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HOM-LIE ALGEBROIDS

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ABSTRACT: We define and give examples of hom-Lie algebroids. Some important properties are outlined.
(Français) Nous introduisons les hom-Lie algébroïdes de Lie, donnons des exemples et soulignons certaines propriétés algébriques de celles-ci.

1. Résumé en français

Un intérêt croissant pour les hom-structures, en particulier les hom-algèbre de Lie suivit [HLS06], qui fit apparaître celles-ci de façon naturelle dans l'étude des cocycles de Virasoro. Makhlouf, Silvestrov et leurs coauteurs [MS1] approfondirent ces études en dégageant l'idée générale que maintes structures algébriques admettent de très naturelles généralisations lorsque l'on suppose donné au plus d'un simple espace vectoriel, un endomorphisme de cet espace supposé préservé les structures introduites. Ainsi furent étudiées des hom-algèbres associatives [MS1], de Jordan [MS4], admissible [MS2], hom-Poisson [MS3] entre autre. Notre but est d'ajouter les hom-algébroïdes de Lie à cette suite.

Une **hom-algèbre de Lie** est un triplet $(\mathfrak{g}, [\,,\,], \alpha)$ où $\mathfrak{g}$ est une espace vectoriel $[\,,\,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ une application bilinéaire et α un endomorphisme linéaire de $\mathfrak{g}$ tel que :

$$[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0$$

$$\text{et } \alpha([x, y]) = [\alpha(x), \alpha(y)], \quad \forall x, y, z \in \mathfrak{g}.$$

Définition 1.1. Une **hom-algèbre de Gerstenhaber** est un quadruplet $(\mathcal{A} = \oplus_{i\in\mathbb{N}}\mathcal{A}_i, \wedge, [\![\,,\,]\!], \alpha)$ où $(\mathcal{A} = \oplus_{i\in\mathbb{N}}\mathcal{A}_i, \wedge)$ est une algèbre commutative associative graduée, α un endomorphisme linéaire de $(\mathcal{A}, \wedge)$ de degré 0 et $[\![\,,\,]\!]$ une application bilinéaire de degré -1 telle que $(\mathcal{A}[-1], [\![\,,\,]\!], \alpha)$ soit une

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hom-algèbre de Lie et vérifiant la règle de **hom-Leibniz** suivante :

$$[\![X, Y \wedge Z]\!] = [\![X, Y]\!] \wedge \alpha(Z) + (-1)^{(i-1)j} \alpha(Y) \wedge [\![X, Z]\!], \ \forall X \in \mathcal{A}_i, Y \in \mathcal{A}_j, Z \in \mathcal{A}.$$

Définition 1.2. Une **hom-algébroïde de Lie** est un quintuplet $(A \to M, \varphi, [\,,\,], \rho, \alpha)$, où $A \to M$ est un fibré vectoriel au-dessus d' une variété M, $\varphi : M \to M$ est une application lisse, $[\,,\,] : \Gamma(A) \otimes \Gamma(A) \to \Gamma(A)$ est une application bilinéaire, appelée **crochet**, $\rho : \varphi^! A \to \varphi^! TM$ est un morphisme de fibré vectoriel, appelé **ancre**, et $\alpha : \Gamma(A) \to \Gamma(A)$ un endomorphisme linéaire, tels que

1. $\alpha(FX) = \varphi^*(F)\alpha(X)$ pour tous $X \in \Gamma(A), F \in C^\infty(M)$;
2. le triplet $(\Gamma(A), [\,,\,], \alpha)$ est une hom-algèbre de Lie;
3. l'identité de hom-Leibniz suivante est satisfaite :

$$[X, FY] = \varphi^*(F)[X, Y] + \rho(X)[F]\, \alpha(Y), \quad \forall X, Y \in \Gamma(A), F \in C^\infty(M).$$

4. (ρ, φ^*) est une représentation (au sens des hom-algèbres de Lie) de $(\Gamma(A), [\,,\,], \alpha)$ sur $C^\infty(M)$.

Théorème 1. Soit $A \to M$ un fibré vectoriel, $\varphi : M \to M$ une application lisse, $\alpha : \Gamma(\wedge^\bullet A) \to \Gamma(\wedge^\bullet A)$ un endomorphisme linéaire vérifiant $\alpha(FX) = \varphi^*(F)\alpha(X)$. Une correspondence biunivoque entre hom-algèbres de Gerstenhaber de la forme $(\Gamma(\wedge^\bullet A), \wedge, [\![\,,\,]\!], \alpha)$ et hom-algébroïdes de Lie de la forme $(A \to M, \varphi, [\,,\,], \rho, \alpha)$ est obtenue comme suit :

(1) A une hom-algèbre de Gerstenhaber $(\Gamma(\wedge^\bullet A), \wedge, [\![\,,\,]\!], \alpha)$, on associe un crochet $[\,,\,]$ sur $\Gamma(A)$ par restriction de $[\![\,,\,]\!]$ à $\Gamma(A)$ et une ancre $\rho : \varphi^! A \to \varphi^! TM$ définie comme étant l'unique morphisme de fibré vectoriel vérifiant $\rho(X)[F] := [\![X, F]\!]$ pour tous $X \in \Gamma(A), F \in C^\infty(M)$.

(2) Inversement, à une hom-algébroïde de Lie $(A \to M, \varphi, [\,,\,], \rho, \alpha)$, on associe un hom-crochet de Gerstenhaber sur $\Gamma(\wedge^\bullet A)$ par la formule suivante, pour tous $X_1, \ldots, X_p, Y_1, \ldots, Y_q \in \Gamma(A)$, :

$$[\![X_1 \wedge \cdots \wedge X_p, Y_1 \wedge \cdots \wedge Y_q]\!] = \\ = \sum_{i=1}^{p} \sum_{j=1}^{q} (-1)^{i+j} [X_i, Y_j] \wedge \alpha(X_1 \wedge \ldots \widehat{X_i} \wedge \cdots \wedge X_p \wedge Y_1 \wedge \ldots \widehat{Y_j} \wedge \cdots \wedge Y_q).$$

Ce théorème permet de construire plusieurs nouveaux exemples, en particulier d'associer une hom-algébroïde de Lie à une variété de hom-Poisson. Les hom-algèbres de Lie et leurs représentations donnent des exemples de hom-algèbres de Gerstenhaber, mais aussi des exemples de hom-algébroïdes de Lie.

2. Introduction

There is an increasing interest for hom-structures, especially hom-Lie algebras, initiated by [HLS06], who showed natural occurrence of this notion while studying cocycles of the Virasoro algebra, then followed by Makhlouf, Silvestrov and their coauthors [MS1], who showed that several classical algebraic structures admit natural generalizations when, instead of just a vector space, we start with a vector space and an automorphism of it, leading to investigate hom-associative algebras [MS1], hom-Jordan algebras [MS4], admissible algebras [MS2], hom-Poisson algebras [MS3] to cite a few.

Our purpose is to introduce hom-Lie algebroids. It is not straightforward at all to see what this definition should be: there is no such a thing as a hom-Lie groupoid that could give us a hint. To derive a definition that makes sense, we indeed had to go through the notion of Gerstenhaber algebra, but even there there was an unexpected phenomenon, for a hom-Gerstenhaber algebra is not a hom-associative, as one could have expected, hence defining a hom-Lie algebroid does not reduce simply adding the prefix hom- to classical definitions and results in a systematic manner.

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3. Hom-Lie algebras and hom-Poisson algebras

Given $\mathfrak{g}$ a vector space and a bilinear map $[\,,\,] : \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}$, we call **automorphism of** $(\mathfrak{g}, [\,,\,])$ a linear map $\alpha : \mathfrak{g} \to \mathfrak{g}$ such that $\alpha([x, y]) = [\alpha(x), \alpha(y)]$ for all $x, y \in \mathfrak{g}$.

Definition 3.1. [HLS06] A **hom-Lie algebra** is a triple $(\mathfrak{g}, [\,,\,], \alpha)$ with $\mathfrak{g}$ a vector space equipped with a skew-symmetric bilinear map $[\,,\,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ and an automorphism α of $(\mathfrak{g}, [\,,\,])$ such that:

$$[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0, \ \forall x, y, z \in \mathfrak{g},$$

(hom-Jacobi identity).

A **morphism** between hom-Lie algebras $(\mathfrak{g}, [\,,\,]_{\mathfrak{g}}, \alpha)$ and $(\mathfrak{h}, [\,,\,]_{\mathfrak{h}}, \beta)$ is a linear map $\psi : \mathfrak{g} \to \mathfrak{h}$ such that $\psi([x, y]_{\mathfrak{g}}) = [\psi(x), \psi(y)]_{\mathfrak{h}}$ and $\psi(\alpha(x)) = \beta(\psi(x))$ for all $x, y \in \mathfrak{g}$. When $\mathfrak{h}$ is a vector subspace of $\mathfrak{g}$ and ψ is the inclusion map, one speaks of **hom-Lie subalgebra**.

In a similar fashion, one defines **graded hom-Lie algebras** to be triples $(\mathfrak{g}, [\,,\,], \alpha)$ with $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}_i$ a graded vector space, $[\,,\,] : \mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$ a graded skew-symmetric bilinear map of degree 0 and $\alpha : \mathfrak{g}\to\mathfrak{g}$ an automorphism of $(\mathfrak{g}, [\,,\,])$ of degree 0 satisfying for all $x\in\mathfrak{g}_i, y\in\mathfrak{g}_j, z\in\mathfrak{g}_k$:

$$(-1)^{ik}\left[\alpha(x), [y,z]\right] + (-1)^{ji}\left[\alpha(y), [z,x]\right] + (-1)^{kj}\left[\alpha(z), [x,y]\right] = 0,$$

(graded hom-Jacobi identity).

Examples 3.2. a. Given a vector space $\mathfrak{g}$ equipped with a skew-symmetric bilinear map $[\,,\,] : \mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$ and an automorphism $\alpha : \mathfrak{g}\to\mathfrak{g}$ of $(\mathfrak{g}, [\,,\,])$, define $[\,,\,]_\alpha : \mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$ by

$$[x,y]_\alpha = \alpha([x,y]), \quad \forall x,y\in\mathfrak{g}.$$

Then $(\mathfrak{g}, [\,,\,]_\alpha, \alpha)$ is a hom-Lie algebra if and only if the restriction of $[\,,\,]$ to the image of α^2 is a Lie bracket. In particular, hom-Lie structures are naturally associated to Lie algebras equipped with a Lie algebra automorphism [Yau1]. Such hom-Lie structures are said to be **obtained by composition**.
b. [MS1] For every hom-associative algebra (A, μ, α) (see definition 3.3 below), the triple $(A, [\,,\,], \alpha)$ is a a hom-Lie algebra, where

$$[x,y] := \mu(x,y) - \mu(y,x), \text{ for all } x,y\in A.$$

Definition 3.3. [MS1] A **hom-associative algebra** is a triple (A, μ, α) consisting of a vector space A, a bilinear map $\mu : A\otimes A\to A$ and an automorphism α of (A, μ) satisfying

$$\mu(\alpha(x), \mu(y,z)) = \mu(\mu(x,y), \alpha(z)), \quad \forall x,y,z\in A \text{ (hom-associativity).}$$

As in example 3.2 **a.**, given (A, μ) an associative algebra and $\alpha : A\to A$ an algebra automorphism, the triple $(A, \mu_\alpha := \alpha\circ\mu, \alpha)$ is a hom-associative algebra, said again to be **obtained by composition**.

Definition 3.4. [MS3] A **hom-Poisson** algebra is a quadruple $(A, \mu, \{\,,\,\}, \alpha)$ consisting of a vector space A, bilinear maps $\mu : A\otimes A\to A$ and $\{\,,\,\} : A\otimes A\to A$ and a linear map $\alpha : A\to A$ such that

(1) (A, μ, α) is a commutative hom-associative algebra,
(2) $(A, \{\,,\,\}, \alpha)$ is a hom-Lie algebra,
(3) $\{\alpha(x), \mu(y,z)\} = \mu(\alpha(y), \{x,z\}) + \mu(\{x,y\}, \alpha(z)), \quad \forall x,y,z\in A.$

Example 3.5. [Yau2] Let $(A, \mu, \{\,,\,\})$ be a Poisson algebra and $\alpha : A \to A$ a Poisson automorphism, then the quadruple

$$(A, \mu_\alpha := \alpha \circ \mu, \{\,,\,\}_\alpha := \alpha \circ \{\,,\,\}, \alpha)$$

is a hom-Poisson algebra, said to be **obtained by composition**. It is indeed enough to assume that $\{\,,\,\}$ (resp. μ) is a Lie bracket (resp. an associative product) when restricted to the image of α^2.

In particular, given (M, π) a manifold equipped with a bivector field π, and $\varphi : M \to M$ a smooth map, then a hom-Poisson structure on $C^\infty(M)$ can be obtained by composition provided that φ preserves the bivector field π (i.e. $\pi_{\varphi(m)} = (\wedge^2 T_m\varphi)(\pi_m)$ for all $m \in M$) and that the Schouten-Nijenhuis bracket $[\pi, \pi]$ is a trivector field that vanishes on $\varphi^2(M) \subset M$. The triple (M, π, φ) is then called a **hom-Poisson manifold**.

Example 3.6. Here are examples that are not obtained by composition in general, see [BEM] for an alternative description. Let $(\mathfrak{g}, [\,,\,], \alpha)$ be a hom-Lie algebra. Equip its symmetric algebra $S(\mathfrak{g})$ with the product $\mu_\alpha := \alpha \circ \mu$, where $\mu(x, y) = x \odot y$ is the symmetric product and $\alpha : S(\mathfrak{g}) \to S(\mathfrak{g})$ stands for the automorphism of $(S(\mathfrak{g}), \mu)$ given by

$$\alpha(x_1 \odot \cdots \odot x_n) := \alpha(x_1) \odot \cdots \odot \alpha(x_n)$$

for all $x_1, \dots, x_n \in \mathfrak{g}$. The quadruple $(S(\mathfrak{g}), \mu_\alpha, \{\,,\,\}, \alpha)$ is a hom-Poisson algebra where:

$$\begin{aligned}&\{x_1 \odot \ldots \cdots \odot x_p, y_1 \odot \cdots \odot y_q\} := \\ &\quad = \sum_{i=1}^p \sum_{j=1}^q [x_i, y_j] \odot \alpha\left(x_1 \odot \ldots \widehat{x_i} \odot \cdots \odot x_p \odot y_1 \odot \ldots \widehat{y_j} \odot \cdots \odot y_q\right)\end{aligned}$$

for all $x_1, \dots, x_p, y_1, \dots, y_q \in \mathfrak{g}$. Identifying $S(\mathfrak{g})$ with polynomial functions on $\mathfrak{g}^*$, we could also write:

$$\{F, G\}(a) = \left\langle \left[\mathrm{dF}|_{\alpha^*(a)}, \mathrm{dG}|_{\alpha^*(a)} \right], a \right\rangle$$

for all polynomial functions F, G on $\mathfrak{g}^*$ and all $a \in \mathfrak{g}^*$, with the understanding that the differential of a function of $\mathfrak{g}^*$, a priori an element in $T^*\mathfrak{g}^*$ is considered as an element in $\mathfrak{g}$.

4. Hom-Gerstenhaber algebra

Definition 4.1. A **hom-Gerstenhaber** algebra is a quadruple $(\mathcal{A} = \oplus_{i \in \mathbb{N}} \mathcal{A}_i, \wedge, [\![\,,\,]\!], \alpha)$ where $(\mathcal{A} = \oplus_{i \in \mathbb{N}} \mathcal{A}_i, \wedge)$ is a graded commutative associative* algebra, α is an automorphism of $(\mathcal{A}, \wedge)$ of degree 0 and $[\![\,,\,]\!]$ is a bilinear map of degree -1 such that, first, $(\mathcal{A}[-1], [\![\,,\,]\!], \alpha)$ is a graded hom-Lie algebra (as usual, $\mathcal{A}[-1]$ refers to the graded vector space whose component of degree i is $\mathcal{A}_{i-1}$) and, second, the **hom-Leibniz rule** holds:

$$[\![X, Y \wedge Z]\!] = [\![X, Y]\!] \wedge \alpha(Z) + (-1)^{(i-1)j} \alpha(Y) \wedge [\![X, Z]\!], \ \forall X \in \mathcal{A}_i, Y \in \mathcal{A}_j, Z \in \mathcal{A}.$$

Example 4.2. Given an automorphism α of a Gerstenhaber algebra $(\mathcal{A}, \wedge, [\![\,,\,]\!])$ (i.e. α is an automophism for both products $\wedge$ and $[\![\,,\,]\!]$), then $(\mathcal{A}, \wedge, \alpha \circ [\![\,,\,]\!], \alpha)$ is a hom-Gerstenhaber algebra, said to be **obtained by composition**. Again, it suffices to assume that $[\![\,,\,]\!]$ satisfies the Jacobi identity on the image of α^2.

Example 4.3. Let $\mathfrak{g}$ be a vector space equipped with a skew-symmetric bilinear map $[\,,\,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ and an automorphism α of $(\mathfrak{g}, [\,,\,])$. The triple $(\mathfrak{g}, [\,,\,], \alpha)$ is a hom-Lie algebra if and only if the quadruple $(\wedge^\bullet \mathfrak{g}, \wedge, [\![\,,\,]\!], \alpha)$ is a hom-Gerstenhaber algebra where

$$[\![x_1 \wedge \cdots \wedge x_p, y_1 \wedge \cdots \wedge y_q]\!] = \sum_{i=1}^{p} \sum_{j=1}^{q} (-1)^{i+j} [x_i, y_j] \wedge \alpha(x_1 \wedge \ldots \widehat{x_i} \wedge \cdots \wedge x_p \wedge y_1 \wedge \ldots \widehat{y_j} \wedge \cdots \wedge y_q)$$

for all $x_1, \ldots, x_p, y_1, \ldots, y_q \in \mathfrak{g}$ and $\alpha(x_1 \wedge \cdots \wedge x_p) = \alpha(x_1) \wedge \cdots \wedge \alpha(x_p)$. The only difficult part is to prove that the graded hom-Jacobi identity holds on $\wedge^\bullet \mathfrak{g}$ if and only if $[\,,\,]$ satisfies the hom-Jacobi identity. This follows from the fact (derived by from a cumbersome but direct computation) that the hom-Jacobiator, defined as

$$\begin{aligned} Jac_\alpha(X, Y, Z) := \ & (-1)^{(i-1)(k-1)} [\![\alpha(X), [\![Y, Z]\!]]\!] + (-1)^{(j-1)(i-1)} [\![\alpha(Y), [\![Z, X]\!]]\!] + \\ & + (-1)^{(k-1)(j-1)} [\![\alpha(Z), [X, Y]]\!] \end{aligned}$$

for all $X \in \wedge^i \mathfrak{g}, Y \in \wedge^j \mathfrak{g}, Z \in \wedge^k \mathfrak{g}$, satisfies

$$Jac_\alpha(XY, Z, T) = \alpha^2(X) Jac_\alpha(Y, Z, T) + (-1)^{ij} \alpha^2(Y) Jac_\alpha(X, Z, T) \qquad (1)$$

and is a graded skew-symmetric map, so that it vanishes if and only if its restriction to $\wedge^0 \mathfrak{g} = \mathbb{R}$ and $\wedge^1 \mathfrak{g} = \mathfrak{g}$ vanishes.

*Notice that $\wedge$ is assumed to be an associative product, not a hom-associative product, so that a hom-Gerstenhaber algebra is not an odd version of a hom-Poisson algebra.

Recall that a **representation** [Sheng] of a hom-Lie algebra $(\mathfrak{g}, [\,,\,], \alpha)$ on a vector space V is a pair (ρ, α_V) of linear maps $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, $\alpha_V : V \to V$ such that, for all $x, y \in \mathfrak{g}$:

$$\rho(\alpha(x)) \circ \alpha_V = \alpha_V \circ \rho(x) \text{ and } \rho([x,y]) \circ \alpha_V = \rho(\alpha(x)) \circ \rho(y) - \rho(\alpha(y)) \circ \rho(x). \tag{2}$$

Example 4.4. Let (ρ, α_V) be a representation of the hom-Lie algebra $(\mathfrak{g}, [\,,\,], \alpha)$ on the vector space V. Then $(\wedge^\bullet \mathfrak{g} \otimes S^\bullet(V), \wedge, [\![\,,\,]\!], \alpha)$ is a hom-Gerstenhaber algebra where $\alpha : \wedge^\bullet \mathfrak{g} \otimes S^\bullet(V) \to \wedge^\bullet \mathfrak{g} \otimes S^\bullet(V)$ is defined as $\alpha(x_1 \wedge \cdots \wedge x_p \otimes v_1 \odot \cdots \odot v_q) = \alpha(x_1) \wedge \cdots \wedge \alpha(x_p) \otimes \alpha_V(v_1) \odot \cdots \odot \alpha_V(v_q)$, the bracket $[\![\,,\,]\!]$ being the hom-Gerstenhaber bracket whose restriction to $\wedge^0 \mathfrak{g} \otimes S(V)$ vanishes, whose restriction to $\wedge^\bullet \mathfrak{g} \otimes S^0(V) \simeq \wedge^\bullet \mathfrak{g}$ is as in example 4.3, and such that:

$$[\![x, v_1 \odot \cdots \odot v_q]\!] = \sum_{i=1}^{q} \rho(x)(v_i) \odot \alpha(v_1 \odot \cdots \odot \hat{v}_i \odot \cdots \odot v_q), \ \forall x \in \mathfrak{g}, v_1, \ldots, v_q \in V.$$

Given an algebra automorphism α of a commutative associative algebra A, we call α**-derivation** a map $\delta : A \to A$ which satisfies

$$\delta(FG) = \alpha(F)\delta(G) + \alpha(G)\delta(F),$$

for all $F, G \in A$.

Proposition 4.5. *For every hom-Gerstenhaber algebra* $(\mathcal{A} = \oplus_{i \in \mathbb{N}} \mathcal{A}_i, \wedge, [\![\,,\,]\!], \alpha)$, *denote by* $\rho : \mathcal{A}_1 \longrightarrow \mathrm{End}(\mathcal{A}_0)$ *the map given by* $\rho(X)[F] := [\![X, F]\!]$ *for all* $F \in \mathcal{A}_0$. *Then* $\mathcal{A}_0$ *is a commutative associative algebra,* $\alpha|_{\mathcal{A}_0}$ *is an algebra automorphism of* $\mathcal{A}_0$, $F \mapsto \rho(X)(F)$ *is, for all* $X \in \mathcal{A}_1$, *a* $\alpha|_{\mathcal{A}_0}$*-derivation of* $\mathcal{A}_0$, *the triple* $(\mathcal{A}_1, [\![\,,\,]\!]|_{\mathcal{A}_1 \times \mathcal{A}_1}, \alpha|_{\mathcal{A}_1})$ *is a hom-Lie algebra, and* $(\rho, \alpha|_{\mathcal{A}_0})$ *is a representation of* $(\mathcal{A}_1, [\![\,,\,]\!]|_{\mathcal{A}_1 \times \mathcal{A}_1}, \alpha|_{\mathcal{A}_1})$ *on* $\mathcal{A}_0$.

Proof: Only the last point needs justification. We recover relations (2) by using the graded hom-Jacobi identity as follows,

$$\begin{aligned} &\alpha|_{\mathcal{A}_0}(\rho(X)[F]) = \rho\left(\alpha|_{\mathcal{A}_1}(X)\right)\left[\alpha|_{\mathcal{A}_0}(F)\right] \\ &\rho([\![X, Y]\!])[\alpha|_{\mathcal{A}_0}(F)] = \left(\rho(\alpha|_{\mathcal{A}_1}(X)) \circ \rho(Y) - \rho(\alpha|_{\mathcal{A}_1}(Y)) \circ \rho(X)\right)[F], \end{aligned}$$

for all $X, Y \in \mathcal{A}_1$ and $F \in \mathcal{A}_0$. ■

5. Definition of hom-Lie algebroid

Definition 5.1. A hom-Lie algebroid is a quintuple $(A \to M, \varphi, [\,,\,], \rho, \alpha)$, where $A \to M$ is a vector bundle over a manifold M, $\varphi : M \to M$ is a smooth map, $[\,,\,] : \Gamma(A) \otimes \Gamma(A) \to \Gamma(A)$ is a bilinear map, called **bracket**, $\rho : \varphi^! A \to \varphi^! TM$ is a vector bundle morphism, called **anchor**, and $\alpha : \Gamma(A) \to \Gamma(A)$ is a linear endomorphism of $\Gamma(A)$ such that

1. $\alpha(FX) = \varphi^*(F)\alpha(X)$ for all $X \in \Gamma(A), F \in C^\infty(M)$ (equivalently †, α is a vector bundle morphism from $\varphi^! A$ to A over the identity of M);
2. the triple $(\Gamma(A), [\,,\,], \alpha)$ is a hom-Lie algebra;
3. the following hom-Leibniz identity holds ‡:

$$[X, FY] = \varphi^*(F)[X, Y] + \rho(X)[F]\,\alpha(Y), \quad \forall X, Y \in \Gamma(A), F \in C^\infty(M).$$

4. (ρ, φ^*) is a representation of $(\Gamma(A), [\,,\,], \alpha)$ on $C^\infty(M)$.

Above, $\rho(X)[F]$ stands for the function on M whose value at $m \in M$ is $\langle \mathrm{d}_{\varphi(\mathrm{m})}\mathrm{F}, \rho_{\mathrm{m}}(\mathrm{X}_{\varphi(\mathrm{m})}) \rangle$ where $\rho_m : (\varphi^! A)_m \simeq A_{\varphi(m)} \to (\varphi^! TM)_m \simeq T_{\varphi(m)}M$ is the anchor map evaluated at $m \in M$ and $X_{\varphi(m)}$ is the value of the section $X \in \Gamma(A)$ at $\varphi(m) \in M$.

Example 5.2. When α (hence φ) is the identity map, a hom-Lie algebroid $(A \to M, \varphi, \alpha, [\,,\,], \rho)$ is simply a Lie algebroid [McK]. A hom-Lie algebra $(\mathfrak{g}, [\,,\,], \alpha)$ is a hom-Lie algebroid over a singleton. More generally, define an **action of a hom-Lie algebra** $(\mathfrak{g}, [\,,\,], \alpha)$ on the manifold M, equipped with a smooth map $\varphi : M \to M$, to be a linear map δ from $\mathfrak{g}$ to the space of φ^*-derivations such that (δ, φ^*) defines a representation of the hom-Lie algebra $(\mathfrak{g}, [\,,\,], \alpha)$ on the vector space $C^\infty(M)$. Then a hom-Lie algebroid is obtained by considering the trivial vector bundle $A = M \times \mathfrak{g} \to M$, the linear map α_A mapping $Fc_v \to \varphi^*(F)c_{\alpha(v)}$ for all $v \in \mathfrak{g}, F \in C^\infty(M)$, the anchor ρ mapping $A_{\varphi(m)} \simeq \mathfrak{g}$ to the element of $T_{\varphi(m)}M$ given by the pointwise derivation $F \mapsto \delta(v)[F]|_m$ and the bracket given by:

$$[Fc_v, Gc_w] = \varphi^*(FG)c_{[v,w]} + \varphi^*(F)\rho(v)[G]c_{\alpha(w)} - \varphi^*(G)\rho(w)[F]c_{\alpha(v)}$$

† Given $X \in \Gamma(A)$, a section of the pull-back bundle $\varphi^! A$ is given by mapping $m \in M$ to $X_{\varphi(m)} \in A_{\varphi(m)} \simeq (\varphi^! A)_m$. Applying a vector bundle morphism from $\varphi^! A$ to A over the identity of M to that section yields a section of A, and the henceforth defined assignment α satisfies $\alpha(FX) = \varphi^*(F)\alpha(X)$ for all $X \in \Gamma(A), F \in C^\infty(M)$. Moreover, every endomorphism of $\Gamma(A)$ satisfying this relation is of that form.

‡ This hom-Leibniz identity implies that, given sections X, Y of A, the value of $[X, Y]$ at a given point $m \in M$ depends only on the first jet of X and Y at $\varphi(m)$.

for all $F, G \in C^\infty(M)$, $v, w \in \mathfrak{g}$. In the previous, c_v, c_w denote the constant sections of $M \times \mathfrak{g} \to M$ given by $m \mapsto (v, m)$ and $m \mapsto (w, m)$ respectively. This hom-Lie algebroid is not obtained by composition in general.

The following theorem is a consequence of proposition 4.5, and will allow us to give more examples.

Theorem 5.3. *Let $A \to M$ be a vector bundle, $\varphi : M \to M$ a smooth map, $\alpha : \Gamma(A) \to \Gamma(A)$ a linear endomorphism satisfying $\alpha(FX) = \varphi^*(F)\alpha(X)$ for all $X \in \Gamma(A), F \in C^\infty(M)$. Denote by α again its extension to $\alpha : \Gamma(\wedge^\bullet A) \to \Gamma(\wedge^\bullet A)$ given by:*

$$\alpha(FX_1 \wedge \dots \wedge X_p) = \varphi^*(F)\alpha(X_1) \wedge \dots \wedge \alpha(X_p) \tag{3}$$

for all $p \in \mathbb{N}$, $X_1, \dots, X_p \in \Gamma(A)$, $F \in C^\infty(M)$.

Then there is a one-to-one correspondence between hom-Gerstenhaber algebra structures on $(\Gamma(\wedge^\bullet A), \wedge, [\![\, ,]\!], \alpha)$ and hom-Lie algebroids structures on $(A \to M, \varphi, [\, ,]\, , \rho, \alpha)$, obtained as follows:

(1) Given a hom-Gerstenhaber algebra structure $(\Gamma(\wedge^\bullet A), \wedge, [\![\, ,]\!], \alpha)$, we define a bracket $[\, ,]$ on $\Gamma(A)$ by restriction of $[\![\, ,]\!]$ to $\Gamma(A)$ and an anchor $\rho : A \to TM$ by $\rho(X)[F] := [\![X, F]\!]$ for all $X \in \Gamma(A)$, $F \in C^\infty(M)$.

(2) Conversely, given a hom-Lie algebroid structure $(A \to M, \varphi, [\, ,], \rho, \alpha)$, we define a hom-Gerstenhaber bracket on $\Gamma(\wedge^\bullet A)$, for all $X_1, \dots, X_p$, $Y_1, \dots, Y_q \in \Gamma(A)$, $F \in C^\infty(M)$, by:

$$[\![X_1 \wedge \dots \wedge X_p, Y_1 \wedge \dots \wedge Y_q]\!] = \\ = \sum_{i=1}^{p} \sum_{j=1}^{q} (-1)^{i+j} [X_i, Y_j] \wedge \alpha(X_1 \wedge \dots \widehat{X_i} \wedge \dots \wedge X_p \wedge Y_1 \wedge \dots \widehat{Y_j} \wedge \dots \wedge Y_q)$$

and by

$$[\![X_1 \wedge \dots \wedge X_p, F]\!] = \sum_{i=1}^{p} (-1)^{i+1} \rho(X_i)[F] \wedge \alpha(X_1 \wedge \dots \wedge \widehat{X_i} \wedge \dots \wedge X_p).$$

Proof: *1*) We first need to justify our definition of ρ. It follows from the hom-Leibniz identity of the hom-Gerstenhaber algebra that $F \mapsto [\![X, F]\!]$ is a φ^*-derivation and that $[\![GX, F]\!] = \varphi^*(G)[\![X, F]\!]$ for all $X \in \Gamma(A)$, $F, G \in C^\infty(M)$. Altogether, these properties imply that there is an unique vector bundle morphism $\rho : \varphi^! A \to \varphi^! TM$ such that $\rho(X)[F] = [\![X, F]\!]$ for all

$X \in \Gamma(A), F \in C^\infty(M)$. Condition 1 in definition 5.1 holds by assumption. Conditions 2 and 4 follows from proposition 4.5. The hom-Leibniz identity of the hom-Gerstenhaber algebra:

$$[\![X, FY]\!] = \varphi^* F[\![X, Y]\!] + [\![X, F]\!]\alpha(Y) \quad \forall X, Y \in \Gamma(A), F \in C^\infty(M)$$

gives the condition 3 and proves the first item, since $[\![X, F]\!] = \rho(X)[F]$ by the very construction of ρ.

2) The hom-Leibniz identity, together with the facts that $\rho(FX)[G] = \varphi^*(F)\rho(X)[G]$ for all $F, G \in C^\infty(M), X \in \Gamma(A)$ and that $F \mapsto \rho(X)[F]$ is an φ^*-derivation imply that $[\![\, ,]\!]$ is well-defined. It also implies that it obeys to the hom-Leibniz identity. By equation (1), it suffices to check the hom-Jacobi identity of $[\![\, ,]\!]$ for triples made of three sections of $\Gamma(A)$ and triples made of two sections of A together with one function on M. In the first case, it simply follows from the hom-Jacobi identity of $[\, ,]$ and in the second case, it is equivalent to the assumption that (ρ, φ^*) is a representation of $(\Gamma(A), [\, ,]\, , \alpha)$. This proves 2). The constructions of both items are clearly inverse one to the other, and the theorem follows. ∎

Example 5.4. Recall [McK] that a Gerstenhaber algebra structure $(\Gamma(\wedge^\bullet A), [\![\, ,]\!], \wedge)$ is naturally associated to every Lie algebroid $(A, [\, ,]\, , \rho)$. Given $\alpha : \Gamma(A) \to \Gamma(A)$ a linear endomorphism of $\Gamma(A)$ satisfying $\alpha(FX) = \varphi^*(F)\alpha(X)$ for all $X \in \Gamma(A), F \in C^\infty(M)$ ($\varphi : M \to M$ being a given smooth map), an algebra endomorphism, again called α, of $(\Gamma(\wedge^\bullet A), \wedge)$ can be construted as in (3). This morphism α preserves the bracket $[\![\, ,]\!]$ provided that $\alpha : \Gamma(A) \to \Gamma(A)$ is a Lie algebra morphism such that $\rho(\alpha(X))[\varphi^* F] = \varphi^*(\rho(X)[F])$ for all $X \in \Gamma(A), F \in C^\infty(M)$. Example 4.2 then allows us to build a hom-Gerstenhaber algebra by composition, which, by theorem 5.3, yields a hom-Lie algebroid. Indeed, it suffices that $[\![\, ,]\!]$ satisfies the Jacobi identity on the image of α^2. For this, it suffices that $[\![\, ,]\!]$ satisfies the Jacobi identity when applied to triples of the form $(\alpha^2(X), \alpha^2(Y), \alpha^2(Z))$ or $(\alpha^2(X), \alpha^2(Y), (\varphi^*)^2(F))$, with $X, Y, Z \in \Gamma(A), F \in C^\infty(M)$. This means that $(A, [\, ,]\, , \rho)$ can be just assumed to be a pre-Lie algebroid (i.e. the Leibniz rule holds, but $[\, ,]$ is not a Lie bracket) that satisfies the Jacobi identity on the image of $\alpha^2 : \Gamma(A) \to \Gamma(A)$ and such that for all $X, Y \in \Gamma(A)$:

$$\rho(\big[\alpha^2(X), \alpha^2(Y)\big]) - \big[\rho(\alpha^2(X)), \rho(\alpha^2(Y))\big] \tag{4}$$

is a vector field on M that vanishes on the image of $\varphi^2 : M \to M$.

We now describe two particular cases of the previous construction.

Example 5.5. A vector field $\mathcal{V} \in \mathfrak{X}(M)$ induces a Lie algebroid as follows: $A_{\mathcal{V}} := M \times \mathbb{R}$ is the trivial line bundle, the Lie bracket of two sections $F, G \in C^\infty(M) = \Gamma(A_{\mathcal{V}})$ is given by $[F, G] := F\mathcal{V}[G] - \mathcal{V}[F]$ and the anchor of $F \in C^\infty(M) = \Gamma(A_{\mathcal{V}})$ is the vector field $F\mathcal{V}$. Let $\varphi : M \to M$ be a smooth map preserving $\mathcal{V}$, i.e. $\varphi^*(\mathcal{V}[F]) = \mathcal{V}[\varphi^* F]$. Then φ^* is a linear endomorphism of $C^\infty(M) = \Gamma(A_{\mathcal{V}})$ which satisfies the required conditions to yield a hom-Lie algebroid by composition.

Example 5.6. Let (M, π, φ) be a hom-Poisson manifold. Let $A = T^*M$, and let α be the pull-back morphism $\varphi^* : \Gamma(T^*M) \to \Gamma(T^*M)$. Then a pre-Lie algebroid structure [Vaisman] is defined on $\Gamma(T^*M)$ by considering the anchor map $\rho := \pi^\# : T^*M \to TM$ together with the bracket

$$[a, b]_\pi := \mathcal{L}_{\pi^\# a} b - \mathcal{L}_{\pi^\# b} a + \mathrm{d}\imath_\pi(\mathrm{a} \wedge \mathrm{b}).$$

It is immediate that $\varphi^* : \Gamma(T^*M) \to \Gamma(T^*M)$ is an automorphism of this bracket and that $\rho(\varphi^* a)[\varphi^* F] = \varphi^*(\rho(a)[F])$ for any 1-form $a \in \Gamma(T^*M)$ and $F \in C^\infty(M)$. Assuming the Schouten-Nijenhuis bracket $[\pi, \pi]$ to vanish on the image of φ^2 amounts to require that $[\ ,\ ]_\pi$ satisfies the Jacobi identity when restricted to the image of $(\varphi^*)^2 : \Gamma(T^*M) \to \Gamma(T^*M)$ and that condition (4) holds. A hom-Lie algebroid can therefore be constructed by composition.

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