Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 12–15

ON THE FULL KOSTANT TODA SYSTEM AND THE DISCRETE KORTEWEG-DE VRIES EQUATIONS

D. BARRIOS ROLANÍA, A. BRANQUINHO AND A. FOULQUIÉ MORENO

ABSTRACT: The relation between the solutions of the full Kostant Toda lattice and the discrete Korteweg-de Vries equation is analyzed. A method for constructing solutions of these systems is given. As a consequence of the matrical interpretation of this method, the transform of Darboux is extended for general Hessenberg banded matrices.

KEYWORDS: Operator theory, orthogonal polynomials, differential equations, recurrence relations.
AMS SUBJECT CLASSIFICATION (2010): 15B57, 47N20, 34K99, 42C05.

1. Introduction

In [8] and [9] the construction of a solution of the Toda lattice

$$\left.\begin{array}{rcl} \dot{a}_n & = & b_n - b_{n-1} \\ \dot{b}_n & = & b_n(a_{n+1} - a_n) \end{array}\right\}, \quad n \in \mathbb{N}, \tag{1}$$

from another given solution was studied. Both solutions of (1) were linked to each other by a Backlünd transformation, also called Miura transformation, given by

$$\left.\begin{array}{l} b_n = \gamma_{2n}\gamma_{2n+1}, \quad a_n = \gamma_{2n-1} + \gamma_{2n} + C \\ \tilde{b}_n = \gamma_{2n+1}\gamma_{2n+2}, \quad \tilde{a}_n = \gamma_{2n} + \gamma_{2n+1} + C \end{array}\right\}$$

where $\{\gamma_n\}$ is a solution of the Volterra lattice

$$\dot{\gamma}_n = \gamma_n\left(\gamma_{n+1} - \gamma_{n-1}\right). \tag{2}$$

Received May 2, 2012.
The work of the first author was partially supported by Dirección General de Investigación, Ministerio de Ciencia e Innovación, under grant MTM2009-14668-C02-02 and by Acciones Integradas, Ministerio de Ciencia e Innovación, under grant PT2009-0031. The work of second and third authors was partially supported by "Acção Integrada Luso-Espanhola" n. E-4/10, Center for Mathematics of the University of Coimbra, and Center for Research and Development in Mathematics and Applications (University of Aveiro), funded by the European Regional Development Fund through the program COMPETE and by the Portuguese Government through the FCT - Fundação para a Ciência e a Tecnologia under the projects PEst-C/MAT/UI0324/2011 and PEst-C/MAT/UI4106/2011 with COMPETE number FCOMP-01-0124-FEDER-022690.

Here and in the following, the dot means differentiation with respect to $t \in \mathbb{R}$. However, we suppress the explicit t-dependence for brevity.

In [2], the first and the second authors generalized the analysis given in [9] to the kind of Toda and Volterra lattice studied in [1]. As a particular case, the results obtained in [2] extend the corresponding of [9] to the case of Toda lattices where $a_n(t)$ and $b_n(t)$ are complex functions of $t \in \mathbb{R}$. Now, our goal is to extend the results of [2] and [9] to the complex full Kostant-Toda lattice, which is given by

$$\left.\begin{array}{rcl} \dot{a}_n^{(1)} & = & a_n^{(2)} - a_{n-1}^{(2)} \\ \dot{a}_n^{(2)} & = & \left(a_{n+1}^{(1)} - a_n^{(1)}\right) a_n^{(2)} + a_n^{(3)} - a_{n-1}^{(3)} \\ \dot{a}_n^{(3)} & = & \left(a_{n+2}^{(1)} - a_n^{(1)}\right) a_n^{(3)} + a_n^{(4)} - a_{n-1}^{(4)} \\ & \vdots & \\ \dot{a}_n^{(p-1)} & = & \left(a_{n+p-2}^{(1)} - a_n^{(1)}\right) a_n^{(p-1)} + a_n^{(p)} - a_{n-1}^{(p)} \\ \dot{a}_n^{(p)} & = & \left(a_{n+p-1}^{(1)} - a_n^{(1)}\right) a_n^{(p)} \end{array}\right\} , \quad n \in \mathbb{N} . \tag{3}$$

In the sequel, for each $n \in \mathbb{N}$ we assume that $a_n^{(i)}$, $i = 1, 2, \ldots, p$, in (3) are continuous functions with complex values defined in the open interval $\mathcal{I}_n$ such that

$$\bigcap_{n=1}^{N} \mathcal{I}_n \neq \emptyset , \quad \text{for all } N \in \mathbb{N} . \tag{4}$$

It is easy to check that these equations can be formally written in a Lax pair form $\dot{J} = [J, J_-]$, where $[M, N] = MN - NM$ is the commutator of the operators M and N, and J, J_- are the operators whose matrical representation is given, respectively, by the banded matrices

$$J = \begin{pmatrix} a_1^{(1)} & 1 & & & & \\ a_1^{(2)} & a_2^{(1)} & 1 & & & \\ \vdots & \vdots & \ddots & \ddots & & \\ a_1^{(p)} & a_2^{(p-1)} & \cdots & a_p^{(1)} & 1 & \\ & \ddots & \ddots & & \ddots & \ddots \end{pmatrix} , \; J_- = \begin{pmatrix} 0 & & & & \\ a_1^{(2)} & 0 & & & \\ \vdots & \vdots & \ddots & & \\ a_1^{(p)} & a_2^{(p-1)} & \cdots & 0 & \\ & \ddots & \ddots & & \ddots \end{pmatrix} , \tag{5}$$

and where J_- is the lower triangular part of J.

In this paper we don't distinguish between each operator and its matrical representation. Moreover, we underline the formal sense of the Lax pair expression. In fact, it could be that there is no open interval where all the entries of J are defined.

Definition 1. We say that J is a *solution* of (3) if its entries verify (3)-(4).

An important tool in the study of these systems is the sequence of polynomials $P_n(z) = P_n(t, z)$ associated with the matrix J, i.e., the polynomials defined by the following recurrence relation, given for $n = 0, 1, \dots$ by

$$\left.\begin{array}{l} \displaystyle\sum_{i=1}^{p-1} a_{n-p+i+1}^{(p-i+1)} P_{n-p+i}(z) + (a_{n+1}^{(1)} - z) P_n(z) + P_{n+1}(z) = 0 \\ P_0(z) = 1\,, \quad P_{-1}(z) = \cdots = P_{-p+1}(z) = 0\,. \end{array}\right\} \tag{6}$$

We will use the following well-known fact,

$$P_n(z) = \det\left(zI_n - J_n\right)\,, \quad n \in \mathbb{N}\,. \tag{7}$$

Here and in the sequel we denote by A_n the finite matrix formed with the first n rows and n columns of each infinite matrix A.

For each $p \in \mathbb{N}$, the system

$$\dot{\gamma}_n = \gamma_n \left(\sum_{i=1}^{p-1} \gamma_{n+i} - \sum_{i=1}^{p-1} \gamma_{n-i} \right)\,, \quad n \in \mathbb{N}\,, \tag{8}$$

called discrete Korteweg-de Vries equations, is an extension of the Volterra lattices (2) studied in [8] and [9] (see also [6]). As in (3)-(4), for each $n \in \mathbb{N}$ we assume that the functions of (8) continuous and defined in an open set $\mathcal{O}_n$ such that

$$\bigcap_{n=1}^{N} \mathcal{O}_n \neq \emptyset\,, \quad \text{for all } N \in \mathbb{N}\,. \tag{9}$$

The system (8) can be rewritten in Lax pair form, $\dot{\Gamma} = [\Gamma, \Gamma_-^p]$, where Γ is given by

$$\Gamma = \begin{pmatrix} 0 & 1 & & & \\ \vdots & \ddots & \ddots & & \\ 0 & \cdots & 0 & 1 & \\ \gamma_1 & 0 & \cdots & 0 & \ddots \\ & \ddots & \ddots & & \ddots \end{pmatrix}\,,$$

being γ_1 in the p-th row, and Γ^p_- is the lower triangular part of Γ^p.

In a different context, and considering Γ as a bounded matrix, this lattice was analyzed for the case $p = 3$ in [3].

Definition 2. We say that Γ is a *solution* of (8) if the sequence $\{\gamma_n\}$ verify (8)-(9).

For the sake of simplicity, in this work, we only consider a four banded matrix J corresponding to $p = 3$ in (3)-(5), but the method can be extended to higher order banded matrices J, as we explain in Section 4. In our particular case (3) becomes the following system,

$$\left.\begin{array}{rcl} \dot{a}_n & = & b_n - b_{n-1} \\ \dot{b}_n & = & b_n(a_{n+1} - a_n) + c_n - c_{n-1} \\ \dot{c}_n & = & c_n(a_{n+2} - a_n) \end{array}\right\} , \quad n \in \mathbb{N} , \tag{10}$$

where we assume $c_n \neq 0$ for each $n \in \mathbb{N}$. Now, the recurrence relation (6) is for $n = 0, 1, \ldots$, given by

$$\left.\begin{array}{l} c_{n-1}P_{n-2}(z) + b_n P_{n-1}(z) + (a_{n+1} - z)P_n(z) + P_{n+1}(z) = 0 \\ P_0(z) = 1 , \quad P_{-1}(z) = P_{-2}(z) = 0 \end{array}\right\} \tag{11}$$

and the matrices J, J_- are, respectively,

$$J = \begin{pmatrix} a_1 & 1 & & \\ b_1 & a_2 & 1 & \\ c_1 & b_2 & a_3 & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix} , \quad J_- = \begin{pmatrix} 0 & & & \\ b_1 & 0 & & \\ c_1 & b_2 & 0 & \\ & \ddots & \ddots & \ddots \end{pmatrix} . \tag{12}$$

We deal with solutions J of (10), i.e., matrices J as in (12) with entries a_n , b_n , c_n , defined in $\mathcal{I}_n$ for each $n \in \mathbb{N}$, verifying (10) and such that (4) holds for each $N \in \mathbb{N}$ (see Definition 1). Consequently, for each of these solutions, the polynomial P_n in (11) is defined in $\bigcap_{j=1}^n \mathcal{I}_j$.

We have the matrix

$$\Gamma = \begin{pmatrix} 0 & 1 & & \\ 0 & 0 & 1 & \\ \gamma_1 & 0 & 0 & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix} \tag{13}$$

and (8) becomes

$$\dot{\gamma}_n = \gamma_n \left(\gamma_{n+1} + \gamma_{n+2} - \gamma_{n-1} - \gamma_{n-2} \right) , \quad n \in \mathbb{N} . \tag{14}$$

Next we have our main result, where we obtain one solution of (14) and two new solutions of (10) from a given solution of (10). This theorem extends the corresponding results of [8, Th. 1] and [9, Section 2].

Theorem 1. *Let J be a solution of (10) and let $C \in \mathbb{C}$ be such that* $\det(J_n - CI_n) \neq 0$, *for all $n \in \mathbb{N}$. Then there exist two solutions $J^{(1)}$, $J^{(2)}$ of (10) given by*

$$J^{(i)} = \begin{pmatrix} a_1^{(i)} & 1 & & \\ b_1^{(i)} & a_2^{(i)} & 1 & \\ c_1^{(i)} & b_2^{(i)} & a_3^{(i)} & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix}, \quad i = 1, 2, \tag{15}$$

and there exists a solution Γ of (14) given by (13) such that the following relations are verified for each $n \in \mathbb{N}$.

$$\left.\begin{array}{lcl} a_n & = & \gamma_{3n-4} + \gamma_{3n-3} + \gamma_{3n-2} + C \\ b_n & = & \gamma_{3n-3}\gamma_{3n-1} + \gamma_{3n-2}\gamma_{3n-1} + \gamma_{3n-2}\gamma_{3n} \\ c_n & = & \gamma_{3n-2}\gamma_{3n}\gamma_{3n+2} \end{array}\right\} \tag{16}$$

$$\left.\begin{array}{lcl} a_n^{(1)} & = & \gamma_{3n-3} + \gamma_{3n-2} + \gamma_{3n-1} + C \\ b_n^{(1)} & = & \gamma_{3n-2}\gamma_{3n} + \gamma_{3n-1}\gamma_{3n} + \gamma_{3n-1}\gamma_{3n+1} \\ c_n^{(1)} & = & \gamma_{3n-1}\gamma_{3n+1}\gamma_{3n+3} \end{array}\right\} \tag{17}$$

$$\left.\begin{array}{lcl} a_n^{(2)} & = & \gamma_{3n-2} + \gamma_{3n-1} + \gamma_{3n} + C \\ b_n^{(2)} & = & \gamma_{3n-1}\gamma_{3n+1} + \gamma_{3n}\gamma_{3n+1} + \gamma_{3n}\gamma_{3n+2} \\ c_n^{(2)} & = & \gamma_{3n}\gamma_{3n+2}\gamma_{3n+4} \end{array}\right\}, \tag{18}$$

where we assume $\gamma_m = 0$ for $m \leq 0$.
Moreover, for $t_0 \in \mathbb{R}$ verifying the following conditions (19)-(20) we have that $\{\gamma_n\}$ is the unique sequence verifying (16) such that (17) and (18) define two solutions of (10):

(1) *If $b_1 \neq 0$ then*

$$\gamma_2 \neq \frac{-b_1}{C - a_1} \tag{19}$$

(2) *For each* $n = 2, 3, \ldots$ *such that* $\delta_n := c_{n-1}P_{n-2}(C) + b_n P_{n-1}(C) \neq 0$, *we have*

$$P_1(C)\gamma_2 \neq -b_1 + \cfrac{c_1 P_0(C) P_2(C)}{-\delta_2 + \cfrac{\ddots}{+\cfrac{c_{n-2}P_{n-3}(C)P_{n-1}(C)}{-\delta_{n-1} + \cfrac{c_{n-1}P_{n-2}(C)P_n(C)}{-\delta_n}}}}\,. \tag{20}$$

We underline that (19)-(20) are referred to $t = t_0$*.*

As a consequence of Theorem 1, in the next result we determine the relation between the solutions of (10) and the solutions of (14).

Theorem 2. *Let* J, Γ *be as in (12)-(13) verifying (16) and let* $C \in \mathbb{C}$ *be such that* $\det(J_n - CI_n) \neq 0$, $\forall n \in \mathbb{N}$. *Then* Γ *is a solution of (14) if, and only if,* J *is a solution of (10) and* $\gamma_2(t)$ *is the solution of the following initial value problem,*

$$\left.\begin{array}{rcl} \dot{y}(t) & = & y(t)(a_2 - a_1 - y(t)) \\ y(t_0) & = & \gamma_2(t_0) \end{array}\right\},$$

with $\gamma_2(t_0)$ *verifying (19)-(20).*

An important tool in the proof of Theorem 1 is the concept of LU-factorization for banded matrices (see [7]). Moreover, it is relevant the fact that some triangular matrices can be factorized using some more simple matrices as factors. This is showed in the following result, of independent interest. Despite of this it can be extended to arbitrary values $p \in \mathbb{N}$. For the sake of simplicity we assume $p = 3$. We underline that this result is verified by matrices whose entries do not depend, in general, on $t \in \mathbb{R}$.

Theorem 3. *We consider the following lower triangular matrix with complex entries,*

$$L = \begin{pmatrix} 1 & & & \\ \ell_{1,1} & 1 & & \\ \ell_{2,1} & \ell_{2,2} & 1 & \\ & \ddots & \ddots & \ddots \end{pmatrix} \tag{21}$$

such that $\ell_{j+1,j} \neq 0$ for each $j \in \mathbb{N}$. Then there exist two lower triangular matrices

$$L^{(i)} = \begin{pmatrix} 1 & & & \\ \gamma_{i+1} & 1 & & \\ & \gamma_{i+4} & 1 & \\ & & \ddots & \ddots \end{pmatrix}, \quad i = 1, 2\,, \tag{22}$$

such that

$$L = L^{(1)} L^{(2)}\,. \tag{23}$$

Moreover, for each fixed $\gamma_2 \in \mathbb{C}$, with $\gamma_2 \neq 0$, verifying the following conditions (24)-(25), the factorization (23) is unique:

(1) *If $\ell_{1,1} \neq 0$ then*

$$\gamma_2 \neq \ell_{1,1} \tag{24}$$

(2) *For each $n = 2, 3, \ldots$ such that $\ell_{n,n} \neq 0$ we have*

$$\gamma_2 \neq \ell_{1,1} - \cfrac{\ell_{2,1}}{\ell_{2,2} - \cfrac{\ddots}{\quad - \cfrac{\ell_{n-1,n-2}}{\ell_{n-1,n-1} - \cfrac{\ell_{n,n-1}}{\ell_{n,n}}}}}\,. \tag{25}$$

That is, if we have (23) and $\widetilde{L}^{(i)}$, $i = 1, 2$, are two lower triangular matrices like (22) such that the entries in the first column and second row of $\widetilde{L}^{(1)}$ and $L^{(1)}$ are γ_2 verifying (24)-(25) and $L = \widetilde{L}^{(1)} \widetilde{L}^{(2)}$, then $\widetilde{L}^{(i)} = L^{(i)}$, $i = 1, 2$.

Section 2 and Section 3 are devoted to prove, respectively, Theorem 1 and Theorem 2. The proof of Theorem 3 and an extension to four banded matrices of the concept of Darboux transformation are given in Section 4. Also a matrical interpretation of Theorem 1 and Theorem 2 is given in this last section.

2. Proof of Theorem 1

Since J is a solution of (10), for each $n \in \mathbb{N}$ the functions a_n, b_n, c_n, verifying (4) are defined in $\mathcal{I}_n$. Let $C \in \mathbb{C}$ be such that $\det(J_n - CI_n) \neq 0$ for all $n \in \mathbb{N}$.

As we have explained in the above section, in the sequel we assume that each polynomial $P_n = P_n(t, z)$ is defined for $t \in \mathcal{S}_m := \bigcap_{j=1}^m \mathcal{I}_j$. In particular,

$\det(J_n - CI_n) \neq 0$ means that C is not a zero of $P_n(t,z)$ for each fixed $t \in \mathcal{S}_n$ (see (7)).

It is well known that, under these conditions, for each $m \in \mathbb{N}$ there exist a lower triangular matrix $L_m = L_m(t)$ and an upper triangular matrix $U_m = U_m(t)$, being both matrices of order m, such that

$$J_m - CI_m = L_m U_m \tag{26}$$

(see for instance [5, Th. 1, pag. 35]). Moreover, for each fixed $m \in \mathbb{N}$ the main section of order m of L_{m+1} (respectively, U_{m+1}) is given by L_m (respectively, U_m). In this sense we can write

$$J - CI = L\,U\,, \tag{27}$$

understanding that (26) is verified for all $m \in \mathbb{N}$ in some open set $\mathcal{S}_m$. With our restrictions, L can be assumed a banded matrix like (21) and U is a two banded matrix,

$$U = \begin{pmatrix} \gamma_1 & 1 & & \\ & \gamma_4 & 1 & \\ & & \ddots & \ddots \end{pmatrix}. \tag{28}$$

Our first purpose is to define the sequence $\{\gamma_n(t)\}\,,\ n \in \mathbb{N}\,.$ First of all, we shall define the sequence $\{\gamma_{3n+1}(t)\}$ and, consequently, the matrix U.

Another way to write the recurrence relation (11) is

$$(J_n - zI_n)\begin{pmatrix} P_0(z) \\ P_1(z) \\ \vdots \\ P_{n-1}(z) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -P_n(z) \end{pmatrix}, \quad n \in \mathbb{N}\,. \tag{29}$$

Then, using (26) and (29) for $z = C$ we have

$$U_n \begin{pmatrix} P_0(C) \\ P_1(C) \\ \vdots \\ P_{n-1}(C) \end{pmatrix} = (L_n)^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -P_n(C) \end{pmatrix}, \quad n \in \mathbb{N}\,, \tag{30}$$

being $(L_n)^{-1}$ a lower triangular matrix whose diagonal entries are equal to 1. Comparing the entries corresponding to the last row in both sides of (30) we arrive to $\gamma_{3n-2}P_{n-1}(C) + P_n(C) = 0$, which is verified for any $n \in \mathbb{N}$. This is,

$$\gamma_{3n+1} = -\frac{P_{n+1}(C)}{P_n(C)}\,, \quad n \geq 0\,, \tag{31}$$

where (31) determines the matrix U gives in (28).

With the aim to determine L we fix $n \in \{-1, 0, 1, \ldots\}$ and we obtain the row $n+2$ in both sides of (27) (see (21)). Then

$$\begin{aligned} a_{n+2} - C &= \ell_{n+1,n+1} + \gamma_{3n+4} && (32)\\ b_{n+1} &= \ell_{n+1,n} + \ell_{n+1,n+1}\gamma_{3n+1} && (33)\\ c_n &= \ell_{n+1,n}\gamma_{3n-2} && (34)\end{aligned}$$

where we assume $\gamma_m = b_m = c_m = \ell_{m+1,m} = \ell_{m,m} = 0$ for $m \leq 0$. Since (34) and (31) we obtain

$$\ell_{n+1,n} = -\frac{c_n P_{n-1}(C)}{P_n(C)}, \quad n \in \mathbb{N}. \tag{35}$$

Moreover, from (32) and again (31),

$$\ell_{n,n} = a_{n+1} - C + \frac{P_{n+1}(C)}{P_n(C)}, \quad n \in \mathbb{N}. \tag{36}$$

On the other hand, we consider the next initial value problem,

$$\left.\begin{array}{rcl} \dot{x}(t) &=& (a_1(t) - a_2(t))\, x(t) + 1 \\ x(t_0) &=& x_0 \end{array}\right\}. \tag{37}$$

The unique solution of (37) is

$$x(t) = e^{-\int_{t_0}^t (a_2 - a_1)ds}\left[\int_{t_0}^t e^{\int_{t_0}^s (a_2-a_1)d\xi} ds + x_0\right],$$

and it is easy to check that $y = 1/x$ is a solution of

$$\dot{y}(t) = y(t)(a_2 - a_1 - y(t)). \tag{38}$$

Due the continuity of $x(t)$, if $x_0 \neq 0$ then there exists some interval $\mathcal{I}$, $t_0 \in \mathcal{I}$, such that $x(t) \neq 0$ for all $t \in \mathcal{I}$. Therefore we can obtain the solution of (38) in $\mathcal{I}$ with the initial condition $y(t_0) = 1/x(t_0)$.

If we take $\gamma_2 := y$, from (38) we arrive to (14) for $n = 2$. In this way, we have defined

$$\gamma_2 = \frac{1}{x} = \frac{e^{\int_{t_0}^t (a_2-a_1)ds}}{\int_{t_0}^t e^{\int_{t_0}^s (a_2-a_1)d\xi} ds + x(t_0)}$$

with an appropriate value of $x(t_0) \neq 0$.

In the sequel we assume $\gamma_2(t_0) = 1/x_0$ verifying the conditions (24)-(25) of Theorem 3 for the matrix $L(t_0)$. Obviously, this is possible because there is only a countable set of complex numbers on the right hand side of (25). Then, from this theorem we obtain

$$L(t_0) = L^{(1)}(t_0)L^{(2)}(t_0)\,,$$

being $L^{(i)}(t_0)\,,\ i = 1,2\,,$ as in (22). Due to this factorization we can express the entries of $L(t_0)$ in terms of the sequences $\{\gamma_{3n+2}(t_0)\}\,,\ \{\gamma_{3n+3}(t_0)\}\,.$ This is,

$$\left.\begin{array}{rcl} \gamma_{3n+2}(t_0)\gamma_{3n}(t_0) & = & \ell_{n+1,n}(t_0) \\ \gamma_{3n-1}(t_0)+\gamma_{3n}(t_0) & = & \ell_{n,n}(t_0) \end{array}\right\}\,, \quad n = 1,2,\ldots\,.$$

Let $N \in \mathbb{N}$ be fixed. Since the continuity of the solutions of (10) given by J we know that there exists an open set, which we again denote $\mathcal{S}_N$, such that $\mathcal{S}_N \subseteq \mathcal{I}$ and (24)-(25) hold for $t \in \mathcal{S}_N$ and $n = 1,2,\ldots,N$, being $t_0 \in \mathcal{S}_N$. Then we assume that the entries a_n, b_n, c_n of J, and also the polynomials, are defined in $\mathcal{S}_N$ for $n \leq N$. We shall obtain the new solutions of (10) and (14) in $\mathcal{S}_N$ for $n \leq N$.

We have defined $\{\gamma_{3n+1}(t)\}$ in (31). Moreover we know the existence of $L^{(i)}(t)\,,\ t \in \mathcal{S}_N\,,\ i = 1,2\,,$ as in Theorem 3. This define the sequence $\gamma_n(t)$ for $n = 1,2,\ldots,3N+2$ verifying

$$\gamma_{3n+2}\gamma_{3n} = -\frac{c_n P_{n-1}(C)}{P_n(C)}\,, \quad n = 1,2,\ldots N\,, \tag{39}$$

$$\gamma_{3n-1}+\gamma_{3n} = -\frac{c_{n-1}P_{n-2}(C)+b_nP_{n-1}(C)}{P_n(C)}\,, \quad n = 1,2,\ldots N. \tag{40}$$

for $t \in \mathcal{S}_N$ (see (35) and (36)). Moreover, an immediate consequence of (31) and (39) is $\gamma_n(t) \neq 0\,,\ n = 1,2,\ldots,3N+2$ in this open set $\mathcal{S}_N$.

We shall prove that these functions $\gamma_1,\ldots,\gamma_{3N+2}$ are solutions of (14) for $t \in \mathcal{I}_N$. Taking $n = 1$ in (40),

$$\gamma_2+\gamma_3 = -\frac{b_1}{P_1(C)}$$

and taking $n = 0\,,\ n = 1\,,$ respectively in (31) we see

$$\gamma_1 = -P_1(C)\,, \quad \gamma_4 = a_2 - C + b_1/P_1(C)\,. \tag{41}$$

Then

$$\gamma_4 = a_2 - a_1 - (C - a_1) - \gamma_2 - \gamma_3\,.$$

Therefore $\gamma_3+\gamma_4-\gamma_1 = a_2-a_1-\gamma_2$ and we arrive to (14) for $n = 2$. For $n = 1$, (14) is directly checked taking derivatives in (41) because $\dot{a}_1 = b_1$ (see (10)).

Now, taking derivatives in $\gamma_3 = \ell_{1,1} - \gamma_2$ and taking into account (14) for $n = 2$ and $\ell_{1,1} = \gamma_3 (\gamma_4 + \gamma_5 - \gamma_1)$ we arrive to (14) for $n = 3$. That is, (14) holds for $n = 1, 2, 3$.

The next auxiliary result is an immediate consequence of [4, Theorem 1]. It will be used to prove that $\Gamma(t)$ is a solution of (14).

Lemma 1. *If $J(t)$ is a solution of (10) then we have for all $n = 0, 1, \ldots,$*

$$\dot{P}_n = -c_{n-1} P_{n-2} - b_n P_{n-1} \, .$$

In fact, in a little more general way, it is easy to prove by induction the following result.

Lemma 2. *If $J(t)$ is a solution of (3) then we have for all $n = 0, 1, \ldots,$*

$$\dot{P}_n = -a^{(p)}_{n-p+2} P_{n-p+1} - \cdots - a^{(3)}_{n-1} P_{n-2} - a^{(2)}_n P_{n-1} \, .$$

Since (31), (40), for $n \le N$ and Lemma 1 we have

$$\begin{aligned}
\dot{\gamma}_{3n+1} &= \frac{d}{dt}\left(-\frac{P_{n+1}(C)}{P_n(C)}\right) \\
&= \frac{P_{n+1}(C)}{P_n(C)} \frac{\dot{P}_{n+1}(C) P_n(C) - P_{n+1}(C) \dot{P}_n(C)}{P_n(C) P_{n+1}(C)} \\
&= -\frac{P_{n+1}(C)}{P_n(C)}\left(-\frac{c_n P_{n-1}(C) + b_{n+1} P_n(C)}{P_{n+1}(C)} + \frac{c_{n-1} P_{n-2}(C) + b_n P_{n-1}(C)}{P_n(C)}\right) \\
&= \gamma_{3n+1} \left(\gamma_{3n+2} + \gamma_{3n+3} - \gamma_{3n} - \gamma_{3n-1}\right) , \quad n \le N , \quad t \in \mathfrak{I}_N \, .
\end{aligned}$$

Then (14) is verified for the subsequence $\{\gamma_{3n+1}\}$ of $\{\gamma_n\}$.

Now we proceed by induction, assuming proved (14) for $n = 1, 2, \ldots, 3k$, $k < N$. From (39)-(40), for $n \le N$, and (32)-(36), for $n \in \mathbb{N}$, we arrive to the expression of a_n and b_n in (16) for $n \le N + 1$, as well as the expression of c_n for $n \le N$. Then

$$\gamma_{3k+2} = \frac{c_k}{\gamma_{3k-2} \gamma_{3k}} \, .$$

Taking derivatives and taking into account (10) for $n = k$ and (14) for $n = 3k - 2, 3k$, we arrive to (14) for $n = 3k + 2$. Now, using the expression of a_{k+2} in (16), and taking into account that $k + 2 \le N + 1$,

$$\gamma_{3k+3} = a_{k+2} - \gamma_{3k+2} - \gamma_{3k+4} - C \, .$$

Taking derivatives in the above expression, using (10) for $n = k+2$ and (14) for $n = 3k+2, 3k+4 = 3(k+1)+1$, and also the expression of b_n, $n = k+1, k+2$, in (16), we arrive to (14) for $n = 3k+3$, as we wanted to show.

For each fixed $N \in \mathbb{N}$ we have defined γ_n in $\mathcal{S}_N$ for $n = 1, 2, \ldots, 3N+2$. Then we have defined the solution Γ of (14).

Remark . We underline that the existence of a solution $\Gamma(t)$ of (14) verifying (16) is independent on the solutions $J^{(i)}(t)$, $i = 1, 2$, of (10). We obtain $\Gamma(t)$ by constructing the sequence $\{\gamma_n\}$ with additional conditions for γ_2. We will use this fact in the proof of Theorem 2.

Now, we define the matrices $J^{(i)}$, $i = 1, 2$, as in (15), where the sequences $\{a_n^{(i)}, b_n^{(i)}\}$, $n \in \mathbb{N}$, $i = 1, 2$, are given in (17)-(18). It is easy to see, after some direct computations, that these sequences verify (10), i.e., are solutions of the full Kostant-Toda lattice.

Finally, we show the uniqueness of the sequence $\{\gamma_n\}$ in the given conditions.

Assume that $\{\widetilde{\gamma}_n\}$ is another sequence verifying (16) such that (19)-(20) hold in $t = t_0$. Taking $n = 1$ in (16) we see $\widetilde{\gamma}_1 = \gamma_1 = a_1$. Moreover, from (16) and (11),

$$\begin{aligned}\widetilde{\gamma}_{3n-5}\widetilde{\gamma}_{3n-3}\widetilde{\gamma}_{3n-1}P_{n-2}(C) + \left(\widetilde{\gamma}_{3n-3}\widetilde{\gamma}_{3n-1} + \widetilde{\gamma}_{3n-2}\widetilde{\gamma}_{3n-1} + \widetilde{\gamma}_{3n-2}\widetilde{\gamma}_{3n}\right) P_{n-1}(C)\\ + \left(\widetilde{\gamma}_{3n-1} + \widetilde{\gamma}_{3n} + \widetilde{\gamma}_{3n+1}\right) P_n(C) + P_{n+1}(C) = 0\,.\end{aligned}$$

We can reorder the above expression as

$$\begin{aligned}&\widetilde{\gamma}_{3n-3}\widetilde{\gamma}_{3n-1}\left(\widetilde{\gamma}_{3n-5}P_{n-2}(C) + P_{n-1}(C)\right) +\\ &\left(\widetilde{\gamma}_{3n-1} + \widetilde{\gamma}_{3n}\right)\left(\widetilde{\gamma}_{3n-2}P_{n-1}(C) + P_n(C)\right) + \left(\widetilde{\gamma}_{3n+1}P_n(C) + P_{n+1}(C)\right) = 0\,. \end{aligned} \tag{42}$$

Taking $\alpha_{-2} = 0$, $\alpha_{-1} = 1$ and $\alpha_n = \widetilde{\gamma}_{3n+1}P_n(C) + P_{n+1}(C)$, $n = 0, 1, \ldots$, we can write (42) as

$$\left.\begin{array}{l}\widetilde{\gamma}_{3n-3}\widetilde{\gamma}_{3n-1}\alpha_{n-2} + \left(\widetilde{\gamma}_{3n-1} + \widetilde{\gamma}_{3n}\right)\alpha_{n-1} + \alpha_n = 0\,, \quad n = 0, 1, \ldots\\ \alpha_{-2} = 0\,, \quad \alpha_{-1} = 1\end{array}\right\}.$$

Because the unique solution of the above recurrence relation is $\alpha_n = 0$, $n = 0, 1, \ldots$, from the definition given for α_n we arrive to

$$\widetilde{\gamma}_{3n+1} = -\frac{P_{n+1}(C)}{P_n(C)} = \gamma_{3n+1}\,, \quad n = 0, 1, \ldots\,. \tag{43}$$

Using (43) and (16),

$$\widetilde{\gamma}_{3n-1}+\widetilde{\gamma}_{3n}=a_{n+1}+\frac{P_{n+1}(C)}{P_n(C)}-C\,,$$

and so (see (11)),

$$\widetilde{\gamma}_{3n-1}+\widetilde{\gamma}_{3n}=-\frac{c_{n-1}P_{n-2}(C)+b_nP_{n-1}(C)}{P_n(C)}\,,\quad n=1,2,\ldots\,.$$

Now, if we take c_n in (16) expressed in terms of $\{\tilde{\gamma}_n\}$ using again (43) we arrive to

$$\widetilde{\gamma}_{3n+2}\widetilde{\gamma}_{3n}=-\frac{c_nP_{n-1}(C)}{P_n(C)}\,.$$

That is, the sequence $\{\widetilde{\gamma}_n\}$ also verifies (39)-(40) in some neighborhood of t_0.

Moreover, if we assume that, using $\{\widetilde{\gamma}_n\}$, we can define in (17) a new solution of (10), then

$$a_1^{(1)}=\widetilde{\gamma}_1+\widetilde{\gamma}_2\,,\quad a_1=\widetilde{\gamma}_1\,,$$

and therefore $\widetilde{\gamma}_2=a_1^{(1)}-a_1$. Taking derivatives in this expression, and taking into account (16)-(17), we arrive to

$$\dot{\widetilde{\gamma}}_2=\widetilde{\gamma}_2(a_2-a_1-\widetilde{\gamma}_2)\,.$$

That is, $\widetilde{\gamma}_2$ is a solution of (38). Then, with the initial condition $\widetilde{\gamma}_2(t_0)=\gamma_2(t_0)$, from the uniqueness of the solution of (37) we deduce $\widetilde{\gamma}_2=\gamma_2$ for some interval $]t_0-\widetilde{\delta},t_0+\widetilde{\delta}[$. We are assuming (19)-(20) for $\widetilde{\gamma}_2$ in $t=t_0$, which is equivalent, in our case, to (24)-(25). In the above construction of $\{\gamma_n\}$ we also assumed (24)-(25) for γ_2 in $t=t_0$ and, under these conditions, we have obtained the unique possible solution $\{\gamma_n\}$ of (14) verifying (16) such that (17)-(18) define two new solutions of (10). Then we deduce that $\{\tilde{\gamma}_n\}$ and $\{\gamma_n\}$ are the same solution of (14). ■

3. Proof of Theorem 2

Firstly, in the required conditions assume that $\Gamma(t)$ is a solution of (14). Taking derivatives in (16) we see that $J(t)$ is a solution of (10). Moreover, the condition $\dot{\gamma}_2=\gamma_2(a_2-a_1-\gamma_2)$ is given in (14) when $n=2$.

Reciprocally, assume that $J(t)$ is a solution of (10) such that $\dot{\gamma}_2=\gamma_2(a_2-a_1-\gamma_2)$ and $\gamma_2(t_0)$ verifies (19)-(20). From Theorem 1 we know that there exists a solution $\{\widetilde{\gamma}_n\}$ of (14) such that (16)-(18) hold. Then, being $\{\gamma_n\}$ the sequence of entries of $\Gamma(t)$, for proving that $\Gamma(t)$ is a solution of (14) it is enough to show that $\{\widetilde{\gamma}_n\} = \{\gamma_n\}$. But this is deduced from the uniqueness of $\{\widetilde{\gamma}_n\}$ when we take $\widetilde{\gamma}_2(t_0) = \gamma_2(t_0)$. ■

We underline that, in the proof of Theorem 1, the uniqueness of $\{\gamma_n\}$ does not depend on (17) and (18), being a consequence of (16) and the conditions assumed for γ_2.

4. Proof of Theorem 3: Matrical interpretation of Theorem 1

We will find two sequences $\{\gamma_{3n-1}\}$, $\{\gamma_{3n}\}$, $n \in \mathbb{N}$, defining the matrices $L^{(1)}$ and $L^{(2)}$ as in (22), such that (23) holds. That is, comparing both hand sides in (23),

$$\left.\begin{array}{rcl} \ell_{j,j} & = & \gamma_{3j-1} + \gamma_{3j} \\ \ell_{j+1,j} & = & \gamma_{3j}\gamma_{3j+2} \end{array}\right\}, \quad j = 1, 2, \ldots . \tag{44}$$

Let $\gamma_2 \neq 0$, $\gamma_2 \in \mathbb{C}$, be verifying (24)-(25). For $j = 1$ in (44) we want

$$\left.\begin{array}{rcl} \ell_{1,1} & = & \gamma_2 + \gamma_3 \\ \ell_{2,1} & = & \gamma_3\gamma_5 \end{array}\right\}. \tag{45}$$

Hence, we can define $\gamma_3 = \ell_{1,1} - \gamma_2$. Obviously, if $\ell_{1,1} = 0$ then we have $\gamma_3 = -\gamma_2 \neq 0$, but if $\ell_{1,1} \neq 0$ then from (24) we also arrive to $\gamma_3 \neq 0$. Therefore, we can also define $\gamma_5 = \ell_{2,1}/\gamma_3$, being $\gamma_5 \neq 0$, and (45) is verified.

Assume that γ_{3j}, γ_{3j+2}, $j = 1, 2, \ldots, m$, are not zero and defined as in (44). Then we define

$$\gamma_{3m+3} = \ell_{m+1,m+1} - \gamma_{3m+2} \tag{46}$$

and we arrive to the first part of (44) for $j = m + 1$.

If $\gamma_{3m+3} \neq 0$, then we can define

$$\gamma_{3m+5} = \frac{\ell_{m+2,m+1}}{\gamma_{3m+3}},$$

and we have $\gamma_{3m+5} \neq 0$. In this way we arrive to the second part of (44) for $j = m+1$. Consequently, we would have defined the sequences $\{\gamma_{3n-1}\}$, $\{\gamma_{3n}\}$ such that (44) and (23) are verified. Therefore, it is enough to show $\gamma_{3m+3} \neq 0$, in order to finish the proof of Theorem 3. Now, we will show that the following expression for γ_2, take place

$$\gamma_2 = \ell_{1,1} - \cfrac{\ell_{2,1}}{\ell_{2,2} - \cfrac{\ddots}{-\cfrac{\ell_{j,j-1}}{\ell_{j,j} - \gamma_{3j}}}} \,, \quad j = 1, 2, \ldots, m+1 \,, \tag{47}$$

where we understand $\gamma_2 = \ell_{1,1} - \gamma_3$ in the case $j = 1$. Indeed, (47) is contained in (44) for $j = 1$. If we assume that (47) holds true for $j \leq m-1$, because we are also assuming (44) with $j \leq m$, we have

$$\gamma_{3j} = \frac{\ell_{j+1,j}}{\gamma_{3j+2}} \,. \tag{48}$$

Moreover, since (46) we know that the first part of (44) is also verified in $m+1$. Then

$$\gamma_{3j+2} = \ell_{j+1,j+1} - \gamma_{3j+3} \,.$$

From this and (48),

$$\gamma_{3j} = \frac{\ell_{j+1,j}}{\ell_{j+1,j+1} - \gamma_{3j+3}} \,.$$

Substituting in (47) we obtain the desired expression for $j+1$.

Now, if $\ell_{m+1,m+1} = 0$, taking into account (46) we have $\gamma_{3m+3} \neq 0$. On the other hand, if $\ell_{m+1,m+1} \neq 0$, then comparing (47) for $j = m+1$ and (25) for $n = m+1$, we arrive to $\gamma_{3m+3} \neq 0$. ■

Theorem 3 provides the key to understanding the relation between Theorem 1 and the Darboux transformation. With the above notation, relations (16) can be written as

$$J - CI = L^{(1)} L^{(2)} U \,.$$

Moreover (17)-(18) can be rewritten, respectively, as

$$J^{(1)} - CI = L^{(2)} U L^{(1)} \,, \quad J^{(2)} - CI = U L^{(1)} L^{(2)} \,. \tag{49}$$

In general, we do not have uniqueness for the matrices $L^{(1)}$ and $L^{(2)}$ such that $L = L^{(1)} L^{(2)}$, but for each $\gamma_2(t_0)$ in the conditions of Theorem 3 determines this factorization. In this way, Theorem 1 provides two new solutions (49) of (10) given by the circular permutations of $L^{(1)}, L^{(2)}, U$. This fact extends the results of [8] and [9], where the initial solution and the obtained solution of (10) are related by the Darboux transformation (see [2]). That is, in the case of a tridiagonal matrix J we have $J - CI = L\,U$, and the new solution is given by $J^{(1)} - CI = UL$.

In the following, we extend the concept of Darboux transformation.

Definition 3. Let $B = (b_{i,j})$, $i, j \in \mathbb{N}$, be a lower Hessenberg banded matrix,

$$B = \begin{pmatrix} b_{1,1} & 1 & & & & \\ b_{2,1} & b_{2,2} & 1 & & & \\ \vdots & \vdots & \ddots & \ddots & & \\ b_{p,1} & b_{p,2} & \cdots & b_{p,p} & 1 & \\ & \ddots & \ddots & & \ddots & \ddots \end{pmatrix},$$

verifying $\det(B_n) \neq 0$ for any $n \in \mathbb{N}$. Let L, U be, respectively two lower and upper triangular matrices, such that the entries in the diagonal of L are $\ell_{i,i} = 1$ and $B = L\,U$ is the unique $L\,U$-factorization of B in these conditions. Assume $L = L^{(1)} L^{(2)} \cdots L^{(p-1)}$, where for $i = 1, \ldots, p-1$,

$$U = \begin{pmatrix} \gamma_1 & 1 & & \\ & \gamma_{p+1} & 1 & \\ & & \ddots & \ddots \end{pmatrix}, \quad L^{(i)} = \begin{pmatrix} 1 & & & \\ \gamma_{i+1} & 1 & & \\ & \gamma_{p+i+1} & 1 & \\ & & \ddots & \ddots \end{pmatrix}. \tag{50}$$

We say that any circular permutation of $L^{(1)} L^{(2)} \cdots L^{(p-1)} U$ is a *Darboux transformation* of B.

With some more computations and the extension of the method used in Section 3, Theorem 1 would be proved for arbitraries values of p. In this case we have $p-1$ new solutions $J^{(i)} - CI$, $i = 1, 2, \ldots, p-1$, of (3) given by the Darboux transformations of

$$J - CI = L^{(1)} L^{(2)} \cdots L^{(p-1)} U, \tag{51}$$

this is,

$$J^{(i)} - CI = L^{(i+1)} \cdots L^{(p-1)} U L^{(1)} \cdots L^{(i)},$$

where $U, L^{(i)}$, $i = 1, 2, \ldots, p-1$, are given in (50) and the sequence $\{\gamma_n\}$ provides a solution of (8). If we fix the initial conditions $\gamma_2(t_0), \ldots, \gamma_{p-1}(t_0)$ verifying the adequate restrictions, then $\{\gamma_n\}$ is the unique sequence verifying (51) such that the Darboux transformations of (51) define $p-1$ new solutions of (3).

References

[1] A.I. Aptekarev and A. Branquinho, F. Marcellán *Toda type differential equation for the recurrence coefficients of orthogonal polynomials and Freud transformations*, J. Comput. Appl. Math. 78 (1997), 139-160.

[2] D. Barrios Rolanía and A. Branquinho, *Complex high order Toda lattices*, J. Difference Equations and Applications, 15 (2) (2009), 197-213.

[3] D. Barrios Rolanía, A. Branquinho, and A. Foulquié Moreno, *Dynamics and interpretation of some integrable systems via multiple orthogonal polynomials*, J. Math. Anal. Appl., 361(2) (2010), 358-370.

[4] D. Barrios Rolanía, A. Branquinho, A. Foulquié Moreno, *On the relation between the full Kostant-Toda lattice and multiple orthogonal polynomials*, J. Math. Anal. Appl., 377 (2011), 228-238.

[5] F. R. Gantmacher, The Theory of Matrices, Vol. 1, AMS Chelsea Pub., Providence, Rhode Island, 1998.

[6] C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, *Method for solving the Korteweg-de Vries equation*, Phys. Rev. Lett. 19 (1967), 1095-1097.

[7] E. Isaacson, H. Bishop Keller, Analysis of Numerical Methods, John Wiley & Sons, Inc., New York, 1994.

[8] F. Peherstorfer, *On Toda lattices and orthogonal polynomials*, J. Comput. Appl. Math. 133 (2001), 519-534.

[9] F. Gesztesy, H. Holden, B. Simon, Z. Zhao, *On the Toda and Kac-van Moerbeke systems*, Trans. Am. Math. Soc. 339(2) (1993), 849- 868.

D. Barrios Rolanía
Facultad de Informática, Universidad Politécnica de Madrid, 28660 Boadilla del Monte, Madrid, Spain.

E-mail address: dbarrios@fi.upm.es

A. Branquinho
CMUC, Department of Mathematics, University of Coimbra, Largo D. Dinis, 3001-454 Coimbra, Portugal.

E-mail address: ajplb@mat.uc.pt

A. Foulquié Moreno
CIDMA, Departamento de Matemática, Universidade de Aveiro, Campus de Santiago 3810, Aveiro, Portugal.

E-mail address: foulquie@ua.pt