

$$\begin{tikzcd} X = \operatorname{Ker} d \arrow[r, "k"] & A \arrow[r, shift left, "d"] & B \arrow[l, shift left, "e"] \end{tikzcd}$$

is Schreier.

We will denote by $SCat(\mathbb{C})$ the category whose objects are Schreier internal categories in $\mathbb{C}$ and whose morphisms are internal functors between them.

Lemma 3.3. *Let (A, B, d, c, e, m) be a Schreier internal category in $\mathbb{C}$, and let $a, a' \in A$ be composable arrows (i.e. $d(a') = c(a)$). Then*

$$m(a', a) = kq(a') + kq(a) + ed(a).$$

Proof: We know that

$$a = kq(a) + ed(a);$$

moreover:

$$d(a') = c(a) = c(kq(a) + ed(a)) = ckq(a) + d(a),$$

and hence

$$a' = kq(a') + ed(a') = kq(a') + e(ckq(a) + d(a)).$$

Since m is a morphism in $\mathbb{C}$ and it preserves identities, we have:

$$m(a', a) = m(kq(a') + e(ckq(a) + d(a)), kq(a) + ed(a)) =$$

$$= m(kq(a'), 0) + m(e(ckq(a) + d(a)), kq(a) + ed(a)) =$$

$$= m(kq(a'), edkq(a')) + m(ec(kq(a) + ed(a)), kq(a) + ed(a)) =$$

$$= kq(a') + kq(a) + ed(a). \qquad \blacksquare$$

Theorem 3.4. *The categories $XMod(\mathbb{C})$ and $SCat(\mathbb{C})$ are equivalent.*

Proof: Let

$$X = \mathrm{Ker}\ d \xrightarrow{k} A \underset{c}{\overset{d}{\underset{\longleftarrow e}{\rightrightarrows}}} B$$

be a Schreier internal category in $\mathbb{C}$, and $q\colon A \to X$ the unique map satisfying the Schreier condition. In the previous Section we proved that q defines an action α of B on X in the following way:

$$\alpha_+(b,x) = q(e(b)+k(x)),$$

$$\alpha_*(b,x) = q(e(b)*k(x)), \text{ for } * \in \Omega_2'.$$

Consider then the morphism $f = ck$. We have to show that (X, B, α, f) is a crossed module:

(i) For any $b \in B$ and $x \in X$ we have that:

$$e(b)+k(x) = k\alpha_+(b,x)+e(b);$$

applying the morphism c on both sides of the equality we get:

$$ce(b)+ck(x) = ck\alpha_+(b,x)+ce(b),$$

and since $ce = 1_B$ we have:

$$b+f(x) = f\alpha_+(b,x)+b.$$

(ii) Applying Schreier condition to the element $eck(x_1)+k(x_2)$, we have, for any $x_1, x_2 \in X$:

$$eck(x_1)+k(x_2) = k\alpha_+(f(x_1),x_2)+ed(eck(x_1)+k(x_2)) = k\alpha_+(f(x_1),x_2)+eck(x_1).$$

It is easy to see that the elements $k\alpha_+(f(x_1),x_2)+eck(x_1)$ and $k(x_1)$ in A are composable; hence, applying Lemma 3.3:

$$k(\alpha_+(f(x_1),x_2)+x_1) = k\alpha_+(f(x_1),x_2)+k(x_1) =$$
$$= m(k\alpha_+(f(x_1),x_2)+eck(x_1),k(x_1)) =$$
$$= m(eck(x_1)+k(x_2),k(x_1)) = m(eck(x_1),k(x_1))+m(k(x_2),0) =$$
$$= m(eck(x_1),k(x_1))+m(k(x_2),edk(x_2)) = k(x_1)+k(x_2) = k(x_1+x_2),$$

and since k is injective, we have that $\alpha_+(f(x_1),x_2)+x_1 = x_1+x_2$.

(iii) We already observed, in the proof of Theorem 2.8, that

$$kq(e(b)*k(x)) = e(b)*k(x) \quad \text{for any } b \in B, x \in X \text{ and } * \in \Omega_2';$$

hence:

$$f(\alpha_*(b,x)) = ckq(e(b)*k(x)) = c(e(b)*k(x)) = ce(b)*ck(x) = b*f(x).$$

So we have a functor

$$G\colon XMod(\mathbb{C}) \to SCat(\mathbb{C}).$$

It is immediate to see that $FG = 1_{XMod(\mathbb{C})}$; let us prove that $GF \simeq 1_{SCat(\mathbb{C})}$. In order to do that consider, for any Schreier internal category

$$X \xrightarrow{k} A \underset{c}{\overset{d}{\underset{\longleftarrow e}{\rightrightarrows}}} B,$$

the following diagram

$$\begin{tikzcd}
X \arrow[r, "k"] \arrow[d, equal] & A \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] \arrow[d, shift right, "\psi"'] & B \arrow[l, "e"' description] \arrow[d, equal] \\
X \arrow[r, "\langle 1,0\rangle"'] & X \rtimes_\alpha B \arrow[r, shift left=2, "\pi_B"] \arrow[r, shift right=2, "c'"'] \arrow[u, shift right, "\varphi"'] & B, \arrow[l, "\langle 0,1\rangle"' description]
\end{tikzcd}$$

where the lower line is the image of the upper one under the functor GF and the morphisms ψ and φ are defined as in the proof of Theorem 2.8:

$$\psi(a) = (q(a), d(a)), \quad \varphi(x,b) = k(x)+e(b).$$

We already know that ψ and φ are isomorphisms in $\mathbb{C}$; it remains to prove that they give rise to internal functors. φ (and hence ψ) is a morphism of internal reflexive graphs, indeed:

$$d\varphi(x,b) = d(k(x)+e(b)) = b = \pi_B(x,b);$$
$$c\varphi(x,b) = c(k(x)+e(b)) = ck(x)+ce(b) = ck(x)+b = c'(x,b);$$
$$\varphi\langle 0,1\rangle(b) = \varphi(0,b) = e(b).$$

Moreover, ψ preserves composition, i.e. $\psi m = m'(\psi \times \psi)$. Indeed:

$$\psi m(a',a) = \psi(kq(a')+kq(a)+ed(a)) = (q(kq(a')+kq(a)+ed(a)), d(a)),$$

while

$$m'(\psi\times\psi)(a',a) = m'((q(a'),d(a')),(q(a),d(a))) = (q(a')+q(a), d(a)),$$

and they are equal, because, applying Schreier condition to the element $k(q(a')+q(a))+ed(a) \in A$ we have:

$$k(q(a')+q(a))+ed(a) = kq(k(q(a')+q(a))+ed(a))+ed(a)$$

and the thesis follows by the uniqueness in Schreier condition. This concludes the proof. ∎

$$= ((x_1' + x_1) * (x_2' + x_2) + \alpha_*(b_1, x_2' + x_2) + \alpha_{*^\circ}(b_2, x_1' + x_1), b_1 * b_2) =$$
$$= ((x_1'+x_1)*(x_2'+x_2)+\alpha_*(b_1, x_2')+\alpha_*(b_1, x_2)+\alpha_{*^\circ}(b_2, x_1')+\alpha_{*^\circ}(b_2, x_1), b_1*b_2) =$$
$$= (x_1'*x_2'+x_1*x_2'+x_1'*x_2+x_1*x_2+\alpha_*(b_1, x_2')+\alpha_*(b_1, x_2)+\alpha_{*^\circ}(b_2, x_1')+\alpha_{*^\circ}(b_2, x_1), b_1*b_2);$$

in order to see that the two pairs are equal, it suffices to apply the bijection $\varphi(x, b) = k(x) + s(b)$ used in the proof of Theorem 2.8, in fact, thanks to Schreier condition, $k\alpha_*(b, x) = s(b) * k(x)$ for any $b \in B$, $x \in X$ and the partial commutativity of $+$ in A gives the result.

It is straightforward to check that m is associative and preserves identities. Hence we have a Schreier internal category. Moreover, given a morphism

$$(\beta, \gamma) \colon (X, B, \alpha, f) \to (X', B', \alpha', f')$$

of crossed modules, we can define a morphism (g_1, g_0) between the corresponding Schreier internal categories by putting

$$g_0 = \gamma, \quad g_1(x, b) = (\beta(x), \gamma(b)).$$

g_1 is a morphism, indeed preservation of unary operations is obvious, and moreover:

$$g_1((x_1, b_1) + (x_2, b_2)) = g_1(x_1 + \alpha_+(b_1, x_2), b_1 + b_2) =$$
$$= (\beta(x_1)+\beta\alpha_+(b_1, x_2), \gamma(b_1+b_2)) = (\beta(x_1)+\alpha_+'(\gamma(b_1), \beta(x_2)), \gamma(b_1)+\gamma(b_2)) =$$
$$= (\beta(x_1), \gamma(b_1)) + (\beta(x_2), \gamma(b_2)) = g_1(x_1, b_1) + g_1(x_2, b_2),$$

and, for any $* \in \Omega_2'$:

$$g_1((x_1, b_1) * (x_2, b_2)) = g_1(x_1 * x_2 + \alpha_*(b_1, x_2) + \alpha_{*^\circ}(b_2, x_1), b_1 * b_2) =$$
$$= (\beta(x_1 * x_2) + \beta\alpha_*(b_1, x_2) + \beta\alpha_{*^\circ}(b_2, x_1), \gamma(b_1 * b_2)) =$$
$$= (\beta(x_1) * \beta(x_2) + \alpha_*'(\gamma(b_1), \beta(x_2)) + \alpha_{*^\circ}'(\gamma(b_2), \beta(x_1)), \gamma(b_1) * \gamma(b_2)) =$$
$$= (\beta(x_1), \gamma(b_1)) * (\beta(x_2), \gamma(b_2)) = g_1(x_1, b_1) * g_1(x_2, b_2).$$

Moreover, we have:

$$g_0 d(x, b) = g_0(b) = \gamma(b) = d'(\beta(x), \gamma(b)) = d' g_1(x, b);$$
$$g_0 c(x, b) = g_0(f(x)+b) = \gamma f(x)+\gamma(b) = f'\beta(x)+\gamma(b) = c'(\beta(x), \gamma(b)) = c' g_1(x, b);$$
$$e' g_0(b) = e'\gamma(b) = (0, \gamma(b)) = g_1(0, b) = g_1 e(b);$$
$$m'(g_1 \times g_1)((x', b'), (x, b)) = m'((\beta(x'), \gamma(b')), (\beta(x), \gamma(b))) =$$
$$= (\beta(x') + \beta(x), \gamma(b)) = g_1(x' + x, b) = g_1 m((x', b'), (x, b)).$$

(iv) We have:

$$k\alpha_*(f(x_1), x_2) = k\alpha_*(ck(x_1), x_2) = kq(eck(x_1) * k(x_2)) = eck(x_1) * k(x_2),$$

and hence, using Lemma 3.3 and the fact that m preserves the binary operation $*$:

$$k\alpha_*(f(x_1), x_2) = m(k\alpha_*(f(x_1), x_2), 0) = m(eck(x_1) * k(x_2), 0) =$$
$$= m(eck(x_1) * k(x_2), k(x_1) * 0) = m(eck(x_1), k(x_1)) * m(k(x_2), 0) =$$
$$= k(x_1) * k(x_2) = k(x_1 * x_2),$$

and, since k is a monomorphism, we have that:

$$\alpha_*(f(x_1), x_2) = x_1 * x_2;$$

the proof that $\alpha_{*^\circ}(f(x_2), x_1) = x_1 * x_2$ is similar.

Consider now the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[d, "\delta"'] & A \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] \arrow[d, "g_1"'] & B \arrow[l, "e" description] \arrow[d, "g_0"] \\ X' \arrow[r, "k'"] & A' \arrow[r, shift left=2, "d'"] \arrow[r, shift right=2, "c'"'] & B', \arrow[l, "e'" description] \end{tikzcd} \tag{2}$$

such that (g_1, g_0) is a morphism of internal categories; we can define a morphism of crossed modules

$$(\beta, \gamma) \colon (X, B, \alpha, f) \to (X', B', \alpha', f')$$

by putting $\beta = \delta$ and $\gamma = g_0$. Indeed:

(a) using Schreier condition we have

$$g_1 k\alpha_+(b, x) + e' g_0(b) = g_1 k\alpha_+(b, x) + g_1 e(b) =$$
$$= g_1(k\alpha_+(b, x) + e(b)) = g_1(e(b) + k(x)) = g_1 e(b) + g_1 k(x) =$$
$$= e' g_0(b) + k'\delta(x) = k'\alpha_+'(g_0(b), \delta(x)) + e' g_0(b);$$

by uniqueness in Schreier condition we obtain that:

$$k'\delta\alpha_+(b, x) = g_1 k\alpha_+(b, x) = k'\alpha_+'(g_0(b), \delta(x)),$$

and since k' is injective we get

$$\delta\alpha_+(b, x) = \alpha_+'(g_0(b), \delta(x)),$$

i.e.

$$\beta\alpha_+(b, x) = \alpha_+'(\gamma(b), \beta(x)).$$

(b) using the fact that $kq(e(b) * k(x)) = e(b) * k(x)$ for any $b \in B$, $x \in X$, we have:

$$k'\delta q(e(b) * k(x)) = g_1 kq(e(b) * k(x)) = g_1(e(b) * k(x)) =$$
$$= g_1 e(b) * g_1 k(x) = e' g_0(b) * k'\delta(x) = k'q'(e' g_0(b) * k'\delta(x)),$$

and since k' is injective, we obtain:

$$\beta\alpha_*(b, x) = \delta q(e(b) * k(x)) = q'(e' g_0(b) * k'\delta(x)) = \alpha'_*(\gamma(b), \beta(x)).$$

(c) The fact that $\gamma f = f'\beta$ comes immediately from the commutativity of diagram (2).

This defines a functor

$$F\colon SCat(\mathbb{C}) \to XMod(\mathbb{C}).$$

In order to show that this functor is an equivalence, we will define another functor

$$G\colon XMod(\mathbb{C}) \to SCat(\mathbb{C}).$$

Given a crossed module (X, B, α, f), we can define $A = X \rtimes_\alpha B$ and we obtain a Schreier point:

$$\begin{tikzcd} X \arrow[r, "\langle 1,0 \rangle"] & X \rtimes_\alpha B \arrow[r, shift left, "\pi_B"] & B. \arrow[l, shift left, "\langle 0,1 \rangle"] \end{tikzcd}$$

Putting $d = \pi_B$, $e = \langle 0, 1\rangle$, $k = \langle 1, 0\rangle$ and defining c by $c(x, b) = f(x) + b$ we obtain a reflexive graph:

$$\begin{tikzcd} X \arrow[r, "k"] & A \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & B; \arrow[l, "e" description] \end{tikzcd}$$

c is a morphism, indeed preservation of unary operations is obvious, and moreover:

$$c((x_1, b_1) + (x_2, b_2)) = c(x_1 + \alpha_+(b_1, x_2), b_1 + b_2) =$$
$$= f(x_1) + f\alpha_+(b_1, x_2) + b_1 + b_2 = f(x_1) + b_1 + f(x_2) + b_2 = c(x_1, b_1) + c(x_2, b_2),$$

and, for any $* \in \Omega'_2$

$$c((x_1, b_1) * (x_2, b_2)) = c(x_1 * x_2 + \alpha_*(b_1, x_2) + \alpha_{*^\circ}(b_2, x_1), b_1 * b_2) =$$
$$= f(x_1 * x_2) + f\alpha_*(b_1, x_2) + f\alpha_{*^\circ}(b_2, x_1) + b_1 * b_2 =$$
$$= f(x_1) * f(x_2) + b_1 * f(x_2) + f(x_1) * b_2 + b_1 * b_2 =$$
$$= (f(x_1) + b_1) * (f(x_2) + b_2) = c(x_1, b_1) * c(x_2, b_2).$$

Now we have to define the composition m. First observe that two pairs (x, b) and (x', b') are composable if and only if $b' = f(x) + b$. Hence we can define m in the following way:

$$m((x', f(x) + b), (x, b)) = (x' + x, b).$$

m is a morphism, indeed preservation of unary operations is obvious, and moreover:

$$m[((x'_1, f(x_1) + b_1), (x_1, b_1)) + ((x'_2, f(x_2) + b_2), (x_2, b_2))] =$$
$$= m[(x'_1, f(x_1) + b_1) + (x'_2, f(x_2) + b_2), (x_1, b_1) + (x_2, b_2)] =$$
$$m[(x'_1 + \alpha_+(f(x_1) + b_1, x'_2), f(x_1) + b_1 + f(x_2) + b_2), (x_1 + \alpha_+(b_1, x_2), b_1 + b_2)] =$$
$$= (x'_1 + \alpha_+(f(x_1) + b_1, x'_2) + x_1 + \alpha_+(b_1, x_2), b_1 + b_2),$$

while

$$m((x'_1, f(x_1) + b_1), (x_1, b_1)) + m((x'_2, f(x_2) + b_2), (x_2, b_2)) =$$
$$= (x'_1 + x_1, b_1) + (x'_2 + x_2, b_2) = (x'_1 + x_1 + \alpha_+(b_1, x'_2 + x_2), b_1 + b_2),$$

and the two pairs are the same, because, thanks to Lemma 2.9, we have:

$$x'_1 + \alpha_+(f(x_1) + b_1, x'_2) + x_1 + \alpha_+(b_1, x_2) =$$
$$= x'_1 + \alpha_+(f(x_1), \alpha_+(b_1, x'_2)) + x_1 + \alpha_+(b_1, x_2) =$$
$$= x'_1 + x_1 + \alpha_+(b_1, x'_2) + \alpha_+(b_1, x_2) = x'_1 + x_1 + \alpha_+(b_1, x'_2 + x_2).$$

Analogously it can be proved that m preserves any $* \in \Omega'_2$:

$$m[((x'_1, f(x_1) + b_1), (x_1, b_1)) * ((x'_2, f(x_2) + b_2), (x_2, b_2))] =$$
$$= m[(x'_1, f(x_1) + b_1) * (x'_2, f(x_2) + b_2), (x_1, b_1)) * (x_2, b_2)] =$$
$$= m[(x'_1 * x'_2 + \alpha_*(f(x_1) + b_1, x'_2) + \alpha_{*^\circ}(f(x_2) + b_2, x'_1), (f(x_1) + b_1) * (f(x_2) + b_2)),$$
$$(x_1 * x_2 + \alpha_*(b_1, x_2) + \alpha_{*^\circ}(b_2, x_1), b_1 * b_2)] =$$
$$= (x'_1 * x'_2 + \alpha_*(f(x_1) + b_1, x'_2) + \alpha_{*^\circ}(f(x_2) + b_2, x'_1) + x_1 * x_2 +$$
$$+ \alpha_*(b_1, x_2) + \alpha_{*^\circ}(b_2, x_1), b_1 * b_2) =$$
$$= (x'_1 * x'_2 + \alpha_*(f(x_1), x'_2) + \alpha_*(b_1, x'_2) + \alpha_{*^\circ}(f(x_2), x'_1) + \alpha_{*^\circ}(b_2, x'_1) + x_1 * x_2 +$$
$$+ \alpha_*(b_1, x_2) + \alpha_{*^\circ}(b_2, x_1), b_1 * b_2) =$$
$$= (x'_1 * x'_2 + x_1 * x'_2 + \alpha_*(b_1, x'_2) + x'_1 * x_2 + \alpha_{*^\circ}(b_2, x'_1) + x_1 * x_2 + \alpha_*(b_1, x_2) + \alpha_{*^\circ}(b_2, x_1), b_1 * b_2),$$

while

$$m((x'_1, f(x_1) + b_1), (x_1, b_1)) * m((x'_2, f(x_2) + b_2), (x_2, b_2)) =$$
$$= (x'_1 + x_1, b_1) * (x'_2 + x_2, b_2) =$$