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WEIGHTED SUMS OF ASSOCIATED VARIABLES

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Abstract: We study the convergence of weighted sums of associated random variables. The convergence for the typical $n^{1/p}$ normalization is proved assuming finiteness of moments somewhat larger than p, but still smaller than 2, together with suitable control on the covariance structure described by a truncation that generates covariances that do not grow too fast. We also consider normalizations of the form $n^{1/q}\log^{1/\gamma} n$, where q is now linked with the properties of the weighting sequence. We prove the convergence under a moment assumption that is weaker that the usual existence of moment generating function. Our results extend analogous characterizations known for sums of independent or negatively dependent random variables.

Keywords: weighted sums, association, almost sure convergence.
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1. Introduction

The interest on characterizing the asymptotics of weighted sums of random variables arises from the fact that many statistical procedures depend on considering such sums: $T_n = \sum_{i=1}^n a_{n,i}X_i$, where the variables X_i are centered. For constant weights and identically distributed variables, it was proved by Baum and Katz [3] that $n^{-1/p}T_n \longrightarrow 0$ almost surely, $p \in [1,2)$, if and only if $\mathrm{E}(|X_1|^p) < \infty$. Naturally this result was extended to dependent and not necessarily identically distributed variables. The case of (positive) associated random variables, with constant weights, was studied by Louhichi [9], who proved that the existence of the moment of order p together with an integral assumption on the covariances of truncated variables implies the almost sure convergence $n^{-1/p}T_n \longrightarrow 0$. Considering independent variables and more general weights, Chow [6] proved that if $\sup n^{-1}\sum_{i=1}^n a_{n,i}^2 < \infty$ and $\mathrm{E}(X_1^2) < \infty$ the almost sure convergence for $p = 1$ holds. This was extended by Cuzick [7] who proved that $n^{-1}T_n \longrightarrow 0$ if $\mathrm{E}(|X_1|^p) < \infty$ and

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$\sup n^{-1} \sum_{i=1}^{n} a_{n,i}^{q} < \infty$ where $p^{-1} + q^{-1} = 1$. Cuzick [7] also studied the convergence of $b_n^{-1} T_n$, characterizing more general normalizing sequences b_n, including $b_n = n^{1/q} \log^{1/\gamma} n$, $q < 2$, considering uniformly bounded weights and a moment condition on the variables involving logarithmic transformations. This type of normalization became more popular after the rate characterizations proved by Cheng [5]. Bai and Cheng [2], for both choices for the b_n's, and Sung [12] for the later choice of the normalizing sequence proved the convergence assuming that $n^{-1} \sum_{i=1}^{n} |a_{ni}|^{\alpha}$ converges and $q = \alpha$, now linking the assumption on the weights with the normalizing sequence b_n, and the existence of the moment generating function. Their proofs relied on suitable versions of exponential inequalities. Naturally, these results were eventually extended to dependent variables where, of course, some extra control on the dependence structure was needed. Some extensions to negatively dependent random variables were proved by Ko and Kim [8], Baek, Park, Chung and Seo [1] or Cai [4] based on assumptions similar to the previous, and also Qiu and Chen [11] now considering more general normalizing sequences b_n. These extensions, in a way, benefited from the particular dependence structure where covariances of sums of random variables tend to be smaller than those in the case of sums of independent random variables. In this paper we will consider positively associated random variables, were covariances of sums tend to be larger than in the case of independent variables, and follow an approach similar to what has been carried by Louhichi [9]. In this way we will be able to prove sufficient conditions using assumptions that separate well restrictions on the weighting coefficients from restrictions on the distributions, expressed in terms of moments and control of the covariance structure. For normalizing sequences $b_n = n^{1/p}$, $p \in (1, 2)$, we will assume the existence of moments of order slightly larger that p, although still smaller than 2, and a control on the covariance structure that strengthens the one used in Louhichi [9] in a way that depends on the behaviour of the weighting coefficients and reduces to Louhichi's condition if the coefficients are constant. For the normalization $b_n = n^{1/q} \log^{1/\gamma} n$, we do not assume the existence of moment generation functions but rather assume the existence of moments of order 2, a moment condition involving logarithmic transformations, in a way similar to some of the conditions used in Cuzick [7], and a control on the covariance structure in the same spirit as for the other normalization, relaxing somewhat the summability assumption.

2. Framework and preliminaries

Let X_n, $n \geq 1$, be a sequence of random variables. Recall that these variables are associated if, for any $m \in \mathbb{N}$ and any two real-valued coordinatewise nondecreasing functions f and g, it holds

$$\mathrm{Cov}\Big(f\left(X_1,\ldots,X_m\right),\, g\left(X_1,\ldots,X_m\right)\Big) \geq 0,$$

whenever this covariance exists. For centered associated variables, it was proved in Theorem 2 of Newman and Wright [10] that

$$\mathrm{E}\left(\max_{k\leq n}\left(X_1+\cdots+X_k\right)^2\right) \leq \mathrm{E}(X_1+\cdots+X_n)^2. \tag{1}$$

This maximal inequality is one of the key ingredients in the approach followed by Louhichi [9] to control tail probabilities of maxima of sums of associated random variables.

We will here be interested in weighted sums of associated random variables. Let us now introduce the coefficients $a_{n,i}$, $i \leq n$, $n \geq 1$, and consider $T_n = \sum_{i=1}^n a_{n,i}X_i$. In order to keep the association we must assume that $a_{n,i} \geq 0$, for every $i \leq n$ and $n \geq 1$. We shall need to strengthen somewhat this assumption to extend (1).

Lemma 2.1. *Let X_n, $n \geq 1$, be centered and associated random variables. Assume that the coefficients are such that*

$$a_{n,i} \geq 0, \quad i \leq n,\ n \geq 1, \qquad \textit{and} \qquad a_{k,j} \geq a_{k-1,j} \quad \textit{for each } k,j \in \mathbb{N}. \tag{2}$$

Then

$$\mathrm{E}\left(\max_{k\leq n} T_k^2\right) \leq \mathrm{E}(T_n^2). \tag{3}$$

Proof: Define $Y_1 = a_{1,1}X_1$ and, for n≥ 2, $Y_n = (a_{n,1}-a_{n-1,1})X_1+\cdots+(a_{n,n-1}-a_{n-1,n-1})X_{n-1}+a_{n,n}X_n$. Given the assumptions on the coefficients, these variables are associated, so the result immediately follows by applying (1) to the Y_n's. ∎

For each $\alpha > 0$, define $A_{n,\alpha} = \frac{1}{n^{1/\alpha}}\left(\sum_{i=1}^n |a_{ni}|^\alpha\right)^{1/\alpha}$. These coefficients are considered in Cuzick [7], where it is assumed they are bounded, and also later in [1, 2, 4, 8, 11, 12], for example, where it is assumed that $A_{n,\alpha}$ converges. In the sequel we will be interested in the case where $\alpha > 1$. Then the following inequalities hold:

$$\max_{i\le n} |a_{n,i}| \le n^{1/\alpha} A_{n,\alpha}, \qquad \text{and} \qquad \sum_{i=1}^n |a_{n,i}| \le n A_{n,\alpha}. \tag{4}$$

Getting back to the random variables, introduce, for notational convenience,

$$\Delta_{i,j}(x,y) = \mathrm{P}(X_i \ge x, X_j \ge y) - \mathrm{P}(X_i \ge x)\mathrm{P}(X_j \ge y).$$

Of course, $\mathrm{Cov}(X_i,X_j) = \int\int \Delta_{i,j}(x,y)\, dxdy$.

Most of the techniques used in the proofs later depend on truncation arguments. As in Louhichi [9], define, for each $M > 0$, the nondecreasing function $g_M(u) = \max(\min(u,M),-M)$, describing the truncation at level M. Moreover, introduce the random variables

$$\bar{X}_n = g_M(X_n), \qquad \text{and} \qquad \tilde{X}_n = X_n - \bar{X}_n, \qquad n \ge 1.$$

It is easily checked that both families of variables $\bar{X}_n$, $n \ge 1$, and $\tilde{X}_n$, $n \ge 1$, are associated, as they are nondecreasing transformations of the original variables. Still, define

$$G_{i,j}(M) = \mathrm{Cov}(\bar{X}_i, \bar{X}_j) = \int\int_{[-M,M]^2} \Delta_{i,j}(x,y)\, dxdy. \tag{5}$$

To complete this section, we introduce some more notation needed to describe our results. Define the partial sums after truncation:

$$\bar{T}_n = \sum_{i=1}^n a_{n,i}(\bar{X}_i - \mathrm{E}\bar{X}_i), \qquad \text{and} \qquad \tilde{T}_n = \sum_{i=1}^n a_{n,i}(\tilde{X}_i - \mathrm{E}\tilde{X}_i), \qquad n \ge 1,$$

and the maxima $T_n^* = \max_{k\le n} |T_k|$ and $\bar{T}_n^* = \max_{k\le n} \left|\bar{T}_k\right|$. It is obvious that

$$T_n^* \le \bar{T}_n^* + \sum_{i=1}^n a_{n,i} \left(\left|\tilde{X}_i\right| + \mathrm{E}\left|\tilde{X}_i\right| \right). \tag{6}$$

Also, assuming the weights verify $a_{k,j} \ge a_{k-1,j}$ for each $k > j$, it follows from Lemma 2.1 that

$$\mathrm{E}\left((T_n^*)^2\right) \le 2\mathrm{E}\left(T_n^2\right) \qquad \text{and} \qquad \mathrm{E}\left((\bar{T}_n^*)^2\right) \le 2\mathrm{E}\left(\bar{T}_n^2\right).$$

3. An auxiliary inequality

In this section we prove a few upper bounds that will be used later for the proof of our main results. First we mention a bound for the second order moments on sums of the truncated variables. Remembering (4), we have that, due to the nonnegativity of each term,

$$\begin{aligned}
\mathrm{E}\left(\bar{T}_n^2\right) &= \sum_{i,j=1}^{n} a_{n,i}a_{n,j}G_{i,j}(M) \\
&\leq \max_{i\leq n} a_{n,i}^2 \sum_{i,j=1}^{n} G_{i,j}(M) \leq n^{2/\alpha}A_{n,\alpha}^2 \sum_{i,j=1}^{n} G_{i,j}(M).
\end{aligned} \tag{7}$$

Lemma 3.1. *Let X_n, $n \geq 1$, be centered and identically distributed associated random variables. Assume the weights satisfy (2). Then, for every $\alpha > 1$, $x \in \mathbb{R}$ and $M > 0$,*

$$\begin{aligned}
&\mathrm{P}(T_n^* > x) \\
&\quad \leq \frac{4}{x^2}n^{1+2/\alpha}A_{n,\alpha}^2\mathrm{E}\left(X_1^2\mathbb{I}_{|X_1|\leq M}\right) + \frac{4}{x^2}n^{1+2/\alpha}A_{n,\alpha}^2M^2\mathrm{P}(|X_1| > M) \\
&\quad +\frac{8}{x^2}n^{2/\alpha}A_{n,\alpha}^2 \sum_{1\leq i<j\leq n} G_{i,j}(M) + \frac{4}{x}nA_{n,\alpha}\mathrm{E}\left(|X_1|\,\mathbb{I}_{|X_1|>M}\right).
\end{aligned} \tag{8}$$

Proof: Taking into account (6) and Markov's inequality, we have

$$\begin{aligned}
\mathrm{P}(T_n^* > x) &\leq \mathrm{P}\left(\bar{T}_n^* > \frac{x}{2}\right) + \frac{4}{x}\sum_{i=1}^{n} a_{n,i}\mathrm{E}\left(\left|\tilde{X}_i\right|\right) \\
&\leq \frac{4}{x^2}\mathrm{E}\left((\bar{T}_n^*)^2\right) + \frac{4}{x}\sum_{i=1}^{n} a_{n,i}\mathrm{E}\left(\left|\tilde{X}_i\right|\right) \\
&\leq \frac{4}{x^2}\mathrm{E}\left(\bar{T}_n^2\right) + \frac{4}{x}\sum_{i=1}^{n} a_{n,i}\mathrm{E}\left(\left|\tilde{X}_i\right|\right).
\end{aligned}$$

The last term is, remembering (4), bounded above by

$$\frac{4}{x}\mathrm{E}\left(\left|\tilde{X}_1\right|\right)\sum_{i} a_{n,i} \leq \frac{8}{x}\mathrm{E}\left(|X_1|\,\mathbb{I}_{|X_1|>M}\right)nA_{n,\alpha}.$$

For the upper bound of $\mathrm{E}\left(\bar{T}_n^2\right)$ use (7) and

$$\sum_{i,j=1}^{n} G_{i,j}(M) \leq n\mathrm{E}\left(\bar{X}_1^2\right) + 2\sum_{1\leq i<j\leq n} G_{i,j}(M)$$
$$= n\mathrm{E}\left(X_1^2 \mathbb{I}_{|X_1|\leq M}\right) + nM^2\mathrm{P}(|X_1|>M) + 2\sum_{1\leq i<j\leq n} G_{i,j}(M).$$

■

As in Louhichi [9] the previous result implies a maximal inequality for weighted sums, that follows immediately from the representation

$$\mathrm{E}\left((T_n^*)^p\right) = p\int_0^\infty x^{p-1}\mathrm{P}(T_n^*>x)\,dx. \tag{9}$$

Lemma 3.2. *Assume all the assumptions of Lemma 3.1 are satisfied and that* $\mathrm{E}\left(|X_1|^p\right)<\infty$, $p\in(1,2)$. *Then, for* $\alpha>1$,

$$\mathrm{E}\left((T_n^*)^p\right) \leq 4pnA_{n,\alpha}\left(\frac{n^{2/\alpha}}{2-p}A_{n,\alpha} + \frac{n^{2/\alpha}}{p}A_{n,\alpha} + \frac{2}{p-1}\right)\mathrm{E}\left(|X_1|^p\right)$$
$$+4pn^{2/\alpha}A_{n,\alpha}^2\int x^{p-3}\sum_{1\leq i<j\leq n} G_{i,j}(x)\,dx.$$

Proof: In (9) use the upper bound (8) choosing $M=x$ and integrate using Fubini's Theorem. ■

This is the counterpart for weighted sums of Proposition 1 in Louhichi [9]. Moreover, if $\alpha\longrightarrow+\infty$ and $\lim_{\alpha\to+\infty}A_{n,\alpha}=A_n<\infty$, the above inequality coincides with the one proved by Louhichi [9].

4. Main results

We have now the tools needed to prove the almost sure convergence of $b_n^{-1}T_n$ based on the Borel-Cantelli Lemma. A direct use of this argument would mean that one should prove $\sum_n \mathrm{P}(T_n>\varepsilon b_n)<\infty$. Instead, we replace T_n by the larger T_n^*, which is then an increasing sequence, as we have now an adequate control on the weighting sequence, taking care of the dependence of this weighting sequence on n. Now, for this increasing sequence T_n^*, the use of the Borel-Cantelli Lemma may be reduced to verification of the convergence of $\sum_n \frac{1}{n}\mathrm{P}(T_n^*>\varepsilon b_n)$, so we will concentrate on this last one. We first consider the normalization $b_n=n^{1/p}$.

Theorem 4.1. *Let X_n, $n \geq 1$, be centered and identically distributed associated random variables such that*

$$\mathrm{E}\left(|X_1|^{p\frac{\alpha+2}{\alpha}}\right) < \infty, \qquad \text{for some } p \in (1,2),\ \alpha > \frac{2p}{2-p},$$

$$\sum_{1\leq i<j<\infty} \int_{j^{1/p}}^{\infty} v^{-3+2\frac{p}{\alpha}}\, G_{i,j}(v)\, dv < \infty.$$

Assume the weights satisfy (2) and $\sup_{n\in\mathbb{N}} A_{n,\alpha} < \infty$. *Then*

$$\frac{1}{n^{1/p}}\, T_n \longrightarrow 0 \qquad \text{almost surely.}$$

Proof: As mentioned above we need to control the behaviour of $\frac{1}{n}\mathrm{P}(T_n^* > \varepsilon n^{1/p})$. Taking into account (8), choosing $M = n^{1/p}$, we find that

$$\begin{aligned}
&\frac{1}{n}\mathrm{P}(T_n^* > \varepsilon n^{1/p}) \\
&\quad \leq \frac{4}{\varepsilon^2 n^{2/p}} n^{2/\alpha} A_{n,\alpha}^2 \mathrm{E}\left(X_1^2 \mathbb{I}_{|X_1|\leq n^{1/p}}\right) + \frac{4}{\varepsilon^2} n^{2/\alpha} A_{n,\alpha}^2 \mathrm{P}(|X_1| > n^{1/p}) \\
&\qquad + \frac{8}{\varepsilon^2 n^{1+2/p}} n^{2/\alpha} A_{n,\alpha}^2 \sum_{1\leq i<j\leq n} G_{i,j}(n^{1/p}) + \frac{4}{\varepsilon n^{1/p}} A_{n,\alpha} \mathrm{E}\left(|X_1|\, \mathbb{I}_{|X_1|>n^{1/p}}\right).
\end{aligned}$$

We prove next that each of the four terms above defines a convergent series. Of course, as the sequence $A_{n,\alpha}$ is bounded, we may discard these terms. The first, second and fourth terms are controlled using Fubini's Theorem.

- The first term:

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{n^{2/p-2/\alpha}} \mathrm{E}\left(X_1^2 \mathbb{I}_{|X_1|^p\leq n}\right) \\
&\qquad = \mathrm{E}\left(X^2 \sum_{n=|X|^p}^{\infty} \frac{1}{n^{2/p-2/\alpha}}\right) \leq c_1 \mathrm{E}\left(|X_1|^{p+2p/\alpha}\right) < \infty,
\end{aligned}$$

  as $2/p - 2/\alpha > 1$, so the series converges.
- The second term:

$$\sum_{n=1}^{\infty} n^{2/\alpha} \mathrm{E}\left(\mathbb{I}_{|X_1|^p>n}\right) \leq c_2 \mathrm{E}\left(|X_1|^{p+2p/\alpha}\right) < \infty.$$

- The fourth term:

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/p}} \mathrm{E}\left(|X_1| \, \mathbb{I}_{|X_1|^p > n}\right) \leq c_3 \mathrm{E}\left(|X_1|^p\right) < \infty.$$

The constants c_1, c_2 and c_3 used above only depend on p and α. Finally, the control of the remaining term requires a little more effort. Again, using Fubini's Theorem we may write

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{n^{1+2/p-2/\alpha}} \sum_{1\leq i<j\leq n} G_{i,j}(n^{1/p}) \\
&\quad = \sum_{1\leq i<j\leq\infty} \iint \sum_{n>j} \frac{1}{n^{1+2/p-2/\alpha}} \, \mathbb{I}_{n>\max(|x|^p,|y|^p,j)} \, \Delta_{i,j}(x,y)\, dxdy \\
&\quad \leq c_4 \sum_{1\leq i<j\leq\infty} \iint \Big(\max(|x|^p, |y|^p, j)\Big)^{-2/p+2/\alpha} \Delta_{i,j}(x,y)\, dxdy,
\end{aligned}$$

where c_4 depends only on p and α. Now, adapting the arguments in Louhichi [9], we have that

$$\Big(\max(|x|^p, |y|^p, j)\Big)^{-2/p+2/\alpha} = \int_0^1 \mathbb{I}_{|x|\leq u^{-\frac{\alpha}{2(\alpha-p)}}} \, \mathbb{I}_{|y|\leq u^{-\frac{\alpha}{2(\alpha-p)}}} \, \mathbb{I}_{u\leq j^{-2/p+2/\alpha}} du. \tag{10}$$

Inserting this representation in the integral above, using Fubini again and remembering (5), we are lead to controlling

$$\begin{aligned}
&\sum_{1\leq i<j\leq\infty} \int_0^{j^{-2/p+2/\alpha}} G_{i,j}\left(u^{-\frac{\alpha}{2(\alpha-p)}}\right) du \\
&\qquad = \frac{2(\alpha-p)}{\alpha} \sum_{1\leq i<j\leq\infty} \int_{j^{1/p}}^{\infty} v^{-3+2\frac{p}{\alpha}} \, G_{i,j}(v)\, dv < \infty,
\end{aligned}$$

so the proof is concluded. ■

Remark 4.2. *Notice that, as assumed in Theorem 4.1, $\alpha > \frac{2p}{2-p}$ implies that the moment of the variables considered in the assumptions is $p + \frac{2p}{\alpha} < 2$.*

Remark 4.3. *If we allow $\alpha \longrightarrow \infty$ in the assumptions of Theorem 4.1, these reduce to the assumptions of Theorem 1 in Louhichi [9]. As mentioned before, Louhichi's framework corresponds to case of constant weights so, for every $\alpha > 0$, $A_{n,\alpha}$ is equal to the constant defining the weight, thus we are really* *allowed to let $\alpha \longrightarrow \infty$. That is, our Theorem 4.1 really extends Louhichi's result.*

We shall now consider the convergence $b_n^{-1}T_n$ with $b_n = n^{1/q}\log^{1/\gamma} n$, where $\gamma \in (0,1)$ and $q > 0$ is suitably chosen taking into account the exponent α used in the definition of the coefficients $A_{n,\alpha}$. This normalization extends the one used in a few recent references, such as [1, 2, 4, 8, 11, 12]. In all these later references the choice $q = \alpha$ is made. However, due to the particular structure of dependence we are considering and the method of approach that follows from the previous maximal inequalities, this later choice for q does not allow to derive the convergences analogous to the ones proved in Theorem 4.1

Theorem 4.4. *Let X_n, $n \geq 1$, be centered and identically distributed associated random variables such that*

$$
\begin{aligned}
&\mathrm{E}\left(X_1^2\right) < \infty, \\
&\mathrm{E}\left(\frac{X_1^2}{\log^{\frac{2}{\gamma}-1}|X_1|}\right), \qquad \textit{for some } \gamma \in (0,2), \\
&\sum_{1\leq i<j<\infty} \int_{j^{1/\beta}}^{\infty} \frac{1}{v^{\beta+1}} G_{i,j}(v)\, dv < \infty,
\end{aligned}
$$

where $\beta > 0$ if $\alpha < 2$ and $\beta \in \left(0, \frac{2\alpha}{\alpha-2}\right)$ if $\alpha > 2$. Assume that $\alpha > 1$ and that the weights satisfy (2) and $\sup_{n\in\mathbb{N}} A_{n,\alpha} < \infty$, and define $q = \frac{2\alpha}{\alpha+2}$. Then

$$
\frac{1}{n^{1/q}\log^{1/\gamma} n} T_n \longrightarrow 0 \qquad \textit{almost surely.}
$$

Proof: We follow the same arguments as before. This time we will be interested in controlling $\frac{1}{n}\mathrm{P}(T_n^* > \varepsilon n^{1/q}\log^{1/\gamma} n)$, again based in (8) and choosing now $M = n^{1/\beta}\log^{1/\gamma} n$, where β is chosen in as described in the theorem statement:

$$\begin{aligned}
&\frac{1}{n}\mathrm{P}(T_n^* > \varepsilon n^{1/q}\log^{1/\gamma} n) \\
&\quad \leq \frac{4n^{2/\alpha}A_{n,\alpha}^2}{\varepsilon^2 n^{2/q}\log^{2/\gamma} n}\mathrm{E}\left(X_1^2\mathbb{I}_{|X_1|\leq n^{1/\beta}\log^{1/\gamma} n}\right) \\
&\qquad +\frac{4n^{2/\alpha+2/\beta}A_{n,\alpha}^2}{\varepsilon^2 n^{2/q}}\mathrm{P}(|X_1|>n^{1/\beta}\log^{1/\gamma} n) \\
&\qquad +\frac{8n^{2/\alpha}A_{n,\alpha}^2}{\varepsilon^2 n^{1+2/q}\log^{2/\gamma} n}\sum_{1\leq i<j\leq n} G_{i,j}(n^{1/\beta}\log^{1/\gamma} n) \\
&\qquad\qquad +\frac{4A_{n,\alpha}}{\varepsilon n^{1/q}\log^{1/\gamma} n}\mathrm{E}\left(|X_1|\,\mathbb{I}_{|X_1|>n^{1/\beta}\log^{1/\gamma} n}\right).
\end{aligned}$$

As the $A_{n,\alpha}$ are bounded we do not need to include them for the proof of the convergence of the terms above. Similarly as in the previous theorem we analyze separately each of the four terms, using Fubini's Theorem.

- First term: we start by remarking that $\frac{2}{q}-\frac{2}{\alpha}=1$, so we have, as $\gamma<2$,

$$\begin{aligned}
&\sum_n \frac{1}{n\log^{2/\gamma} n}\mathrm{E}\left(X_1^2\mathbb{I}_{|X_1|\leq n^{1/\beta}\log^{1/\gamma} n}\right) \\
&\quad \leq \mathrm{E}\left(X_1^2 \sum_{n:\,|X_1|\leq n^{1/\beta}\log^{1/\gamma} n}\frac{1}{n\log^{2/\gamma} n}\right) \\
&\quad \leq \mathrm{E}\left(X_1^2 \sum_{n=|X_1|^\beta}^{\infty}\frac{1}{n\log^{2/\gamma} n}\right) \leq c'\mathrm{E}\left(\frac{X_1^2}{\log^{\frac{2}{\gamma}-1}|X_1|}\right)<\infty.
\end{aligned}$$

- Second term:

$$\begin{aligned}
&\sum_n n^{2/\beta-1}\mathrm{E}\left(\mathbb{I}_{|X_1|>n^{1/\beta}\log^{1/\gamma} n}\right) \\
&\quad \leq \mathrm{E}\left(\sum_n n^{2/\beta-1}\mathbb{I}_{|X_1|>n^{1/\beta}}\right) \leq c''\mathrm{E}\left(X_1^2\right)<\infty.
\end{aligned}$$

- Fourth term: reasoning as in the previous case, we have
$$\sum_n \frac{1}{n^{1/q}\log^{1/\gamma} n} \mathrm{E}\left(|X_1| \mathbb{I}_{|X_1|>n^{1/\beta}\log^{1/\gamma} n}\right) \leq c''' \mathrm{E}\left(|X_1|^{\beta+1-\beta/q}\right).$$
Given the choice for β and q it follows that $\beta+1-\beta/q<2$, so this expectation is also finite.

Remark that, as in the proof of Theorem 4.1, the constants c', c'' and c''' do not depend on n. The control of the remaining term follow arguments similar to those on the proof of Theorem 4.1. Taking again into account that $\frac{2}{q}-\frac{2}{\alpha}=1$ and using Fubini's Theorem we need to bound:

$$\begin{aligned}
&\sum_n \frac{1}{n^2\log^{2/\gamma} n} \sum_{1\leq i<j\leq n} G_{i,j}(n^{1/\beta}\log^{1/\gamma} n) \\
&\quad \leq \sum_{1\leq i<j\leq\infty} \iint \sum_{n>j} \frac{1}{n^2} \mathbb{I}_{n>\max(|x|^\beta,|y|^\beta,j)} \Delta_{i,j}(x,y)\,dxdy \qquad (11)\\
&\quad \leq c^* \sum_{1\leq i<j\leq\infty} \iint \left(\max(|x|^\beta,|y|^\beta,j)\right)^{-1} \Delta_{i,j}(x,y)\,dxdy,
\end{aligned}$$

where $c^*>0$ is independent from n. We need now an analogous to the representation (10), which is easily found as:

$$\left(\max(|x|^\beta,|y|^\beta,j)\right)^{-1} = \int_0^1 \mathbb{I}_{|x|\leq u^{-1/\beta}} \mathbb{I}_{|y|\leq u^{-1/\beta}} \mathbb{I}_{u\leq j^{-1}}\,du. \qquad (12)$$

Inserting this on the expression above, using the same arguments as in the final part of the proof of Theorem 4.1, and making the change of variable $v=u^{1/\beta}$, (11) is, up to the multiplication by a constant depending only on β, bounded above by

$$\sum_{1\leq i<j<\infty} \int_{j^{1/\beta}}^\infty \frac{1}{v^{\beta+1}} G_{i,j}(v)\,dv<\infty.$$

■

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