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SECOND ORDER DIFFERENTIAL EQUATIONS IN THE LAGUERRE-HAHN CLASS

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Abstract: Laguerre-Hahn families on the real line are characterized in terms of second order differential equations with matrix coefficients for vectors involving the orthogonal polynomials and their associated polynomials, as well as in terms of second order differential equation for the functions of the second kind. Some characterizations of the classical families are derived.

Keywords: Orthogonal polynomials on the real line, Riccati differential equation, semi-classical functionals, classical orthogonal polynomials.
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1. Introduction

The study of real orthogonal polynomial sequences, $\{P_n\}$, that are solutions of differential equations

$$\sum_{j=0}^{N} A_j y^{(j)} = 0 \tag{1}$$

where A_j are polynomials, is connected to measure perturbation theory and spectral theory of differential operators (see [6]). The minimal order of a differential equation (1) having orthogonal polynomial solutions is $N = 2$ or $N = 4$ (see [14]). For the case $N = 2$ in (1) and and $A_0 = \lambda$, where λ is some spectral (eigenvalue) parameter,

$$A_2 y'' + A_1 y' + \lambda y = 0\,, \tag{2}$$

it is known the classification of sequences of orthogonal polynomial solutions: $\{P_n\}$ must be, up to a linear change of variable, a member of the classical families, i.e., the Hermite, Laguerre, Jacobi and Bessel orthogonal polynomials (see [3] and also [11], for an overview on the problem of determination of orthogonal polynomial families that are solutions of (2)).

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In the present paper we focus our attention on differential equations satisfied by Laguerre-Hahn orthogonal polynomials on the real line. These polynomials are related to Stieltjes functions satisfying Riccati type differential equations with polynomial coefficients [10, 18, 19, 21, 22]

$$AS' = BS^2 + CS + D\,. \tag{3}$$

Note that the Laguerre-Hahn orthogonal polynomials are a generalization of the semi-classical orthogonal polynomials, since the later ones are related to (3) with $B \equiv 0$, the classical families appearing if, in addiction, $\deg(A) \leq 2$ and $\deg(C) = 1$. Laguerre-Hahn orthogonal polynomials can be generated by performing a perturbation on the Stieltjes function of semi-classical orthogonal polynomials or by doing a modification on the three term recurrence relation coefficients of semi-classical orthogonal polynomials (see [1, 4, 9, 22]). Thus, it turns out that the associated polynomials of semi-classical orthogonal polynomials constitute a well-known example of Laguerre-Hahn polynomials (see [1, 4, 7, 25]). Other examples include the co-recursive, co-dilated and co-modified polynomials (see [4, 15, 16]).

Laguerre-Hahn families of orthogonal polynomials are solutions of differential equations (1), where the minimal order is $N = 4$ (see [2, 10, 14, 20, 23]), thus when no simplification occurs, Laguerre-Hahn orthogonal polynomials satisfy

$$A_4 P_n^{(4)} + A_3 P_n^{(3)} + A_2 P_n'' + A_1 P_n' + A_0 P_n = 0\,,$$

where the A_j's are polynomials.

In this work we start by reinterpreting a result of [10], by showing an equivalence between (3) and differential-difference equations with matrix coefficients

$$A\Psi_n' = \mathcal{M}_n \Psi_n + \mathcal{N}_n \Psi_{n-1}\,, \quad \Psi_n = \begin{bmatrix} P_{n+1} & P_n^{(1)} \end{bmatrix}\,, \;\; n \geq 0\,,$$

with $\{P_n\}$ the sequence of monic orthogonal polynomials related to (3) and $\{P_n^{(1)}\}$ the sequence of first order associated polynomials (cf. Theorem 2). We prove the equivalence between (3) and differential-difference equations for the sequence of functions of the second kind $\{q_n\}$ (cf. Section 2),

$$Aq_n' = (l_{n-1} + \frac{C}{2} + BS)q_n + \Theta_{n-1} q_{n-1}\,, \quad n \geq 0\,.$$

Next, we prove the equivalence between (3) and a second order differential equation with matrix coefficients having polynomial entries,

$$\tilde{\mathcal{A}}_n \Psi_n'' + \tilde{\mathcal{B}}_n \Psi_n' + \tilde{\mathcal{C}}_n \Psi_n = 0_{2\times 1}\,, \quad n \geq 1 \tag{4}$$

as well as a second order differential equation for the sequence of functions of the second kind $\{q_n\}$

$$\tilde{A}_n q_n'' + \tilde{B}_n q_n' + \tilde{C}_n q_n = 0\,, \quad n \geq 1\,, \tag{5}$$

where $\tilde{A}_n$ is a polynomial and $\tilde{B}_n, \tilde{C}_n$ are functions. These equivalences are the analogue for orthogonality on the real line of [5, Theorems 1 and 2]. Taking into account the above referred equivalence between (3) and (4), we deduce a characterization of the sequences $\{\Psi_n\}$ corresponding to the Laguerre-Hahn class zero (i.e., $\max\{\deg(A), \deg(B)\} \leq 2$ and $\deg(C) = 1$ in (3) [4, 10]) as solutions of second order matrix operators,

$$\mathbb{L}_n(\Psi_n) = 0\,, \quad \mathbb{L}_n = \mathcal{A}\,\mathbb{D}^2 + \Psi\,\mathbb{D} + \Lambda_n\,\mathbb{I}\,, \quad n \geq 0\,, \tag{6}$$

with $\mathcal{A}, \Psi, \Lambda_n$ 2×2 matrices explicitly given in terms of the polynomials A, B, C, D in (3) (cf. Theorem 4).

Finally, the last part of the paper is devoted to the analysis of the classical families. As a consequence of the above referred results some characterizations for the classical orthogonal polynomials are shown, from which we emphasize the characterizations in term of:

– the hypergeometric-type differential equation for the sequence of functions of the second kind;

– the differential equation that links the associated polynomials $P_n^{(1)}$ and the derivative of P_{n+1},

$$A\left(P_n^{(1)}\right)'' + (A' - C)\left(P_n^{(1)}\right)' + \lambda_{n+1}^* P_n^{(1)} = 2DP_{n+1}'\,, \quad n \geq 0\,,$$

where λ_{n+1}^* are constants, explicitly given in terms of the polynomials A, B, C, D in (3);

– the Rodrigues-type formulas for $\{q_n\}$.

This paper is organized as follows. In Section 2 we give the definitions and state the basic results which will be used in the forthcoming sections. In Section 3 we establish the equivalence between (3) and the second order differential equations (4) and (5). In Section 4 we establish a characterization of Laguerre-Hahn orthogonal polynomials of class zero as solutions of (6). In

Section 5 we present characterizations of the classical families of orthogonal polynomials.

2. Preliminary Results

Let $\mathbb{P} = \text{span}\{z^k : k \in \mathbb{N}_0\}$ be the space of polynomials with complex coefficients, and let $\mathbb{P}'$ be its algebraic dual space, i.e., the linear space of linear functionals defined on $\mathbb{P}$. We will denote by $\langle u, f\rangle$ the action of $u \in \mathbb{P}'$ on $f \in \mathbb{P}$. We consider a linear functional $u \in \mathbb{P}'$ and $\langle u, x^n\rangle = u_n,\ n \geq 0$, its *moments*. We will take u normalized, i.e., $u_0 = 1$.

Given the sequence of moments (u_n) of u, the principal minors of the corresponding Hankel matrix are defined by $H_n = \det((u_{i+j})_{i,j=0}^n), n \geq 0$. By convention, $H_{-1} = 1$. The linear functional u is said to be *quasi-definite* (respectively, *positive-definite*) if $H_n \neq 0$ (respectively, $H_n > 0$), for all integer $n \geq 0$. If u is positive-definite, then it has an integral representation in terms of a positive Borel measure, μ, supported on an infinite set of points of the real line, I, such that

$$\langle u, x^n\rangle = \int_I x^n\, d\mu\,, \quad n \geq 0\,.$$

Definition 1. Let $u \in \mathbb{P}'$. A sequence $\{P_n\}_{n\geq 0}$ is said to be *orthogonal with respect to* u if the following two conditions hold:
(i) $\deg(P_n) = n,\ n \geq 0\,,$
(ii) $\langle u, P_nP_m\rangle = k_n\delta_{n,m},\ k_n = \langle u, P_n^2\rangle \neq 0,\ n \geq 0\,.$
If the leading coefficient of each P_n is 1, then $\{P_n\}$ is said to be a sequence of *monic orthogonal polynomials with respect to* u, and it will be denoted by SMOP.

The equivalence between the quasi-definiteness of $u \in \mathbb{P}'$ and the existence of a SMOP with respect to u is well-known in the literature of orthogonal polynomials (see [8, 24]).

Monic orthogonal polynomials satisfy a three term recurrence relation (see [24])

$$P_{n+1}(x) = (x - \beta_n)P_n(x) - \gamma_n P_{n-1}(x),\ n = 1, 2, \ldots \tag{7}$$

with $P_0(x) = 1,\ P_1(x) = x - \beta_0$ and $\gamma_n \neq 0,\ n \geq 1,\ \gamma_0 = u_0 = 1$. Conversely, given a SMOP $\{P_n\}$ satisfying a three-term recurrence relation as above, there exists a unique quasi-definite linear functional u such that $\{P_n\}$ is the SMOP with respect to u (see [8, 24]).

Definition 2. Let $\{P_n\}$ be the SMOP with respect to a linear functional u. The sequence of *first order associated polynomials* is defined by

$$P_n^{(1)}(x) = \langle u_t, \frac{P_{n+1}(x) - P_{n+1}(t)}{x-t}\rangle , \quad n \geq 0 ,$$

where u_t denotes the action of u on the variable t.

Note that the sequence $\{P_n^{(1)}\}$ also satisfies a three term recurrence relation,

$$P_n^{(1)}(x) = (x-\beta_n)P_{n-1}^{(1)}(x) - \gamma_n P_{n-2}^{(1)}(x), \; n = 1, 2, \ldots$$

with $P_{-1}^{(1)}(x) = 0, \; P_0^{(1)}(x) = 1$.

Definition 3. Let $u \in \mathbb{P}'$ be quasi-definite and (u_n) its sequence of moments. The *Stieltjes function* of u is defined by

$$S(z) = \sum_{n=0}^{+\infty} \frac{u_n}{z^{n+1}} .$$

Given a SMOP $\{P_n\}$ and $\{P_n^{(1)}\}$ its sequence of associated polynomials, let S and and $S^{(1)}$ denote the corresponding Stieltjes functions, respectively. One has

$$\gamma_1 S^{(1)} = -\frac{1}{S} - (z-\beta_0) . \tag{8}$$

The sequence of *functions of the second kind* corresponding to $\{P_n\}$ is defined as follows:

$$q_n(z) = \langle u_t, \frac{P_n(t)}{z-t}\rangle , \quad n \geq 1 , \; q_0 = S ,$$

thus

$$q_{n+1} = P_{n+1}S - P_n^{(1)} , \quad n \geq 0 , \; q_0 = S . \tag{9}$$

Definition 4. Let $u \in \mathbb{P}'$ be quasi-definite and let S be its Stieltjes function. u (or S) is said to be *Laguerre-Hahn* if there exist polynomials A, B, C, D, with $A \neq 0$, such that S satisfies a Riccati differential equation

$$AS' = BS^2 + CS + D . \tag{10}$$

The corresponding sequence of orthogonal polynomials is called Laguerre-Hahn. If $B = 0$, then S is said to be *semi-classical* or *Laguerre-Hahn affine.*

Note that if u is semi-classical, with the corresponding Stieltjes function satisfying $AS' = CS + D$, then, taking into account (8), there follows that $S^{(1)}$ is Laguerre-Hahn, since it satisfies a Riccati type differential equation,

$$A\left(S^{(1)}\right)' = B_1\left(S^{(1)}\right)^2 + D_1 S^{(1)} + D_1\,,$$

where

$$B_1 = \gamma_1 D,\ C_1 = -C + 2(z-\beta_0)D,\ D_1 = \frac{-A}{\gamma_1} - \frac{(z-\beta_0)}{\gamma_1}\left(C - (z-\beta_0)D\right)\,.$$

Equation (10) is equivalent to the distributional equation for the corresponding linear functional u,

$$\mathcal{D}(Au) = \psi u + B(x^{-1}u^2)\,, \tag{11}$$

where A, B are the same as in (10), $\psi = A' + C$ (cf. [22]), being the left product of u by a polynomial defined as

$$\langle g\,u, p\rangle = \langle u, g\,p\rangle\,,\ p \in \mathbb{P}\,,$$

the derivative $\mathcal{D}u$ defined as

$$\langle \mathcal{D}u, p\rangle = -\langle u, p'\rangle\,,\ p \in \mathbb{P}\,,$$

and the functional $x^{-1}u$ and the product of two linear functionals defined, respectively, as follows:

$$\langle x^{-1}u, p\rangle = \langle u, \theta_0 p\rangle\,, \quad (\theta_0 p)(x) = \frac{p(x) - p(0)}{x}\,,\ \ \langle u\,v, p\rangle = \langle u, v\,p\rangle\,,\ \ p \in \mathbb{P}\,,$$

with the right product given by

$$v\,p = \sum_{m=0}^{n}\left(\sum_{j=m}^{n} p_j v_{j-m}\right)x^m\,, \quad p(x) = \sum_{j=0}^{n} p_j x^j\,.$$

Note that the distributional equation (11) is not unique, many triples of polynomials can be associated with such an equation, but only one canonical set of minimal degree exists. The *class* of u is defined as the minimum value of $\max\{\deg(\psi) - 1, d - 2\}$, $d = \max\{\deg(A), \deg(B)\}$, for all triples of polynomials satisfying (11). When $B \equiv 0$ and the class of u is zero, i.e., $\deg(\psi) = 1$ and $\deg(A) \leq 2$, u is called a *classical functional*, and the corresponding orthogonal polynomials are the so-called classical orthogonal polynomials.

In the sequel we will use the following matrices:

$$\Psi_n = \begin{bmatrix} P_{n+1} \\ P_n^{(1)} \end{bmatrix}, \quad Q_n = \begin{bmatrix} q_{n+1} \\ q_n \end{bmatrix}, \quad n \geq 0 \,. \tag{12}$$

Hereafter I denotes the 2×2 identity matrix.

Lemma 1. *Let $u \in \mathbb{P}'$ be quasi-definite, let $\{P_n\}$ be the corresponding SMOP and β_n, γ_n the coefficients of the three term recurrence relation (7). Let $\{\Psi_n\}$, $\{Q_n\}$ be the sequences defined in (12). Then,*
(a) Ψ_n satisfies

$$\Psi_n = (x - \beta_n)\Psi_{n-1} - \gamma_n \Psi_{n-2} \,, \quad n \geq 1 \,, \tag{13}$$

with initial conditions $\Psi_{-1} = \begin{bmatrix} P_0 \\ P_{-1}^{(1)} \end{bmatrix}, \Psi_0 = \begin{bmatrix} P_1 \\ P_0^{(1)} \end{bmatrix}$;
(b) $\varphi_n = \begin{bmatrix} \Psi_{n+1} \\ \Psi_n \end{bmatrix}$ *satisfies*

$$\varphi_n = \mathcal{K}_n \varphi_{n-1} \,, \quad \mathcal{K}_n = \begin{bmatrix} (x - \beta_{n+1})I & -\gamma_{n+1} I \\ I & 0_{2\times 2} \end{bmatrix}, \quad n \geq 1 \,, \tag{14}$$

with initial conditions $\varphi_0 = \begin{bmatrix} P_2 & P_1^{(1)} & P_1 & P_0^{(1)} \end{bmatrix}^T$;
(c) Q_n satisfies

$$Q_n = \mathcal{A}_n Q_{n-1} \,, \quad n \geq 1 \,,$$

with $\mathcal{A}_n = \begin{bmatrix} x - \beta_n & -\gamma_n \\ 1 & 0 \end{bmatrix}$ *and initial conditions* $Q_0 = \begin{bmatrix} (x - \beta_0)S - 1 \\ S \end{bmatrix}$.

Theorem 1 (see [17]). *Let $\{f_n\}$ be a sequence of functions satisfying a three term recurrence relation*

$$x f_n(x) = f_{n+1}(x) + \beta_n f_n(x) + \gamma_n f_{n-1}$$

with $\gamma_n \neq 0$, $n \geq 1$, $f_{-1} = 0$, $f_0(x) = 1$. If $g_n = \dfrac{f_{n+1}}{f_n}$ satisfies

$$a_n(x) g_n'(x) = b_n(x) g_n^2(x) + c_n g_n(x) + d_n(x), \quad n \geq 0 \,,$$

where a_n, b_n, c_n and d_n are bounded degree polynomials, then, for all $n \geq 0$, the following relations hold:

$$\begin{aligned} a_{n+1} &= a_n\,, \\ b_{n+1} &= \frac{d_n}{\gamma_{n+1}}\,, \\ c_{n+1} &= -c_n - 2(x-\beta_{n+1})\frac{d_n}{\gamma_{n+1}}\,, \qquad (15) \\ d_{n+1} &= a_n + \gamma_{n+1} b_n + (x-\beta_{n+1})c_n + (x-\beta_{n+1})^2 \frac{d_n}{\gamma_{n+1}}. \qquad (16) \end{aligned}$$

3. Second order differential equations with matrix coefficients

Theorem 2. *Let $u \in \mathbb{P}'$ be quasi-definite and S its Stieltjes function. Let $\{\Psi_n\}$ be the corresponding sequence defined in (12), and let $\{q_n\}$ be the sequence of functions of the second kind. The following statements are equivalent:*
(a) S satisfies

$$AS' = BS^2 + CS + D\,, \quad A, B, C, D \in \mathbb{P}\,;$$

(b) Ψ_n satisfies

$$A\Psi_n' = \mathcal{M}_n \Psi_n + \mathcal{N}_n \Psi_{n-1}\,, \quad n \geq 0\,, \qquad (17)$$

$\mathcal{M}_n = \begin{bmatrix} l_n - \frac{C}{2} & -B \\ D & l_n + \frac{C}{2} \end{bmatrix}$, $\mathcal{N}_n = \Theta_n I$, *$\Theta_n, l_n$ bounded degree polynomials, with initial conditions* $A = (l_0 - C/2)(x-\beta_0) - B + \Theta_0\,,\ 0 = D(x-\beta_0) + (l_0 + C/2)\,;$
(c) q_n satisfies

$$Aq_n' = (l_{n-1} + \frac{C}{2} + BS)q_n + \Theta_{n-1} q_{n-1}\,, \quad n \geq 0\,, \qquad (18)$$

with $q_{-1} = 1,\ \Theta_{-1} = D,\ l_{-1} = C/2$.
Moreover, the following relations hold, for all $n \geq 0$:

$$\begin{aligned} l_{n+1} + l_n &= -\frac{(x-\beta_{n+1})}{\gamma_{n+1}}\Theta_n\,, \qquad (19) \\ \Theta_{n+1} &= A + \frac{\gamma_{n+1}}{\gamma_n}\Theta_{n-1} + (x-\beta_{n+1})(l_n - l_{n+1})\,. \qquad (20) \end{aligned}$$

Proof: $(a) \Leftrightarrow (b)$.
This equivalence was proven in [10, 17]. Note that equation (17) is the matrix form of the equations in [10, 17].
$(b) \Rightarrow (c)$.
Take derivatives in (9), $P_{n+1}S - P_n^{(1)} = q_{n+1}\,, n \geq 0$, then multiply the resulting expression by A and use the equations enclosed in (17), to get

$$Aq'_{n+1} = (l_n + \frac{C}{2} + BS)q_{n+1} + \Theta_n q_n\,,\ n \geq 0\,,$$

thus

$$Aq'_n = (l_{n-1} + \frac{C}{2} + BS)q_n + \Theta_{n-1}q_{n-1}\,,\ n \geq 1\,.$$

Furthermore, since (17) holds and since $q_{-1} = 1,\ q_0 = S$, taking into account $\Theta_{-1} = D,\ l_{-1} = C/2$, there follows that the above equation also holds for $n = 0$. Hence we obtain (18).
$(c) \Rightarrow (a)$
Take $n = 0$ in (18), with $q_{-1} = 1,\ q_0 = S,\ \Theta_{-1} = D,\ l_{-1} = C/2$.

Finally, to obtain (19) and (20) we proceed as follows. From (18) we get

$$A(q'_{n+1}q_n - q'_n q_{n+1}) = \Theta_n q_n^2 + (l_n - l_{n-1})q_n q_{n+1} - \Theta_{n-1}q_{n-1}q_{n+1}\,, \quad n \geq 0\,.$$

The division of both members by q_n^2 and the use of the three term recurrence relation for q_n yields

$$A\left(\frac{q_{n+1}}{q_n}\right)' = \frac{\Theta_{n-1}}{\gamma_n}\frac{q_{n+1}^2}{q_n^2} + \left(l_n - l_{n-1} - \frac{(x-\beta_n)}{\gamma_n}\Theta_{n-1}\right)\frac{q_{n+1}}{q_n} + \Theta_n\,.$$

Taking into account Theorem 1 we get, using (15),

$$l_{n+1} - l_{n-1} = -\frac{(x-\beta_{n+1})}{\gamma_{n+1}}\Theta_n + \frac{(x-\beta_n)}{\gamma_n}\Theta_{n-1}\,, \quad n \geq 0\,. \tag{21}$$

Therefore, we get, for all $n \geq 0$,

$$m_n + \frac{(x-\beta_{n+1})}{\gamma_{n+1}}\Theta_n = m_{n-1} + \frac{(x-\beta_n)}{\gamma_n}\Theta_{n-1}\,, \quad m_n = l_{n+1} + l_n\,, \tag{22}$$

from which there follows

$$m_n + \frac{(x-\beta_{n+1})}{\gamma_{n+1}}\Theta_n = m_{-1} + \frac{(x-\beta_0)}{\gamma_0}\Theta_{-1}\,, \quad n \geq 0\,,$$

that is,

$$l_{n+1} + l_n + \frac{(x-\beta_{n+1})}{\gamma_{n+1}}\Theta_n = l_0 + l_{-1} + \frac{(x-\beta_0)}{\gamma_0}\Theta_{-1}\,, \quad n \geq 0\,,$$

Using the initial conditions $D = \Theta_{-1}, C/2 = l_{-1}$ we get $l_0+l_{-1}+\frac{(x-\beta_0)}{\gamma_0}\Theta_{-1} = 0$, thus (19) follows.

Eq. (20) follows taking into account (16) and the use of (19). ■

Remark . If the class of u is s, then $\deg(\Theta_n) \leq s$, $\deg(l_n) \leq s+1$.

If we take $B = 0$ in the previous theorem we obtain differential relations in the semi-classical class.

Corollary 1. *Let $u \in \mathbb{P}'$ be quasi-definite and let S be its Stieltjes function. Let $\{\Psi_n\}$ be the corresponding sequence defined in (12), and let $\{q_n\}$ be the sequence of functions of the second kind. The following statements are equivalent:*
(a) S is semi-classical and it satisfies $AS' = CS + D$;
(b) Ψ_n satisfies

$$A\Psi_n' = \mathcal{M}_n\Psi_n + \mathcal{N}_n\Psi_{n-1}\,, \quad n \geq 0\,,$$

where $\mathcal{M}_n = \begin{bmatrix} l_n - \frac{C}{2} & 0 \\ D & l_n + \frac{C}{2} \end{bmatrix}$, $\mathcal{N}_n = \Theta_n I$, and Θ_n, l_n are bounded degree polynomials;
(c) q_n satisfies the differential-difference equation with polynomial coefficients

$$Aq_n' = (l_{n-1} + \frac{C}{2})q_n + \Theta_{n-1}q_{n-1}\,, \quad n \geq 0\,.$$

Theorem 3. *Let $u \in \mathbb{P}'$ be quasi-definite and let S be its Stieltjes function. Let $\{\Psi_n\}$ be the corresponding sequence defined in (12), and let $\{q_n\}$ be the sequence of functions of the second kind. The following statements are equivalent:*
(a) S satisfies

$$AS' = BS^2 + CS + D\,, \quad A, B, C, D \in \mathbb{P}\,;$$

(b) Ψ_n satisfies the second order differential equation

$$\tilde{\mathcal{A}}_n\Psi_n'' + \tilde{\mathcal{B}}_n\Psi_n' + \tilde{\mathcal{C}}_n\Psi_n = 0_{2\times 1}\,, \quad n \geq 1\,, \tag{23}$$

where $\tilde{\mathcal{A}}_n, \tilde{\mathcal{B}}_n, \tilde{\mathcal{C}}_n$ *are matrices, with polynomial entries, given by*

$$\tilde{\mathcal{A}}_n = A^2\Theta_n I \,, \tag{24}$$

$$\tilde{\mathcal{B}}_n = A\Theta_n(A'I - \mathcal{M}_n - \mathcal{M}_{n-1}) - A\Theta_{n-1}\Theta_n \frac{(x-\beta_n)}{\gamma_n} I - A^2\Theta_n' I \,, \tag{25}$$

$$\begin{aligned}\tilde{\mathcal{C}}_n &= \Theta_n\left(\frac{\Theta_{n-1}\Theta_n}{\gamma_n} I - A\mathcal{M}_n'\right) \\ &\quad + \left\{\Theta_n\left(\mathcal{M}_{n-1} + \frac{(x-\beta_n)}{\gamma_n}\Theta_{n-1} I\right) + A\Theta_n' I\right\}\mathcal{M}_n \,;\end{aligned} \tag{26}$$

(c) q_n *satisfies the second order differential equation*

$$\tilde{A}_n q_{n+1}'' + \tilde{B}_n q_{n+1}' + \tilde{C}_n q_{n+1} = 0 \,, \quad n \geq 0 \,, \tag{27}$$

where $\tilde{A}_n$, $\tilde{B}_n, \tilde{C}_n$ *are functions given by*

$$\tilde{A}_n = A^2\Theta_n \,, \tag{28}$$

$$\tilde{B}_n = A\Theta_n(A' - C - 2BS) - A^2\Theta_n' \,, \tag{29}$$

$$\begin{aligned}\tilde{C}_n &= \Theta_n\left(\frac{\Theta_{n-1}\Theta_n}{\gamma_n} - A(l_n + \frac{C}{2} + BS)'\right) \\ &\quad + (l_n + \frac{C}{2} + BS)\left(\Theta_n(-l_n + \frac{C}{2} + BS) + A\Theta_n'\right) .\end{aligned} \tag{30}$$

Corollary 2. *Let* $u \in \mathbb{P}'$ *be quasi-definite and* S *its Stieltjes function. Let* $\{\Psi_n\}$ *be the corresponding sequence defined in (12), and let* $\{q_n\}$ *be the sequence of functions of the second kind. The following statements are equivalent:*
(a) S *is semi-classical and satisfies*

$$AS' = CS + D \,, \quad A, C, D \in \mathbb{P} \,;$$

(b) $\{\Psi_n\}$ *satisfies the second order differential equation* (23) *with matrix coefficients of polynomials entries given by (24)-(26);*
(c) $\{q_n\}$ *satisfies the second order differential equation (27) with polynomial*

coefficients $\tilde{A}_n$, $\tilde{B}_n$, $\tilde{C}_n$ *given by*

$$\begin{aligned}
\tilde{A}_n &= A\Theta_n \,, &(31)\\
\tilde{B}_n &= \Theta_n(A'-C) - A\Theta_n' \,, &(32)\\
\tilde{C}_n &= \Theta_n\left(\sum_{k=1}^{n}\frac{\Theta_{k-1}}{\gamma_k} + D - (l_n+\frac{C}{2})'\right) + \Theta_n'(l_n+\frac{C}{2})\,. &(33)
\end{aligned}$$

Proof: If u is semi-classical then we take $B\equiv 0$ in the previous theorem, thus we get the second order differential equation (27) with polynomial coefficients $\tilde{A}_n$, $\tilde{B}_n$, $\tilde{C}_n$ given by

$$\begin{aligned}
\tilde{A}_n &= A^2\Theta_n \,,\\
\tilde{B}_n &= A\left(\Theta_n(A'-C) - A\Theta_n'\right)\,,\\
\tilde{C}_n &= \Theta_n\left(\frac{\Theta_{n-1}\Theta_n}{\gamma_n} - l_n^2 + (C/2)^2\right) - A\left(\Theta_n(l_n+C/2)' - \Theta_n'(l_n+C/2)\right)\,.
\end{aligned}$$

Let $\tau_n = \dfrac{\Theta_{n-1}\Theta_n}{\gamma_n} - l_n^2 + (C/2)^2$.

Using (19) and (20) we obtain

$$\frac{\Theta_{n-1}\Theta_n}{\gamma_n} - l_n^2 = A\frac{\Theta_{n-1}}{\gamma_n} + \frac{\Theta_{n-2}\Theta_{n-1}}{\gamma_{n-1}} - l_{n-1}^2\,, \quad n\geq 1\,,$$

thus,

$$\frac{\Theta_{n-1}\Theta_n}{\gamma_n} - l_n^2 = A\sum_{k=1}^{n}\frac{\Theta_{k-1}}{\gamma_k} + \frac{\Theta_{-1}\Theta_0}{\gamma_0} - l_0^2\,, \quad n\geq 1\,. \tag{34}$$

The initial conditions

$$\Theta_{-1}=D,\quad D(x-\beta_0)+(l_0+C/2)=0,\quad A=(l_0-C/2)(x-\beta_0)-B+\Theta_0$$

yield

$$\frac{\Theta_{-1}\Theta_0}{\gamma_0} - l_0^2 = AD + BD - (C/2)^2\,. \tag{35}$$

From (34) and (35) there follows

$$\tau_n = A\sum_{k=1}^{n}\frac{\Theta_{k-1}}{\gamma_k} + AD + BD\,. \tag{36}$$

Note that we are assuming $B \equiv 0$, thus we obtain $\tilde{C}_n$ given by

$$\tilde{C}_n = A\left\{\Theta_n\left(\sum_{k=1}^{n}\frac{\Theta_{k-1}}{\gamma_k} + D - (l_n + \frac{C}{2})'\right) + \Theta_n'(l_n + \frac{C}{2})\right\}.$$

Consequently, $\{q_n\}$ satisfies a second order differential equation (27) with polynomial coefficients (31)-(33). ■

In the sequel $X^{(i,j)}$ will denote the (i,j) entry in the matrix X.

Corollary 3. *Let S be a Laguerre-Hahn Stieltjes function satisfying $AS' = BS^2+CS+D$. The SMOP related to S, $\{P_n\}$, as well as the sequence of first order associated polynomials, $\{P_n^{(1)}\}$, satisfy fourth order linear differential equations with polynomial coefficients.*

Proof: Let us consider the differential equations enclosed in (23),

$$A^2\Theta_n P_{n+1}'' + \tilde{\mathcal{B}}_n^{(1,1)} P_{n+1}' + \tilde{\mathcal{C}}_n^{(1,1)} P_{n+1} + \tilde{\mathcal{B}}_n^{(1,2)}\left(P_n^{(1)}\right)' + \tilde{\mathcal{C}}_n^{(1,2)} P_n^{(1)} = 0\,, \quad (37)$$

$$A^2\Theta_n\left(P_n^{(1)}\right)'' + \tilde{\mathcal{B}}_n^{(2,2)}\left(P_n^{(1)}\right)' + \tilde{\mathcal{C}}_n^{(2,2)} P_n^{(1)} + \tilde{\mathcal{B}}_n^{(2,1)} P_{n+1}' + \tilde{\mathcal{C}}_n^{(2,1)} P_{n+1} = 0\,. \quad (38)$$

Take derivatives in (37), multiply the resulting equation by $A^2\Theta_n$ and use (38) to eliminate $\left(P_n^{(1)}\right)''$, thus obtaining an equation of the following type:

$$(A^2\Theta_n)^2 P_{n+1}''' + b_n P_{n+1}'' + c_n P_{n+1}' + d_n P_{n+1} + e_n\left(P_n^{(1)}\right)' + f_n P_n^{(1)} = 0\,. \quad (39)$$

Now take derivatives in (39), multiply the resulting equation by $A^2\Theta_n$ and again use (38) to eliminate $\left(P_n^{(1)}\right)''$, thus obtaining an equation of the following type:

$$(A^2\Theta_n)^3 P_{n+1}^{(4)} + g_n P_{n+1}''' + h_n P_{n+1}'' + j_n P_{n+1}' + k_n P_{n+1} + m_n\left(P_n^{(1)}\right)' + r_n P_n^{(1)} = 0\,. \quad (40)$$

The elimination of $\left(P_n^{(1)}\right)'$ and $P_n^{(1)}$ between (37), (39) and (40) yields a fourth order differential equation with polynomial coefficients for P_n,

$$A_4 P_n^{(4)} + A_3 P_n^{(3)} + A_2 P_n'' + A_1 P_n' + A_0 P_n = 0\,.$$

The fourth order differential equation with polynomial coefficients for $P_n^{(1)}$, can be obtained analogously, starting by taking derivatives to (38) and eliminating P''_{n+1}, P'_{n+1}, as well as P_{n+1}. ■

Now we will prove the Theorem 3, by using the lemmas that follow.

Lemma 2. *Let $u \in \mathbb{P}'$ be quasi-definite and let S be its Stieltjes function. Let $\{\Psi_n\}$ be the corresponding sequence defined in (12), and let $\{q_n\}$ be the sequence of functions of the second kind. If S satisfies $AS' = BS^2 + CS + D\,,\ \ A, B, C, D \in \mathbb{P}\,,$ then $\{\Psi_n\}$ satisfies (23) with coefficients (24)-(26) and $\{q_n\}$ satisfies (27) with coefficients (28)-(30).*

Proof: If we take derivatives in (17) and multiply the resulting equation by A we get

$$A^2\Psi''_n = A(\mathcal{M}_n - A'I)\Psi'_n + \mathcal{N}_n A\Psi'_{n-1} + A\mathcal{M}'_n\Psi_n + A\mathcal{N}'_n\Psi_{n-1}\,. \tag{41}$$

If we use (17) to $n-1$ and the recurrence relation (13) for Ψ_n we obtain

$$A\Psi'_{n-1} = \left(\mathcal{M}_{n-1} + \frac{(x-\beta_n)}{\gamma_n}\Theta_{n-1}\right)\Psi_{n-1} - \frac{\Theta_{n-1}}{\gamma_n}\Psi_n\,. \tag{42}$$

The substitution of (42) into (41) yields

$$\begin{aligned} A^2\Psi''_n = A(\mathcal{M}_n - A'I)\Psi'_n &+ \left(A\mathcal{M}'_n - \frac{\Theta_{n-1}}{\gamma_n}\mathcal{N}_n\right)\Psi_n \\ &+ \left[\mathcal{N}_n\left(\mathcal{M}_{n-1} + \frac{(x-\beta_n)}{\gamma_n}\Theta_{n-1}\right) + A\mathcal{N}'_n\right]\Psi_{n-1}\,. \end{aligned}$$

The multiplication of the above equation by Θ_n and the use of (17) gives us (23) with coefficients (24)-(26).

To get (27) we proceed analogously as before, starting by taking derivatives in (18), thus obtaining $\tilde{A}_n q''_{n+1} + \tilde{B}_n q'_{n+1} + \tilde{C}_n q_{n+1} = 0$, with $\tilde{A}_n = A^2\Theta_n$ and

$$\begin{aligned} \tilde{B}_n &= -A\Theta_n(l_n + l_{n-1} + C + 2BS - A') - A\frac{(x-\beta_n)}{\gamma_n}\Theta_{n-1}\Theta_n - A^2\Theta'_n\,, \\ \tilde{C}_n &= \Theta_n\left(\frac{\Theta_{n-1}\Theta_n}{\gamma_n} - A(l_n + \frac{C}{2} + BS)'\right) \\ &+ (l_n + \frac{C}{2} + BS)\left[\Theta_n\left(\frac{(x-\beta_n)}{\gamma_n}\Theta_{n-1} + l_{n-1} + \frac{C}{2} + BS\right) + A\Theta'_n\right]. \end{aligned}$$

The use of $l_n + l_{n-1} = -\frac{(x-\beta_n)}{\gamma_n}\Theta_{n-1}$ (cf. (19)) in the above equations yields $\tilde{B}_n$ and $\tilde{C}_n$ given by (29) and (30). ■

Lemma 3. *Let $u \in \mathbb{P}'$ be quasi-definite and let $\{\Psi_n\}$ be the corresponding sequence defined in (12). If $\{\Psi_n\}$ satisfies the second order differential equation (23) with coefficients (24)-(26), then the following equation holds:*

$$\hat{A}_n \Psi_n' = \mathcal{M}_n \Psi_n + \mathcal{N}_n \Psi_{n-1}\,, \quad n \geq 1\,,$$

where $\hat{A}_n \in \mathbb{P}$, $\mathcal{M}_n$ is a matrix of order two with polynomial entries, and $\mathcal{N}_n$ is a scalar matrix.

Proof: We write the equation (23) in the form

$$\mathcal{D}_n \varphi_n'' + \mathcal{E}_n \varphi_n' + \mathcal{F}_n \varphi_n = 0_{4\times 1} \tag{43}$$

where, $\varphi_n = \begin{bmatrix} \Psi_{n+1} & \Psi_n \end{bmatrix}^T$, $n \geq 1$, and $\mathcal{D}_n, \mathcal{E}_n, \mathcal{F}_n$ are block matrices given by

$$\mathcal{D}_n = A^2 \begin{bmatrix} \Theta_{n+1} I & 0_{2\times 2} \\ 0_{2\times 2} & \Theta_n I \end{bmatrix}, \ \mathcal{E}_n = \begin{bmatrix} \tilde{\mathcal{B}}_{n+1} & 0_{2\times 2} \\ 0_{2\times 2} & \tilde{\mathcal{B}}_n \end{bmatrix}, \ \mathcal{F}_n = \begin{bmatrix} \tilde{\mathcal{C}}_{n+1} & 0_{2\times 2} \\ 0_{2\times 2} & \tilde{\mathcal{C}}_n \end{bmatrix}.$$

Taking $n+1$ in (43) and using the recurrence relations for φ_n (cf. (14)) we obtain

$$\begin{aligned} \mathcal{D}_{n+1}\mathcal{K}_{n+1}\varphi_n'' + (2\mathcal{D}_{n+1}\mathcal{K}_{n+1}' + \mathcal{E}_{n+1}\mathcal{K}_{n+1})\varphi_n' \\ + (\mathcal{E}_{n+1}\mathcal{K}_{n+1}' + \mathcal{F}_{n+1}\mathcal{K}_{n+1})\varphi_n = 0_{4\times 1}\,. \end{aligned} \tag{44}$$

To eliminate φ_n'' between (43) and (44) we proceed in two steps: firstly we multiply (43) by $\Theta_{n+2}\mathcal{K}_{n+1}\mathcal{G}_n$, $\mathcal{G}_n = \begin{bmatrix} \Theta_n I & 0 \\ 0 & \Theta_{n+1} I \end{bmatrix}$, thus obtaining

$$A^2\Theta_n\Theta_{n+1}\Theta_{n+2}\mathcal{K}_{n+1}\varphi_n'' + \Theta_{n+2}\mathcal{K}_{n+1}\mathcal{G}_n\mathcal{E}_n\varphi_n' + \Theta_{n+2}\mathcal{K}_{n+1}\mathcal{G}_n\mathcal{F}_n\varphi_n = 0_{4\times 1}\,, \tag{45}$$

and we multiply (44) by $\Theta_n\mathcal{G}_{n+1}$, thus obtaining

$$\begin{aligned} A^2\Theta_n\Theta_{n+1}\Theta_{n+2}\mathcal{K}_{n+1}\varphi_n'' + \Theta_n\mathcal{G}_{n+1}(2\mathcal{D}_{n+1}\mathcal{K}_{n+1}' + \mathcal{E}_{n+1}\mathcal{K}_{n+1})\varphi_n' \\ + \Theta_n\mathcal{G}_{n+1}(\mathcal{E}_{n+1}\mathcal{K}_{n+1}' + \mathcal{F}_{n+1}\mathcal{K}_{n+1})\varphi_n = 0_{4\times 1}\,. \end{aligned} \tag{46}$$

Then we subtract (46) to (45), thus obtaining

$$\mathcal{H}_n\varphi_n' = \mathcal{J}_n\varphi_n\,, \tag{47}$$

with

$$\begin{aligned}\mathcal{H}_n &= \Theta_{n+2}\mathcal{K}_{n+1}\mathcal{G}_n\mathcal{E}_n - \Theta_n\mathcal{G}_{n+1}(2\mathcal{D}_{n+1}\mathcal{K}'_{n+1} + \mathcal{E}_{n+1}\mathcal{K}_{n+1})\\ \mathcal{J}_n &= \Theta_n\mathcal{G}_{n+1}(\mathcal{E}_{n+1}\mathcal{K}'_{n+1} + \mathcal{F}_{n+1}\mathcal{K}_{n+1}) - \Theta_{n+2}\mathcal{K}_{n+1}\mathcal{G}_n\mathcal{F}_n\,.\end{aligned}$$

The multiplication of (47) by $\mathrm{adj}(\mathcal{H}_n)$ yields

$$\hat{A}_n\varphi'_n = \hat{\mathcal{L}}_n\varphi_n\,, \tag{48}$$

with

$$\hat{A}_n = \det(\mathcal{H}_n),\ \hat{\mathcal{L}}_n = \mathrm{adj}(\mathcal{H}_n)\mathcal{J}_n\,.$$

Thus, the assertion follows. ∎

Now we study the coefficients of the structure relations obtained in the preceding lemma.

Lemma 4. *Let $u \in \mathbb{P}'$ be quasi-definite and let $\{\Psi_n\}$ be the corresponding sequence defined in (12). Let $\varphi_n = \begin{bmatrix}\Psi_{n+1} & \Psi_n\end{bmatrix}^T$ satisfy*

$$\hat{A}_n\varphi'_n = \hat{\mathcal{L}}_n\varphi_n, \quad n \geq 1\,, \tag{49}$$

where $\hat{A}_n$ are bounded degree polynomials and $\hat{\mathcal{L}}_n$, $n \geq 1$, are block matrices of order two whose entries are bounded degree polynomials. Then, (49) is equivalent to

$$\hat{A}\varphi'_n = \mathcal{L}_n\varphi_n\,, \quad n \geq 1\,. \tag{50}$$

Furthermore, it holds that

$$\hat{A}\mathcal{K}'_{n+1} = \mathcal{L}_{n+1}\mathcal{K}_{n+1} - \mathcal{K}_{n+1}\mathcal{L}_n\,, \quad n \geq 1\,, \tag{51}$$

where $\mathcal{K}_n$ are the matrices of the recurrence relation (14).

Proof: If we take $n+1$ in (49) and use the recurrence relation for φ_n (cf. (14)) we get

$$\hat{A}_{n+1}\varphi'_n = \mathcal{K}^{-1}_{n+1}\left(\hat{\mathcal{L}}_{n+1}\mathcal{K}_{n+1} - \hat{A}_{n+1}\mathcal{K}'_{n+1}\right)\varphi_n \tag{52}$$

From (49) and (52) we conclude that there exists a polynomial L_n such that, for all $n \geq 1$,

$$\begin{cases}\hat{A}_{n+1} = L_n\hat{A}_n\\ \mathcal{K}^{-1}_{n+1}\left(\hat{\mathcal{L}}_{n+1}\mathcal{K}_{n+1} - \hat{A}_{n+1}\mathcal{K}'_{n+1}\right) = L_n\hat{\mathcal{L}}_n.\end{cases}$$

because the first order differential equation for φ_n is unique, up to a multiplicative factor. But from $\hat{A}_{n+1} = L_n \hat{A}_n$ we obtain

$$\hat{A}_{n+1} = (L_n \cdots L_2)\, \hat{A}_1 \,, \ \forall n \geq 1 \,.$$

Since, for all $n \geq 1$, the degree of $\hat{A}_n$ is bounded by a number independent of n, then the degree of the L_n's must be zero, i.e., L_n is constant, for all $n \geq 1$. Hence we obtain (50) with $\hat{A} = \hat{A}_1$ and

$$\mathcal{L}_n = \mathcal{K}_{n+1}^{-1} \left(\hat{\mathcal{L}}_{n+1} \mathcal{K}_{n+1} - \hat{A}_{n+1} \mathcal{K}'_{n+1} \right) / (L_n \cdots L_2) \,.$$

To obtain (51) we take derivatives on $\varphi_{n+1} = \mathcal{K}_{n+1} \varphi_n$ and multiply the result by $\hat{A}$, to get

$$\hat{A}\, \varphi'_{n+1} = \hat{A}\, \mathcal{K}'_{n+1} \varphi_n + \mathcal{K}_{n+1} \hat{A}\, \varphi'_n \,.$$

Using (50) in the previous equation and the recurrence relation (14) there follows

$$\mathcal{L}_{n+1} \mathcal{K}_{n+1} \varphi_n = \hat{A} \mathcal{K}'_{n+1} \varphi_n + \mathcal{K}_{n+1} \mathcal{L}_n \varphi_n \,,$$

thus (51). ∎

Corollary 4. *Let $\{\varphi_n\}$ satisfy (50), $\hat{A}\varphi'_n = \mathcal{L}_n \varphi_n \,, \ n \geq 1 \,,$ where $\mathcal{L}_n$ are block matrices of order two whose entries are bounded degree polynomials. Then, the following assertions take place:*
(a) $\mathcal{L}_n^{(1,2)}$ is a scalar matrix if, and only if, $\mathcal{L}_n^{(2,1)}$ is scalar.
(b) If $\mathcal{L}_n^{(2,1)}$ is a scalar matrix, then there exist polynomials p_i, $i = 1, \ldots, 3$, such that

$$\mathcal{L}_n^{(1,1)} = \begin{bmatrix} l_{n+1} - p_1 & p_2 \\ p_3 & l_{n+1} + p_1 \end{bmatrix} \,, \quad n \geq 1 \,. \tag{53}$$

Proof: Taking into account the definition of $\mathcal{K}_n$, (50) is equivalent to, $\forall n \geq 1$,

$$\hat{A} I = (x - \beta_{n+2})(\mathcal{L}_{n+1}^{(1,1)} - \mathcal{L}_n^{(1,1)}) + \mathcal{L}_{n+1}^{(1,2)} + \gamma_{n+2} \mathcal{L}_n^{(2,1)} \,, \tag{54}$$

$$-\gamma_{n+2} \mathcal{L}_{n+1}^{(1,1)} - (x - \beta_{n+2}) \mathcal{L}_n^{(1,2)} + \gamma_{n+2} \mathcal{L}_n^{(2,2)} = 0 \,, \tag{55}$$

$$(x - \beta_{n+2}) \mathcal{L}_{n+1}^{(2,1)} + \mathcal{L}_{n+1}^{(2,2)} - \mathcal{L}_n^{(1,1)} = 0 \,, \tag{56}$$

$$-\gamma_{n+2} \mathcal{L}_{n+1}^{(2,1)} - \mathcal{L}_n^{(1,2)} = 0 \,. \tag{57}$$

Assertion (a) follows taking into account (57), i.e., $\mathcal{L}_n^{(1,2)} = -\gamma_{n+2} \mathcal{L}_{n+1}^{(2,1)}$.

Let us prove assertion (b).

Since $\mathcal{L}_n^{(1,2)}$ and $\mathcal{L}_n^{(2,1)}$ are diagonal, from (54) there follows that the entries $(1,2)$ and $(2,1)$ of the matrix $\mathcal{L}_n^{(1,1)}$ are independent of n.

Further, from (54) we obtain that $\left[\mathcal{L}_n^{(1,1)}\right]^{(1,1)} - \left[\mathcal{L}_n^{(1,1)}\right]^{(2,2)}$ is independent of n. Hence, (53) follows. ■

Remark . Note that eq. (48) in Lemma 3, combined with the independence of n in the polynomials $\hat{A}_n$ (cf. Lemma 4) reads as

$$\begin{cases} \hat{A}\Psi'_{n+1} = \mathcal{L}_n^{(1,1)}\Psi_{n+1} + \mathcal{L}_n^{(1,2)}\Psi_n \\ \hat{A}\Psi'_n = \mathcal{L}_n^{(2,1)}\Psi_{n+1} + \mathcal{L}_n^{(2,2)}\Psi_n\,. \end{cases} \tag{58}$$

Using the recurrence relation (13) in the second equation of (58) we obtain

$$\hat{A}\Psi'_n = \left[(x-\beta_{n+1})\mathcal{L}_n^{(2,1)} + \mathcal{L}_n^{(2,2)}\right]\Psi_n - \gamma_{n+1}\mathcal{L}_n^{(2,1)}\Psi_{n-1}\,. \tag{59}$$

Taking into account (56) as well as (57), there follows that (59) is the first equation of (58) for $n-1$, that is,

$$\hat{A}\Psi'_n = \mathcal{L}_{n-1}^{(1,1)}\Psi_n + \mathcal{L}_{n-1}^{(1,2)}\Psi_{n-1}\,.$$

Lemma 5. *Let $u \in \mathbb{P}'$ be quasi-definite and let $\{q_n\}$ be the corresponding sequence of functions of the second kind. If $\{q_n\}$ satisfies the second order differential equation (27)*

$$\tilde{A}_n q''_{n+1} + \tilde{B}_n q'_{n+1} + \tilde{C}_n q_{n+1} = 0$$

with coefficients (28)*-(30), then the Q_n's given in* (12) *satisfy*

$$\widehat{A}_n Q'_n = \widehat{\mathcal{L}}_n Q_n, \quad n \geq 1\,,$$

with $\widehat{A}_n \in \mathbb{P}$ and $\widehat{\mathcal{L}}_n$ a matrix of order two with analytic entries.

Proof: Analogous to the proof of the Lemma 3. ■

Lemma 6. *Let $u \in \mathbb{P}'$ be quasi-definite and let $\{Q_n\}$ be the corresponding sequence given in* (12)*. If*

$$\widehat{A}_n Q'_n = \widehat{\mathcal{L}}_n Q_n, \quad n \geq 1\,,$$

with $\widehat{A}_n \in \mathbb{P}$ and $\widehat{\mathcal{L}}_n$ a matrix of order two with analytic entries, then $\widehat{A}_n$ does not depend on n.

Proof: Analogous to the proof of the Lemma 4, using the Theorem 1. ■

Proof of the Theorem 3:
Lemma 2 proves $(a) \Rightarrow (b)$ and $(a) \Rightarrow (c)$. Using the Lemmas 3 and 4 and Corollary 4 we prove $(b) \Rightarrow (a)$. Lemmas 5 and 6 prove $(c) \Rightarrow (a)$.

4. Second order matrix operators in the Laguerre-Hahn class zero

Theorem 4. *Let u be a Laguerre-Hahn Stieltjes functional satisfying $\mathcal{D}(Au) = \psi u + B(x^{-1}u^2)$, with $\deg(\psi) = 1$, $\max\{\deg(A), \deg(B)\} \leq 2$. Let $\{P_n\}$ be the SMOP related to u and let $\{P_n^{(1)}\}$ be the sequence of first order associated polynomials. It holds that*

$$\mathbb{L}_n(\Psi_n) = 0\,, \quad \Psi_n = \begin{bmatrix} P_{n+1} \\ P_n^{(1)} \end{bmatrix}, \quad n \geq 0\,, \tag{60}$$

where $\mathbb{L}_n$ is a matrix operator given by

$$\mathbb{L}_n = A\mathbb{D}^2 + \Psi\mathbb{D} + \Lambda_n\mathbb{I}\,, \quad \Psi = \begin{bmatrix} \psi & 2B \\ -2D & 2A' - \psi \end{bmatrix}, \; \Lambda_n = \begin{bmatrix} \lambda_{n+1} & B' \\ 0 & \lambda_{n+1}^* \end{bmatrix} \tag{61}$$

where $\mathbb{D}^k$ denotes the derivative operator, $\mathbb{D}^0 = \mathbb{I}$, and

$$\lambda_{n+1} = \lambda_{n+1}^* - A'' + \psi'\,, \; \lambda_{n+1}^* = 2(n+1)D - n(n+3)\frac{A''}{2} + n\psi'\,,$$
$$D = \frac{A''}{2} - \psi' - \frac{B''}{2}\,.$$

Moreover, the coefficients of the three term recurrence relation of the MOP sequences $\{P_n\}$ satisfying (60) are given by

$$\gamma_n = \frac{(2D - 2(n-1)a_2 + \psi_1)\,\nu_{n-1} + (\lambda_n^* - \lambda_{n+1}^*)\,\nu_n}{\lambda_{n-1}^* - \lambda_n^*}$$
$$+ \frac{(2\beta_0 D + 2(n-1)a_1 - \psi_0)\,\alpha_{n-1} - 2\gamma_1 D - 2(n-1)a_0}{\lambda_{n-1}^* - \lambda_n^*}\,, \quad n \geq 2\,, \tag{62}$$

$$\beta_n = \alpha_n - \alpha_{n-1}\,, \quad n \geq 1\,, \tag{63}$$

with for $n \geq 1$,

$$\alpha_n = \frac{n\,[-(n+1)a_1 + \psi_0 - 2\beta_0 D]}{-(n-1)(n+2)a_2 + (n-1)\psi_1 - \lambda_{n+1}^* + 2nD}\,, \; \alpha_0 = 0\,, \tag{64}$$

$$\nu_n = \frac{(n-1)\,[\alpha_n\,(na_1 - \psi_0 + 2\beta_0 D) - na_0 - 2\gamma_1 D]}{(n-2)(n+1)a_2 - (n-2)\psi_1 + \lambda_{n+1}^* - 2(n-1)D}\,. \tag{65}$$

Remark . We emphasize the equation enclosed by (60),

$$\mathcal{L}_n^*(P_n^{(1)}) = 2DP'_{n+1}\,, \quad n \geq 0\,, \tag{66}$$

where $\mathcal{L}_n^*$ is the operator defined by

$$\mathcal{L}_n^* = A\mathcal{D}^2 + (2A' - \psi)\mathcal{D} + \lambda_{n+1}^*\mathcal{I}\,.$$

The preceding theorem gives us the formulas for the three term recurrence relation coefficients of the SMOP $\{P_n\}$ satisfying (66). This result is an extension of [13].

Remark . The Theorem 4 gives us a characterization of the sequences of monic orthogonal polynomials of the Laguerre-Hahn class zero. The full description of the three term recurrence relation coefficients of such family was given in [4].

Proof: The Stieltjes function of u satisfies

$$AS' = BS^2 + CS + D\,, \;\; C = \psi - A'\,, \;\; D \text{ is constant.}$$

Since the class of u is zero, the Θ's involved in the structure relation (17) are constant. If we use the notation $\tau_n = \dfrac{\Theta_{n-1}\Theta_n}{\gamma_n} - l_n^2 + (C/2)^2$, then, taking into account (cf. (36)) $\tau_n = A\sum_{k=1}^{n}\Theta_{k-1}/\gamma_k + AD + BD$, the second order differential equation (23) can be written as (60) with the operator $\mathbb{L}_n$ given by (61).

To obtain the three term recurrence coefficients of $\{P_n\}$ we start by writing

$$\begin{aligned} P_n^{(1)}(x) &= x^n - \alpha_n x^{n-1} + \nu_n x^{n-2} + \cdots \\ P_{n+1}(x) &= x^{n+1} - (\alpha_n + \beta_0)x^n + (\nu_n + \beta_0\alpha_n - \gamma_1)x^{n-1} + \cdots \end{aligned}$$

with

$$\alpha_n = \sum_{k=1}^{n}\beta_k\,, \;\; \nu_n = \sum_{1\leq i<j\leq n}\beta_i\beta_j - \sum_{k=2}^{n}\gamma_k\,, \quad n \geq 1\,. \tag{67}$$

Equating coefficients of x^{n-1} and x^{n-2} in (66) we get (64) and (65).

Taking derivatives in (7) we get

$$P_n = P'_{n+1} - (x - \beta_n)P'_n + \gamma_n P'_{n-1}\,.$$

If we multiply the above equation by $2D$ and use the equation enclosed by (60),

$$\mathcal{L}_n^*(P_n^{(1)}) = 2DP'_{n+1}\,, \quad n \geq 0\,,$$

where $\mathcal{L}_n^*$ is the operator defined by

$$\mathcal{L}_n^* = A\mathcal{D}^2 + (2A' - \psi)\mathcal{D} + \lambda_{n+1}^*\mathcal{I}\,,$$

as well as the recurrence relation, we get

$$\begin{aligned} 2DP_n = 2A\left(P_{n-1}^{(1)}\right)' \\ + (2A' - \psi)P_{n-1}^{(1)} + (\lambda_{n+1}^* - \lambda_n^*)P_n^{(1)} + (\lambda_{n-1}^* - \lambda_n^*)\gamma_n P_{n-2}^{(1)}\,. \end{aligned} \tag{68}$$

Equating coefficients of x^{n-2} in (68) we get (62). (63) follows from (67). ■

5. Characterizations of classical orthogonal polynomials

Taking into account the results of the preceding sections, we will deduce characterizations of the classical families. Hereafter we consider the distributional equation $\mathcal{D}(Au) = \psi u$ with the canonical expressions for A and ψ given in Table 1. We denote the corresponding orthogonal polynomials, Hermite, Laguerre, Jacobi and Bessel, by H_n, L_n^α, $P_n^{(\alpha,\beta)}$ and B_n^α, respectively.

	A	ψ
H_n	1	$-2x$
$L_n^{(\alpha)}$	x	$-x+\alpha+1$
$P_n^{(\alpha,\beta)}$	$1-x^2$	$-(\alpha+\beta+2)x+\beta-\alpha$
$B_n^{(\alpha)}$	x^2	$(\alpha+2)x+2$

Table 1

We also present the three term recurrence relation coefficients β_n, γ_{n+1}, $n \geq 0$.

	β_n	γ_{n+1}
H_n	0	$\frac{n+1}{2}$
$L_n^{(\alpha)}$	$2n+\alpha+1$	$(n+1)(n+\alpha+1)$
$P_n^{(\alpha,\beta)}$	$\frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}$	$\frac{4(n+1)(n+\alpha+1)(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)^2(2n+\alpha+\beta+3)}$
$B_n^{(\alpha)}$	$\frac{-2\alpha}{(n+\alpha)(2n+\alpha+2)}$	$\frac{-4(n+1)(n+\alpha+1)}{(2n+\alpha+1)(2n+\alpha+2)^2(2n+\alpha+3)}$

Table 2

Theorem 5. *Let $u \in \mathbb{P}'$ be regular, let $\{P_n\}$ be the SMOP with respect to u, let $\{P_n^{(1)}\}$ be the sequence of associated polynomials, and let $\{q_n\}$ be the sequence of functions of the second kind. The following statements are equivalent:*
(a) u is classical and it satisfies $\mathcal{D}(Au) = \psi u$;
(b) $\{P_n\}$ satisfies

$$AP_n'' + \psi P_n' + \lambda_n P_n = 0\,, \quad n \geq 0\,; \tag{69}$$

(c) q_n satisfies the second order differential equation

$$Aq_n'' + (2A' - \psi)q_n' + (\lambda_n + A'' - \psi')q_n = 0\,, \quad n \geq 0\,, \tag{70}$$

(d) the derivative P_n' is linked to the associated polynomial $P_n^{(1)}$ through a relation of the same type as (66),

$$A\left(P_n^{(1)}\right)'' + (2A' - \psi)\left(P_n^{(1)}\right)' + \lambda_{n+1}^* P_n^{(1)} = 2DP_{n+1}'\,, \quad n \geq 0\,, \tag{71}$$

where, for all $n \geq 0$,

$$\lambda_{n+1}^* = 2(n+1)D - n(n+3)\frac{A''}{2} + n\psi'\,,\ \lambda_{n+1} = \lambda_{n+1}^* - A'' + \psi'\,,\ D = \frac{A''}{2} - \psi'\,,$$

and, by convention, $\lambda_0 = 0$.

Proof: Note that $\mathcal{D}(Au) = \psi u$ is equivalent to the first order differential equation for the corresponding Stieltjes function

$$AS' = CS + D\,,\ \ C = \psi - A'\,.$$

Since u is classical, that is, $\deg(A) \leq 2, \deg(\psi) = 1$, then the Θ_n's and the l_n's involved in the coefficients of the second order differential equations (23) and (27) satisfy $\deg(\Theta_n) = 0, \deg(l_n) \leq 1$. Thus, (23) yields (69) and (71), and (27) yields (70) for all $n \geq 1$. Notice that (70) for $n = 0$, with $\lambda_0 = 0$, reads as $AS'' + (A' - C)S' - C'S = 0$, which is the derivative of $AS' = CS + D$.

To prove $(d) \Rightarrow (a)$ we use the equations (62) and (63) (cf. Remark 3) with the values of β_0 and γ_1 given in Table 2, thus recovering the expressions for γ_{n+1} and β_n for all $n \geq 1$, thus obtaining the classical families of orthogonal polynomials. ∎

The preceding theorem gives a characterization of the classical families in terms of hypergeometric-type differential equations for the orthogonal polynomials, as well as for the functions of the second kind. Note that given any hypergeometric-type differential equation, that is,

$$Ay'' + Hy' + k_n y = 0, \quad \deg(A) \leq 2,\ \deg(H) \leq 1,\ k_n \text{ constant},$$

and given a nonnegative integer n, the above differential equation has a unique polynomial solution P_n of degree exactly n if, and only if,

$$\frac{n(n-1)}{2} A'' + nH' + k_n = 0\,, \quad n \geq 0\,. \tag{72}$$

Under the hypothesis (72), the existence of $w(x)$ satisfying

$$\frac{d}{dx}(A(x)w(x)) = H(x)w(x)$$

allows the representation of the solution in terms of a Rodrigues formula (see [12])

$$P_n(x) = \frac{1}{w(x)} \frac{d^n}{dx^n}(A^n(x)w(x))\,.$$

Combining the previous theorem with the above referred results we obtain the characterization that follows.

Theorem 6. *Let $u \in \mathbb{P}'$ be regular, let $\{P_n\}$ be the SMOP with respect to u and let $\{q_n\}$ be the sequence of functions of the second kind. The following statements are equivalent:*

(a) u is classical and it satisfies $\mathcal{D}(Au) = \psi u$;

(b) $\{P_n\}$ has a Rodrigues representation

$$P_n(x) = \frac{1}{w(x)} \frac{d^n}{dx^n}(A^n(x)w(x))\,, \quad n \geq 0\,; \tag{73}$$

(c) $\{q_n\}$ has a Rodrigues-type representation

$$q_n(x) = \frac{d^n}{dx^n}\left(A^n(x)w(x)\right)\,, \quad n \geq 1\,, \tag{74}$$

where $w = \frac{1}{A}e^{\Phi}\,, \Phi = \int_{x_0}^{x} \frac{\psi(t)}{A(t)}dt$.

Proof: $(a) \Rightarrow (b)$ and $(a) \Rightarrow (c)$.

From Theorem 5 we have (69),

$$AP_n'' + \psi P_n' + \lambda_n P_n = 0\,, \quad n \geq 0\,.$$

Note that by equating the leading coefficients in (69) we get

$$\frac{A''}{2}n(n-1) + n\psi' + \lambda_n = 0\,, \quad n \geq 0\,,$$

which is the assumption (72). Hence, there follows the Rodrigues representation (73) for P_n.

To deduce a Rodrigues-type formula for $\{q_n\}$ we recall, from Theorem 5, the second order differential equation (70),

$$Aq_n'' + \tilde{\psi}q_n' + \tilde{\lambda}_n q_n = 0\,, \quad \tilde{\psi} = 2A' - \psi,\ \tilde{\lambda}_n = \lambda_n + A'' - \psi'\,.$$

If we multiply the above equation by $\tilde{w} = \frac{1}{A}e^{\tilde{\Phi}}$, $\tilde{\Phi} = \int_{x_0}^{x} \frac{\tilde{\psi}(t)}{A(t)}dt$, we get

$$(A\tilde{w}q_n')' + \tilde{\lambda}_n \tilde{w} q_n = 0\,. \tag{75}$$

Let

$$Z = \tilde{w}q_n\,. \tag{76}$$

Then, (75) becomes

$$AZ'' + (2A' - \tilde{\psi})Z' + (\tilde{\lambda}_n + A'' - \tilde{\psi}')Z = 0\,,$$

which yields

$$AZ'' + \psi Z' + \lambda_n Z = 0\,.$$

Therefore, there holds a Rodrigues representation for Z,

$$Z = \frac{1}{w(x)}\frac{d^n}{dx^n}(A^n(x)w(x))\,. \tag{77}$$

Using $q_n = \tilde{w}^{-1}Z$ (cf. (76)), (77) yields

$$q_n = \frac{1}{w\tilde{w}}\frac{d^n}{dx^n}(A^n(x)w(x))\,.$$

Since $\tilde{w} = w^{-1}$, there follows (74).

$(b) \Rightarrow (a)$ and $(c) \Rightarrow (a)$.

From (73) and (74) we obtain the second order differential equations (69) and (70), respectively, thus from the Theorem 5 there follows the distributional equation for the corresponding u. ■

References

[1] R. Askey and J. Wimp, *Associated Laguerre and Hermite polynomials*, Proc. Royal Soc. Edinburgh 96 A (1984), 15-37.

[2] S. Belmehdi, J. Dini, P. Maroni, and A. Ronveaux, *4th order differential equation for the co-modified of semi-classical orthogonal polynomials*, J. Comp. Appl. Math. 29 (2) (1989), 225-231.

[3] S. Bochner, *Über Sturm-Liouvillesche polynomsysteme*, Math. Zeit. 29 (1929), 730-736.

[4] H. Bouakkaz and P. Maroni, *Description des polynomes orthogonaux de Laguerre-Hahn de classe zero*, in "Orthogonal Plynomials and their applications", (C. Brezinski, L. Gori and A. Ronveaux Eds.) J. C. Baltzer A. G. Basel IMACS Annals on Computing and Applied Mathematics, (1991), 189-194.

[5] A. Branquinho and M. N. Rebocho, *On differential equations for orthogonal polynomials on the unit circle*, J. Math. Anal. Appl. 356 (1), (2009), 242-256.

[6] E. Buendia, J. S. Dehesa, and F. J. Galvez, *The distribution of the zeros of the polynomial eigenfunctions of ordinary differential operators of arbitrary order*, in Orthogonal polynomials and their applications, M. Alfaro et al Editors, Lecture Notes in Mathematics, Vol. 1329, Springer-Verlag, Berlin 1988, 222-235.

[7] J. Bustoz and M. E. H. Ismail, *The associated ultraspherical polynomials and their q-analogues*, Canad. J. Math. 34 (1982), 718-736.

[8] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.

[9] J. S. Dehesa, F. Marcellán, and A. Ronveaux, *On orthogonal polynomials with perturbed recurrence relations*, J. Comput. Appl. Math. 30 (1990) 203-212.

[10] J. Dini, *Sur les formes linéaires et les polynômes orthogonaux de Laguerre-Hahn*, Thèse de doctorat, Univ. Pierre et Marie Curie, Paris, 1988 (in French).

[11] W. N. Everitt and L. L. Littlejohn, *Orthogonal polynomials and spectral theory: a survey*, in "Orthogonal Plynomials and their applications", (C. Brezinski, L. Gori and A. Ronveaux Eds.) J. C. Baltzer A. G. Basel IMACS Annals on Computing and Applied Mathematics, (1991), 21-55.

[12] J. V. Gonçalves, *Sur la formule de Rodrigues*, Portugaliae Math. 4 (1), (1943), 52-64.

[13] C. C. Grosjean, *Theory of recursive generation of systems of orthogonal polynomials: an illustrative example*, J. Comput. Appl. Math. 12-13 (1985), 299-318.

[14] W. Hahn, *On Differential Equations for Orthogonal Polynomials*, Funkcialaj Ek., **21** (1978), 1-9.

[15] J. Letessier, *On co-recursive associated Laguerre polynomials*, J. Comput. Appl. Math. 49 (1993), 127-136.

[16] J. Letessier, *Some results on co-recursive associated Laguerre and Jacobi polynomials*, SIAM J. Math. Anal. (25) (1994), 528-548.

[17] A. P. Magnus, *Riccati acceleration of the Jacobi continued fractions and Laguerre-Hahn polynomials*, in "Padé Approximation and its Applications, Proc., Bad Honnef 1883", Lect. Notes in Math. 1071 (H. Werner e H. T. Bunger, eds.), Springer Verlag, Berlin, (1984), 213-230.

[18] F. Marcellán and E. Prianes, *Perturbations of Laguerre-Hahn linear funtionals*, J. Comput. Appl. Math. 105 (1999), 109-128.

[19] F. Marcellán and E. Prianes, *Orthogonal Polynomials and Stieltjes functions: The Laguerre-Hahn case*, Rendiconti di Matematica e le sue Applicazione Serie VII, 16 (1996), 117-141.

[20] F. Marcellán and A. Ronveaux, *Co-recursive orthogonal polynomials and fourth order differential equation*, J. Comp. Appl. Math. 25 (1) (1989), 105-109.

[21] P. Maroni, *Prolégomènes à l'etude des polynômes orthogonaux semi-classiques*, Ann. Mat. Pura ed Appli. 149 (1987), 165-184.

[22] P. Maroni, *Une théorie algébrique des polynômes orthogonaux. Application aux polynômes orthogonaux semi-classiques*, in "Orthogonal Plynomials and their applications", (C. Brezinski, L. Gori and A. Ronveaux Eds.) J. C. Baltzer A. G. Basel IMACS Annals on Computing and Applied Mathematics, (1991), 95-130.

[23] A. Ronveaux, *Fourth-order differential equations for the numerator polynomials*, J. Phys. A.: Math. Gen. 21 (1988), 749-753.

[24] G. Szegő, *Orthogonal Polynomials*, fourth ed., Amer. Math. Soc. Colloq. Publ., vol. 23, Providence Rhode Island, 1975.

[25] J. Wimp, *Explicit formulas for the associated Jacobi polynomials and some applications*, Can. J. Math. 39 (4) (1987), 983-1000.

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