Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 12–04

SYMMETRY OF REGULAR DIAMONDS, THE GOURSAT PROPERTY, AND SUBTRACTIVITY

MARINO GRAN, ZURAB JANELIDZE, DIANA RODELO AND ALDO URSINI

Abstract: We investigate 3-permutability, in the sense of universal algebra, in an abstract categorical setting which unifies the pointed and the non-pointed contexts in categorical algebra. This leads to a unified treatment of regular subtractive categories and of regular Goursat categories, as well as of E-subtractive varieties (where E is the set of constants in a variety) recently introduced by the fourth author. As an application, we show that "ideals" coincide with "clots" in any regular subtractive category, which can be considered as a pointed analogue of a known result for regular Goursat categories.

Keywords: Ideal of null morphisms, Goursat category, 3-permutable variety, subtractive category, subtractive variety, ideal, clot, ideal determined category, good theory of ideals.
AMS Subject Classification (2000): 18D99, 18C99, 18C05, 08B05.

Introduction

The concept of a category equipped with an ideal $\mathcal{N}$ of morphisms in the sense of C. Ehresmann [6], which was used by M. Grandis in [9] in his "categorical foundation of homological and homotopical algebra", turns out to have yet another interesting use in modern categorical algebra, where it gives a suitable general context for comparing and unifying results from pointed and non-pointed contexts. The *pointed context* is captured by choosing $\mathcal{N}$ to be the class of zero morphisms of a pointed category, while the non-pointed context, which we call the *total context*, is given when $\mathcal{N}$ is the class of all morphisms of a category. In [7] it was shown that the notion of an ideal determined category [12] can be conveniently extended from the pointed context to the context of a general $\mathcal{N}$, so that in the total context it becomes the notion of a Barr exact [2] Goursat category [5, 4]. Such an extension is

Received January 29, 2012.
Supported by
F.N.R.S. grant *Crédit aux chercheurs* 1.5.016.10F and Université catholique de Louvain,
South African National Research Foundation and Georgian National Science Foundation (GNSF/ST09_730_3-105),
CMUC and FCT (Portugal), through the European Program COMPETE/FEDER.

based on replacing the notion of a kernel from the pointed context, not with the standard notion of a kernel with respect to a class $\mathcal{N}$ (used in e.g. [9]), which trivializes in the total context, but with the notion of a "star-kernel" introduced in [7] (which was called a "kernel star" there), which in the total context becomes the notion of a kernel pair.

In the present paper we study the Goursat property beyond Barr exactness and show that its pointed counterpart is precisely subtractivity [15]. In this process we establish a unified characterization theorem for regular Goursat and regular subtractive categories. In particular, it gives the following equivalent reformulation of the Goursat property: in a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "e"'] \arrow[dr, "f"] & \\ W \arrow[dr, "d"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

of regular epimorphisms, if the e-image of the kernel pair of f is the kernel pair of d, then symmetrically, the f-image of the kernel pair of e must be the kernel pair of g (in our terminology, if the above *regular diamond is left saturated*, then it is *right saturated*) — we call this the *symmetric saturation property*. We also observe that requiring the class of left saturated regular diamonds to coincide with the class of pushouts of regular epimorphisms gives a characterization of Barr exact Goursat categories. In the pointed case this distinction corresponds to the one between regular subtractive categories and ideal determined categories.

One of the properties of an ideal determined category is that in it kernels coincide with "ideals", i.e. direct images of kernels along regular epimorphisms, which also coincide with "clots", i.e. those ideals which appear as "0-classes" of reflexive relations (see [14] and the references there). In the present paper we show that the symmetric saturation property still implies the coincidence of clots and ideals.

We also show that in the case of a variety of universal algebras, where $\mathcal{N}$ is chosen to be the class of homomorphisms whose image is generated by constants, the symmetric saturation property becomes "E-subtractivity" (where E is the set of constants of the variety) introduced in [21].

1. Preliminaries

Let $\mathbb{C}$ denote a category with finite limits, and $\mathcal{N}$ a distinguished class of morphisms that forms an *ideal*, i.e. for any diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

in $\mathbb{C}$, if either $f \in \mathcal{N}$ or $g \in \mathcal{N}$, then $gf \in \mathcal{N}$. The class $\mathcal{N}$ is often referred to as the class of *null* morphisms. An $\mathcal{N}$*-kernel* of a morphism $f : X \to Y$ is defined as a morphism $k : K \to X$ such that $fk \in \mathcal{N}$ and k is universal with this property, i.e. for any other morphism k' with $fk' \in \mathcal{N}$ there is a unique morphism u such that $ku = k'$; notice that an $\mathcal{N}$-kernel is a monomorphism. A pair of morphisms, written as $\sigma = (\sigma_1, \sigma_2) : S \rightrightarrows X$ and with $\sigma_1 \in \mathcal{N}$, is called a *star*; it is called a *monic* star when σ_1, σ_2 are jointly monomorphic (i.e. when $\sigma = (\sigma_1, \sigma_2) : S \rightrightarrows X$ is a relation from X to X).

A commutative diagram of stars and morphisms

$$\begin{tikzcd} S \arrow[r, shift left, "\sigma"] \arrow[r, shift right] \arrow[d, "g"'] & X \arrow[d, "f"] \\ T \arrow[r, shift left] \arrow[r, shift right, "\tau"'] & Y \end{tikzcd}$$

(where the commutativity $f\sigma = \tau g$ means that $f\sigma_1 = \tau_1 g$ and $f\sigma_2 = \tau_2 g$) is called a *star-pullback* when given another such commutative (outer) diagram

$$\begin{tikzcd} S' \arrow[dr, dotted, "h"] \arrow[drr, bend left, shift left, "\sigma'"] \arrow[drr, bend left, shift right] \arrow[ddr, bend right, "g'"'] & & \\ & S \arrow[r, shift left] \arrow[r, shift right, "\sigma"'] \arrow[d, "g"'] & X \arrow[d, "f"] \\ & T \arrow[r, shift left] \arrow[r, shift right, "\tau"'] & Y \end{tikzcd}$$

there exists a unique morphism $h : S' \to S$ such that $gh = g'$ and $\sigma h = \sigma'$.

Given a relation $\varrho = (\varrho_1, \varrho_2) : R \rightrightarrows X$ on an object X, by ϱ^* we denote the biggest subrelation of ϱ which is a (monic) star, provided it exists; it indeed exists when $\mathcal{N}$-kernels exist and can be constructed as $\varrho^* = (\varrho_1 k, \varrho_2 k)$, where k is the $\mathcal{N}$-kernel of ϱ_1. In particular, $\Delta_X^* = (k_X, k_X)$ where Δ_X denotes the discrete (equivalence) relation $\Delta_X = (1_X, 1_X) : X \rightrightarrows X$ on an object X, and k_X denotes the $\mathcal{N}$-kernel of 1_X. When $\varrho = (\varrho_1, \varrho_2)$ is a kernel pair, we get the following notion: the *star-kernel* of a morphism $f : X \to Y$ is a universal (monic) star $\kappa = (\kappa_1, \kappa_2) : K \rightrightarrows X$ with the property $f\kappa_1 = f\kappa_2$; it can be equivalently defined as the star κ_f^* of the kernel pair of f, which we denote by $\kappa_f : K_f \rightrightarrows X$.

In the pointed context, the first morphism σ_1 in a star $\sigma = (\sigma_1, \sigma_2) : S \rightrightarrows X$ is the unique null morphism $S \to X$ and hence a star σ can be identified with a morphism (its second component σ_2); then, $\mathcal{N}$-kernels and star-kernels become the usual kernels. In the total context stars are pairs of parallel morphisms, $\mathcal{N}$-kernels are isomorphisms and star-kernels are kernel pairs.

Convention 1.1. *Throughout the paper we work in a regular category* $\mathbb{C}$ *equipped with an ideal* $\mathcal{N}$ *such that every morphism admits an* $\mathcal{N}$*-kernel. Following the terminology used in* [7]*, such a category will be called a regular multi-pointed category with kernels.*

Definition 1.2. [7] *A regular multi-pointed category* $\mathbb{C}$ *with kernels is said to be* star-regular *when every regular epimorphism in* $\mathbb{C}$ *is a coequalizer of a star.*

In the total context a star-regular category is precisely a regular category. In the pointed context a star-regular category is the same as a normal category [17], i.e. a regular category in which any regular epimorphism is a normal epimorphism.

By a *diamond* we mean a commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "e"'] \arrow[dr, "f"] & \\ W \arrow[dr, "d"'] & & Y \arrow[dl, "g"] \\ & Z. & \end{tikzcd} \tag{1}$$

We say that the diamond (1) is

- *left saturated* if the direct image $e\langle\kappa_f^*\rangle$ along e of the star-kernel κ_f^* of f is the star-kernel of d:
$$e\langle\kappa_f^*\rangle = \kappa_d^*;$$
- *right saturated* if, symmetrically, $f\langle\kappa_e^*\rangle = \kappa_g^*$;
- *saturated* if it is both left and right saturated;
- a *regular diamond* if all morphisms in the diamond are regular epimorphisms.

Theorem 1.3. *A regular multi-pointed category $\mathbb{C}$ with kernels is star-regular if and only if the following conditions hold:*

(a) $\mathbb{C}$ admits coequalizers of star-kernels;
(b) every left saturated regular diamond in $\mathbb{C}$ is a pushout.

Proof: Suppose that $\mathbb{C}$ is star-regular. Consider a star-kernel κ_f^* of a morphism f. Decompose $f = me$ as a regular epimorphism e followed by a monomorphism m. Then, κ_f^* is a star-kernel of e. By star-regularity, e is a coequalizer of a star, which implies that it is a coequalizer of its own star-kernel. Now consider a regular diamond (1). By Theorem 2.14 in [7], such a diamond is a pushout if and only if d is a coequalizer of $e\langle\kappa_f^*\rangle$. If the diamond is left saturated, then $e\langle\kappa_f^*\rangle$ is a star-kernel of d. Since d is a regular epimorphism, star-regularity gives that d is a coequalizer of its star-kernel.

Now suppose star-kernels have coequalizers and every left saturated regular diamond is a pushout. For a regular epimorphism $d : X \twoheadrightarrow Z$, consider the following commutative diagram

$$\begin{tikzcd} & X \arrow[dl, two heads, "1_X"'] \arrow[dr, two heads, "f"] & \\ X \arrow[dr, two heads, "d"'] & & Y \arrow[dl, two heads, "g"] \\ & Z & \end{tikzcd}$$

where f is the coequalizer of the star-kernel κ_d^* of d and g is the canonical morphism arising from the universal property of f. The above regular diamond is trivially left saturated (κ_d^* is a star-kernel of f), and hence must be a pushout. Then, g must be an isomorphism, which implies that d is a coequalizer of κ_d^*. ∎

2. 3-star-permutability and the symmetric saturation property

Given a morphism $f : X \to Y$, by f^+ we denote the relation from X to Y

$$\begin{tikzcd} & X \arrow[dl, "1_X"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

and by f^- the opposite relation from Y to X

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "1_X"] & \\ Y & & X. \end{tikzcd}$$

In particular, $1_X^+ = 1_X^- = \Delta_X$.

For the results of this section we need to develop a "calculus of star relations". First of all we note that for any relation $\varrho : R \rightrightarrows X$ we have

$$\varrho^* = \varrho \circ \Delta_X^*.$$

Inspired by this formula, for any relation ϱ from X to an object Y, we define ϱ^* as $\varrho \circ \Delta_X^*$, and we define ${}^*\varrho$ as $\Delta_Y^* \circ \varrho$. Note that associativity of composition yields

$${}^*(\varrho^*) = ({}^*\varrho)^*$$

and so we can write ${}^*\varrho^*$ for the above. For any relation σ (from some object Z to X), the associativity of composition also gives

$$(\varrho^*) \circ \sigma = \varrho \circ ({}^*\sigma),$$

which suggests to write

$$\varrho * \sigma$$

for the above equal composites. It is easy to verify that for any morphism $f : X \to Y$ we have

$$(f^+)^* = {}^*(f^+)^*,$$
$${}^*(f^-) = {}^*(f^-)^*.$$

These "techniques" can be used to establish the following basic properties of direct and inverse images

$$f\langle\varrho^*\rangle = f^+ \circ \varrho * f^- = f^+ \circ \varrho * (f^-)^* = f\langle\varrho^*\rangle^*,$$
$$f^{-1}\langle\varrho\rangle^* = f^- \circ \varrho \circ (f^+)^* = f^- \circ \varrho * (f^+)^* = f^{-1}\langle\varrho^*\rangle^*.$$

For any span

$$\begin{tikzcd} & X \arrow[dl, two heads, "e"'] \arrow[dr, two heads, "f"] & \\ W & & Y \end{tikzcd} \qquad (2)$$

of regular epimorphisms, we have

$$\kappa_e \circ \kappa_f * \kappa_e^* = (e^- \circ e^+ \circ \kappa_f * e^- \circ e^+)^* = e^{-1}\langle e\langle\kappa_f^*\rangle\rangle^*,$$

and, symmetrically,

$$\kappa_f \circ \kappa_e * \kappa_f^* = f^{-1}\langle f\langle \kappa_e^*\rangle\rangle^*.$$

If $\kappa_e = (\varepsilon_1, \varepsilon_2)$, then $\kappa_e = \varepsilon_2^+ \circ \varepsilon_1^- = \varepsilon_1^+ \circ \varepsilon_2^-$ and so we get

$$\kappa_e \circ \kappa_f * \kappa_e^* = (\varepsilon_1^+ \circ \varepsilon_2^- \circ \kappa_f * \varepsilon_2^+ \circ \varepsilon_1^-)^* = \varepsilon_1\langle \varepsilon_2^{-1}\langle \kappa_f^*\rangle\rangle^*.$$

We write $\dot{\kappa}_e$ for

$$\dot{\kappa}_e = \varepsilon_2^+ * \varepsilon_1^-.$$

Then, similarly as above, we get

$$\kappa_e \circ \kappa_f \circ \dot{\kappa}_e = \varepsilon_1^+ \circ \varepsilon_2^- \circ \kappa_f \circ \varepsilon_2^+ * \varepsilon_1^- = \varepsilon_1\langle \varepsilon_2^{-1}\langle \kappa_f\rangle^*\rangle.$$

Notice that we have

$$\dot{\kappa}_e = {}^*(\dot{\kappa}_e)^*$$

and, since $\dot{\kappa}_e \leqslant \kappa_e$, we get

$$\kappa_e \circ \kappa_f \circ \dot{\kappa}_e = \kappa_e \circ \kappa_f * \dot{\kappa}_e^* \leqslant \kappa_e \circ \kappa_f * \kappa_e^*. \tag{3}$$

Definition 2.1. *A regular multi-pointed category $\mathbb{C}$ with kernels is said to be*

(a) 2-star-permutable *if for any span (2) of regular epimorphisms, we have*

$$\kappa_e \circ \kappa_f^* = \kappa_f \circ \kappa_e^*;$$

(b) 3*-star-permutable if for any span (2) of regular epimorphisms, we have*

$$\kappa_e \circ \kappa_f * \kappa_e^* \leqslant \kappa_f \circ \kappa_e * \kappa_f^*$$

(and, consequently, the equality holds) or equivalently,

$$e^{-1}\langle e\langle \kappa_f^*\rangle\rangle^* \leqslant f^{-1}\langle f\langle \kappa_e^*\rangle\rangle^*$$

(and, consequently, the equality holds);

(c) nearly 3*-star-permutable if for any span (2) of regular epimorphisms, we have*

$$\kappa_e \circ \kappa_f \circ \dot{\kappa}_e \leqslant \kappa_f \circ \kappa_e * \kappa_f^*.$$

Remark 2.2. In the total context we always have

$$\kappa_e \circ \kappa_f \circ \dot{\kappa}_e = \kappa_e \circ \kappa_f \circ \kappa_e = \kappa_e \circ \kappa_f * \kappa_e^*$$

and so both the 3-star-permutability and the near 3-star-permutability become the usual 3-permutability which defines Goursat categories, while 2-star-permutability defines precisely the Mal'tsev categories [5]. In the pointed context we always have

$$\kappa_e \circ \kappa_f \circ \dot{\kappa}_e = \kappa_e \circ \kappa_f^* = \kappa_e \circ \kappa_f * \kappa_e^*$$

and so 3-star-permutability and near 3-star-permutability coincide with 2-star-permutability. Pointed categories having these equivalent properties are precisely the regular subtractive categories (this follows easily from the characterization of subtractivity given in Theorem 6.9 in [16]).

As in the total and pointed contexts, in general we have:

Proposition 2.3. *For any regular multi-pointed category with kernels,* 2*-star-permutability implies* 3*-star-permutability, which in turn implies near* 3*-star-permutability.*

Proof: The following calculation shows that 2-star-permutability implies 3-star-permutability:

$$\kappa_e \circ \kappa_f * \kappa_e^* \leqslant \kappa_e \circ \kappa_f \circ \kappa_e^* = \kappa_e \circ \kappa_f^* = \kappa_f \circ \kappa_e^* = \kappa_f \circ \kappa_e * \Delta^* \leqslant \kappa_f \circ \kappa_e * \kappa_f^*.$$

3-star-permutability implies near 3-star-permutability by (3). ■

Remark 2.4. We do not have an example which would show that near 3-star-permutability is strictly weaker than 3-star-permutability.

The aim of the rest of this section is to examine intermediate properties between 3-star-permutability and near 3-star-permutability, which, in view of Remark 2.2, yield characterizations of regular subtractive and of Goursat categories.

Lemma 2.5. *In a regular multi-pointed category with kernels, for a saturated regular diamond* (1) *we have*

$$\kappa_e \circ \kappa_f * \kappa_e^* = \kappa_f \circ \kappa_e * \kappa_f^*,$$

or equivalently,

$$e^{-1}\langle e\langle \kappa_f^* \rangle\rangle^* = f^{-1}\langle f\langle \kappa_e^* \rangle\rangle^*.$$

Proof: Suppose a regular diamond (1) is saturated. Then

$$e^{-1}\langle e\langle \kappa_f^* \rangle\rangle^* = e^{-1}\langle \kappa_d^* \rangle^* = e^{-1}\langle \kappa_d \rangle^*,$$

$$f^{-1}\langle f\langle \kappa_e^* \rangle\rangle^* = f^{-1}\langle \kappa_g^* \rangle^* = f^{-1}\langle \kappa_g \rangle^*.$$

By the commutativity of the diamond we see that $e^{-1}\langle\kappa_d\rangle^* = f^{-1}\langle\kappa_g\rangle^*$. ■

A diamond (1) is said to be

- *left split* if e and g are split epimorphisms with right inverses e' and g' such that $fe' = g'd$;
- *right split* if, symmetrically, f and d are split epimorphisms having right inverses f' and d' satisfying $ef' = d'g$.

Lemma 2.6. *In a regular multi-pointed category with kernels, a diamond* (1) *which is left split is always left saturated.*

Proof: The splittings of the diamond obviously induce a splitting between the kernel pairs of f and d, thus between their star-kernels. ■

If a regular diamond is left split, then it is a pushout. In [8], in the total context, such a left split pushout of regular epimorphisms was called a "Goursat pushout" when it is right saturated, inspired by the following result: a regular category is a Goursat category if and only if every left split pushout of regular epimorphisms is right saturated (and hence saturated, due to the above lemma). We revisit this result in our more general context (see Theorem 2.12 below), where we use the same construction of a left split regular diamond from a span, which was used in [8] for the proof of the above characterization of Goursat categories. Namely, any span (2) gives rise to a left split regular diamond as follows: consider the "right slice" of the diagram which specifies the image of $\kappa_e = (\varepsilon_1, \varepsilon_2)$ under f

$$\begin{tikzcd} & K_e \arrow[dl, "\varepsilon_2"'] \arrow[dr, "\varphi"] & \\ X \arrow[dr, "f"'] & & f\langle K_e\rangle \arrow[dl, "\gamma_2"] \\ & Y. & \end{tikzcd} \tag{4}$$

The diamond (4) will be called the *right derived diamond* of the span (2) (the *left derived diamond* is defined symmetrically). A right derived diamond is always left split, and hence it is always left saturated by Lemma 2.6 (similarly, a left derived diamond is always right split and hence right saturated).

Proposition 2.7. *In a regular multi-pointed category with kernels, for any span* (2) *whose right derived diamond is right saturated, we have:*

$$\kappa_e \circ \kappa_f \circ \dot{\kappa}_e \leqslant \kappa_f \circ \kappa_e * \kappa_f^*.$$

Proof: The right saturation of the right derived diamond (4) of the span (2) is the identity

$$\kappa^*_{\gamma_2} = \varphi\langle\kappa^*_{\varepsilon_2}\rangle.$$

Taking the inverse image along φ followed by direct image along ε_1, of both sides of the above equality, we get

$$\varepsilon_1\langle\varphi^{-1}\langle\kappa^*_{\gamma_2}\rangle\rangle = \varepsilon_1\langle\varphi^{-1}\langle\varphi\langle\kappa^*_{\varepsilon_2}\rangle\rangle\rangle.$$

We are going to show that we always have the following, which will complete the proof:

$$\kappa_e \circ \kappa_f \circ \dot{\kappa}_e \leqslant \varepsilon_1\langle\varphi^{-1}\langle\kappa^*_{\gamma_2}\rangle\rangle^*,$$
$$\varepsilon_1\langle\varphi^{-1}\langle\varphi\langle\kappa^*_{\varepsilon_2}\rangle\rangle\rangle^* \leqslant \kappa_f \circ \kappa_e * \kappa^*_f.$$

We begin by proving the first inequality:

$$\begin{aligned}
\kappa_e \circ \kappa_f \circ \dot{\kappa}_e &= \varepsilon_1\langle\varepsilon_2^{-1}\langle\kappa_f\rangle^*\rangle \\
&= \varepsilon_1\langle\varphi^{-1}\langle\kappa_{\gamma_2}\rangle^*\rangle \\
&= \varepsilon_1\langle\varphi^{-1}\langle\kappa^*_{\gamma_2}\rangle^*\rangle^* \\
&\leqslant \varepsilon_1\langle\varphi^{-1}\langle\kappa^*_{\gamma_2}\rangle\rangle^*.
\end{aligned}$$

Now, the second inequality:

$$\begin{aligned}
\varepsilon_1\langle\varphi^{-1}\langle\varphi\langle\kappa^*_{\varepsilon_2}\rangle\rangle\rangle^* &= (\varepsilon_1^+ \circ \varphi^- \circ \varphi\langle\kappa^*_{\varepsilon_2}\rangle \circ \varphi^+ \circ \varepsilon_1^-)^* \\
&\leqslant (\varepsilon_1^+ \circ \varphi^- \circ \varphi\langle\kappa^*_{\varepsilon_2}\rangle \circ \gamma_1^- \circ f^+)^* \\
&= (\varepsilon_1^+ \circ \varphi^- \circ \varphi^+ \circ \kappa_{\varepsilon_2} * \varphi^- \circ \gamma_1^- \circ f^+)^* \\
&= (\varepsilon_1^+ \circ \varphi^- \circ \varphi^+ \circ \kappa_{\varepsilon_2} * \varepsilon_1^- \circ f^- \circ f^+)^* \\
&= \varepsilon_1^+ \circ \varphi^- \circ \varphi^+ \circ \kappa_{\varepsilon_2} * \varepsilon_1^- * \kappa^*_f \\
&\leqslant \varepsilon_1^+ \circ \varphi^- \circ \varphi^+ \circ \varepsilon_1^- \circ \varepsilon_1^+ \circ \kappa_{\varepsilon_2} \circ \varepsilon_1^- * \kappa^*_f \\
&= \varepsilon_1^+ \circ \varphi^- \circ \varphi^+ \circ \varepsilon_1^- \circ \kappa_e * \kappa^*_f \\
&\leqslant \varepsilon_1^+ \circ \varphi^- \circ \gamma_1^- \circ f^+ \circ \kappa_e * \kappa^*_f \\
&= \varepsilon_1^+ \circ \varepsilon_1^- \circ f^- \circ f^+ \circ \kappa_e * \kappa^*_f \\
&= \kappa_f \circ \kappa_e * \kappa^*_f.
\end{aligned}$$

■

Definition 2.8. *A morphism $f : X \to Y$ is said to be saturating if the following condition holds: the right derived diamond*

$$\begin{tikzcd} & X \arrow[dl, "1_X"'] \arrow[dr, "f"] & \\ X \arrow[dr, "f"'] & & Y \arrow[dl, "1_Y"] \\ & Y & \end{tikzcd}$$

associated to the span

$$\begin{tikzcd} & X \arrow[dl, "1_X"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

is right saturated.

In other words, f is saturating when the induced morphism from the $\mathcal{N}$-kernel of 1_X to the $\mathcal{N}$-kernel of 1_Y is a regular epimorphism.

In the pointed context, all morphisms are saturating. In the total context, any regular epimorphism is saturating.

The proof of the following result is straightforward:

Lemma 2.9. *For a regular epimorphism $f : X \twoheadrightarrow Y$ the following conditions are equivalent:*

(a) f is saturating.
(b) $\Delta_Y^* = (f^+ * f^-)^*$.
(c) For any relation $\varrho : R \rightrightarrows Y$ we have $f\langle f^{-1}\langle \varrho\rangle^*\rangle = \varrho^*$.

Proposition 2.10. *In a regular multi-pointed category with kernels, a left saturated regular diamond* (1) *with saturating f is saturated if and only if $\kappa_e \circ \kappa_f * \kappa_e^* = \kappa_f \circ \kappa_e * \kappa_f^*$, or, equivalently, if and only if*

$$e^{-1}\langle e\langle \kappa_f^*\rangle\rangle^* = f^{-1}\langle f\langle \kappa_e^*\rangle\rangle^*.$$

Proof: The "only if" part is exactly Lemma 2.5 (which does not require the fact that f is saturating). To prove the "if" part, suppose that for a left saturated regular diamond (1) we have the 3-star-permutability property.

Then, the following calculation shows that the diamond is right saturated:

$$\begin{aligned}
f\langle\kappa_e^*\rangle &= f\langle\kappa_e^*\rangle^* \\
&= f\langle f^{-1}\langle f\langle\kappa_e^*\rangle\rangle^*\rangle \\
&= f\langle e^{-1}\langle e\langle\kappa_f^*\rangle\rangle^*\rangle \\
&= f\langle e^{-1}\langle\kappa_d^*\rangle^*\rangle \\
&= f\langle e^{-1}\langle\kappa_d\rangle^*\rangle \\
&= f\langle f^{-1}\langle\kappa_g\rangle^*\rangle \\
&= \kappa_g^*
\end{aligned}$$

(we use Lemma 2.9(c) in the second and last equalities). ∎

Definition 2.11. *We say that $\mathbb{C}$ has the symmetric saturation property if the following equivalent conditions hold:*

(a) *any left saturated regular diamond is right saturated;*
(b) *any right saturated regular diamond is left saturated;*
(c) *left/right saturated regular diamonds are the same as the saturated ones.*

Theorem 2.12. *For a regular multi-pointed category $\mathbb{C}$ with kernels, each of the conditions below implies the subsequent one:*

(a) $\mathbb{C}$ is 3-star-permutable and has saturating regular epimorphisms;
(b) $\mathbb{C}$ has the symmetric saturation property;
(c) any left split regular diamond in $\mathbb{C}$ is saturated;
(d) $\mathbb{C}$ is nearly 3-star-permutable and has saturating regular epimorphisms.

Both in the pointed and total contexts these conditions are equivalent and characterize regular subtractive and Goursat categories, respectively.

Proof: (a)⇒(b) follows from Proposition 2.10.
(b)⇒(c) follows from Lemma 2.6.
(c)⇒(d) follows from Proposition 2.7.
The final claim in the theorem follows from Remark 2.2. ∎

Recall from [7] that a *proto-pointed context* refers to the context of a regular multi-pointed category where null morphisms $w : W \to X$ are precisely those whose regular image is the smallest subobject of X. When the category is a variety of universal algebras such morphisms are those whose image is the subalgebra of constants of the algebra X (by a "constant" we mean a nullary operation/term): this latter situation will be referred to as the *algebraic proto-pointed context.* In this context, an $\mathcal{N}$-kernel of a morphism $f : X \to Y$ is given by the subalgebra of X consisting of those elements $x \in X$ which are mapped by f to a constant in Y. In particular, if the set E of constants of the variety is empty, then $\mathcal{N}$-kernels are empty subalgebras; in this case, stars are parallel morphisms whose domain is the empty algebra and the conditions in Theorem 2.12 hold trivially. In the case when E is non-empty, any regular epimorphism (i.e. a surjective homomorphism) is still trivially saturating, since saturating morphisms are those homomorphisms which are surjective on constants. The conditions of Theorem 2.12 are then still equivalent and define *E-subtractive varieties* in the sense of [21], as we are now going to prove:

Theorem 2.13. *In the algebraic proto-pointed context, each of the conditions* 2.12(a)-(d) *is equivalent to* 2*-star-permutability and is also equivalent to the following syntactic condition: for every constant c, there exists a binary term s_c such that $s_c(x, x) = c$ and $s_c(x, c) = x$.*

Proof: Suppose that near 3-star-permutability holds. For a given constant c, we apply the inequality

$$\kappa_e \circ \kappa_f \circ \dot{\kappa}_e \leqslant \kappa_f \circ \kappa_e * \kappa_f^*$$

in the case when e and f are the following algebra homomorphisms

$$e : \mathrm{Fr}\{x, y\} \to \mathrm{Fr}\{x\},\ x \mapsto x,\ y \mapsto x$$

and

$$f : \mathrm{Fr}\{x, y\} \to \mathrm{Fr}\{x\},\ x \mapsto x,\ y \mapsto c,$$

where $\mathrm{Fr}\{x\}$ and $\mathrm{Fr}\{x, y\}$ are the free algebras over one and two generators, respectively. Then the chain

$$c \xrightarrow{\dot{\kappa}_e} c \xrightarrow{\kappa_f} y \xrightarrow{\kappa_e} x$$

must give rise to a chain

$$c \xrightarrow{\kappa_f^*} c' \xrightarrow{\kappa_e^*} s(x, y) \xrightarrow{\kappa_f} x$$

where each arrow yields, respectively, the equalities $c = c'$, $s(x, x) = c'$, and $s(x, c) = x$. It follows that $s_c = s$ is the required binary term.

Suppose now that the syntactic condition given in the theorem holds. To deduce from it 2-star-permutability we observe that every chain

$$c \xrightarrow{\kappa_f^*} x \xrightarrow{\kappa_e} y$$

of elements of an algebra X, where c is a constant, produces the chain

$$c = s_c(x,x) \xrightarrow{\kappa_e^*} s_c(y,x) \xrightarrow{\kappa_f} s_c(y,c) = y.$$

Since 2-star-permutability implies 3-star-permutability (see Proposition 2.3), and any regular epimorphism is saturating, Theorem 2.12 completes the proof. ■

3. Some remarks on clots and ideals

The notion of an ideal has been extended from varieties of universal algebras [11, 19, 20] to pointed regular categories (with finite coproducts) in [14] (see also [13]). First recall that a subobject $c : C \rightarrowtail X$ is a *clot* when it is the "0-class" of a reflexive relation, i.e. there exists an internal reflexive relation

$$\begin{tikzcd} R \arrow[r, shift left=1.5, "\varrho_1"] \arrow[r, shift right=1.5, "\varrho_2"'] & X \arrow[l] \end{tikzcd}$$

such that $c = \varrho_2 \circ \ker(\varrho_1)$ (see [1, 14, 18]). As proposed in [13], in a categorical setting an *ideal* should be defined as a direct image of a clot along a regular epimorphism: a subobject $i : I \rightarrowtail Y$ is an ideal when there exists a commutative square

$$\begin{tikzcd} C \arrow[r, two heads, "q"] \arrow[d, tail, "c"'] & I \arrow[d, tail, "i"] \\ X \arrow[r, two heads, "p"'] & Y \end{tikzcd}$$

with p and q regular epimorphisms and c a clot. It was later observed in [14] that ideals can be equivalently defined as those subobjects which are direct images of kernels along regular epimorphisms; indeed, as clots themselves are a particular type of direct images of kernels along regular epimorphisms, direct images of clots along regular epimorphisms will coincide with those of kernels. Writing $\mathrm{N}(X)$, $\mathrm{C}(X)$ and $\mathrm{I}(X)$ for the classes of kernels, clots and ideals of an object X, respectively, the inclusions

$$\mathrm{N}(X) \subset \mathrm{C}(X) \subset \mathrm{I}(X)$$

are strict, in general. A crucial axiom in the definition of an ideal determined category [12] states that these inclusions are, in fact, equalities. More precisely, an ideal determined category is a normal category with finite colimits, in which every ideal is a kernel. Recall also that ideal determined varieties [10] were introduced in [20] under the name of BIT (for "Buona Teoria degli Ideali") varieties. This explains the choice of the term "category with a good theory of ideals" for the notion we are now going to recall.

The wish of unifying the pointed and non-pointed contexts led the authors of [7] to introduce a more general notion of ideal in the context of a regular multi-pointed category $\mathbb{C}$, which in the pointed case gives the one recalled above. Since in a regular multi-pointed category the role of kernels is played by star-kernels, it is natural to say that a monic star $\varrho : R \rightrightarrows Y$ is an *ideal* when there exists a commutative diagram

$$\begin{tikzcd}
K \arrow[r, two heads, "q"] \arrow[d, shift left, "\kappa"'] \arrow[d, shift right] & R \arrow[d, shift left, "\varrho"] \arrow[d, shift right] \\
X \arrow[r, two heads, "p"'] & Y
\end{tikzcd}$$

with $\kappa : K \rightrightarrows X$ a star-kernel, and p, q regular epimorphisms (thus, $\varrho = p\langle\kappa\rangle$). Now, a star-regular category $\mathbb{C}$ (with coequalizers of ideals) has *a good theory of ideals* if any ideal is a star-kernel [7]. From what we said above it is then evident that, in the pointed context, this gives precisely the notion of an ideal determined category (under the further assumption of the existence of finite colimits, since this is required in the definition of an ideal determined category given in [12]). In the total context, categories with a good theory of ideals are precisely the Barr exact Goursat categories, since these latter ones can be characterized as those regular categories in which the direct image of an effective equivalence relation (i.e. of a kernel pair) along a regular epimorphism is an effective equivalence relation (see [4]).

We observe that in view of Theorem 1.3 above, Theorem 3.8 of [7] can be refined as follows:

Theorem 3.1. *$\mathbb{C}$ has a good theory of ideals if and only if the following conditions hold:*

- *(a) $\mathbb{C}$ admits coequalizers of ideals;*
- *(b) pushouts of regular epimorphisms in $\mathbb{C}$ are the same as left saturated regular diamonds.*

As it follows from this theorem, any category with a good theory of ideals has the symmetric saturation property.

We now extend the notion of a clot to an arbitrary regular multi-pointed category with kernels:

Definition 3.2. *A monic star $\beta : B \rightrightarrows X$ is said to be a* clot *if there is a reflexive relation $\varrho = (\varrho_1, \varrho_2) : R \rightrightarrows X$ in $\mathbb{C}$ such that $\beta = \varrho_2\langle \kappa^*_{\varrho_1} \rangle$.*

Thus, any star-kernel is a clot, and any clot is an ideal.

In the pointed context, the above notion of a clot becomes the one recalled earlier.

In the total context, a clot is the same as a relation β which has the form $\beta = \varrho \circ \varrho^\circ$ for some reflexive relation ϱ (where ϱ° denotes the opposite relation of ϱ). In particular, any equivalence relation ε is a clot since $\varepsilon = \varepsilon \circ \varepsilon^\circ$. Observe that in general a clot is always reflexive and symmetric. By Theorem 3.5 in [4], the coincidence of equivalence relations and clots (which is equivalent to every clot being transitive) is equivalent to 3-permutability and hence to the symmetric saturation property by Theorem 2.12. At the same time, by Theorem 6.8 in [4], it is further equivalent to the stability of equivalence relations under direct images along regular epimorphisms, and hence to the coincidence of equivalence relations and ideals. This readily gives that, in the total context, the symmetric saturation property implies the coincidence of clots and ideals. More generally, we have:

Proposition 3.3. *If $\mathbb{C}$ is a regular multi-pointed category with kernels satisfying* 2.12(c)*, then clots are stable under direct images along regular epimorphism in $\mathbb{C}$, i.e. clots coincide with ideals: for any object X in $\mathbb{C}$,*

$$\mathrm{C}(X) = \mathrm{I}(X).$$

Proof: Consider a clot $\beta : B \rightrightarrows X$ and a reflexive relation $\varrho = (\varrho_1, \varrho_2) : R \rightrightarrows X$ such that $\beta = \varrho_2\langle \kappa^*_{\varrho_1} \rangle$. We are going to show that its direct image $p\langle \beta \rangle$ along a regular epimorphism $p : X \twoheadrightarrow Y$ is a clot. For this, consider a regular-image decomposition (the bottom square in the following diagram), and the induced square of star-kernels (the top square in the same diagram):

$$\begin{tikzcd}
K_1 \arrow[r, dotted, "q"] \arrow[d, shift left] \arrow[d, shift right, "\kappa^*_{\varrho_1}"'] & K_2 \arrow[d, shift left, "\kappa^*_{\pi_1}"] \arrow[d, shift right] \\
R \arrow[r, two heads, "r"] \arrow[d, shift left] \arrow[d, shift right, "\varrho"'] & R' \arrow[d, shift left, "p\langle\varrho\rangle=(\pi_1,\pi_2)"] \arrow[d, shift right] \\
X \arrow[r, two heads, "p"'] & Y.
\end{tikzcd}$$

Since ϱ is reflexive it follows that $p\langle\varrho\rangle$ is reflexive, and further, the regular diamond

$$\begin{tikzcd}
& R \arrow[dl, two heads, "\varrho_1"'] \arrow[dr, two heads, "r"] & \\
X \arrow[dr, two heads, "p"'] & & R' \arrow[dl, two heads, "\pi_1"] \\
& Y &
\end{tikzcd}$$

is left split. Then it is right saturated by the assumption on $\mathbb{C}$, and consequently, q is a regular epimorphism. This implies

$$\pi_2\langle\kappa^*_{\pi_1}\rangle = (p\varrho_2)\langle\kappa^*_{\varrho_1}\rangle = p\langle\varrho_2\langle\kappa^*_{\varrho_1}\rangle\rangle = p\langle\beta\rangle$$

which shows that $p\langle\beta\rangle$ is a clot, as desired. ■

In the pointed context the above result says that clots and ideals coincide in any regular subtractive category. This result extends a well known one for subtractive varieties (see [1]). Note however that, unlike subtractive varieties where kernels, clots and ideals coincide, in general a regular subtractive category may have ideals which are not kernels. In fact, there are regular subtractive categories where every monomorphism is an ideal, but not every ideal is a kernel: any non-abelian regular additive category is such. Indeed, as shown in [3], a regular subtractive category in which every monomorphism is a kernel is the same as an abelian category. Consequently, suitable counterexamples are given here by the category of torsion-free abelian groups, as already pointed out in [12], and by the category of topological abelian groups.

References

[1] P. Agliano and A. Ursini, Ideals and other generalizations of congruence classes, J. Austal. Math. Soc. 53 (Ser. A) (1992), 103-115.

[2] M. Barr, P. A. Grillet and D. H. van Osdol, Exact categories and categories of sheaves, Springer Lecture Notes in Mathematics 236, 1971.

[3] D. Bourn and Z. Janelidze, A note on the abelianization functor, submitted for publication.

[4] A. Carboni, G. M. Kelly, and M. C. Pedicchio, Some remarks on Maltsev and Goursat categories, Appl. Cat. Struct. 1 (1993) 385-421.

[5] A. Carboni, J. Lambek, and M. C. Pedicchio, Diagram chasing in Malcev categories, J. Pure Appl. Alg. 69 (1991) 271-284.

[6] C. Ehresmann, Sur une notion générale de cohomologie, C. R. Acad. Sci. Paris 259 (1964) 2050-2053.

[7] M. Gran, Z. Janelidze, and A. Ursini, A good theory of ideals in a regular multi-pointed category, accepted for publication in J. Pure Appl. Algebra.

[8] M. Gran and D. Rodelo, A new characterisation of Goursat categories, Appl. Categ. Structures, published online 27 September 2010.

[9] M. Grandis, On the categorical foundations of homological and homotopical algebra, Cah. Top. Géom. Diff. Catég. 33 (1992) 135-175.

[10] H. P. Gumm and A. Ursini, Ideals in universal algebra, Alg. Univers. 19 (1984) 45-54.

[11] P. J. Higgins, Groups with multiple operators, Proc. London Math. Soc. (1956), 366-416.

[12] G. Janelidze, L. Márki, W. Tholen, and A. Ursini, Ideal determined categories, Cah. Top. Géom. Diff. Catég. 51 (2010) 113-124.

[13] G. Janelidze, L. Márki, and A. Ursini, Ideals and clots in universal algebra and in semi-abelian categories, Journal of Algebra 307 (2007), 191-208.

[14] G. Janelidze, L. Márki, and A. Ursini, Ideals and clots in pointed regular categories, Appl. Categ. Structures 17 (4) (2009) 345-350.

[15] Z. Janelidze, Subtractive categories, Appl. Categ. Struct. 13 (2005) 343-350.

[16] Z. Janelidze, Closedness properties of internal relations I: A unified approach to Mal'tsev, unital and subtractive categories, Theory and Applications of Categories 16 (2006) 236-261.

[17] Z. Janelidze, The pointed subobject functor, 3×3 lemmas and subtractivity of spans, Th. Appl. Categ. 23 (2010) 221-242.

[18] Z. Janelidze and A. Ursini, Split Short Five Lemma for Clots and Subtractive Categories, Appl. Categ. Structures 19 (2011) 233-255.

[19] R. Magari, Su una classe equazionale di algebre, Ann. Mat. Pura Appl. 75(4) (1967) 277-312.

[20] A. Ursini, Sulle varietà di algebre con una buona teoria degli ideali, Boll. Unione Mat. Ital. (4) 7 (1972) 90-95.

[21] A. Ursini, Normal subalgebras I, Appl. Categ. Structures, published online 14 July 2011.

Marino Gran

Institut de Recherche en Mathématique et Physique, Université catholique de Louvain, Chemin du Cyclotron 2, 1348 Louvain-la-Neuve, Belgium

E-mail address: marino.gran@uclouvain.be

Zurab Janelidze

Department of Mathematical Sciences, Stellenbosch University, Private Bag X1, Matieland, 7602, South Africa

E-mail address: zurab@sun.ac.za

Diana Rodelo
Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade do Algarve, Campus de Gambelas, 8005-139 Faro, Portugal; Centre for Mathematics of the University of Coimbra, 3001-454 Coimbra, Portugal

E-mail address: drodelo@ualg.pt

Aldo Ursini
Dipartimento di Scienze Matematiche ed Informatiche, Universita' di Siena, Pian dei Mantellini 44, 53100 Siena, Italy

E-mail address: ursini@unisi.it