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THE DEDEKIND ORDER COMPLETION OF $\mathrm{C}(L)$

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ABSTRACT: This paper introduces the frame of partially defined real numbers and the lattice-ordered ring of partial real functions on a frame. This is then used to construct the order completion of rings of pointfree continuous real functions. The bounded and integer-valued cases are also analysed. The application of this pointfree approach to the classical case $\mathrm{C}(X)$ of the ring of continuous real-valued functions on a topological space X yields a new construction for the Dedekind completion of $\mathrm{C}(X)$, considerably more direct and natural than the known procedure using Hausdorff continuous functions.

KEYWORDS: Frame of reals, frame of partial reals, continuous real function, partial real function, order complete, Dedekind order completion, completely regular frame, extremally disconnected frame, zero-dimensional frame.

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Introduction

Our main goal with this paper is to construct the Dedekind order completion of the ring $\mathrm{C}(X)$ in the most direct and transparent way, avoiding the use of Hausdorff continuous functions in [1]. For that, we approach the problem from a pointfree viewpoint, replacing spaces by an abstraction of their lattices of open sets. The lattices involved here are the *frames* (or *locales*), which form the object of study of pointfree topology.

Let $\mathfrak{L}(\mathbb{R})$ denote the *frame of reals* [2], that is, the frame generated by all ordered pairs (p,q) of rationals, subject to the relations

(R1) $(p,q)\wedge(r,s)=(p\vee r,q\wedge s)$,
(R2) $(p,q)\vee(r,s)=(p,s)$ whenever $p\le r<q\le s$,
(R3) $(p,q)=\bigvee\{(r,s)\mid p<r<s<q\}$,
(R4) $\bigvee\{(p,q)\mid p,q\in\mathbb{Q}\}=1$.

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For any frame L the *real continuous functions* on L are the frame homomorphisms $\mathfrak{L}(\mathbb{R}) \to L$. They form a lattice-ordered ring (ℓ-ring) [2] that we shall denote by $\mathrm{C}(L)$. The correspondence $L \mapsto \mathrm{C}(L)$ extends that for spaces: if $L = \mathcal{O}X$ (the frame of open sets of a space X) then the classical function ring $\mathrm{C}(X)$ is naturally isomorphic to $\mathrm{C}(L)$ [2].

Order completeness of an ℓ-ring [5], it may be recalled, means that each non-void set of elements that is bounded from above has a supremum and dually each non-void set of elements that is bounded from below has an infimum. What can one say about the Dedekind order completion of $\mathrm{C}(L)$ and, in particular, of $\mathrm{C}(X)$ for any space X? In general, due to axiom (R2) above, $\mathrm{C}(L)$ fails to be order complete. The best known result is a theorem of Banaschewski and Hong [5] that extends familiar facts concerning topological spaces that go back to Nakano [12] and Stone [15]: for a completely regular L, $\mathrm{C}(L)$ is order complete iff L is extremally disconnected iff L is zero-dimensional and the Boolean part of L is complete.

Our aim is to construct the order completion of $\mathrm{C}(L)$. In order to achieve it we must find in some way the smallest order complete lattice containing $\mathrm{C}(L)$. A natural idea is to avoid the problem caused by (R2) by deleting it from the list of axioms. So our main device will be the frame

$$\mathfrak{L}(\mathbb{IR})$$

of *partially defined real numbers*, presented by the same generators as $\mathfrak{L}(\mathbb{R})$ and by all relations except (R2). Of course, this is a bigger frame in which $\mathfrak{L}(\mathbb{R})$ embeds canonically. Then $\mathrm{C}(L)$ also embeds canonically in the class

$$\mathrm{IC}(L) = \mathsf{Frm}(\mathfrak{L}(\mathbb{IR}), L)$$

of *partial real functions* on L. We prove that $\mathrm{IC}(L)$ is Dedekind order complete (Section 2) and describe (in Section 3) the Dedekind order completion $\mathrm{C}(L)^{\#}$ of $\mathrm{C}(L)$ inside $\mathrm{IC}(L)$ by

$$\begin{aligned}\mathrm{C}(L)^{\#} = \{h \in \mathrm{IC}(L) \mid \ & \text{(a) there exist } f, g \in \mathrm{C}(L) \text{ such that } f \le h \le g \\ & \text{(b) } h(p,\text{—})^* \le h(\text{—}, q) \text{ and } h(\text{—}, q)^* \le h(p, \text{—}) \text{ for any } p < q\}.\end{aligned}$$

We also show that, alternatively, the elements of $\mathrm{C}(L)^{\#}$ are precisely the maximal elements of $\mathrm{IC}(L)$ (with respect to a certain partial order $\sqsubseteq$) that satisfy condition (a). The bounded and integer-valued cases are then analysed (Sections 4 and 5). In particular, the aforementioned result of Banaschewski-Hong [5] follows as an immediate corollary from our construction. Finally, in Section 6, we show that the application of these ideas to the classical case of the ring C(X) of continuous real-valued functions on a topological space X provides a new construction for its order completion. In particular, the results of Anguelov [1] and Danet [6] (see also [7]) follow easily from our approach.

1. Preliminaries

1.1. Order completion. For any subset A of a partially ordered set $(P, \leq)$ we denote by

$$\bigvee^{P} A \quad (\text{resp. } \bigwedge^{P} A)$$

the supremum (resp. infimum) of A in P in case it exists (we shall omit the superscript if it is clear from the context).

A partially ordered set $(P, \leq)$ is called *Dedekind order complete* if every non-void subset A of P which is bounded from above has a supremum in P and dually every non-void subset B of P which is bounded from below has a infimum in P.

A *Dedekind order completion* of a poset P is a pair $(P^{\#}, \Phi)$ where $P^{\#}$ is a Dedekind order complete poset and $\Phi\colon P \to P^{\#}$ is an order embedding that preserves all suprema and infima that exist in P and satisfies

$$\hat{p} = \bigvee^{P^{\#}} \{\Phi(p) \in \Phi(P) \mid \Phi(p) \leq \hat{p}\} = \bigwedge^{P^{\#}} \{\Phi(p) \in \Phi(P) \mid \Phi(p) \geq \hat{p}\}$$

for every $\hat{p} \in P^{\#}$.

1.2. Frames. A *frame* (or *locale*) L is a complete lattice such that

$$a \wedge \bigvee B = \bigvee \{a \wedge b \mid b \in B\}$$

for all $a \in L$ and $B \subseteq L$; equivalently, it is a complete Heyting algebra with Heyting operation $\to$ satisfying the standard equivalence $a \wedge b \leq c$ if and only if $a \leq b \to c$. The *pseudocomplement* of an $a \in L$ is the element $a^* = a \to 0 = \bigvee\{b \in L \mid a \wedge b = 0\}$. An element a is *regular* if $a^{**} = a$ (equivalently, if $a = b^*$ for some b). A *frame homomorphism* is a map $h\colon L \to M$ between frames which preserve finitary meets (including the top element 1) and arbitrary joins (including the bottom element 0). Frm is then the corresponding category of frames and their homomorphisms.

The most typical example of a frame is the lattice $\mathcal{O}X$ of open subsets of a topological space X. The correspondence $X \mapsto \mathcal{O}X$ is clearly functorial (by taking inverse images), and consequently we have a contravariant functor $\mathcal{O}\colon \mathsf{Top} \to \mathsf{Frm}$ where Top is the category of topological spaces and continuous maps. There is also a functor in the opposite direction, the *spectrum functor* $\Sigma\colon \mathsf{Frm} \to \mathsf{Top}$ which assigns to each frame L its spectrum ΣL, the space of all homomorphisms $\xi\colon L \to \{0,1\}$ with open sets $\Sigma_a = \{\xi \in \Sigma L \mid \xi(a) = 1\}$ for any $a \in L$, and to each frame homomorphism $h\colon L \to M$ the continuous map $\Sigma h\colon \Sigma M \to \Sigma L$ such that $\Sigma h(\xi) = \xi h$. The spectrum functor is right adjoint to $\mathcal{O}$, with adjunction maps

$$\eta_L\colon L \to \mathcal{O}\Sigma L,\ \eta_L(a) = \Sigma_a$$

and

$$\epsilon_X\colon X \to \Sigma\mathcal{O}X,\ \epsilon_X(x) = \hat{x},\ \hat{x}(U) = 1 \text{ iff } x \in U$$

(the former is the *spatial reflection* of the frame L).

For general notions and results concerning frames we refer to Johnstone [11] or the recent Picado-Pultr [13]. The particular notions we will need are the following: a frame L is

completely regular if $a = \bigvee\{b \in L \mid b \prec\!\!\prec a\}$ for each $a \in L$, where $b \prec\!\!\prec a$ means that there is $\{c_r \mid r \in \mathbb{Q} \cap [0,1]\} \subseteq L$ such that $a \leq c_0$, $c_1 \leq b$ and $c_r \prec c_s$ (i.e. $c_r^* \vee c_s = 1$) whenever $r < s$;
extremally disconnected if $a^* \vee a^{**} = 1$ for every $a \in L$; and
zero-dimensional if each element of L is a join of complemented elements.

1.3. Real functions. It will be useful here (as it has been also in [4]) to adopt the equivalent description of $\mathfrak{L}(\mathbb{R})$ with the elements $(r,\text{—}) = \bigvee_{s\in\mathbb{Q}}(r,s)$ and $(\text{—},s) = \bigvee_{r\in\mathbb{Q}}(r,s)$ as primitive notions. Specifically, the *frame of reals* $\mathfrak{L}(\mathbb{R})$ is equivalently defined by generators $(r,\text{—})$ and $(\text{—},r)$ for $r \in \mathbb{Q}$ and the following relations

(r1) $(r,\text{—}) \wedge (\text{—},s) = 0$ whenever $r \geq s$,
(r2) $(r,\text{—}) \vee (\text{—},s) = 1$ whenever $r < s$,
(r3) $(r,\text{—}) = \bigvee_{s>r}(s,\text{—})$, for every $r \in \mathbb{Q}$,
(r4) $(\text{—},r) = \bigvee_{s<r}(\text{—},s)$, for every $r \in \mathbb{Q}$,
(r5) $\bigvee_{r\in\mathbb{Q}}(r,\text{—}) = 1$,
(r6) $\bigvee_{r\in\mathbb{Q}}(\text{—},r) = 1$.

With $(p,q) = (p,\text{—}) \wedge (\text{—},q)$ one goes back to (R1)–(R4).

Regarding the frame homomorphisms $\mathfrak{L}(\mathbb{R}) \to L$, for a general frame L, as the *continuous real functions* on L provides a natural extension of the classical notion since continuous real functions on a space X may be represented as frame homomorphisms $h\colon \mathfrak{L}(\mathbb{R}) \to \mathcal{O}X$ (see [2] for a detailed account).

There is a useful way of specifying continuous real functions on L with the help of scales. A *scale* in L is a map $\sigma\colon \mathbb{Q}\to L$ such that
(1) $\sigma(r) \vee \sigma(s)^* = 1$ whenever $r < s$;
(2) $\bigvee_{r\in\mathbb{Q}} \sigma(r) = 1 = \bigvee_{r\in\mathbb{Q}} \sigma(r)^*$.
For any scale σ the formulas

$$f(r,\text{—}) = \bigvee\{\sigma(s) \mid s > r\} \quad \text{and} \quad f(\text{—},r) = \bigvee\{\sigma(s)^* \mid s < r\} \quad (r \in \mathbb{Q})$$

determine a continuous real function $f\colon \mathfrak{L}(\mathbb{R}) \to L$.

Note. The meaning of the term *scale* used here differs from its use in [11] where it refers to maps from the unit interval of $\mathbb{Q}$ (and not all of $\mathbb{Q}$) into L. In [2] the term *descending trail* is used instead.

$\mathsf{C}(L) = \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), L)$ is partially ordered by

$$\begin{aligned} f \le g \quad &\text{iff} \quad f(r,\text{—}) \le g(r,\text{—}) \quad \text{for all } r \in \mathbb{Q} \\ &\text{iff} \quad g(\text{—},r) \le f(\text{—},r) \quad \text{for all } r \in \mathbb{Q}. \end{aligned} \tag{1.3.1}$$

Examples. (1) For each $r \in \mathbb{Q}$, the *constant function* $\boldsymbol{r}$ determined by r is defined by

$$\boldsymbol{r}(s,\text{—}) = \begin{cases} 0 & \text{if } s \ge r \\ 1 & \text{if } s < r \end{cases} \qquad \text{and} \qquad \boldsymbol{r}(\text{—},s) = \begin{cases} 1 & \text{if } s > r \\ 0 & \text{if } s \le r \end{cases}$$

for every $s \in \mathbb{Q}$.

(2) For each complemented $a \in L$, the *characteristic function* χ_a determined by a is given by

$$\chi_a(s,\text{—}) = \begin{cases} 0 & \text{if } s \ge 1 \\ a & \text{if } 0 \le s < 1 \\ 1 & \text{if } s < 0 \end{cases} \qquad \text{and} \qquad \chi_a(\text{—},s) = \begin{cases} 1 & \text{if } s > 1 \\ a* & \text{if } 0 < s \le 1 \\ 0 & \text{if } s \le 0 \end{cases}$$

for every $s \in \mathbb{Q}$.

An $f \in \mathrm{C}(L)$ is said to be *bounded* if there exist $p, q \in \mathbb{Q}$ such that $\boldsymbol{p} \le f \le \boldsymbol{q}$. Equivalently, f is bounded iff there is some rational r such that $f\left((\text{—},-r) \vee (r,\text{—})\right) = 0$, that is, $f(-r,r) = 1$. We shall denote by $\mathrm{C}^*(L)$ the set of all bounded members of $\mathrm{C}(L)$. Obviously, all constant functions and all characteristic functions are in $\mathrm{C}^*(L)$.

As it is well known, in general neither $\mathrm{C}(L)$ nor $\mathrm{C}^*(L)$ are Dedekind order complete [5].

The operations on the algebra $\mathrm{C}(L)$ are determined by the operations of $\mathbb{Q}$ as lattice-ordered ring as follows (see [2] and [10] for more details):

(1) For $\diamond = +, \cdot, \wedge, \vee$:

$$(f \diamond g)(p, q) = \bigvee\{f(r, s) \wedge g(t, u) \mid \langle r, s\rangle \diamond \langle t, u\rangle \subseteq \langle p, q\rangle\}$$

where $\langle \cdot, \cdot\rangle$ stands for open interval in $\mathbb{Q}$ and the inclusion on the right means that $x \diamond y \in \langle p, q\rangle$ whenever $x \in \langle r, s\rangle$ and $y \in \langle t, u\rangle$.

(2) $(-f)(p, q) = f(-q, -p)$.

(3) For each $r \in \mathbb{Q}$, the nullary operation $\mathbf{r}$ is defined as in Example 1.3 (1) above.

(4) For each $0 < \lambda \in \mathbb{Q}$, $(\lambda \cdot f)(p, q) = f\left(\frac{p}{\lambda}, \frac{q}{\lambda}\right)$.

These operations satisfy all the identities which hold for their counterparts in $\mathbb{Q}$ and hence they determine an f-ring structure in $\mathrm{C}(L)$.

2. Partial real functions

Let $\mathbb{IR}$ denote the set of compact intervals $\boldsymbol{a} = [\underline{a}, \overline{a}]$ of the real line ordered by reverse inclusion (which we denote by $\sqsubseteq$):

$$\boldsymbol{a} \sqsubseteq \boldsymbol{b} \quad \text{iff} \quad [\underline{a}, \overline{a}] \supseteq [\underline{b}, \overline{b}] \quad \text{iff} \quad \underline{a} \leq \underline{b} \leq \overline{b} \leq \overline{a}.$$

The pair $(\mathbb{IR}, \sqsubseteq)$ is a domain, referred to as the *partial real line* (also *interval-domain*). The interval domain was proposed by Dana Scott in [14] as a domain-theoretic model for the real numbers. It is a successful idea that has inspired a number of computational models for real numbers.

The way-below relation of $\mathbb{IR}$ is given by

$$\boldsymbol{a} \ll \boldsymbol{b} \quad \text{iff} \quad \underline{a} < \underline{b} \leq \overline{b} < \overline{a}$$

and we denote the set $\{\boldsymbol{b} \in \mathbb{IR} \mid \boldsymbol{a} \ll \boldsymbol{b}\}$ by $\twoheaduparrow\boldsymbol{a}$. The family

$$\{\twoheaduparrow\boldsymbol{a} \mid \boldsymbol{a} \in \mathbb{IR}, \underline{a}, \overline{a} \in \mathbb{Q}\}$$

forms a countable basis of the *Scott topology* $\mathcal{O}\mathbb{IR}$ on $(\mathbb{IR}, \sqsubseteq)$.

Remarks 2.1. (1) Let $\pi_1, \pi_2 \colon \mathbb{IR} \to \mathbb{R}$ denote the projections defined for each $\boldsymbol{a} \in \mathbb{IR}$ by $\pi_1(\boldsymbol{a}) = \underline{a}$ and $\pi_2(\boldsymbol{a}) = \overline{a}$. Then for each $r \in \mathbb{Q}$

$$\begin{aligned}\pi_1^{-1}(r, +\infty) &= \{\boldsymbol{a} \in \mathbb{IR} \mid r < \underline{a}\} \\ &= \bigcup_{\beta \in \mathbb{R}, \beta > r} \{\boldsymbol{a} \in \mathbb{IR} \mid r < \underline{a} \leq \overline{a} < \beta\} = \bigcup_{\beta \in \mathbb{R}, \beta > r} \Uparrow [r, \beta]\end{aligned}$$

and

$$\begin{aligned}\pi_2^{-1}(-\infty, r) &= \{\boldsymbol{a} \in \mathbb{IR} \mid \overline{a} < r\} \\ &= \bigcup_{\alpha \in \mathbb{R}, \alpha < r} \{\boldsymbol{a} \in \mathbb{IR} \mid \alpha < \underline{a} \leq \overline{a} < r\} = \bigcup_{\alpha \in \mathbb{R}, \alpha < r} \Uparrow [\alpha, r]\,.\end{aligned}$$

It follows that for the upper τ_u and lower τ_l topologies in $\mathbb{R}$, $\pi_1 \colon \mathbb{IR} \to (\mathbb{R}, \tau_u)$ is continuous, (i.e. π_1 is lower semicontinuous) and $\pi_2 \colon \mathbb{IR} \to (\mathbb{R}, \tau_l)$ is continuous, (i.e. π_2 is upper semicontinuous). Hence, for any $f \in \mathrm{C}(X, \mathbb{IR})$, $\pi_1 \circ f \in \mathrm{LSC}(X, \mathbb{R})$, $\pi_2 \circ f \in \mathrm{USC}(X, \mathbb{R})$ and $\pi_1 \circ f \leq \pi_2 \circ f$.

Note further that for each $\boldsymbol{a} \in \mathbb{IR}$,

$$\Uparrow \boldsymbol{a} = \pi_1^{-1}(\underline{a}, +\infty) \cap \pi_2^{-1}(-\infty, \overline{a}).$$

Consequently, the Scott topology on $\mathbb{IR}$ is the *initial topology* w.r.t. $\pi_1 \colon \mathbb{IR} \to (\mathbb{R}, \tau_u)$ and $\pi_2 \colon \mathbb{IR} \to (\mathbb{R}, \tau_l)$.

(2) Let $e \colon \mathbb{R} \to \mathbb{IR}$ be given by $e(a) = [a, a]$ for each $a \in \mathbb{R}$. It is easy to check that e is an embedding of $\mathbb{R}$ endowed with the usual topology into $(\mathbb{IR}, \mathcal{O}\mathbb{IR})$. Sometimes we shall identify $\mathbb{R}$ with its homeomorphic copy $e(\mathbb{R}) \subseteq \mathbb{IR}$. Similarly, a real-valued function $f \colon X \to \mathbb{R}$ will be identified with the function $e \circ f \colon X \to \mathbb{IR}$.

When investigating the existence of suprema of families of continuous real functions on a frame one immediately realizes that the problem lies on the defining relation (r2) (or (R2)). This urged us to consider the *partial* variant of $\mathfrak{L}(\mathbb{R})$ defined by generators $(r, \text{—})$ and $(\text{—}, r)$ for $r \in \mathbb{Q}$ and relations (r1), (r3)–(r6). We call it the *frame of partial reals*

$$\mathfrak{L}(\mathbb{IR}).$$

There is of course a *basic homomorphism*

$$\varrho \colon \mathfrak{L}(\mathbb{IR}) \to \mathfrak{L}(\mathbb{R})$$

defined on generators by $(r, \text{—}) \mapsto (r, \text{—})$ and $(\text{—}, r) \mapsto (\text{—}, r)$.

Proposition 2.2. *The space of partial reals with the Scott topology is homeomorphic to $\Sigma\mathfrak{L}(\mathbb{IR})$. The homeomorphism $\tau\colon \Sigma\mathfrak{L}(\mathbb{IR}) \to \mathbb{IR}$ is such that*

$$\underline{\tau(\xi)} = \bigvee\{r \in \mathbb{Q} \mid \xi(r,\text{—}) = 1\} \quad \textit{and} \quad \overline{\tau(\xi)} = \bigwedge\{s \in \mathbb{Q} \mid \xi(\text{—},s) = 1\}$$

for each $\xi \in \Sigma\mathfrak{L}(\mathbb{IR})$.

Proof: Let $\xi \in \Sigma\mathfrak{L}(\mathbb{IR})$. We first note that by (r1), (r5) and (r6) there exists a pair of rationals $r_1 < r_2$ such that

$$\xi(\text{—},r_1) = \xi(r_2,\text{—}) = 0 \text{ and } \xi(r_1,\text{—}) = \xi(\text{—},r_2) = 1.$$

Indeed, if $\xi(r,\text{—}) = 0$ for every $r \in \mathbb{Q}$, then $\xi(\bigvee_{r\in\mathbb{Q}}(r,\text{—})) = \bigvee_{r\in\mathbb{Q}}\xi(r,\text{—}) = 0$, contradicting (r5) by the compactness of $\{0,1\}$. Therefore there exists some $r_1 \in \mathbb{Q}$ such that $\xi(r_1,\text{—}) = 1$ and then, by (r1), $0 = \xi(0) = \xi((r_1,\text{—}) \wedge (\text{—},r_1)) = \xi(\text{—},r_1)$. By a similar argument, using (r1) and (r6), we may conclude that $\xi(\text{—},r_2) = 1$ and $\xi(r_2,\text{—}) = 1$ for some $r_2 \in \mathbb{Q}$. Finally, $1 = \xi(r_1,\text{—}) \wedge \xi(\text{—},r_2) = \xi((r_1,\text{—}) \wedge (\text{—},r_2))$ implies $r_1 < r_2$, by (r1).

It now follows that we have

$$\underline{\tau(\xi)} = \bigvee\{r \in \mathbb{Q} \mid \xi(r,\text{—}) = 1\} \in \mathbb{R} \quad \text{and} \quad \overline{\tau(\xi)} = \bigwedge\{s \in \mathbb{Q} \mid \xi(\text{—},s) = 1\} \in \mathbb{R}.$$

For any such r, s, $\xi((r,\text{—}) \wedge (\text{—},s)) = \xi(r,\text{—}) \wedge \xi(\text{—},s) = 1$ and thus, by (r1), $r < s$. Hence

$$\underline{\tau(\xi)} \leq \overline{\tau(\xi)}$$

and $\tau(\xi) = \left[\underline{\tau(\xi)}, \overline{\tau(\xi)}\right]$ belongs in fact to $\mathbb{IR}$.

In order to show that τ is one-one, let $\xi_1 \neq \xi_2$. Then there exists $r \in \mathbb{Q}$ such that, say, $\xi_1(r,\text{—}) = 1$ and $\xi_2(r,\text{—}) = 0$. Then, by (r3), $1 = \xi_1(r,\text{—}) = \xi_1(\bigvee_{p>r}(p,\text{—}))$. Thus there exists $p > r$ such that $\xi_1(p,\text{—}) = 1$, and hence $r < p \leq \underline{\tau(\xi_1)}$. On the other hand, since $\xi_2(q,\text{—}) = 0$ for each $q \geq r$, it follows that

$$\underline{\tau(\xi_2)} = \bigvee\{q \in \mathbb{Q} \mid \xi_2(q,\text{—}) = 1\} \leq r.$$

Hence $\underline{\tau(\xi_2)} \leq r < p \leq \underline{\tau(\xi_1)}$. The arguments for the other cases are analogous.

τ is also surjective. Indeed, given $\boldsymbol{a} \in \mathbb{IR}$, let $\xi_{\boldsymbol{a}}\colon \mathfrak{L}(\mathbb{IR}) \to \{0,1\}$ be given by $\xi_{\boldsymbol{a}}(r,\text{—}) = 1$ iff $r < \underline{a}$ and $\xi_{\boldsymbol{a}}(\text{—},r) = 1$ iff $\overline{a} < r$ for every $r \in \mathbb{Q}$. It is easy to check that this correspondence turns the defining relations (r1), (r3)–(r6) into identities in $\{0,1\}$ and so each $\xi_{\boldsymbol{a}}$ is a frame homomorphism. Moreover

$$\underline{\tau(\xi_{\boldsymbol{a}})} = \bigvee\{r \in \mathbb{Q} \mid \xi_{\boldsymbol{a}}(r,\text{—}) = 1\} = \bigvee\{r \in \mathbb{Q} \mid r < \underline{a}\} = \underline{a}$$

and

$$\overline{\tau(\xi_{\boldsymbol{a}})} = \bigwedge\{r \in \mathbb{Q} \mid \xi_{\boldsymbol{a}}(\text{—}, r) = 1\} = \bigwedge\{r \in \mathbb{Q} \mid \overline{a} < r\} = \overline{a}.$$

Hence $\tau(\xi_{\boldsymbol{a}}) = \boldsymbol{a}$.

It remains to show τ is a homeomorphism. Now, for each basic Scott open set $\Uparrow\boldsymbol{a}$ (with $\boldsymbol{a} \in \mathbb{IR}$ and $\underline{a}, \overline{a} \in \mathbb{Q}$) we have that,

$$\begin{aligned}\tau^{-1}(\Uparrow\boldsymbol{a}) &= \{\xi_{\boldsymbol{b}} \in \Sigma\mathfrak{L}(\mathbb{IR}) \mid \boldsymbol{a} \ll \boldsymbol{b}\} = \left\{\xi_{\boldsymbol{b}} \in \Sigma\mathfrak{L}(\mathbb{IR}) \mid \underline{a} < \underline{b} \leq \overline{b} < \overline{a}\right\} \\ &= \{\xi_{\boldsymbol{b}} \in \Sigma\mathfrak{L}(\mathbb{IR}) \mid \xi_{\boldsymbol{b}}(\underline{a}, \text{—}) = 1 \text{ and } \xi_{\boldsymbol{b}}(\text{—}, \overline{a}) = 1\} = \Sigma_{(\underline{a},\text{—})} \cap \Sigma_{(\text{—},\overline{a})}.\end{aligned}$$

Hence τ is continuous. On the other hand, for any opens $\Sigma_{(r,\text{—})}$ or $\Sigma_{(\text{—},r)}$ of $\Sigma\mathfrak{L}(\mathbb{IR})$,

$$\begin{aligned}\tau\left(\Sigma_{(r,\text{—})}\right) &= \{\tau(\xi) \mid \xi \in \Sigma\mathfrak{L}(\mathbb{IR}) \text{ and } \xi(r, \text{—}) = 1\} \\ &= \{\tau(\xi_{\boldsymbol{a}}) \mid \boldsymbol{a} \in \mathbb{IR} \text{ and } \xi_{\boldsymbol{a}}(r, \text{—}) = 1\} = \{\boldsymbol{a} \in \mathbb{IR} \mid r < \underline{a}\} \\ &= \bigcup_{\beta\in\mathbb{R},\beta>r} \{\boldsymbol{a} \in \mathbb{IR} \mid r < \underline{a} \leq \overline{a} < \beta\} = \bigcup_{\beta\in\mathbb{R},\beta>r} \Uparrow[r, \beta].\end{aligned}$$

and

$$\begin{aligned}\tau\left(\Sigma_{(\text{—},r)}\right) &= \{\tau(\xi) \mid \xi \in \Sigma\mathfrak{L}(\mathbb{IR}) \text{ and } \xi(\text{—}, r) = 1\} \\ &= \{\tau(\xi_{\boldsymbol{a}}) \mid \boldsymbol{a} \in \mathbb{IR} \text{ and } \xi_{\boldsymbol{a}}(\text{—}, r) = 1\} = \{\boldsymbol{a} \in \mathbb{IR} \mid \overline{a} < r\} \\ &= \bigcup_{\alpha\in\mathbb{R},\alpha<r} \{\boldsymbol{a} \in \mathbb{IR} \mid \alpha < \underline{a} \leq \overline{a} < r\} = \bigcup_{\alpha\in\mathbb{R},\alpha<r} \Uparrow[\alpha, r]\end{aligned}$$

are Scott open sets. ■

Remark 2.3. The homeomorphism $\tau^{-1}\colon \mathbb{IR} \to \Sigma\mathfrak{L}(\mathbb{IR})$ induces an isomorphism

$$\mathcal{O}\Sigma\mathfrak{L}(\mathbb{IR}) \to \mathcal{O}\mathbb{IR}, \quad \Sigma_{(r,\text{—})} \mapsto \pi_1^{-1}(r, +\infty), \quad \Sigma_{(\text{—},r)} \mapsto \pi_2^{-1}(-\infty, r).$$

Thus the homomorphism $\mathfrak{L}(\mathbb{IR}) \to \mathcal{O}\mathbb{IR}$ taking $(r, \text{—})$ to $\pi_1^{-1}(r, +\infty)$ and $(\text{—}, r)$ to $\pi_2^{-1}(-\infty, r)$ is the spatial reflection map $\eta_{\mathfrak{L}(\mathbb{IR})}$ of the frame of partial real numbers. Note that this homomorphism is an isomorphism. Indeed, $\eta_{\mathfrak{L}(\mathbb{IR})}$ is onto, since for each $\boldsymbol{a} \in \mathbb{IR}$ with $\underline{a}, \overline{a} \in \mathbb{Q}$,

$$\eta_{\mathfrak{L}(\mathbb{IR})}\big((\underline{a}, \text{—}) \wedge (\text{—}, \overline{a})\big) = \{\boldsymbol{b} \in \mathbb{IR} \mid \underline{a} < \underline{b} \text{ and } \overline{b} < \overline{a}\} = \Uparrow\boldsymbol{a},$$

and $\Psi\colon \mathcal{O}\mathbb{IR} \to \mathfrak{L}(\mathbb{IR})$, given by

$$\Psi(\Uparrow\boldsymbol{a}) = (\underline{a}, \text{—}) \wedge (\text{—}, \overline{a}) \quad \text{for each } \boldsymbol{a} \in \mathbb{IR} \text{ such that } \underline{a}, \overline{a} \in \mathbb{Q},$$

is a left inverse of $\eta_{\mathfrak{L}(\mathbb{IR})}$:

$$\begin{aligned}
\Psi \circ \eta_{\mathfrak{L}(\mathbb{IR})}(r,\text{—}) &= \Psi\big(\{\boldsymbol{a} \in \mathbb{IR} \mid r < \underline{a}\}\big) \\
&= \Psi\big(\bigcup\{\Uparrow\boldsymbol{b} \mid \underline{b} = r, \overline{b} \in \mathbb{Q}\}\big) \\
&= \bigvee\big\{(r,\text{—}) \wedge (\text{—},\overline{b}) \mid \overline{b} \in \mathbb{Q}\big\} \\
&= (r,\text{—}) \wedge \bigvee\big\{(\text{—},\overline{b}) \mid \overline{b} \in \mathbb{Q}\big\} = (r,\text{—}), \\
\Psi \circ \eta_{\mathfrak{L}(\mathbb{IR})}(\text{—},r) &= \Psi\big(\{\boldsymbol{a} \in \mathbb{IR} \mid \overline{a} < r\}\big) \\
&= \Psi\big(\bigcup\{\Uparrow\boldsymbol{b} \mid \underline{b} \in \mathbb{Q}, \overline{b} < r\}\big) \\
&= \bigvee\big\{(\underline{b},\text{—}) \wedge (\text{—},r) \mid \underline{b} \in \mathbb{Q}\big\} \\
&= (\text{—},r) \wedge \bigvee\big\{(\underline{b},\text{—}) \mid \overline{b} \in \mathbb{Q}\big\} = (\text{—},r).
\end{aligned}$$

Definition 2.4. A *continuous partial real function* on a frame L is a frame homomorphism $h\colon \mathfrak{L}(\mathbb{IR}) \to L$.

As in the case of continuous real functions on a space X, one can easily show that continuous functions $X \to \mathbb{IR}$ may be represented as frame homomorphisms $h\colon \mathfrak{L}(\mathbb{IR}) \to \mathcal{O}X$, which justifies the preceding definition:

Corollary 2.5. *For each topological space* $(X, \mathcal{O}X)$ *there is a natural isomorphism*

$$\Phi\colon \mathsf{Frm}(\mathfrak{L}(\mathbb{IR}), \mathcal{O}X) \xrightarrow{\sim} \mathsf{Top}(X, \mathbb{IR}).$$

Proof: By the (dual) adjunction between contravariant functors $\mathcal{O}\colon \mathsf{Top} \to \mathsf{Frm}$ and $\Sigma\colon \mathsf{Frm} \to \mathsf{Top}$ there is a natural isomorphism

$$\mathsf{Frm}(L, \mathcal{O}X) \xrightarrow{\sim} \mathsf{Top}(X, \Sigma L)$$

for all L and X. Combining this for $L = \mathfrak{L}(\mathbb{IR})$ with the homeomorphism $\tau\colon \Sigma(\mathfrak{L}(\mathbb{IR})) \to \mathbb{IR}$ from Proposition 2.2 one obtains the isomorphism. ■

Specifically, Φ is given by the correspondence $h \mapsto \tilde{h}$ where

$$\tilde{h}(x) = [\bigvee\{r \in \mathbb{Q} \mid x \in h(r,\text{—})\}, \bigwedge\{r \in \mathbb{Q} \mid x \in h(\text{—},r)\}] \quad \text{for every } x \in X.$$

In the opposite direction, given $f \in \mathrm{C}(X, \mathbb{IR})$ the corresponding h is defined by

$$h(r,\text{—}) = (\pi_1 \circ f)^{-1}(r, +\infty) \text{ and } h(\text{—},r) = (\pi_2 \circ f)^{-1}(-\infty, r) \text{ for every } r \in \mathbb{Q}.$$

We shall denote by $\mathrm{IC}(L)$ the set $\mathsf{Frm}(\mathfrak{L}(\mathbb{IR}), L)$, partially ordered by

$$f \leq g \quad \text{iff} \quad f(r,\text{—}) \leq g(r,\text{—}) \text{ and } g(\text{—},r) \leq f(\text{—},r) \quad \text{for all } r \in \mathbb{Q}. \quad (2.5.1)$$

Remarks 2.6. (1) Note that using the basic homomorphism $\varrho\colon \mathfrak{L}(\mathbb{IR}) \to \mathfrak{L}(\mathbb{R})$, continuous real maps $h \in \mathrm{C}(L)$ are in a one-to-one correspondence with the $\hat{h} \in \mathrm{IC}(L)$ such that $\hat{h}(r,\text{—}) \vee \hat{h}(\text{—},s) = 1$ whenever $r < s$ (just take $\hat{h} = h \cdot \varrho$). So in what follows we will keep the notation $\mathrm{C}(L)$ to denote also the class inside $\mathrm{IC}(L)$ of the h's such that $h(r,\text{—}) \vee h(\text{—},s) = 1$ whenever $r < s$.

(2) In case $f \in \mathrm{C}(L)$, as in (1.3.1), the second condition on f and g in (2.5.1) is needless because it is equivalent to the first one:

$$g(\text{—},r) = g(\bigvee_{s<r}(\text{—},s)) = \bigvee_{s<r} g(\text{—},s) \leq \bigvee_{s<r} g(s,\text{—})^* \leq \bigvee_{s<r} f(s,\text{—})^* \leq f(\text{—},r),$$

the last inequality because f being in $\mathrm{C}(L)$ then, by (r2), $f(s,\text{—}) \vee f(\text{—},r) = 1$ (the argument for the converse is analogous).

A continuous partial real function $h \in \mathrm{IC}(L)$ is said to be *bounded* if there exist $p, q \in \mathbb{Q}$ such that $\boldsymbol{p} \leq h \leq \boldsymbol{q}$. Equivalently,

$$h \text{ is bounded} \quad \text{iff} \quad \exists r \in \mathbb{Q} \text{ such that } h(-r,r) = 1.$$

We shall denote by $\mathrm{IC}^*(L)$ the set of bounded functions in $\mathrm{IC}(L)$.

Example 2.7. For each $a, b \in L$ such that $a \wedge b = 0$ let $\chi_{a,b}$ denote the bounded continuous partial real function given by

$$\chi_{a,b}(r,\text{—}) = \begin{cases} 0 & \text{if } r \geq 1 \\ a & \text{if } 0 \leq r < 1 \\ 1 & \text{if } r < 0 \end{cases} \quad \text{and} \quad \chi_{a,b}(\text{—},r) = \begin{cases} 1 & \text{if } r > 1 \\ b & \text{if } 0 < r \leq 1 \\ 0 & \text{if } r \leq 0 \end{cases}$$

for each $r \in \mathbb{Q}$. Clearly, $\chi_{a,b} \in \mathrm{C}^*(L)$ iff $a \vee b = 1$, i.e. iff a is complemented with complement b.

Remark 2.8. There is an order reversing isomorphism $-(\cdot)\colon \mathrm{IC}(L) \to \mathrm{IC}(L)$ defined by

$$(-h)(\text{—},r) = h(-r,\text{—}) \quad \text{and} \quad (-h)(r,\text{—}) = h(\text{—},-r) \quad \text{for all } r \in \mathbb{Q}.$$

When restricted to $\mathrm{C}(L)$ it yields an isomorphism $\mathrm{C}(L) \to \mathrm{C}(L)$.

Using these isomorphisms, the study of the Dedekind order completeness of $\mathrm{IC}(L)$ and $\mathrm{C}(L)$ boils down to the discussion of the existence of non-void bounded suprema. We start with $\mathrm{IC}(L)$.

Proposition 2.9. $\mathrm{IC}(L)$ *is closed under non-void bounded suprema.*

Proof: Let $\{h_i\}_{i\in I} \subseteq \mathrm{IC}(L)$ and $h \in \mathrm{IC}(L)$ such that $h_i \le h$ for all $i \in I$. For each $r, s \in \mathbb{Q}$ we define $h_\vee \colon \mathfrak{L}(\mathbb{IR}) \to L$ on generators by

$$h_\vee(r,\text{—}) = \bigvee_{i\in I} h_i(r,\text{—}) \quad \text{and} \quad h_\vee(\text{—},s) = \bigvee_{q<s} \bigwedge_{i\in I} h_i(\text{—},q).$$

This is a frame homomorphism since it turns the defining relations (r1) and (r3)–(r6) of $\mathfrak{L}(\mathbb{IR})$ into identities in L:

(r1) $h_\vee(r,\text{—}) \wedge h_\vee(\text{—},s) \le \bigvee_{i\in I}\bigvee_{q<s} h_i(r,\text{—}) \wedge h_i(\text{—},q) \le \bigvee_{i\in I} h_i(r,\text{—}) \wedge h_i(\text{—},s) = 0$, whenever $r \ge s$.
(r3) $\bigvee_{s>r} h_\vee(s,\text{—}) = \bigvee_{i\in I}\bigvee_{s>r} h_i(s,\text{—}) = \bigvee_{i\in I} h_i(r,\text{—}) = h_\vee(r,\text{—})$ for each $r \in \mathbb{Q}$.
(r4) $\bigvee_{s<r} h_\vee(\text{—},s) = \bigvee_{s<r}\bigvee_{q<s}\bigwedge_{i\in I} h_i(\text{—},q) = \bigvee_{q<r}\bigwedge_{i\in I} h_i(\text{—},q) = h_\vee(\text{—},r)$ for each $r \in \mathbb{Q}$.
(r5) $\bigvee_{r\in\mathbb{Q}} h_\vee(r,\text{—}) = \bigvee_{r\in\mathbb{Q}}\bigvee_{i\in I} h_i(r,\text{—}) = \bigvee_{i\in I}\bigvee_{r\in\mathbb{Q}} h_i(r,\text{—}) = 1$.
(r6) $\bigvee_{s\in\mathbb{Q}} h_\vee(\text{—},s) = \bigvee_{s\in\mathbb{Q}}\bigvee_{q<s}\bigwedge_{i\in I} h_i(\text{—},q) \ge \bigvee_{q\in\mathbb{Q}} h(\text{—},q) = 1$.

Hence $h_\vee \in \mathrm{IC}(L)$. In addition, for each $i \in I$ and $r, s \in \mathbb{Q}$,

$$\begin{aligned} &h_i(r,\text{—}) \le h_\vee(r,\text{—}) \le h(r,\text{—}) \quad \text{and} \\ &h(\text{—},s) = \bigvee_{q<s} h(\text{—},q) \le h_\vee(\text{—},s) \le \bigvee_{q<s} h_i(\text{—},q) = h_i(\text{—},s) \end{aligned}$$

and thus $h_i \le h_\vee \le h$ for every $i \in I$. Finally, if $g \in \mathrm{IC}(L)$ is such that $h_i \le g$ for every $i \in I$, then we have, for each $r, s \in \mathbb{Q}$,

$$\begin{aligned} &g(r,\text{—}) \ge \bigvee_{i\in I} h_i(r,\text{—}) = h_\vee(r,\text{—}) \quad \text{and} \\ &g(\text{—},s) = \bigvee_{q<s} g(\text{—},q) \le \bigvee_{q<s}\bigwedge_{i\in I} h_i(\text{—},q) = h_\vee(\text{—},s) \end{aligned}$$

and so $h_\vee \le g$. Hence $h_\vee$ is in fact the supremum of $\{h_i\}_{i\in I}$ in $\mathrm{IC}(L)$. ∎

Corollary 2.10. $\mathrm{IC}(L)$ *is Dedekind order complete.*

Proof: It follows easily from Proposition 2.9 and Remark 2.8. Indeed, let $\{h_i\}_{i\in I} \subseteq \mathrm{IC}(L)$ and $h \in \mathrm{IC}(L)$ such that $h \le h_i$ for all $i \in I$. Then $\{-h_i\}_{i\in I} \subseteq \mathrm{IC}(L)$, $-h \in \mathrm{IC}(L)$ and $-h_i \le -h$ for all $i \in I$. It follows that $\overset{\mathrm{IC}(L)}{\bigvee}_{i\in I}(-h_i)$ do exist. It is easy to check that $\overset{\mathrm{IC}(L)}{\bigwedge}_{i\in I} h_i = -\overset{\mathrm{IC}(L)}{\bigvee}_{i\in I}(-h_i)$. ∎

3. The Dedekind order completion of $\mathrm{C}(L)$

Now, since $\mathrm{IC}(L)$ is Dedekind order complete it follows that it contains the Dedekind order completion of all its subposets, in particular $\mathrm{C}(L)$. Our next task will be to determine the Dedekind order completion of $\mathrm{C}(L)$. As a by-product we shall also determine the order completion of $\mathrm{C}(L)$ in the sense of [5].

We first note that, as explained in ([5], Section 2), there is no essential loss of generality if we restrict ourselves to *completely regular* frames. So, in the sequel, all frames will be taken as completely regular. We start by establishing a couple of lemmas:

Lemma 3.1. *Let L be a completely regular frame and let $h \in \mathrm{IC}(L)$ be such that*

(1) $\{f \in \mathrm{C}(L) \mid f \leq h\} \neq \varnothing$ *and*
(2) $h(p,\text{—})^* \leq h(\text{—},q)$ *whenever* $p < q$.

Then $h = \bigvee^{\mathrm{IC}(L)} \{f \in \mathrm{C}(L) \mid f \leq h\}$.

Proof: Let $\mathcal{F} = \{f \in \mathrm{C}(L) \mid f \leq h\}$. By (1), $\mathcal{F} \neq \varnothing$. Since $\mathrm{IC}(L)$ is Dedekind order complete, the supremum $\bigvee^{\mathrm{IC}(L)} \mathcal{F}$ exists. We shall prove that

$$\bigvee^{\mathrm{IC}(L)} \mathcal{F} = h.$$

We only need to show that, for any $h' \in \mathrm{IC}(L)$ such that $f \leq h'$ for all $f \in \mathcal{F}$, $h \leq h'$ i.e.
(a) $h(p,\text{—}) \leq h'(p,\text{—})$ for every $p \in \mathbb{Q}$ and
(b) $h(\text{—},q) \geq h'(\text{—},q)$ for every $q \in \mathbb{Q}$.

(a): We fix $p \in \mathbb{Q}$ and consider $p' \in \mathbb{Q}$ such that $p < p'$. Since L is completely regular, then $h(p',\text{—}) = \bigvee\{a \in L \mid a \prec\!\prec h(p',\text{—})\}$. Let $a \in L$ such that $a \prec\!\prec h(p',\text{—})$. Then there exists a family $\{c_r : r \in \mathbb{Q} \cap [0,1]\} \subseteq L$ such that $a \leq c_0$, $c_1 \leq h(p',\text{—})$ and $c_r \prec c_s$ whenever $r < s$. Hence the map $\sigma_{a,p'} \colon \mathbb{Q} \to L$ given by

$$\sigma_{a,p'}(r) = \begin{cases} 0 & \text{if } r > 1 \\ c_{1-r} & \text{if } 0 \leq r \leq 1 \\ 1 & \text{if } r < 0 \end{cases}$$

is a scale and generates a $g_{a,p'} \in \mathrm{C}(L)$ given by

$$g_{a,p'}(r,\text{—}) = \begin{cases} 0 & \text{if } r \leq 1 \\ \bigvee_{r'>r} c_{1-r'} & \text{if } 0 \leq r < 1 \\ 1 & \text{if } r < 0 \end{cases}$$

and

$$g_{a,p'}(\text{—},s) = \begin{cases} 1 & \text{if } s > 1 \\ \bigvee_{s'<s} c^*_{1-s'} & \text{if } 0 < s \leq 1 \\ 0 & \text{if } s \leq 0. \end{cases}$$

Evidently $\mathbf{0} \leq g_{a,p'} \leq \mathbf{1}$. Let

$$f_{a,p'} = f + (((\boldsymbol{p'} - f) \vee \mathbf{0}) \cdot g_{a,p'}) \in \mathrm{C}(L).$$

We have $f_{a,p'} \leq h$; indeed, for each $r \in \mathbb{Q}$,

$$\begin{aligned} f_{a,p'}(r,\text{—}) &= \bigvee_{r' \in \mathbb{Q}} f(r - r', \text{—}) \wedge \left(((\boldsymbol{p'} - f) \vee \mathbf{0}) \cdot g_{a,p'}\right)(r',\text{—}) \\ &= \bigvee_{r'<0} f(r-r',\text{—}) \vee \bigvee_{r' \geq 0} f(r - r', \text{—}) \wedge \left(((\boldsymbol{p'} - f) \vee \mathbf{0}) \cdot g_{a,p'}\right)(r',\text{—}) \\ &= f(r,\text{—}) \vee \bigvee_{r' \geq 0} \bigvee_{r''>0} f(r-r',\text{—}) \wedge ((\boldsymbol{p'} - f) \vee \mathbf{0})(r'',\text{—}) \wedge g_{a,p'}\left(\frac{r'}{r''},\text{—}\right) \\ &= f(r,\text{—}) \vee \bigvee_{r' \geq 0} \bigvee_{r''>0} f(r-r', p'-r'') \wedge g_{a,p'}\left(\frac{r'}{r''},\text{—}\right) \\ &= f(r,\text{—}) \vee \bigvee_{r' \geq 0} \bigvee_{r'<r''<p'-r+r'} f(r-r', p'-r'') \wedge g_{a,p'}\left(\frac{r'}{r''},\text{—}\right) \\ &= f(r,\text{—}) \vee \bigvee_{r' \geq 0} \bigvee_{r'<r''<p'-r+r'} \bigvee_{r'''>\frac{r'}{r''}} f(r-r', p'-r'') \wedge c_{1-r'''}. \end{aligned}$$

Now, if $r \geq p'$ then $p' - r + r' \leq r'$ for each $r' \geq 0$ and thus $f_{a,p'}(r,—) = f(r,—) \leq h(r,—)$. Otherwise, if $r < p'$ then

$$\begin{aligned}
f_{a,p'}(r,—) &\leq f(r,—) \vee \bigvee_{r' \geq 0} \bigvee_{r' < r'' < p' - r + r'} f(r - r', p' - r'') \wedge c_1 \\
&= f(r,—) \vee \bigvee_{r' \geq 0} f(r - r', p' - r') \wedge c_1 \\
&= f(r,—) \vee (f(—, p') \wedge c_1) \\
&= (f(r,—) \vee f(—, p')) \wedge (f(r,—) \vee c_1) \\
&= f(r,—) \vee c_1 \\
&\leq h(r,—) \vee h\,(p',—) = h(r,—).
\end{aligned}$$

Hence $f_{a,p'}(r,—) \leq h(r,—)$ for every $r \in \mathbb{Q}$ and since $f_{a,p'} \in \mathrm{C}(L)$, it follows that $f_{a,p'} \leq h$, by Remark 2 of 2.6, and we may conclude that $f_{a,p'} \in \mathcal{F}$.

Finally, since $p < p'$ it follows that

$$\begin{aligned}
f_{a,p'}(p,—) &= f(p,—) \vee \bigvee_{r' \geq 0} \bigvee_{r' < r'' < p' - p + r'} \bigvee_{r''' > \frac{r'}{r''}} f(p - r', p' - r'') \wedge c_{1 - r'''} \\
&\geq f(p,—) \vee \bigvee_{r' \geq 0} \bigvee_{r' < r'' < p' - p + r'} f(p - r', p' - r'') \wedge c_0 \\
&= f(p,—) \vee \bigvee_{r' \geq 0} f(p - r', p' - r') \wedge c_0 \\
&= f(p,—) \vee (f(—, p') \wedge c_0) \\
&= (f(p,—) \vee f(—, p')) \wedge (f(r,—) \vee c_0) \\
&= f(p,—) \vee c_0 \geq c_0
\end{aligned}$$

and thus $a \leq c_0 \leq f_{a,p'}(p,—) \leq h'(p,—)$. Hence

$$h(p,—) = \bigvee_{p' > p} h(p',—) = \bigvee_{p' > p} \bigvee_{a \prec\!\!\prec h(p',—)} a \leq h'(p,—).$$

(b): Using (2) it follows that

$$\begin{aligned}
h(—, q) &= \bigvee_{s' < s} \bigvee_{s < q} h(—, s') \geq \bigvee_{s < q} h(s,—)^* \\
&\geq \bigvee_{s < q} h'(s,—)^* \geq \bigvee_{s < q} h'(—, s) = h'(—, q). \quad \blacksquare
\end{aligned}$$

Then, it follows from Lemma 3.1 and Remark 2.8 that:

Lemma 3.2. *Let L be a completely regular frame and let $h \in \mathrm{IC}(L)$ be such that*

(1) $\{g \in \mathrm{C}(L) \mid h \leq g\} \neq \varnothing$ *and*
(2) $h(\text{—}, q)^* \leq h(p, \text{—})$ *whenever* $p < q$.

Then $h = \bigwedge^{\mathrm{IC}(L)} \{g \in \mathrm{C}(L) \mid h \leq g\}$. ■

We introduce now the following classes:

$\mathrm{C}(L)^{\vee} = \{h \in \mathrm{IC}(L) \mid \exists f, g \in \mathrm{C}(L) : f \leq h \leq g \text{ and } h(p, \text{—})^* \leq h(\text{—}, q) \text{ if } p < q\}$,

$\mathrm{C}(L)^{\wedge} = \{h \in \mathrm{IC}(L) \mid \exists f, g \in \mathrm{C}(L) : f \leq h \leq g \text{ and } h(\text{—}, q)^* \leq h(p, \text{—}) \text{ if } p < q\}$,

$\mathrm{C}(L)^{\vee\!\!\!\wedge} = \mathrm{C}(L)^{\vee} \cap \mathrm{C}(L)^{\wedge}$.

Next result is an immediate consequence of Lemmas 3.1 and 3.2.

Proposition 3.3. *Let L be a completely regular frame and let $h \in \mathrm{C}(L)^{\vee\!\!\!\wedge}$. Then*

$$h = \bigvee^{\mathrm{IC}(L)} \{f \in \mathrm{C}(L) \mid f \leq h\} = \bigwedge^{\mathrm{IC}(L)} \{g \in \mathrm{C}(L) \mid h \leq g\}. \quad ■$$

The following diagram depicts the inclusions between those classes (each arrow represents a strict inclusion):

$$\begin{tikzcd}
 & & \mathrm{C}(L)^{\vee} \arrow[dr] & \\
\mathrm{C}(L) \arrow[r] & \mathrm{C}(L)^{\vee\!\!\!\wedge} \arrow[ur] \arrow[dr] & & \mathrm{IC}(L) \\
 & & \mathrm{C}(L)^{\wedge} \arrow[ur] &
\end{tikzcd}$$

The only non trivial inclusion, $\mathrm{C}(L) \subseteq \mathrm{C}(L)^{\vee\!\!\!\wedge}$, follows from the fact that $h(p, \text{—}) \vee h(\text{—}, q) = 1$ implies $h(p, \text{—})^* \leq h(\text{—}, q)$ and $h(\text{—}, q)^* \leq h(p, \text{—})$. Further, the inclusions are strict. Indeed, for each $a, b \in L$ such that $a \wedge b = 0$ recall the bounded $\chi_{a,b}$ from Example 2.7. Then:

(1) $\chi_{a,b} \in \mathrm{C}(L)^{\vee}$ if and only if $a^* = b$;
(2) $\chi_{a,b} \in \mathrm{C}(L)^{\wedge}$ if and only if $b^* = a$;
(3) $\chi_{a,b} \in \mathrm{C}(L)^{\vee\!\!\!\wedge}$ if and only if $a^* = b$ and $b^* = a$, i.e. if and only if a is regular and $b = a^*$.

Consequently,

- if a is regular but not complemented then $\chi_{a,a^*} \in \mathrm{C}(L)^{\vee\!\!\!\wedge} \smallsetminus \mathrm{C}(L)$;

- if $a^* = b$ but $b^* \neq a$ then $\chi_{a,a^*} \in \mathrm{C}(L)^\vee \smallsetminus \mathrm{C}(L)^\wedge$ (for instance, take $L = \mathcal{O}\mathbb{R}$, $a = \mathbb{R} \smallsetminus \{0\}$ and $b = \varnothing$);
- if $b^* = a$ but $a^* \neq b$ then $\chi_{b^*,b} \in \mathrm{C}(L)^\wedge \smallsetminus \mathrm{C}(L)^\vee$;
- if $a^* \neq b$ and $b^* \neq a$ then $\chi_{a,b} \in \mathrm{IC}(L) \smallsetminus (\mathrm{C}(L)^\vee \cup \mathrm{C}(L)^\wedge)$ (for instance, take $a = b = 0$).

Remark 3.4. The order reversing isomorphism $-(\cdot)\colon \mathrm{IC}(L) \to \mathrm{IC}(L)$ introduced in Remark 2.8 induces an isomorphism from $\mathrm{C}(L)^\vee$ onto $\mathrm{C}(L)^\wedge$ (and hence an isomorphism from $\mathrm{C}(L)^{\vee\!\!\!\wedge}$ onto $\mathrm{C}(L)^{\vee\!\!\!\wedge}$).

We can consider on $\mathrm{IC}(L)$ a further partial order given by

$$f \sqsubseteq g \quad \text{if } f(r,\text{—}) \leq g(r\text{—}) \text{ and } f(\text{—},r) \leq g(\text{—},r) \text{ for all } r \in \mathbb{Q}.$$

Now we shall see how the classes $\mathrm{C}(L)^\vee$, $\mathrm{C}(L)^\wedge$ and $\mathrm{C}(L)^{\vee\!\!\!\wedge}$ can be alternatively described in terms of $\sqsubseteq$:

Proposition 3.5. *The following are equivalent for any $h \in \mathrm{IC}(L)$.*

(i) $h(p,\text{—})^* \leq h(\text{—},q)$ *whenever $p < q$ in $\mathbb{Q}$.*

(ii) $g(\text{—},r) = h(\text{—},r)$ *for all $r \in \mathbb{Q}$ and all $g \in \mathrm{IC}(L)$ such that $h \sqsubseteq g$.*

Proof: In order to check that (i) $\implies$ (ii), let $g \in \mathrm{IC}(L)$ such that $h \sqsubseteq g$. By (i),

$$g(p,\text{—})^* \leq h(p,\text{—})^* \leq h(\text{—},q) \leq g(\text{—},q)$$

for all $p < q$ in $\mathbb{Q}$. Consequently, $g(\text{—},q) = \bigvee_{p<q} g(p,\text{—})^*$ for all $q \in \mathbb{Q}$. Thus we get $g(\text{—},q) = \bigvee_{p<q} g(p,\text{—})^* \leq h(\text{—},q) \leq g(\text{—},q)$ and so $g(\text{—},q) = h(\text{—},q)$.

For the reverse implication let $g \in \mathrm{IC}(L)$ be defined as follows:

$$g(r,\text{—}) = \bigvee_{s>r} h(s,\text{—})^{**} \quad \text{and} \quad g(\text{—},r) = \bigvee_{s<r} h(s,\text{—})^*.$$

It is straightforward to check that g is indeed a partial continuous functions and that $h \sqsubseteq g$. Therefore, by hypothesis, $h(\text{—},r) = g(\text{—},r)$ for all $r \in \mathbb{Q}$. Consequently, $\bigvee_{s<r} h(s,\text{—})^* = h(\text{—},r)$ which implies $h(s,\text{—})^* \leq h(\text{—},r)$ for all $s < r$ in $\mathbb{Q}$. ∎

Proposition 3.6. *The following are equivalent for any $h \in \mathrm{IC}(L)$.*

(i) $h(\text{—},q)^* \leq h(p,\text{—})$ *whenever $p < q$ in $\mathbb{Q}$;*

(ii) $g(r,\text{—}) = h(r,\text{—})$ *for all $r \in \mathbb{Q}$ and all $g \in \mathrm{IC}(L)$ such that $h \sqsubseteq g$.*

Proof: Clearly, $h(\text{—},q)^* \leq h(p,\text{—})$ for all $p < q$ if and only if $(-h(-q,\text{—}))^* \leq -h(\text{—},-p)$ for all $-q < -p$, which is equivalent to $-g(\text{—},r) = -h(\text{—},r)$ for all $r \in \mathbb{Q}$ and all $g \in \mathrm{IC}(L)$ such that $-h \sqsubseteq -g$ (by Proposition 3.5). ∎

Then we may conclude that the elements h of $\mathrm{C}(L)^{\vee\!\!\!\wedge}$ are precisely the maximal elements of $(\mathrm{IC}(L), \sqsubseteq)$ for which there exist $f, g \in \mathrm{C}(L)$ satisfying $f \le h \le g$:

Corollary 3.7. *Let L be a frame. Then*

$$\mathrm{C}(L)^{\vee\!\!\!\wedge} = \{h \in \mathrm{IC}(L) \mid \begin{array}{l}\text{(a) } \textit{there exist } f, g \in \mathrm{C}(L) \textit{ such that } f \le h \le g \\ \text{(b) } h \sqsubseteq h' \in \mathrm{IC}(L) \implies h = h'\}. \quad \blacksquare\end{array}$$

We pick up now the thread of the narrative of the article by studying the completeness properties of $\mathrm{C}(L)^{\vee}$, $\mathrm{C}(L)^{\wedge}$ and $\mathrm{C}(L)^{\vee\!\!\!\wedge}$.

Proposition 3.8. $\mathrm{C}(L)^{\vee}$ *is closed under non-void bounded suprema and* $\mathrm{C}(L)^{\wedge}$ *is closed under non-void bounded infima.*

Proof: Let $\{h_i\}_{i\in I} \subseteq \mathrm{C}(L)^{\vee}$ and $h \in \mathrm{C}(L)^{\vee}$ such that $h_i \le h$ for all $i \in I$. On one hand, since $\mathrm{IC}(L)$ is Dedekind order complete, the supremum $\overset{\mathrm{IC}(L)}{\bigvee}_{i\in I} h_i$ exists and it is given by

$$\left(\bigvee_{i\in I}^{\mathrm{IC}(L)} h_i\right)(p,\text{—}) = \bigvee_{i\in I} h_i(p,\text{—}) \quad \text{and} \quad \left(\bigvee_{i\in I}^{\mathrm{IC}(L)} h_i\right)(\text{—},q) = \bigvee_{s<q}\bigwedge_{i\in I} h_i(\text{—},s)$$

for every $p, q \in \mathbb{Q}$. On the other hand, for each $i \in I$, since $h_i \in \mathrm{C}(L)^{\vee}$, there exists $f_i \in \mathrm{C}(L)$ such that $f_i \le h_i$, and since $h \in \mathrm{C}(L)^{\vee}$, there exists $g \in \mathrm{C}(L)$ such that $h \le g$. Consequently, $f_i \le h_i \le \overset{\mathrm{IC}(L)}{\bigvee}_{i\in I} h_i \le g$. Further, let $p < q$ in $\mathbb{Q}$ and $p < r < q$. Then

$$\left(\left(\bigvee_{i\in I}^{\mathrm{IC}(L)} h_i\right)(p,\text{—})\right)^{*} = \bigwedge_{i\in I} h_i(p,\text{—})^{*} \le \bigwedge_{i\in I} h_i(\text{—},r) \le \left(\bigvee_{i\in I}^{\mathrm{IC}(L)} h_i\right)(\text{—},q),$$

which shows that $\overset{\mathrm{IC}(L)}{\bigvee}_{i\in I} h_i \in \mathrm{C}(L)^{\vee}$.

The second assertion follows immediately by Remark 3.4. $\blacksquare$

Finally, we establish the main result of the paper.

Theorem 3.9. $\mathrm{C}(L)^{\vee\!\!\!\wedge}$ *is Dedekind order complete.*

Proof: (a) Let $\{h_i\}_{i\in I} \subseteq \mathrm{C}(L)^{\vee\!\!\!\wedge}$ and $h \in \mathrm{C}(L)^{\vee\!\!\!\wedge}$ such that $h_i \le h$ for all $i \in I$. For each $r, s \in \mathbb{Q}$ we define $h_{\vee} \colon \mathfrak{L}(\mathbb{IR}) \to L$ on generators by

$$h_{\vee}(r,\text{—}) = \bigvee_{p>r}\left(\bigwedge_{i\in I} h_i(p,\text{—})^{*}\right)^{*} \quad \text{and} \quad h_{\vee}(\text{—},s) = \bigvee_{q<s}\bigwedge_{i\in I} h_i(q,\text{—})^{*}.$$

This is a frame homomorphism since it turns the defining relations (r1) and (r3)–(r6) of $\mathfrak{L}(\mathbb{IR})$ into identities in L:

(r1) $h_\vee(r,\text{—}) \wedge h_\vee(\text{—},s) \leq \left(\bigwedge_{i\in I} h_i(r,\text{—})^*\right)^* \wedge \bigwedge_{i\in I} h_i(r,\text{—})^* = 0$ whenever $r \geq s$.

(r3) $\bigvee_{s>r} h_\vee(s,\text{—}) = \bigvee_{s>r} \bigvee_{p>s} \left(\bigwedge_{i\in I} h_i(p,\text{—})^*\right)^* = h_\vee(r,\text{—})$ for each $r \in \mathbb{Q}$.

(r4) $\bigvee_{s<r} h_\vee(\text{—},s) = \bigvee_{s<r} \bigvee_{q<s} \bigwedge_{i\in I} h_i(q,\text{—})^* = \bigvee_{q<r} \bigwedge_{i\in I} h_i(q,\text{—})^* = h_\vee(\text{—},r)$ for each $r \in \mathbb{Q}$.

(r5) $\bigvee_{r\in\mathbb{Q}} h_\vee(r,\text{—}) = \bigvee_{r\in\mathbb{Q}} \bigvee_{p>r} \left(\bigwedge_{i\in I} h_i(p,\text{—})^*\right)^* \geq \bigvee_{p\in\mathbb{Q}} \bigvee_{i\in I} h_i(p,\text{—}) = 1$.

(r6) $\bigvee_{s\in\mathbb{Q}} h_\vee(\text{—},s) = \bigvee_{s\in\mathbb{Q}} \bigvee_{q<s} \bigwedge_{i\in I} h_i(q,\text{—})^* = \bigvee_{q\in\mathbb{Q}} h(q,\text{—})^* \geq \bigvee_{q\in\mathbb{Q}} h(\text{—},q) = 1$.

Moreover, for each $r < s$ in $\mathbb{Q}$ and $r < t < s$,

$$h_\vee(r,\text{—})^* = \bigwedge_{p>r} \left(\bigwedge_{i\in I} h_i(p,\text{—})^* \right)^{**} = \bigwedge_{p>r} \left(\bigvee_{i\in I} h_i(p,\text{—}) \right)^{***} \leq$$
$$\leq \left(\bigvee_{i\in I} h_i(t,\text{—}) \right)^* \leq h_\vee(\text{—},s) \quad \text{and}$$
$$h_\vee(\text{—},s)^* = \bigwedge_{q<s} \left(\bigwedge_{i\in I} h_i(q,\text{—})^* \right)^* \leq \left(\bigwedge_{i\in I} h_i(t,\text{—})^* \right)^* \leq h_\vee(r,\text{—}).$$

Further, for each $r, s \in \mathbb{Q}$ and $i \in I$, we have

$$h_i(r,\text{—}) = \bigvee_{p>r} h_i(p,\text{—}) \leq \bigvee_{p>r} h_i(p,\text{—})^{**} \leq h_\vee(r,\text{—}) \leq$$
$$\leq \bigvee_{p>r} h(p,\text{—})^{**} \leq h(r,\text{—}) \quad \text{and}$$
$$h(\text{—},s) = \bigvee_{q<s} h(\text{—},q) \leq \bigvee_{q<s} h(q,\text{—})^* \leq h_\vee(\text{—},s) \leq \bigvee_{q<s} h_i(\text{—},q) = h_i(\text{—},s)$$

and thus $h_i \leq h_\vee \leq h$ for all $i \in I$. Since $h_i \in \mathrm{C}(L)^\vee$, there exists $f_i \in \mathrm{C}(L)$ such that $f_i \leq h_i$, and since $h \in \mathrm{C}(L)^\wedge$, there exists $g \in \mathrm{C}(L)$ such that $h \leq g$. Consequently $h_\vee \in \mathrm{C}(L)^{\vee\!\wedge}$. Finally, if $g \in \mathrm{C}(L)^{\vee\!\wedge}$ is such that $h_i \leq g$ for every $i \in I$, then

$$g(r,\text{—}) \geq \bigvee_{p>r} g(p,\text{—})^{**} \geq \bigvee_{p>r} \left(\bigvee_{i\in I} h_i(p,\text{—}) \right)^{**} = h_\vee(r,\text{—}) \quad \text{and}$$
$$g(\text{—},s) = \bigvee_{q<s} g(\text{—},q) \leq \bigvee_{q<s} \bigwedge_{i\in I} h_i(\text{—},q) \leq \bigvee_{q<s} \bigwedge_{i\in I} h_i(q,\text{—})^* = h_\vee(\text{—},s)$$

for every $r, s \in \mathbb{Q}$ and therefore $h_\vee \leq g$. Hence $h_\vee$ is the supremum of $\{h_i\}_{i\in I}$ in $\mathrm{C}(L)^{\vee\!\wedge}$.

(b) If $\{h_i\}_{i\in I} \subseteq \mathrm{C}(L)^{\vee\!\!\wedge}$ and $h \in \mathrm{C}(L)^{\vee\!\!\wedge}$ is such that $h \leq h_i$ for all $i \in I$, then $\{-h_i\}_{i\in I} \subseteq \mathrm{C}(L)^{\vee\!\!\wedge}$ and $-h \in \mathrm{C}(L)^{\vee\!\!\wedge}$ is such that $-h_i \leq -h$. By (a), the supremum $\overset{\mathrm{C}(L)^{\vee\!\!\wedge}}{\bigvee_{i\in I}}(-h_i)$ exists. It is easy to check that $\overset{\mathrm{C}(L)^{\vee\!\!\wedge}}{\bigwedge}_{i\in I} h_i = -\overset{\mathrm{C}(L)^{\vee\!\!\wedge}}{\bigvee_{i\in I}}(-h_i)$. ∎

Corollary 3.10. *Let L be a frame. Then the Dedekind order completion $\mathrm{C}(L)^{\#}$ of $\mathrm{C}(L)$ coincides with $\mathrm{C}(L)^{\vee\!\!\wedge}$, i.e.*

$$\begin{aligned}\mathrm{C}(L)^{\#} = \{h \in \mathrm{IC}(L) \mid &\text{(a) } \textit{there exist } f, g \in \mathrm{C}(L) \textit{ such that } f \leq h \leq g\\ &\text{(b) } h(p,\text{—})^* \leq h(\text{—},q) \textit{ and } h(\text{—},q)^* \leq h(p,\text{—}) \textit{ for any } p < q\}\\ = \{h \in \mathrm{IC}(L) \mid &\text{(a) } \textit{there exist } f, g \in \mathrm{C}(L) \textit{ such that } f \leq h \leq g\\ &\text{(b) } h \sqsubseteq h' \in \mathrm{IC}(L) \implies h = h'\}. \quad \blacksquare\end{aligned}$$

4. The bounded case

In this section we show that if we restrict the preceding statements to bounded functions most results remain essentially the same.

Proposition 4.1. $\mathrm{IC}^*(L)$ *is Dedekind order complete.*

Proof: Let $\{h_i\}_{i\in I} \subseteq \mathrm{IC}^*(L)$ and $h \in \mathrm{IC}^*(L)$ such that $h_i \leq h$ for all $i \in I$. Since $\mathrm{IC}(L)$ is Dedekind order complete, there exists $\overset{\mathrm{IC}(L)}{\bigvee_{i\in I}} h_i$. Let $j \in I$. Then both h_j and h are bounded and so there are $p, q \in \mathbb{Q}$ such that $\mathbf{p} \leq h_j \leq \overset{\mathrm{IC}(L)}{\bigvee_{i\in I}} h_i \leq h \leq \mathbf{q}$. Consequently, $\overset{\mathrm{IC}^*(L)}{\bigvee_{i\in I}} h_i = \overset{\mathrm{IC}(L)}{\bigvee_{i\in I}} h_i$. Dually, if $h \leq h_i$ for all $i \in I$ and some $h \in \mathrm{IC}^*(L)$, one has $\overset{\mathrm{IC}^*(L)}{\bigwedge}_{i\in I} h_i = \overset{\mathrm{IC}(L)}{\bigwedge}_{i\in I} h_i$. ∎

Let

$$\mathrm{C}^*(L)^{\vee} = \mathrm{C}(L)^{\vee} \cap \mathrm{IC}^*(L), \quad \mathrm{C}^*(L)^{\wedge} = \mathrm{C}(L)^{\wedge} \cap \mathrm{IC}^*(L)$$

and

$$\mathrm{C}^*(L)^{\vee\!\!\wedge} = \mathrm{C}(L)^{\vee\!\!\wedge} \cap \mathrm{IC}^*(L).$$

Proposition 4.2. *For any completely regular frame L and $h \in \mathrm{C}^*(L)^{\vee}$,*

$$h = \overset{\mathrm{IC}(L)}{\bigvee} \{f \in \mathrm{C}^*(L) \mid f \leq h\}.$$

Proof: Since h is bounded, there exist $p, q \in \mathbb{Q}$ such that $\mathbf{p} \le h \le \mathbf{q}$. Note that $f \vee \mathbf{p} \in \mathrm{C}^*(L)$ for any $f \in \mathrm{C}(L)$ such that $f \le h$, since $\mathbf{p} \le f \vee \mathbf{p} \le \mathbf{q}$. Then, by Lemma 3.1, one has

$$\begin{aligned} h = & \overset{\mathrm{IC}(L)}{\bigvee} \{f \in \mathrm{C}(L) \mid f \le h\} \le \overset{\mathrm{IC}(L)}{\bigvee} \{f \vee \mathbf{p} \mid f \in \mathrm{C}(L), f \le h\} \le \\ & \le \overset{\mathrm{IC}(L)}{\bigvee} \{f \in \mathrm{C}^*(L) \mid f \le h\} \le h, \end{aligned}$$

and, consequently, $h = \overset{\mathrm{IC}(L)}{\bigvee} \{f \in \mathrm{C}^*(L) \mid f \le h\}$. ■

Proposition 4.3. *Let L be a completely regular frame and $h \in \mathrm{C}^*(L)^{\wedge}$. Then*

$$h = \overset{\mathrm{IC}(L)}{\bigwedge} \{g \in \mathrm{C}^*(L) \mid h \le g\}.$$

Proof: It follows from Lemma 3.2, in a similar way as the preceding proposition follows from Lemma 3.1. ■

Corollary 4.4. *Let L be a completely regular frame and $h \in \mathrm{C}^*(L)^{\vee\!\wedge}$. Then*

$$h = \overset{\mathrm{IC}^*(L)}{\bigvee} \{f \in \mathrm{C}^*(L) \mid f \le h\} = \overset{\mathrm{IC}^*(L)}{\bigwedge} \{g \in \mathrm{C}^*(L) \mid h \le g\}. \quad ■$$

Proposition 4.5. $\mathrm{C}^*(L)^{\vee}$ *is closed under non-void bounded suprema and* $\mathrm{C}^*(L)^{\wedge}$ *is closed under non-void bounded infima.*

Proof: Let $\{h_i\}_{i \in I} \subseteq \mathrm{C}^*(L)^{\vee}$ and $h \in \mathrm{C}^*(L)^{\vee}$ such that $h_i \le h$ for all $i \in I$. Since $\mathrm{C}(L)^{\vee}$ is closed under non-void bounded suprema, there exists $\overset{\mathrm{C}(L)^{\vee}}{\bigvee_{i \in I}} h_i$. As h is bounded from above and each h_i is bounded from below, then

$$\overset{\mathrm{C}(L)^{\vee}}{\bigvee_{i \in I}} h_i \in \mathrm{C}^*(L)^{\vee}$$

and thus $\mathrm{C}^*(L)^{\vee}$ is closed under non-void bounded suprema. ■

Proposition 4.6. *For any completely regular frame L, $\mathrm{C}^*(L)^{\vee\!\wedge}$ is Dedekind order complete.*

Proof: Let $\{h_i\}_{i \in I} \subseteq \mathrm{C}^*(L)^{\vee\!\wedge}$ and $h \in \mathrm{C}^*(L)^{\vee\!\wedge}$ such that $h_i \le h$ for all $i \in I$. Then, since $\mathrm{C}(L)^{\vee\!\wedge}$ is Dedekind order complete, $\overset{\mathrm{C}(L)^{\vee\!\wedge}}{\bigvee_{i \in I}} h_i$ exists. As each h_i is bounded from below and h is bounded from above, $\overset{\mathrm{C}(L)^{\vee\!\wedge}}{\bigvee_{i \in I}} h_i$ is bounded.

Consequently, $\overset{\mathrm{C}^*(L)^{\vee\!\wedge}}{\bigvee_{i\in I}} h_i = \overset{\mathrm{C}(L)^{\vee\!\wedge}}{\bigvee_{i\in I}} h_i$. The second assertion follows in a similar way. ∎

Corollary 4.7. *For any completely regular frame L, $\mathrm{C}^*(L)^{\vee\!\wedge}$ is the Dedekind order completion of $\mathrm{C}^*(L)$.* ∎

We close this section with a corollary that augments a characterization of Banaschewski-Hong [5, Proposition 1].

Corollary 4.8. *For any completely regular frame L, the following are equivalent:*

(1) *L is extremally disconnected.*
(2) $\mathrm{C}(L) = \mathrm{C}(L)^{\vee\!\wedge}$.
(3) $\mathrm{C}(L)$ *is Dedekind order complete.*
(4) $\mathrm{C}(L)$ *is closed under non-void bounded suprema.*
(5) $\mathrm{C}^*(L) = \mathrm{C}^*(L)^{\vee\!\wedge}$.
(6) $\mathrm{C}^*(L)$ *is Dedekind order complete.*
(7) $\mathrm{C}^*(L)$ *is closed under non-void bounded suprema.*

Proof: (1) $\Longrightarrow$ (2): Let L be extremally disconnected, $h \in \mathrm{C}(L)^{\vee\!\wedge}$ and $p < r < q$. Then $h(r,\text{—})^* \le h(\text{—},q)$ and $h(r,\text{—})^{**} \le h(\text{—},r)^* \le h(p,\text{—})$. Hence $h(p,\text{—}) \vee h(\text{—},q) \ge h(r,\text{—})^{**} \vee h(r,\text{—})^* = 1$. Consequently, $\mathrm{C}(L) = \mathrm{C}(L)^{\vee\!\wedge}$.

(3) $\Longrightarrow$ (1): For each $a \in L$, let $\mathcal{F}_a = \{f \in \mathrm{C}(L) \mid f \le \chi_{a^*,a^{**}}\}$ and $\mathcal{G}_a = \{g \in \mathrm{C}(L) \mid \chi_{a^*,a^{**}} \le g\}$. By Lemmas 3.1 and 3.2,

$$\chi_{a^*,a^{**}} = \overset{\mathrm{IC}(L)}{\bigvee} \mathcal{F}_a = \overset{\mathrm{IC}(L)}{\bigwedge} \mathcal{G}_a.$$

On the other hand, since $\mathbf{0} \in \mathcal{F}_a$, $\mathbf{1} \in \mathcal{G}_a$, $f \le \mathbf{1}$ for all $f \in \mathcal{F}_a$ and $\mathbf{0} \le g$ for all $g \in \mathcal{G}_a$, $\overset{\mathrm{C}(L)}{\bigvee} \mathcal{F}_a$ and $\overset{\mathrm{C}(L)}{\bigwedge} \mathcal{G}_a$ do exist. Therefore

$$\chi_{a^*,a^{**}} = \overset{\mathrm{IC}(L)}{\bigvee} \mathcal{F}_a \le \overset{\mathrm{C}(L)}{\bigvee} \mathcal{F}_a \le \overset{\mathrm{C}(L)}{\bigwedge} \mathcal{G}_a \le \overset{\mathrm{IC}(L)}{\bigwedge} \mathcal{G}_a = \chi_{a^*,a^{**}}$$

and we may conclude that $\chi_{a^*,a^{**}} \in \mathrm{C}(L)$, that is, $a^* \vee a^{**} = 1$.

Finally, the implication (2) $\Longrightarrow$ (3) follows from Theorem 3.9, (3) $\Longleftrightarrow$ (4) is obvious and the equivalences (1) $\Longleftrightarrow$ (5) $\Longleftrightarrow$ (6) $\Longleftrightarrow$ (7) can be proved in a similar way. ∎

5. The integer-valued case

Recall from [3] and [5] that the ring $\mathfrak{Z}L$ of integer-valued continuous functions on a frame L has as its elements the maps $\alpha, \beta, \gamma, \ldots : \mathbb{Z} \to L$ such that

$$\alpha(n) \wedge \alpha(m) = 0 \quad \text{for } n \neq m \quad \text{and} \quad \bigvee\{\alpha(n) \mid n \in \mathbb{Z}\} = 1.$$

The elements of $\mathfrak{Z}L$ can be easily identified with those elements of $f \in \mathrm{C}(L)$ such that

$$f(p,\text{—}) = f(\lfloor p \rfloor, \text{—}) \text{ and } f(\text{—}, q) = f(\text{—}, \lceil q \rceil) \quad \text{for all } p, q \in \mathbb{Q}, \quad (\mathbb{Z}\text{-valued})$$

(where $\lfloor p \rfloor$ denotes the biggest integer $\leq p$ and $\lceil q \rceil$ the smallest integer $\geq q$). Denoting the subclass of $\mathrm{C}(L)$ of all $\mathbb{Z}$-valued functions by $\mathrm{C}(L, \mathbb{Z})$, the correspondence $\mathfrak{Z}L \simeq \mathrm{C}(L, Z)$ is given by

$$\alpha \in \mathfrak{Z}L \longmapsto f_\alpha(p,\text{—}) = \bigvee\{\alpha(n) \mid p < n\}, \quad f_\alpha(\text{—}, q) = \bigvee\{\alpha(n) \mid n < q\}$$

$$f \in \mathrm{C}(L, Z) \longmapsto \alpha_f(n) = f(n-1, \text{—}) \wedge f(\text{—}, n+1).$$

From this it follows that the Dedekind order completion of $\mathfrak{Z}L$ is isomorphic to the Dedekind order completion of $\mathrm{C}(L, \mathbb{Z})$, which is included in $\mathrm{C}(L)^{\vee\!\!\!\wedge}$.

In the same vein, we shall also denote by $\mathrm{IC}(L, \mathbb{Z})$, $\mathrm{C}(L, \mathbb{Z})^\vee$, $\mathrm{C}(L, \mathbb{Z})^\wedge$ and $\mathrm{C}(L, \mathbb{Z})^{\vee\!\!\!\wedge}$ the $\mathbb{Z}$-valued subsets of $\mathrm{IC}(L)$, $\mathrm{C}(L)^\vee$, $\mathrm{C}(L)^\wedge$ and $\mathrm{C}(L)^{\vee\!\!\!\wedge}$, respectively.

Example 5.1. The bounded continuous partial real function $\chi_{a,b}$ ($a, b \in L$, $a \wedge b = 0$) from Example 2.7 is clearly $\mathbb{Z}$-valued. Moreover:

(1) $\chi_{a,b} \in \mathrm{IC}(L, \mathbb{Z})$.
(2) $\chi_{a,b} \in \mathrm{C}(L, \mathbb{Z})^\vee$ if and only if $a^* = b$.
(3) $\chi_{a,b} \in \mathrm{C}(L, \mathbb{Z})^\wedge$ if and only if $b^* = a$.
(4) $\chi_{a,b} \in \mathrm{C}(L, \mathbb{Z})^{\vee\!\!\!\wedge}$ if and only if $a^* = b$ and $b^* = a$, i.e. if and only if a is regular and $b = a^*$.
(5) $\chi_{a,b} \in \mathrm{C}(L, \mathbb{Z})$ iff a is complemented with complement b.

Proposition 5.2. $\mathrm{IC}(L, \mathbb{Z})$ *is Dedekind order complete.*

Proof: Let $\{h_i\}_{i \in I} \subseteq \mathrm{IC}(L, \mathbb{Z})$, $h \in \mathrm{IC}(L, \mathbb{Z})$, $h_i \underset{\mathrm{IC}(L)}{\leq} h$ for all $i \in I$. Since $\mathrm{IC}(L)$ is Dedekind order complete, there exists $\bigvee_{i \in I} h_i$ in $\mathrm{IC}(L)$. In addition, for each $r, s \in \mathbb{Q}$,

$$\begin{aligned}
&\bigvee_{i\in I}^{\mathrm{IC}(L)} h_i(r,\text{—}) = \bigvee_{i\in I} h_i(r,\text{—}) = \bigvee_{i\in I} h_i(\lfloor r\rfloor,\text{—}) = h_\vee(\lfloor r\rfloor,\text{—}) \quad \text{and}\\
&\bigvee_{i\in I}^{\mathrm{IC}(L)} h_i(\text{—},s) = \bigvee_{q<s}\bigwedge_{i\in I} h_i(\text{—},q) = \bigvee_{q<s}\bigwedge_{i\in I} h_i(\text{—},\lceil q\rceil) =\\
&\qquad\qquad = \bigvee_{q<\lceil s\rceil}\bigwedge_{i\in I} h_i(\text{—},\lceil q\rceil) = h_\vee(\text{—},\lceil s\rceil)
\end{aligned}$$

which ensures that $\bigvee^{\mathrm{IC}(L)}_{i\in I} h_i$ is $\mathbb{Z}$-valued. Dually, if $h \le h_i$ for all $i \in I$, one gets that $\bigwedge^{\mathrm{IC}(L)}_{i\in I} h_i$ is $\mathbb{Z}$-valued. ■

Proposition 5.3. *Let L be a zero-dimensional frame and let $h \in \mathrm{C}(L,\mathbb{Z})^\vee$. Then $h = \bigvee^{\mathrm{IC}(L,\mathbb{Z})} \{f \in \mathrm{C}(L,\mathbb{Z}) \mid f \le h\}$.*

Proof: Let $\mathcal{F} = \{f \in \mathrm{C}(L,\mathbb{Z}) \mid f \le h\}$. Since $\mathrm{IC}(L,\mathbb{Z})$ is Dedekind order complete, $\bigvee^{\mathrm{IC}(L,\mathbb{Z})} \mathcal{F}$ exists. We shall prove that

$$\bigvee^{\mathrm{IC}(L,\mathbb{Z})} \mathcal{F} = h.$$

For that we only need to check that $h \le h'$ for any $h' \in \mathrm{IC}(L,\mathbb{Z})$ such that $f \le h'$ for all $f \in \mathcal{F}$, i.e.
(a) $h(p,\text{—}) \le h'(p,\text{—})$ for every $p \in \mathbb{Q}$ and
(b) $h(\text{—},q) \ge h'(\text{—},q)$ for every $q \in \mathbb{Q}$.

(a): Fix $p \in \mathbb{Q}$, let $n = \lfloor p\rfloor$ and $f \in \mathrm{C}(L,\mathbb{Z})$ such that $f \le h$. Since L is zero-dimensional, then

$$h(p,\text{—}) = h(n,\text{—}) = \bigvee\{a \in L \mid a \text{ is complemented and } a \le h(n,\text{—})\}.$$

For each such complemented a, define $\sigma_{a,n}\colon \mathbb{Q}\to L$ by

$$\sigma_{a,n}(r) = \begin{cases} f(r,\text{—}) & \text{if } r \ge n+1\\ f(r,\text{—}) \vee a & \text{if } r < n+1.\end{cases}$$

This is a scale in L. Indeed, $\bigvee_{r\in\mathbb{Q}}\sigma_{a,n}(r) \geq \bigvee_{r\in\mathbb{Q}} f(r,—) = 1$, $\bigvee_{r\in\mathbb{Q}}\sigma_{a,n}(r)^* \geq \bigvee_{r\geq n+1} f(r,—)^* = 1$ and if $r, s \in \mathbb{Q}$ are such that $r < s$, then

$$\sigma_{a,n}(r) \vee \sigma_{a,n}(s)^* = \begin{cases} f(r,—) \vee f(s,—)^* = 1 & \text{if } r, s \geq n+1 \\ f(r,—) \vee a \vee f(s,—)^* = 1 & \text{if } s \geq n+1 > r \\ f(r,—) \vee a \vee (f(s,—)^* \wedge a^*) & \\ \geq (f(r,—) \vee f(s,—)^*) \wedge (a \vee a^*) = 1 & \text{if } r, s < n+1. \end{cases}$$

Consequently, it defines an $f_{a,n} \in \mathrm{C}(L)$ by

$$f_{a,n}(r,—) = \begin{cases} f(r,—) & \text{if } r \geq n+1 \\ f(r,—) \vee a & \text{if } r < n+1 \end{cases}$$

and

$$f_{a,n}(—,s) = \begin{cases} f(—,s) & \text{if } s > n+1 \\ f(—,s) \wedge a^* & \text{if } s \leq n+1. \end{cases}$$

It is easy to check that $f_{a,n}$ is $\mathbb{Z}$-valued. Moreover, $f_{a,n} \leq h$:

- If $r \geq n+1$ then $f_{a,n}(r,—) = f(r,—) \leq h(r,—)$.
- If $r < n+1$ then $\lfloor r \rfloor \leq n$ and so $f_{a,n}(r,—) = f(r,—) \vee a \leq h(r,—) \vee h(n,—) = h(\lfloor r \rfloor,—) \vee h(n,—) = h(\lfloor r \rfloor,—) = h(r,—)$.

Hence $f_{a,n}(r,—) \leq h(r,—)$ for each $r \in \mathbb{Q}$ and since $f_{a,n} \in \mathrm{C}(L)$, it follows that $f_{a,n} \leq h$. We conclude that $f_{a,n} \in \mathcal{F}$.

Finally, we have also that

$$a \leq f(n,—) \vee a = f_{a,n}(n,—) \leq h'(n,—) = h'(p,—).$$

Hence

$$h(p,—) = h(n,—) = \bigvee\{a \in L \mid a \text{ is complemented and } a \leq h(n,—)\} \leq h'(p,—).$$

(b): Since $h \in \mathrm{C}(L,\mathbb{Z})^{\vee}$, we have

$$h(—,q) = \bigvee_{s'<s}\bigvee_{s<q} h(—,s') \geq \bigvee_{s<q} h(s,—)^* \geq \bigvee_{s<q} h'(s,—)^* \geq \bigvee_{s<q} h'(—,s) = h'(—,q).$$

Then $\bigvee^{\mathrm{IC}(L,\mathbb{Z})} \mathcal{F} = h$. ■

Similarly, we have:

Proposition 5.4. *Let L be a zero-dimensional frame and let $h \in \mathrm{C}(L,\mathbb{Z})^{\wedge}$. Then* $h = \bigwedge^{\mathrm{IC}(L,\mathbb{Z})} \{g \in \mathrm{C}(L,\mathbb{Z}) \mid h \leq g\}$. ■

Corollary 5.5. *Let L be a zero-dimensional frame and let $h \in \mathrm{C}(L,\mathbb{Z})(L)^{\vee\!\wedge}$. Then*

$$h = \bigvee^{\mathrm{IC}(L,\mathbb{Z})} \{f \in \mathrm{IC}(L,\mathbb{Z}) \mid f \leq h\} = \bigwedge^{\mathrm{IC}(L,\mathbb{Z})} \{g \in \mathrm{IC}(L,\mathbb{Z}) \mid h \leq g\}. \quad \blacksquare$$

Now we have the following analogues of Propositions 4.5 and 4.6 in the integer-valued case, which can be proved in a similar way.

Proposition 5.6. $\mathrm{C}(L,\mathbb{Z})^{\vee}$ *is closed under non-void bounded suprema and* $\mathrm{C}(L,\mathbb{Z})^{\wedge}$ *is closed under non-void bounded infima.* $\blacksquare$

Proposition 5.7. *For any zero-dimensional frame L,* $\mathrm{C}(L,\mathbb{Z})^{\vee\!\wedge}$ *is Dedekind order complete.* $\blacksquare$

Corollary 5.8. *For any zero-dimensional frame L,* $\mathrm{C}(L,\mathbb{Z})^{\vee\!\wedge}$ *is the Dedekind order completion of* $\mathrm{C}(L,\mathbb{Z})$. $\blacksquare$

Finally, we have a corollary that augments Proposition 3 of [5] (the proof goes very similar to that of Corollary 4.8 so we omit it).

Corollary 5.9. *For any zero-dimensional frame L, the following are equivalent:*

(1) *L is extremally disconnected.*
(2) $\mathrm{C}(L,\mathbb{Z}) = \mathrm{C}(L,\mathbb{Z})^{\vee\!\wedge}$.
(3) $\mathrm{C}(L,\mathbb{Z})$ *is Dedekind order complete.*
(4) $\mathrm{C}(L,\mathbb{Z})$ *is is closed under non-void bounded suprema.* $\blacksquare$

It is quite evident now that we could also consider the case of bounded integer-valued continuous function. We omit the details.

6. The classical case

In this final section we show that the pointfree approach pursued in this paper sheds new light on the classical case of $C(X)$ (for a space X) and provides a new construction that we believe is more natural than that given by Anguelov in [1]. The construction in [1] works with Hausdorff continuous functions, whereas our construction hinges only on a direct lattice-theoretical approach to the problem.

To begin with, recall from Corollary 2.5 the natural isomorphism

$$\Phi\colon \mathrm{IC}(\mathcal{O}X) \longrightarrow \mathrm{C}(X,\mathbb{IR})$$

given, for each $h \in \mathrm{IC}(\mathcal{O}X)$, by

$$\Phi(h)(x) = [\bigvee\{r \in \mathbb{Q} \mid x \in h(r,\text{—})\}, \bigwedge\{r \in \mathbb{Q} \mid x \in h(\text{—},r)\}] \quad \text{for all } x \in X.$$

Composing $\Phi(h)$ with projections π_1 and π_2 we get a couple of real-valued functions $\pi_1 \circ \Phi(h), \pi_2 \circ \Phi(h) \colon X \to \mathbb{R}$ such that

(1) $\pi_1 \circ \Phi(h) \leq \pi_2 \circ \Phi(h)$,
(2) $\pi_1 \circ \Phi(h) \in \mathrm{LSC}\,(X, \mathbb{R})$, and
(3) $\pi_2 \circ \Phi(h) \in \mathrm{USC}\,(X, \mathbb{R})$ (recall Remark 1 of 2.1).

Lemma 6.1. *Let $f, g \in \mathrm{IC}(\mathcal{O}X)$. Then:*
(1) $\pi_1 \circ \Phi(f) \leq \pi_1 \circ \Phi(g)$ *iff* $f(r,\text{—}) \leq g(r,\text{—})$ *for all* $r \in \mathbb{Q}$.
(2) $\pi_2 \circ \Phi(f) \geq \pi_2 \circ \Phi(g)$ *iff* $f(\text{—},r) \leq g(\text{—},r)$ *for all* $r \in \mathbb{Q}$.

Proof: To check (1), first consider $f, g \in \mathrm{IC}(\mathcal{O}X)$ such that $\pi_1 \circ \Phi(f) \leq \pi_1 \circ \Phi(g)$ and let $r \in \mathbb{Q}$. Then, for any $s > r$ in $\mathbb{Q}$ and $x \in f(s,\text{—})$ one has

$$r < s \leq \bigvee\{p \in \mathbb{Q} \mid x \in f(p,\text{—})\} \leq \bigvee\{p \in \mathbb{Q} \mid x \in g(p,\text{—})\}$$

and thus there exists a $p > r$ in $\mathbb{Q}$ such that $x \in g(p,\text{—}) \leq g(r,\text{—})$. Consequently, $f(r,\text{—}) = \bigvee_{s>r} f(s,\text{—}) \leq g(r,\text{—})$. The reverse implication is straightforward.

Finally, in order to check (2) note first that $\Phi(-f) = -\Phi(f)$, $\pi_1(-f) = -\pi_2(f)$ and $\pi_2(-f) = -\pi_1(f)$. Thus $f(\text{—},r) \leq g(\text{—},r)$ for any $r \in \mathbb{Q}$ if and only if $-f(r,\text{—}) \leq -g(r,\text{—})$ for any $r \in \mathbb{Q}$. Then, by (1), this is equivalent to $\pi_1 \circ \Phi(-f) \leq \pi_1 \circ \Phi(-g)$, that is, $-(\pi_2 \circ \Phi(f)) \leq -(\pi_2 \circ \Phi(g))$. ■

In particular, this implies that Φ is an order isomorphism for both $\leq$ and $\sqsubseteq$. Furthermore, its restriction to $\mathrm{C}(\mathcal{O}X)$ and $\mathrm{C}(X)$ is also an order isomorphism. Then, using Lemma 6.1, the following facts follow immediately.

Fact 6.2. *Let $h \in \mathrm{IC}(\mathcal{O}X)$ and let $f, g \in \mathrm{C}(\mathcal{O}X)$ such that $f \leq h \leq g$. Then:*
(1) $h \in \mathrm{C}(\mathcal{O}X)^{\vee}$ *if and only if*

$$\Phi(h) \sqsubseteq h' \implies \pi_2 \circ \Phi(h) = \pi_2(h') \;\; \textit{in } \mathrm{C}(X, \mathbb{IR}). \tag{P$^{\vee}$}$$

(2) $h \in \mathrm{C}(\mathcal{O}X)^{\wedge}$ *if and only if*

$$\Phi(h) \sqsubseteq h' \implies \pi_1 \circ \Phi(h) = \pi_1(h') \;\; \textit{in } \mathrm{C}(X, \mathbb{IR}). \tag{P$^{\wedge}$}$$

(3) $h \in \mathrm{C}(\mathcal{O}X)^{\vee\!\!\!\wedge}$ *if and only if*

$$\Phi(h) \sqsubseteq h' \implies \Phi(h) = h' \;\; \textit{in } \mathrm{C}(X, \mathbb{IR}).$$

This ensures that Φ yields order isomorphisms between $\mathrm{C}(\mathcal{O}X)^{\vee}$, $\mathrm{C}(\mathcal{O}X)^{\wedge}$ and $\mathrm{C}(\mathcal{O}X)^{\vee\!\!\!\wedge}$ (ordered by $\leq$), respectively, and classes

$$\mathrm{C}(X)^{\vee} = \{h \in \mathrm{C}(X, \mathbb{IR}) \mid \text{(a) there exist } f, g \in \mathrm{C}(X) \text{ such that } f \leq h \leq g \\ \text{(b) } h \sqsubseteq h' \implies \pi_2(h) = \pi_2(h')\}.$$

$$\mathrm{C}(X)^{\wedge} = \{h \in \mathrm{C}(X, \mathbb{IR}) \mid \text{(a) there exist } f, g \in \mathrm{C}(X) \text{ such that } f \leq h \leq g \\ \text{(b) } h \sqsubseteq h' \implies \pi_1(h) = \pi_1(h')\}.$$

$$\mathrm{C}(X)^{\vee\!\!\!\wedge} = \{h \in \mathrm{C}(X, \mathbb{IR}) \mid \text{(a) there exist } f, g \in \mathrm{C}(X) \text{ such that } f \leq h \leq g \\ \text{(b) } h \sqsubseteq h' \implies h = h'\}.$$

Additionally, notice that $h \in \mathrm{IC}(\mathcal{O}X)$ is constant if and only if $\Phi(h)$ is constant in $\mathrm{C}(X, \mathbb{IR})$ and that $h \in \mathrm{IC}(\mathcal{O}X)$ is $\mathbb{Z}$-valued if and only if both $\pi_1 \circ \Phi(h)$ and $\pi_2 \circ \Phi(h)$ take values in $\mathbb{Z}$.

For the sake of completeness, let us also introduce the following classes:

$$\mathrm{C}^*(X)^{\vee} = \{h \in \mathrm{C}(X)^{\vee} \mid \exists p, q \in \mathbb{Q} \text{ such that } h(x) \subseteq [p, q] \text{ for all } x \in X\},$$
$$\mathrm{C}^*(X)^{\wedge} = \{h \in \mathrm{C}(X)^{\vee} \mid \exists p, q \in \mathbb{Q} \text{ such that } h(x) \subseteq [p, q] \text{ for all } x \in X\},$$
$$\mathrm{C}^*(X)^{\vee\!\!\!\wedge} = \{h \in \mathrm{C}(X)^{\vee\!\!\!\wedge} \mid \exists p, q \in \mathbb{Q} \text{ such that } h(x) \subseteq [p, q] \text{ for all } x \in X\},$$
$$\mathrm{C}(X, \mathbb{Z})^{\vee} = \{h \in \mathrm{C}(X)^{\vee} \mid \pi_1(h(x)), \pi_2(h(x)) \in \mathbb{Z} \text{ for all } x \in X\},$$
$$\mathrm{C}(X, \mathbb{Z})^{\wedge} = \{h \in \mathrm{C}(X)^{\vee} \mid \pi_1(h(x)), \pi_2(h(x)) \in \mathbb{Z} \text{ for all } x \in X\},$$
$$\mathrm{C}(X, \mathbb{Z})^{\vee\!\!\!\wedge} = \{h \in \mathrm{C}(X)^{\vee\!\!\!\wedge} \mid \pi_1(h(x)), \pi_2(h(x)) \in \mathbb{Z} \text{ for all } x \in XZ\}.$$

Analogously, they are order isomorphic to $\mathrm{C}^*(\mathcal{O}X)^{\vee}$, $\mathrm{C}^*(\mathcal{O}X)^{\wedge}$, $\mathrm{C}^*(\mathcal{O}X)^{\vee\!\!\!\wedge}$, $\mathrm{C}(\mathcal{O}X, \mathbb{Z})^{\vee}$, $\mathrm{C}(\mathcal{O}X, \mathbb{Z})^{\wedge}$ and $\mathrm{C}(\mathcal{O}X, \mathbb{Z})^{\vee\!\!\!\wedge}$ (ordered by $\leq$), respectively.

Finally, recall that $\mathcal{O}X$ is completely regular (resp. extremally disconnected, zero-dimensional) as a frame if and only if the space X is completely regular (resp. extremally disconnected, zero-dimensional). Then, from Corollaries 3.10, 4.7, 4.8, 5.8 and 5.9 it follows immediately that:

Proposition 6.3. *For any completely regular topological space* $(X, \mathcal{O}X)$*,*
(1) $\mathrm{C}(X)^{\vee\!\!\!\wedge}$ *is the Dedekind order completion of* $\mathrm{C}(X)$*.*
(2) $\mathrm{C}^*(X)^{\vee\!\!\!\wedge}$ *is the Dedekind order completion of* $\mathrm{C}^*(X)$*.* ∎

Corollary 6.4. *For any completely regular topological space* $(X, \mathcal{O}X)$, *the following are equivalent:*

(1) X *is extremally disconnected.*
(2) $\mathrm{C}(X) = \mathrm{C}(X)^{\curlyvee\!\curlywedge}$.
(3) $\mathrm{C}(X)$ *is Dedekind order complete.*
(4) $\mathrm{C}(X)$ *is is closed under non-void bounded suprema.*
(5) $\mathrm{C}^*(X) = \mathrm{C}^*(X)^{\curlyvee\!\curlywedge}$.
(6) $\mathrm{C}^*(X)$ *is Dedekind order complete;.*
(7) $\mathrm{C}^*(X)$ *is is closed under non-void bounded suprema.* ■

Proposition 6.5. *For each zero-dimensional topological space* $(X, \mathcal{O}X)$,

$$\mathrm{C}(X, \mathbb{Z})^{\curlyvee\!\curlywedge}$$

is the Dedekind order completion of $\mathrm{C}(X, \mathbb{Z})$. ■

Corollary 6.6. *For any zero-dimensional topological space* $(X, \mathcal{O}X)$, *the following are equivalent:*

(1) X *is extremally disconnected.*
(2) $\mathrm{C}(X, \mathbb{Z}) = \mathrm{C}(X, \mathbb{Z})^{\curlyvee\!\curlywedge}$.
(3) $\mathrm{C}(X, \mathbb{Z})$ *is Dedekind order complete.*
(4) $\mathrm{C}(X, \mathbb{Z})$ *is is closed under non-void bounded suprema.* ■

We close with a comment regarding the relation of our results above to the construction of Anguelov [1]. For that we need to recall the well known fact that each real-valued function $f \colon X \to \mathbb{R}$ on a space X admits an upper regularization $f^- \in \mathrm{USC}(X, \overline{\mathbb{R}})$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, defined by

$$f^-(x) = \bigwedge\{\bigvee f(U) \mid x \in U \in \mathcal{O}X\} \quad \text{for all } x \in X.$$

This is the smallest upper semicontinuous majorant of f, i.e.,

$$f^- = \bigwedge \left\{g \in \mathrm{USC}(X, \overline{\mathbb{R}}) \mid f \leq g\right\}.$$

Dually, f admits a lower regularization $f^\circ \in \mathrm{LSC}(X, \overline{\mathbb{R}})$ defined by

$$f^\circ(x) = \bigvee\{\bigwedge f(U) \mid x \in U \in \mathcal{O}X\} \quad \text{for all } x \in X,$$

and f° is the biggest lower semicontinuous minorant of f, i.e.,

$$f^\circ = \bigvee \left\{g \in \mathrm{LSC}(X, \overline{\mathbb{R}}) \mid g \leq f\right\}.$$

It is then not hard to check that

$$\mathrm{C}(X)^{\vee} = \{h \in \mathrm{C}(X, \mathbb{IR}) \mid \exists f, g \in \mathrm{C}(X) : f \leq h \leq g \text{ and } \pi_1(h)^- = \pi_2(h)\},$$

$$\mathrm{C}(X)^{\wedge} = \{h \in \mathrm{C}(X, \mathbb{IR}) \mid \exists f, g \in \mathrm{C}(X) : f \leq h \leq g \text{ and } \pi_2(h) = \pi_1(h)^{\circ}\}$$

and

$$\mathrm{C}(X)^{\vee\!\wedge} = \{h \in \mathrm{C}(X, \mathbb{IR}) \mid \exists f, g \in \mathrm{C}(X) : f \leq h \leq g,\ \pi_1(h) = \pi_1(h)^{\circ}, \pi_1(h) = \pi_1(h)^{\circ}\}.$$

For instance, for the first, given $h \in \mathrm{IC}(X)$ and $f, g \in \mathrm{C}(X)$ such that $f \leq h \leq g$ and $\pi_2(h) = \pi_2(j)$ whenever $h \sqsubseteq j$, since $h \sqsubseteq [\pi_1(h), \pi_1(h)^-]$ it follows that $\pi_2(h) = \pi_1(h)^-$. Conversely, let $h \in \mathrm{IC}(X)$ be such that $\pi_2(h) = \pi_1(h)^-$ and $h \sqsubseteq j$, i.e. $\pi_1(h) \leq \pi_1(j) \leq \pi_2(j) \leq \pi_2(h)$, then $\pi_2(h) = \pi_1(h)^- \leq \pi_1(j)^- \leq \pi_2(j) \leq \pi_2(h)$ and so $\pi_2(h) = \pi_2(j)$. The other identities follow similarly.

This description of the Dedekind order completion of $\mathrm{C}(X)$ is precisely the one given by the construction of Anguelov in terms of Hausdorff continuous functions presented in [1].

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