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OBSTRUCTION THEORY IN ACTION ACCESSIBLE CATEGORIES

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ABSTRACT: We show that, in semi-abelian action accessible categories (such as the categories of groups, Lie algebras, rings, associative algebras and Poisson algebras), the obstruction to the existence of extensions is classified by the second cohomology group in the sense of Bourn. Moreover, we describe explicitly the obstruction to the existence of extensions in the case of Leibniz algebras, comparing Bourn cohomology with Loday-Pirashvili cohomology of Leibniz algebras.

KEYWORDS: obstruction theory, action accessible categories, Leibniz algebras.
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1. Introduction

Given an extension of groups as below, the conjugation in X determines an action of X on K, and consequently a homomorphism $\phi\colon Y \to \frac{\mathrm{Aut}K}{\mathrm{Inn}K} = \mathrm{Out}K$, called the *abstract kernel* of the extension:

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "k"] \arrow[d] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d, "\phi"] & 0 \\ 0 \arrow[r] & \mathrm{Inn}K \arrow[r] & \mathrm{Aut}K \arrow[r] & \mathrm{Out}K \arrow[r] & 0 \end{tikzcd} \tag{1}$$

It is a classical problem to establish whether, given a morphism ϕ as above, there exists an extension having ϕ as its abstract kernel. A first solution to this problem was given by Schreier in [31, 32]: he associated with any abstract kernel ϕ an *obstruction*, that vanishes if and only if there exists an extension inducing ϕ.

Twenty years later, Eilenberg and Mac Lane in [17] described the obstruction in terms of cohomology (for a more detailed account see also [26]): with

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any abstract kernel ϕ it is possible to associate an element of the third cohomology group $H^3_{\overline{\phi}}(Y, ZK)$, where ZK is the centre of K and $\overline{\phi}$ is the (unique) action of Y on ZK induced by ϕ. They proved that an extension with abstract kernel ϕ exists if and only if the corresponding element in $H^3_{\overline{\phi}}(Y, ZK)$ is 0.

This result was then generalized to other algebraic structures, such as rings [25], associative algebras [18] and Lie algebras [19]. Analogous results, expressing the obstruction in terms of triple cohomology, were obtained by Barr for commutative associative algebras [1] and by Orzech for categories of interest [28].

Then the natural question arose whether it is possible to unify the different treatments of obstruction theory for all these algebraic examples. A first answer in this direction was given by Bourn in [10]: in that paper the author developed obstruction theory in Barr-exact action representative categories [3] in terms of the cohomology theory expressed via n-groupoids [6]. In action representative categories, actions over a fixed object K can be described as morphisms into a representative object (such as $\mathrm{Aut}K$ for groups), and there exists a canonical exact sequence determined by K (as the bottom row of diagram (1)). This is what happens, for example, in the categories of groups and Lie algebras (explaining the strong analogies between the cohomology theories of these two structures), but not in the other examples mentioned above.

In fact, it turns out that the representability of actions is not necessary in order to develop obstruction theory, provided we replace the canonical exact sequence and the notion of abstract kernel with suitable ones. The present article extends the results of [10] about obstruction theory to the context of action accessible categories [13]. As showed in [27], this context includes all the algebraic structures above (even those which are not covered by the action representative case, such as rings and associative algebras) and some new ones, such as Poisson algebras, Leibniz algebras [21], associative dialgebras [22] and trialgebras [24]. As an example, we describe explicitly obstruction theory for Leibniz algebras, comparing, in this context, Bourn cohomology with the Leibniz algebras cohomology described by Loday and Pirashvili in [23].

The paper is organized as follows. After recalling some background material in Section 2, Section 3 is devoted to recalling the categorical theory

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of extensions, as developed in [14]. In Section 4 we describe the obstruction to the existence of extensions with fixed abstract direction in the context of action accessible categories. In section 5 we describe obstruction theory for Leibniz algebras over a field.

2. Background material

2.1. Pretorsors. In this section we recall from [10] some definitions and results that will be used in the subsequent sections.

Definition 2.1. *A regular category $\mathbb{C}$ is said to be efficiently regular when any equivalence relation T on an object X, with T a subobject of an* effective *equivalence relation R on X by a regular monomorphism (i.e. an equalizer in $\mathbb{C}$), is itself effective.*

This categorical setting was introduced by Bourn in [9], as an intermediate notion between those of regular and Barr-exact category. Efficiently regular categories are stable under formation of slice and coslice categories. As a leading example, the category GpTop (resp. AbTop) of topological groups (resp. topological abelian groups) is efficiently regular, but not Barr-exact. The main point here is that when $\mathbb{C}$ is efficiently regular and when there is a discrete fibration $S \to R$ between two equivalence relations, then S is effective as soon as R is effective. Let us also observe that any Barr-exact category is efficiently regular.

We recall here the notion of pretorsor, following [29], where they are called pregroupoids (see also [10], Definition 1.3).

Definition 2.2 ([29])**.** *A pretorsor in an efficiently regular category $\mathbb{C}$ is a pair of regular epimorphisms $X \overset{f}{\twoheadleftarrow} W \overset{g}{\twoheadrightarrow} Y$ such that $[R[f], R[g]] = 0$, i.e. $R[f]$ and $R[g]$ centralize each other in the sense of Smith (see also* [12] *for details).*

From now on in this section, let us suppose that $\mathbb{C}$ is efficiently regular and Mal'tsev.

Pretorsors owe their name to the fact that they canonically determine categorical bitorsors. Bourn showed in [11] that these can be described in terms of regularly fully faithful profunctors. Actually, given a pretorsor $X \overset{f}{\twoheadleftarrow} W \overset{g}{\twoheadrightarrow} Y$, we can consider the following diagram, where the upper left-hand side part

is the centralizing double equivalence relation of the pair $([R[f], R[g]])$:

$$\begin{tikzcd}[column sep=large, row sep=large]
R[g] \times_W R[f] \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "\pi_0"'] \arrow[d, shift right=2, "p_0"'] \arrow[d, shift left=2, "\pi_1"] & R[f] \arrow[l] \arrow[r, dashed, two heads, "g_1"] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & Y_1 \arrow[d, shift right=2, "y_0"'] \arrow[d, shift left=2, "y_1"] \\
R[g] \arrow[u] \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] \arrow[d, dashed, "f_1"'] & W \arrow[u] \arrow[l] \arrow[r, two heads, "g"] \arrow[d, "f"] & Y \arrow[u] \\
X_1 \arrow[r, shift left=2, "x_1"] \arrow[r, shift right=2, "x_0"'] & X. \arrow[l] &
\end{tikzcd} \tag{2}$$

One can show that the upper horizontal equivalence relation and the vertical one on the left-hand side are effective and admit quotients g_1 and f_1. This construction produces two groupoids $\delta_0(f,g) = \underline{X}_1$ and $\delta_1(f,g) = \underline{Y}_1$, which are called the *domain* and the *codomain* of the pretorsor. They are in fact the domain and codomain of a (regularly fully faithful) profunctor $(f,g)\colon \underline{X}_1 \nrightarrow \underline{Y}_1$. By abuse of notation, we indicate with the same symbol the pretorsor and the corresponding profunctor.

Actually, as shown in [11], pretorsors (or, to be more precise, their counterpart: regularly fully faithful profunctors) can be seen as morphisms of a bigroupoid $\mathcal{RF}(\mathbb{C})$ whose objects are internal groupoids. Let us denote by $\mathsf{RF}(\mathbb{C})$ its classifying groupoid, in the sense of Bénabou [2].

Let $\underline{Z}_1$ be any internal groupoid. The canonical (regular epi, mono) factorization of the map $\langle z_0, z_1\rangle\colon Z_1 \to Z_0 \times Z_0$ gives rise to an equivalence relation $\Sigma\underline{Z}_1$:

$$Z_1 \twoheadrightarrow \Sigma\underline{Z}_1 \rightarrowtail Z_0 \times Z_0$$

which is called the *support* of the object $\underline{Z}_1$ in the fibre $\mathsf{Gpd}_{Z_0}(\mathbb{C})$ with respect to the fibration $()_0\colon \mathsf{Gpd}(\mathbb{C}) \to \mathbb{C}$. Following [10], we say that the groupoid $\underline{Z}_1$ has effective support when the equivalence relation $\Sigma\underline{Z}_1$ is effective.

When the Mal'tsev category $\mathbb{C}$ is not only efficiently regular, but also Barr-exact, any groupoid has effective support. When $\underline{Z}_1$ is a groupoid with effective support, we denote by $q_{\underline{Z}_1}\colon Z_0 \twoheadrightarrow \pi_0\underline{Z}_1$ the coequalizer of this effective support.

Proposition 2.3 ([10], Proposition 1.5)**.** *Suppose $\mathbb{C}$ is Mal'tsev and efficiently regular. Let $(f,g)\colon \underline{X}_1 \nrightarrow \underline{Y}_1$ be a pretorsor. Then $\underline{X}_1$ has effective support if and only if $\underline{Y}_1$ has effective support. If $\underline{Y}_1$ has effective support,*

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inclusion and projection, we obtain an extension in k-Leib:

$$0 \longrightarrow K \xrightarrow{j} F \longrightarrow Y \longrightarrow 0 \ .$$

Moreover, the map $u\colon F \to E$ (where E is as in diagram (10)), with $u(k,y) = \mu k + \phi y$, is a morphism of Leibniz algebras, which induces a morphism of crossed modules

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & K \arrow[r, "j"] \arrow[d, equal] & F \arrow[r] \arrow[d, "u"] & Y \arrow[d, "\phi"] \\
ZK \arrow[r] & K \arrow[r, "\mu"'] & E \arrow[r] & Q
\end{tikzcd}$$

showing that the abstract kernel associated with the extension above is the original one.

If $g\phi^{\otimes 3}$ is not identically zero, but still a 3-cocycle cohomologous to 0, i.e. $g\phi^{\otimes 3} = d^2_{\phi}\alpha$ for some 2-cochain α in $C^2(Y, ZK)$, then it is sufficient to replace $tf(\phi \otimes \phi)$ with $tf(\phi \otimes \phi) - \alpha$ to turn F into a Leibniz algebra and to have again an extension of Y by K.

Conversely, let an extension in k-Leib be given:

$$0 \longrightarrow K \xrightarrow{i} X \xrightarrow{p} Y \longrightarrow 0 \ .$$

Since the category is groupoid accessible, there is a morphism from the kernel pair $R[p]$ of p to a canonical faithful K-groupoid, which yields a crossed module morphism between i and a faithful crossed module μ:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & K \arrow[r, "i"] \arrow[d, equal] & X \arrow[r, "p"] \arrow[d] & Y \arrow[d, "\phi"] \\
ZK \arrow[r] & K \arrow[r, "\mu"'] & E \arrow[r] & Q
\end{tikzcd}$$

So we obtain the abstract kernel associated with the given extension. The corresponding cochain $g\phi^{\otimes 3} : Y^{\otimes 3} \to ZK$ actually lifts to the upper row, thus being identically 0. ■

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there is a unique dashed arrow which makes the following square commutative:

$$\begin{tikzcd}
W \arrow[r, two heads, "g"] \arrow[d, "f"'] & Y \arrow[d, "q_{\underline{Y}_1}"] \\
X \arrow[r, dashed, two heads, "q_{\underline{X}_1}"'] & \pi_0\underline{X}_1 = \pi_0\underline{Y}_1
\end{tikzcd}$$

It is the quotient map $q_{\underline{X}_1}$, and it produces a regular pushout (i.e. such that the factorization of the pair (f,g) through the pullback is a regular epimorphism).

In [7], Bourn observed that, given a finitely complete Barr-exact category $\mathbb{E}$ (but the same considerations hold if $\mathbb{E}$ is efficiently regular), it is possible to define a *direction* functor $d\colon \mathsf{AutM}(\mathbb{E}_g) \to \mathsf{Ab}(\mathbb{E})$ from the category of objects in $\mathbb{E}$ with global support endowed with an autonomous Mal'tsev operation to the category of abelian group objects in $\mathbb{E}$. The fibers of d are endowed with a closed symmetric monoidal structure. In [8] the same author showed that, if $\mathbb{C}$ is finitely complete and Barr-exact, then the category $\mathbb{E} = \mathsf{Gpd}_{X_0}(\mathbb{C})$ of internal groupoids in $\mathbb{C}$ with fixed object of objects X_0 is finitely complete and Barr-exact, too. Moreover, if $\mathbb{C}$ is Mal'tsev, then any internal groupoid in $\mathbb{C}$ is endowed with an autonomous Mal'tsev operation. Therefore, the direction functor $d\colon \mathsf{AutM}(\mathsf{Gpd}_{X_0}(\mathbb{C})_g) \to \mathsf{Ab}(\mathsf{Gpd}_{X_0}(\mathbb{C}))$ associates with any connected groupoid $\underline{X}_1$ (i.e. an object with global support in $\mathsf{Gpd}_{X_0}(\mathbb{C})$) an abelian group object $d(\underline{X}_1)$ in $\mathsf{Gpd}_{X_0}(\mathbb{C})$. Furthermore, if the object of objects X_0 of $\underline{X}_1$ has global support (and in this case $\underline{X}_1$ is said to be *aspherical*), then there is an equivalence of categories (see Theorem 9 in [8]):

$$\mathsf{Ab}(\mathsf{Gpd}_{X_0}(\mathbb{C})) \cong \mathsf{Ab}(\mathbb{C})\,,$$

and then the direction functor gives rise to a functor

$$d_1\colon \mathsf{AsphGpd}(\mathbb{C}) \to \mathsf{Ab}(\mathbb{C})$$

from the category of aspherical groupoids in $\mathbb{C}$ to the category of abelian groups in $\mathbb{C}$.

This construction can be applied to groupoids with effective support. Indeed any such groupoid $\underline{Z}_1$ is aspherical when considered as an internal groupoid in the slice category $\mathbb{C} \downarrow \pi_0\underline{Z}_1$. The above functor d_1, applied to this situation, yields a groupoid which is called *global direction* in [10], Definition

1.5. The global direction of $\underline{Z}_1$ is the totally disconnected groupoid $d_1\underline{Z}_1$ produced on the right hand side by the following pushout of solid arrows:

$$\begin{tikzcd}
R[\langle z_0, z_1\rangle] \arrow[r, two heads] \arrow[d, dashed, shift right=2, "p_0"'] \arrow[d, dashed, shift left=2, "p_1"] & d_1\underline{Z}_1 \arrow[d, dashed, shift left] \\
Z_1 \arrow[r, two heads] \arrow[u] \arrow[d, "\langle z_0, z_1\rangle"'] & \pi_0\underline{Z}_1 \arrow[u, shift left] \\
Z_0 \times Z_0 &
\end{tikzcd} \tag{3}$$

Notice that the maps p_0 and p_1 provide the same retraction of $\pi_0\underline{Z}_1 \to d_1\underline{Z}_1$ since they are coequalized by the lower horizontal map. The following strong property relates the domain and the codomain of a pretorsor:

Proposition 2.4 ([10], Proposition 1.6)**.** *Suppose $\mathbb{C}$ is Mal'tsev and efficiently regular. Let (f, g) be a pretorsor such that $\underline{Y}_1$ (or, equivalently, $\underline{X}_1$) has effective support. Then the global directions of $\underline{X}_1$ and $\underline{Y}_1$ are the same.*

2.2. Action accessible categories. Most of the notions and the results of this section are borrowed from [13].

Let $\mathbb{C}$ be a pointed protomodular category. Fixed an object $K \in \mathbb{C}$, a split extension with kernel K is a diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "k"] & A \arrow[r, shift left, "p"] & B \arrow[l, shift left, "s"] \arrow[r] & 0
\end{tikzcd}$$

such that $ps = 1_B$ and $k = \ker(p)$. We denote such a split extension by (B, A, p, s, k). Given another split extension (D, C, q, t, l) with the same kernel K, a morphism of split extensions

$$(g, f)\colon (B, A, p, s, k) \longrightarrow (D, C, q, t, l) \tag{4}$$

is a pair (g, f) of morphisms:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "k"] \arrow[d, "1_K"'] & A \arrow[r, shift left, "p"] \arrow[d, "f"'] & B \arrow[l, shift left, "s"] \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & K \arrow[r, "l"] & C \arrow[r, shift left, "q"] & D \arrow[l, shift left, "t"] \arrow[r] & 0
\end{tikzcd} \tag{5}$$

Suppose now that an abstract K-kernel $Y \xrightarrow{\phi} Q$ is given, where Q is the cokernel of a faithful crossed module μ:

$$\begin{tikzcd}
ZK \arrow[r, "n"] & K \arrow[rr, "\mu"] \arrow[dr, shift left, "q"] & & E \arrow[r, shift left, "p"] & Q \arrow[l, shift left, dashed, "s"] \\
& & \mu(K) \arrow[ul, shift left, dashed, "t"] \arrow[ur, "m"'] & &
\end{tikzcd} \tag{10}$$

Then, as in Proposition 5.6, we can associate with μ a linear map $g : Q^{\otimes 3} \to ZK$, which represents an element of $HL^3_\xi(Q, ZK)$. Simply composing with ϕ we obtain a 3-cocycle $g\phi^{\otimes 3} : Y^{\otimes 3} \to ZK$, whose cohomology class is independent from the choice of s and t. In this way we have associated with the abstract kernel ϕ an equivalence class in $HL^3_{\overline{\phi}}(Y, ZK)$ (where $\overline{\phi}$ is the action of Y on ZK induced by ϕ).

The following theorem is the counterpart of Theorem 4.2 for the special case of Leibniz algebras. Here obstructions to the existence of extensions of Y by K are classified by means of Loday-Pirashvili cohomology group $HL^3_{\overline{\phi}}(Y, ZK)$, while in Theorem 4.2 they are classified by groupoid cohomology. This shows that these two approaches give the same classification of obtructions to extensions in k-Leib.

Theorem 5.10. *An abstract K-kernel $Y \xrightarrow{\phi} Q$ in k-Leib gives rise to an extension of Y by K if and only if the associated equivalence class in $HL^3_{\overline{\phi}}(Y, ZK)$ is* 0.

Proof: With the notation of the previous paragraph, consider the vector space $F = K \oplus Y$ endowed with a bracket operation:

$$[(k_1, y_1), (k_2, y_2)] = ([k_1, k_2] + [s\phi y_1, k_2] + [k_1, s\phi y_2] + tf(\phi y_1, \phi y_2), [y_1, y_2])$$

A simple calculation shows that:

$$\begin{aligned}
[[(k_1, y_1), (k_2, y_2)], (k_3, y_3)] = {} & [[(k_1, y_1), (k_3, y_3)], (k_2, y_2)] \\
& + [(k_1, y_1), [(k_2, y_2), (k_3, y_3)]] \\
& + (ng(\phi y_1, \phi y_2, \phi y_3), 0)
\end{aligned}$$

thus F is a Leibniz algebra if and only if $g\phi^{\otimes 3} \equiv 0$, or, in other words, if and only if $tf(\phi \otimes \phi)$ satisfies an equation like (7). In that case, with the obvious

Lemma 5.8. *The crossed module associated with any faithful internal groupoid on a Leibniz algebra K has the centre ZK of K as its kernel.*

We call faithful *any crossed module associated with a faithful internal groupoid.*

Proof: Let be given an internal K-groupoid in k-Leib:

$$\begin{tikzcd} K \arrow[r, "i"] & A \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & B \arrow[l, "e" description] \end{tikzcd}$$

Then, by the already mentioned equivalence between internal categories and crossed modules, the composite ci gives rise to a crossed module. By the Peiffer identity, the kernel of ci is contained in the centre ZK of K.

Suppose now the groupoid (A, B, d, c, e) to be faithful. This implies that the action of B on K induced by conjugation in A is faithful (see Proposition 4.6 in [27]), i.e.:

$$(\text{for all } x \in K \quad [b,x] = [e(b), i(x)] = 0 = [i(x), e(b)] = [x,b]) \iff b = 0$$

Since for all $z \in ZK$ and $x \in K$:

$$[ci(z), x] = [z, x] = 0 = [x, z] = [x, ci(z)]$$

where the first equality depends on the Peiffer identity and the second one holds because $z \in ZK$ (and similarly for the other two equalities), then $ci(z) = 0$ and this proves that $ZK = \ker(ci)$. ■

Thanks to the previous lemma, the following one is a special case of Definition 3.1:

Definition 5.9. *An* abstract K-kernel *in k-Leib is a pair*

$$(\mu, \phi),$$

where μ is a faithful crossed module of domain K, and ϕ is a regular epimorphism $Y \to \mathrm{coker}(\mu)$.

We will eventually refer simply to ϕ to denote the abstract kernel (μ, ϕ), when this is not ambiguous.

such that $l = fk$, $qf = gp$ and $fs = tg$. Let us notice that, since the category $\mathbb{C}$ is protomodular, the pair (k, s) is jointly (strongly) epimorphic, and then the morphism f in (5) is uniquely determined by g.

Split extensions with fixed kernel K form a category, denoted by $\mathsf{SplExt}_{\mathbb{C}}(K)$, or simply by $\mathsf{SplExt}(K)$.

In many algebraic contexts, a split extension as above induces an action of B on K. By considering the faithful actions one can obtain a notion of *faithful extension*, as introduced in [13]:

Definition 2.5 ([13])**.**
- *An object in $\mathsf{SplExt}(K)$ is said to be* faithful *if any object in $\mathsf{SplExt}(K)$ admits at most one morphism into it.*
- *Split extensions with a morphism into a faithful one are called* accessible.
- *If, for any $K \in \mathbb{C}$, every object in $\mathsf{SplExt}(K)$ is accessible, we say that the category $\mathbb{C}$ is* action accessible.

Example 2.6. In the case of groups, faithful extensions are those inducing a group action of B on K (via conjugation in A) which is faithful. Every split extension in Gp is accessible and a morphism into a faithful one can be performed by taking the quotients of B and A over the centralizer $C(K, B)$, i.e. the (normal) subobject of A given by those elements of B that commute in A with every element of K.

The notion of action accessible category generalizes that of action representative category ([4]). In fact, in an action representative category every category $\mathsf{SplExt}(K)$ has a terminal object:

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & K \rtimes [K] \arrow[r, shift left] & {[K]} \arrow[l, shift left] \arrow[r] & 0, \end{tikzcd}$$

where the object $[K]$ is called the *actor* of K. Examples of this situation are the categories Gp of groups (where the actor is $\mathrm{Aut}K$) and R-Lie of R-Lie algebras, with R a commutative ring (where the actor is the Lie algebra $\mathrm{Der}K$ of derivations). The category Rng of (not necessarily unitary) rings is action accessible [13] but not action representative, as shown in [4]. In [27] it is shown that every category of interest, in the sense of [28], is action accessible. This family of examples includes Poisson algebras, Leibniz algebras [21], associative dialgebras [22] and trialgebras [24].

A variation of the notion of action accessible category is that of *groupoid accessible* category. We recall that, in a Mal'tsev category, a reflexive graph

(B, A, d_0, s_0, d_1) is a groupoid if and only if $[R[d_0], R[d_1]] = 0$, i.e. $R[d_0]$ and $R[d_1]$ centralize each other in the sense of Smith (see [12]).

Fixed $K \in \mathbb{C}$, by a groupoid structure on an object (B, A, p, s, k) in $\mathsf{SplExt}(K)$ we mean a morphism $u\colon A \longrightarrow B$ such that $us = 1_B$ and $[R[p], R[u]] = 0$; the system (B, A, p, s, k, u) is then called a K-groupoid. K-groupoids form a category $\mathsf{Gpd}(K)$, in which a morphism

$$(g, f)\colon (B, A, p, s, k, u) \longrightarrow (D, C, q, t, l, v)$$

is a morphism $(g, f)\colon (B, A, p, s, k) \longrightarrow (D, C, q, t, l)$ in $\mathsf{SplExt}(K)$ such that $vf = gu$.

As for the corresponding notions concerning split extensions, we introduce the following ones for internal groupoids.

Definition 2.7. *Let K be an object in $\mathbb{C}$. We denote by K-$\mathsf{Gpd}(\mathbb{C})$ (or simply K-Gpd) the category of K-groupoids in $\mathbb{C}$, morphisms in K-$\mathsf{Gpd}(\mathbb{C})$ are called K-morphisms.*

- *A K-groupoid is said to be* faithful *if any K-groupoid admits at most one K-morphism into it.*
- *A K-groupoid is said to be* accessible *if it admits a K-morphism into a faithful K-groupoid.*
- *If, for any $K \in \mathbb{C}$, every K-groupoid is accessible, then we say that $\mathbb{C}$ is* groupoid accessible.

In [13] it is shown that if $\mathbb{C}$ is homological, then it is action accessible if and only if it is groupoid accessible.

Moreover, when $\mathbb{C}$ is a homological action accessible category, given a morphism

$$(g, f)\colon (B, R, r_0, s, k, r_1) \longrightarrow (D, C, q, t, l, v)$$

in K-$\mathsf{Gpd}(\mathbb{C})$, where the domain is an equivalence relation and the codomain is faithful, then the kernel pair $R[g]$ of g is the centralizer of the relation R, i.e. the largest equivalence relation S on B such that $[R, S] = 0$ (see [13], Theorem 4.1). The normalization of $R[g]$ is the centralizer $C(K, B)$ of K in B. In particular, the normalization of the centralizer of the total relation ∇K is the centre ZK of K.

3. Extensions in action accessible categories

In this section we recall the categorical theory of extensions, as developed in [14]. The reader can refer to that paper and to the references therein for

and satisfying the following equation for all $(y_1, y_2, y_3) \in Y^{\otimes 3}$:

$$\begin{aligned}&[sy_1, if(y_2, y_3)] + [if(y_1, y_3), sy_2] - [if(y_1, y_2), sy_3] + \\ &-f([y_1, y_2], y_3) + f([y_1, y_3], y_2) + f(y_1, [y_2, y_3]) = 0\end{aligned} \tag{9}$$

Observe that if the section s is a morphism in k-Leib, then $f \equiv 0$, Y acts on K by conjugation in X via s and X is isomorphic to the semi-direct product $K \rtimes Y$ in k-Leib, that is the Leibniz algebra with underlying vector space $K \oplus Y$ and with bracket operation:

$$[(k_1, y_1), (k_2, y_2)] = ([k_1, k_2] + [k_1, sy_2] + [sy_1, k_2], [y_1, y_2])$$

In the general case, s is not a morphism and X is isomorphic to a Leibniz algebra whose underlying vector space is again $K \oplus Y$ and the bracket operation is perturbed by f:

$$[(k_1, y_1), (k_2, y_2)] = ([k_1, k_2] + [k_1, sy_2] + [sy_1, k_2] + f(y_1, y_2), [y_1, y_2])$$

The conjugation in X via s does not induce any action of Y on K, but an action of Y on the centre ZK of K.

Indeed, given an *abstract K-kernel* (μ, ϕ) (see Definition 5.9 below), with the induced action $\overline{\phi}$ of Y on ZK, the following theorem, which turns out to be a particular case of Theorem 4.1 in [14] (see also Theorem 3.3 herein), holds:

Theorem 5.7 (Schreier-Mac Lane extension Theorem)**.** *There is a simply transitive action of the group $HL^2_{\overline{\phi}}(Y, ZK)$ on the set of equivalence classes of extensions of Y by K inducing the abstract kernel (μ, ϕ).*

Proof: We give here only a sketch of the proof, explaining the (very simple) way $HL^2_{\overline{\phi}}(Y, ZK)$ acts on the set of extensions.

As above, any extension $\underline{E}$ of Y by K is associated with a bilinear map $f : Y \otimes Y \to K$ for any chosen section of p. An element of $HL^2_{\overline{\phi}}(Y, ZK)$ acts on $\underline{E}$ by the sum $f + g$, where g is a 2-cocycle in the given class of $HL^2_{\overline{\phi}}(Y, ZK)$. Indeed, $f + g : Y \otimes Y \to K$ is a bilinear map satisfying an equation like (9) and, with the same construction of the previous paragraph, it allows to turn $K \oplus Y$ into a Leibniz algebra. ∎

In order to formulate the inverse problem of constructing an extension of Y by K we need some tools.

where $n = \ker(\mu)$, $p = \operatorname{coker}(\mu)$, (q, m) is the (regular epi, mono) factorization of μ, s and t are any fixed k-linear sections of p and q respectively (i.e. $ps = 1_Q$ and $qt = 1_{\mu(A)}$ as linear maps). As a consequence of the definition of crossed module, q is a central extension and $m = \ker(p)$. The action of B on A induces an action ξ of Q on N:

$$\text{for all } (x, y) \in Q \times N: \quad [x, y] := [sx, ny] \quad \text{and} \quad [y, x] := [ny, sx]$$

The choice of s determines a k-linear map $f : Q \otimes Q \to \mu(A)$ defined by the following equality for all $(x_1, x_2) \in Q \otimes Q$:

$$[sx_1, sx_2] = mf(x_1, x_2) + s[x_1, x_2]$$

which measures the extent to which s is not a morphism in k-Leib. By the Leibniz identity it is easy to show that for all $(x_1, x_2, x_3) \in Q^{\otimes 3}$:

$$\begin{array}{l} [sx_1, mf(x_2, x_3)] + [mf(x_1, x_3), sx_2] - [mf(x_1, x_2), sx_3] + \\ -mf([x_1, x_2], x_3) + mf([x_1, x_3], x_2) + mf(x_1, [x_2, x_3]) = 0 \end{array} \tag{7}$$

Notice that this "cocycle" equation does not mean that f is a "true" 2-cocycle since the bracket operation in B does not induce any action of Q on $\mu(A)$, unless $\mu(A)$ is abelian. By lifting f via the chosen t, the equality above no longer holds, but the distance of tf from being a (true) 2-cocycle is measured by an element of N (apply q to obtain (7)):

$$\begin{aligned} ng(x_1, x_2, x_3) = \; & [sx_1, tf(x_2, x_3)] + [tf(x_1, x_3), sx_2] - [tf(x_1, x_2), sx_3] + \\ & -tf([x_1, x_2], x_3) + tf([x_1, x_3], x_2) + tf(x_1, [x_2, x_3]) \end{aligned}$$

In fact, some calculation shows that $d^3 g \equiv 0$, so that the equation above defines a 3-cocycle $g : Q^{\otimes 3} \to N$. Moreover, it is possible to show that different choices of the k-linear sections s and t give rise to a 3-cocycle cohomologous to g. ∎

We are now ready to deal with the problem of extensions with non-abelian kernel. Given an extension of Leibniz algebras:

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "i"] & X \arrow[r, shift left, "p"] & Y \arrow[l, dashed, shift left, "s"] \arrow[r] & 0 \end{tikzcd} \tag{8}$$

that is a pair of morphisms as above, such that $i = \ker(p)$ and $p = \operatorname{coker}(i)$, it is always possible to choose a k-linear section s of p. As in the proof of Proposition 5.6, we can see that this choice produces a k-linear map $f : Y \otimes Y \to K$ defined by the following equality for all $(y_1, y_2) \in Y \otimes Y$:

$$[sy_1, sy_2] = if(y_1, y_2) + s[y_1, y_2]$$

a more detailed account. Throughout this section, $\mathbb{C}$ will be a Barr-exact action accessible category, which is then also groupoid accessible.

In this setting there exists a canonical faithful groupoid associated with any equivalence relation. More precisely, it is possible to show that given a K-morphism with faithful codomain and an equivalence relation as domain, then it factors through a specified regular epimorphism with faithful codomain.

Consider now any extension

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "k"] & X \arrow[r, "f"] & Y \arrow[r] & 0 \end{tikzcd} .$$

Denoting by $\underline{T}_1 f$ the canonical faithful K-groupoid associated with the kernel relation $R[f]$, let $(q_f, Q_f) = (q_{\underline{T}_1 f}, \pi_0 \underline{T}_1 f)$ and consider the diagram:

$$\begin{tikzcd} R[f] \arrow[r, two heads, "k_{1f}"] \arrow[d, shift right=2, "f_0"'] \arrow[d, shift left=2, "f_1"] & T_1 f \arrow[d, shift right=2, "\tau_0"'] \arrow[d, shift left=2, "\tau_1"] \\ X \arrow[u] \arrow[r, two heads, "k_f"] \arrow[d, "f"'] & T_0 f \arrow[u] \arrow[d, "q_f"] \\ Y \arrow[r, dashed, two heads, "\phi"'] & Q_f \end{tikzcd} \tag{6}$$

Since $q_f k_f f_0 = q_f k_f f_1$, there exists a unique arrow $\phi \colon Y \to Q_f$ making the lower square commutative. It is immediate to show that this square is a pushout.

Definition 3.1. *We call the pair $(\underline{T}_1 f, \phi)$ the* abstract direction *of the extension f (an* indexation *in* [14]*), and we denote by*

$$\operatorname{Ext}_{(\underline{T}_1 f, \phi)}(Y, K)$$

the set of (isomorphism classes of) extensions of Y by K inducing the abstract direction $(\underline{T}_1 f, \phi)$.

Remark 3.2. For the reader who is not familiar with the categorical theory of extensions, it may be useful to briefly examine the situation in the action representative category of groups. In fact, the discussion below is *element free*, so that it applies to any Barr-exact action representative category.

In the case of groups, diagram (6) above can be obtained as a factorization of the one involving the automorphisms group.

Actually, the normal subgroup K determines a (conjugation) action

$$X \longrightarrow \mathrm{Aut}K \ ,$$

that is the object component of the internal functor into the action groupoid of K, i.e. the groupoid with group of objects $\mathrm{Aut}K$, and with group of arrows the semidirect product $K \rtimes \mathrm{Aut}K$ (the *holomorph* group of K). This induces a homomorphism Φ into the connected components of that groupoid, the group $\mathrm{Out}K$ of outer homomorphisms, i.e. Φ is the classical abstract kernel of [26]. Hence diagram (6) above takes part in the following factorization:

$$\begin{tikzcd}
R[f] \arrow[r, two heads, "k_{1f}"] \arrow[d, shift right=2, "f_0"'] \arrow[d, shift left=2, "f_1"] & T_1 f \arrow[r, tail] \arrow[d, shift right=2, "\tau_0"'] \arrow[d, shift left=2, "\tau_1"] & K \rtimes \mathrm{Aut}K \arrow[d, shift right=2] \arrow[d, shift left=2] \\
X \arrow[r, two heads, "k_f"] \arrow[d, "f"'] \arrow[u] & T_0 f \arrow[r, tail] \arrow[d, "q_f"] \arrow[u] & \mathrm{Aut}K \arrow[d, "q_K"] \arrow[u] \\
Y \arrow[r, dashed, two heads, "\phi"] \arrow[rr, bend right, "\Phi"'] & Q_f \arrow[r] & \mathrm{Out}K
\end{tikzcd}$$

When K is abelian, the map q_K is an isomorphism, so that the abstract kernel gives an actual action of Y on K. The totally disconnected groupoid given by this action is called the *direction of the extension* (see [10]). From this comes the name *abstract direction* of Definition 3.1.

Let us return to the general case. The pullback along ϕ clearly induces a *change of base*. This yields the groupoid $\underline{D}_1\phi := \phi^*(\underline{T}_1 f)$ and the factorization $\underline{k}_{1f} = \underline{d}_{1\phi} \cdot \underline{f}_{1\phi}$ of K-discrete fibrations:

$$\begin{tikzcd}
R[f] \arrow[r, dashed, "f_{1\phi}"'] \arrow[rr, bend left, "k_{1f}"] \arrow[d, shift right=2, "f_0"'] \arrow[d, shift left=2, "f_1"] & D_1\phi \arrow[r, two heads, "d_{1\phi}"'] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & T_1 f \arrow[d, shift right=2, "\tau_0"'] \arrow[d, shift left=2, "\tau_1"] \\
X \arrow[r, dashed, "f_\phi"'] \arrow[rr, bend left, "k_f"] \arrow[dr, "f"'] & D\phi \arrow[r, two heads, "d_\phi"'] \arrow[d, "q_\phi"] & T_0 f \arrow[d, "q_f"] \\
& Y \arrow[r, two heads, "\phi"'] & Q_f
\end{tikzcd}$$

operation. So it becomes clear that the properties listed above are inherited from the Leibniz identity.

Definition 5.4 ([23], Definition (1.6))**.** *A crossed module in k-*Leib *is a morphism $A \xrightarrow{\mu} B$, together with an action of B on A, such that, for all $a, a_1, a_2 \in A$ and $b \in B$:*

$$\begin{cases} [\mu(a), b] = [a, b] \\ [b, \mu(a)] = [b, a] \end{cases} \qquad \textit{(precrossed module condition)}$$

$$[\mu(a_1), a_2] = [a_1, a_2] = [a_1, \mu(a_2)] \quad \textit{(Peiffer identity)}$$

Definition 5.5 ([23], Definition (1.8))**.** *Let two Leibniz algebras A and B be given, together with an action ξ of B on A. Denote:*

$$C^n(B, A) := \mathrm{Hom}_k(B^{\otimes n}, A)$$

$$(d^n_\xi f)(x_1, \ldots, x_{n+1}) := [x_1, f(x_2, \ldots, x_{n+1})] + \sum_{i=2}^{n+1} (-1)^i [f(x_1, \ldots, \widehat{x}_i, \ldots, x_{n+1}), x_i] +$$

$$+ \sum_{1 \le i < j \le n+1} (-1)^{j+1} f(x_1, \ldots, x_{i-1}, [x_i, x_j], x_{i+1}, \ldots, \widehat{x}_j, \ldots, x_{n+1})$$

Then $(C^*(B, A), d_\xi)$ is a cochain complex, whose cohomology is called the cohomology of the Leibniz algebra B *with coefficients in A:*

$$HL^*_\xi(B, A) := H^*(C^*(B, A), d_\xi)\,.$$

Notice that, differently from the original notation adopted in [23] for the cohomology groups HL, we use the subscript ξ in order to keep track of the action.

Proposition 5.6. *Every crossed module $A \xrightarrow{\mu} B$ in k-*Leib *is associated with an equivalence class of cocycles in* $HL^3_\xi(\mathrm{coker}(\mu), \ker(\mu))$.

Proof: Consider the following diagram:

$$\begin{tikzcd}
N \arrow[r, "n"] & A \arrow[rr, "\mu"] \arrow[dr, shift left, "q"] & & B \arrow[r, shift left, "p"] & Q \arrow[l, shift left, dashed, "s"] \\
& & \mu(A) \arrow[ul, shift left, dashed, "t"] \arrow[ur, "m"'] & &
\end{tikzcd}$$

This example also shows that, if we define the actions of G on M as morphisms $G \to \mathrm{Bider}(M)$, we can construct crossed modules which are not crossed modules in the sense of [23], hence their equivalence classes do not correspond to elements of the third cohomology group. For example, if K is the Leibniz algebra defined above, the zero morphism $K \to K$, with the non-derived action defined above, gives rise to a crossed module which is not a crossed module in the sense of Definition 5.4 below and it cannot be seen as an element of Loday-Pirashvili third cohomology group.

In conclusion, for a Leibniz algebra M, the linear subspace S of $\mathrm{Bider}(M)$ that classifies derived actions is not a subalgebra in general, as shown in Example 5.1. In fact, it satisfies the following chain of inclusions:

$$\underline{\mathrm{Bider}}(M) \subseteq S \subseteq \mathrm{Bider}(M),$$

which are all proper in general. It follows from Theorem 5.5 in [16] that S is a subalgebra if and only if $\underline{\mathrm{Bider}}(M) = \mathrm{Bider}(M)$, and in this case $\mathrm{Bider}(M)$ is an actor for M.

Our approach permits to define abstract kernels without using actors, and it fixes the problems highlighted by Examples 5.1 and 5.2, because it allows to associate an abstract kernel with any extension, and to prove that an abstract kernel has trivial obstruction class if and only if there is an extension associated with it.

5.2. Obstruction theory.

Definition 5.3 ([23], Definition (1.6))**.** *An action of a Leibniz algebra B on another Leibniz algebra A is given by a pair of bilinear maps:*

$$\begin{array}{l}[-,-] : B \times A \to A \\ [-,-] : A \times B \to A\end{array}$$

satisfying the following identities (for all $a, a_1, a_2 \in A$ and $b, b_1, b_2 \in B$):

$$\begin{array}{ll}
[[a_1,a_2],b] = [[a_1,b],a_2] + [a_1,[a_2,b]] & [[a,b_1],b_2] = [[a,b_2],b_1] + [a,[b_1,b_2]] \\
[[a_1,b],a_2] = [[a_1,a_2],b] + [a_1,[b,a_2]] & [[b_1,a],b_2] = [[b_1,b_2],a] + [b_1,[a,b_2]] \\
[[b,a_1],a_2] = [[b,a_2],a_1] + [b,[a_1,a_2]] & [[b_1,b_2],a] = [[b_1,a],b_2] + [b_1,[b_2,a]]
\end{array}$$

In the previous definition we use the same symbol to denote the action and the bracket operation. This choice is justified by the fact that, in the semi-abelian case, actions can always be interpreted as conjugations (in the semi-direct product) and in k-Leib the conjugation is exactly the bracket

The map f_ϕ is a regular epimorphism whose kernel is the centre ZK of K, and the kernel pair of $f_{1\phi}$ is a centralizing double relation for $R[f]$ and $R[f_\phi]$, so that $[R[f], R[f_\phi]] = 0$. Hence the extension we started with determines a pretorsor (f_ϕ, f). This can be identified with a profunctor

$$\underline{D}_1\phi \looparrowright \underline{E}_1\phi\,,$$

where $\underline{E}_1\phi$ is the direction of $\underline{D}_1\phi$ in $\mathbb{C} \downarrow Y$ (notice that the groupoid $\underline{D}_1\phi$ is aspherical in $\mathbb{C} \downarrow Y$).

The codomain of the profunctor can be constructed as follows. First we observe that K is a subobject of the centralizer $C(ZK, X)$, which is, by contruction, the kernel of k_{f_ϕ}. This implies that there is a regular epimorphism

$$c_f \colon Y = \frac{X}{K} \twoheadrightarrow \frac{X}{C(ZK,X)} = T_0 f_\phi$$

such that $c_f \cdot f = k_{f_\phi}$. Then the pullback of the totally disconnected groupoid $\underline{T}_1 f_\phi$ gives $\underline{E}_1\phi$, and then the desired discrete fibration, as shown in the following diagram:

$$\begin{tikzcd}
R[f_\phi] \arrow[r, dashed, "\lambda"'] \arrow[rr, bend left, "k_{1f_\phi}"] \arrow[d, shift right=2] \arrow[d, shift left=2] & E_1\phi \arrow[r] \arrow[d, shift right, "e_\phi"'] & T_1 f_\phi \arrow[d, shift right=2] \arrow[d, shift left=2] \\
X \arrow[u] \arrow[r, two heads, "f"] \arrow[rr, bend right, "k_{f_\phi}"'] & Y \arrow[u, shift right, "s_\phi"'] \arrow[r, "c_f"] & T_0 f_\phi \arrow[u]
\end{tikzcd}$$

As a matter of fact the profunctor

$$\begin{tikzcd}
D \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, shift left=2] \arrow[d, shift right=2] & R[f_\phi] \arrow[l] \arrow[r, "\lambda"] \arrow[d, shift left=2] \arrow[d, shift right=2] & E_1\phi \arrow[d, shift right, "e_\phi"'] \\
R[f] \arrow[u] \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "f_{1\phi}"'] & X \arrow[u] \arrow[l] \arrow[r, two heads, "f"'] \arrow[d, "f_\phi"] & Y \arrow[u, shift right, "s_\phi"'] \\
D_1\phi \arrow[r, shift left] \arrow[r, shift right] & D\phi, \arrow[l] &
\end{tikzcd}$$

determined by the pretorsor (f_ϕ, f) is characteristic of the isomorphism class of the extension.

Moreover one can verify that profunctor composition induces a simply transitive action

$$\mathsf{RF}(\mathbb{C})(\underline{D}_1\phi, \underline{E}_1\phi) \times \mathsf{RF}(\mathbb{C})(\underline{E}_1\phi, \underline{E}_1\phi) \to \mathsf{RF}(\mathbb{C})(\underline{D}_1\phi, \underline{E}_1\phi)$$

of the abelian group $\mathsf{RF}(\mathbb{C})(\underline{E}_1\phi, \underline{E}_1\phi)$ onto the set $\mathsf{RF}(\mathbb{C})(\underline{D}_1\phi, \underline{E}_1\phi)$.

Finally, the connection with the classical theory of extensions is given by the fact that the kernel of e_ϕ is (isomorphic to) ZK, and that one can identify $\mathsf{RF}(\mathbb{C})(\underline{E}_1\phi, \underline{E}_1\phi)$ with the abelian group $\mathrm{Ext}_{\overline{\phi}}(Y, ZK)$, where $\overline{\phi}$ is the abstract direction determined by the split extension (e_ϕ, s_ϕ), and the group operation is the usual Baer sum. This gives the following result (see [14], Theorem 4.1, and [11], Theorem 4.11).

Theorem 3.3 (Schreier-Mac Lane extension Theorem)**.** *Let $\mathbb{C}$ be a semi-abelian action accessible category. Given an abstract direction $(\underline{T}_1 f, \phi)$, on the set $\mathrm{Ext}_{(\underline{T}_1 f, \phi)}(Y, K)$ there is a simply transitive action of the abelian group $\mathrm{Ext}_{\overline{\phi}}(Y, ZK)$.*

4. Obstruction to extensions in action accessible categories

Given any faithful K-groupoid

$$K \longrightarrow T_1 \underset{\tau_0}{\overset{\tau_1}{\rightleftarrows}} T_0$$

and any morphism $\phi\colon Y \to Q$, where Q is the coequalizer of τ_0 and τ_1, we want to characterize the situations where the set $\mathrm{Ext}_{(\underline{T}_1, \phi)}(Y, K)$ is not empty. In the case of groups, with any such morphism ϕ is associated a cohomology class in $H^3_{\overline{\phi}}(Y, ZK)$, $\overline{\phi}$ being the action on ZK induced by the abstract kernel.

Intrinsically, $H^3_{\overline{\phi}}(Y, ZK)$ corresponds to a second cohomology group in the sense of Bourn, as explained below.

In a finitely complete Barr-exact category $\mathbb{E}$, Bourn cohomology is constructed using n-groupoids (following [6]). In particular, given an abelian group object A in $\mathbb{E}$, we are interested in the group $H^2_{\mathbb{E}}A$, which is given by the component classes of aspherical groupoids $\underline{X}_1$ with global direction $K_1(A)$ (aspherical means a connected groupoids such that X_0 has global support, while $K_1(A)$ is the groupoid $A \rightleftarrows 1$). In this way, any internal groupoid necessarily determines an element in the second cohomology group with coefficients in its global direction. It turns out that $H^3_{\overline{\phi}}(Y, ZK)$ corresponds to

$$-[a, [b, (x, y, z)]] = (-ab(x + z), 0, ab(x + z))\,.$$

Hence, denoting

$$[-, a] = -d_a\,, \quad [a, -] = D_a\,,$$
$$[-, b] = -d'_b\,, \quad [b, -] = D'_b\,,$$

we have that $D_a d'_b \neq D_a D'_b$ if a or b are not 0.

This means that the action of A on L defined above is a derived action that cannot be expressed as a morphism into $\underline{\mathrm{Bider}}(L)$.

This example shows that there are extensions of Leibniz algebras with which it is impossible to associate an abstract kernel in the sense of [15]. Indeed, considering the semi-direct product $L \rtimes A$ of L and A defined by the action of A on L above, we obtain an extension of Leibniz algebras

$$0 \longrightarrow L \longrightarrow L \rtimes A \longrightarrow A \longrightarrow 0\,,$$

actually a split extension, such that the conjugation action of $L \rtimes A$ on L does not give rise to a morphism $L \rtimes A \to \underline{\mathrm{Bider}}(L)$.

On the other hand, every derived action of a Leibniz algebra G on a Leibniz algebra M can be seen as a morphism $G \to \mathrm{Bider}(M)$. However, there are in general morphisms into $\mathrm{Bider}(M)$ that do not give rise to derived actions (see Example 5.2 below). This implies that $\mathrm{Bider}(M)$ is not an actor, as observed in [16], where the authors give necessary and sufficient conditions for a Leibniz algebra to have an actor.

Example 5.2. Given a field k with characteristic different from 2, consider the Leibniz algebra K, whose underlying k-vector space is k itself, and whose bracket is the trivial one: $[x_1, x_2] = 0$ for all $x_1, x_2 \in K$. Define a morphism $\varphi\colon K \to \mathrm{Bider}(K)$ in the following way:

$$\varphi(a) = (d_a, D_a), \quad \text{where } d_a(x) = -ax,\ D_a(x) = ax.$$

This defines an action of K on itself which is not derived: indeed, denoting

$$[-, a] = -d_a, \quad [a, -] = D_a,$$

we have:

$$[a, [b, x]] = abx \neq -abx = -[a, [x, b]].$$

Hence, according to Theorem 5.5 in [16], K does not admit an actor.

Bourn second cohomology group $H^2(Y, ZK)$ give the same classification of obstructions to the existence of extensions with non-abelian kernel K.

Obstruction theory for Leibniz algebras over a field was already studied by Casas in [15]. In that paper, the author defined an action of a Leibniz algebra G on a Leibniz algebra M as a morphism of Leibniz algebras

$$\sigma \colon G \to \underline{\mathrm{Bider}}(M),$$

where $\underline{\mathrm{Bider}}(M)$ is the subalgebra of the Leibniz algebra $\mathrm{Bider}(M)$ of biderivations of M (see [21]) defined by:

$$\underline{\mathrm{Bider}}(M) = \{(d, D) \in \mathrm{Bider}(M) \mid Dd' = DD', \text{ for all } (d', D') \in \mathrm{Bider}(M)\}.$$

We recall that a biderivation of a Leibniz algebra M is a pair (d, D) of k-linear maps satisfying, for any $x, y \in M$, the following conditions:

i) $d([x, y]) = [d(x), y] + [x, d(y)]$;
ii) $D([x, y] = [D(x), y] - [D(y), x]$;
iii) $[x, d(y)] = [x, D(y)]$.

However, Casas's definition does not include all derived actions, as the following example shows.

Example 5.1. Let k be any field. It is easy to see that the following operation in $L = k^3$:

$$[(x_1, y_1, z_1), (x_2, y_2, z_2)] = (0, (x_1 + z_1)(x_2 + z_2), 0)$$

satisfies the Leibniz identity, so $(L, +, [\,,])$ is a Leibniz algebra.

Let now A and B be the same Leibniz algebra, given by k as a vector space, with bracket operation $[x, y] = 0$ for all $x, y \in k$. We can construct two derived actions, of A and B respectively, on L in the following way:

$$\begin{array}{ll} [-,-] : A \times L \to L & [a, (x, y, z)] = (-ax, 0, ax) \\ [-,-] : L \times A \to L & [(x, y, z), a] = (ax, 0, -ax) \end{array}$$

$$\begin{array}{ll} [-,-] : B \times L \to L & [b, (x, y, z)] = (-b(x + z), 0, -b(x + z)) \\ [-,-] : L \times B \to L & [(x, y, z), b] = (2bx, 4by, 2bz) \end{array}$$

These two actions induce biderivations that are not compatible with each other, indeed it is not true that for all $a \in A$, $b \in B$ and $(x, y, z) \in L$:

$$[a, [(x, y, z), b]] = -[a, [b, (x, y, z)]]\,,$$

since

$$[a, [(x, y, z), b]] = (-2abx, 0, 2abx)\,,$$

$H^2_{\mathbb{C}\downarrow Y}(\underline{A})$, where $\underline{A}$ is the abelian group in $\mathbb{C} \downarrow Y$ given by the split extension $ZK \rtimes_{\overline{\phi}} Y \rightleftarrows Y$. Now we can make more precise Proposition 2.4:

Proposition 4.1 ([10], Proposition 3.5)**.** *Suppose $\mathbb{C}$ is Mal'tsev and efficiently regular. Let (f, g) be a pretorsor such that $\underline{Y}_1$ has effective support. Then not only the global directions of $\underline{X}_1$ and $\underline{Y}_1$ are the same (let us say (v, u), with $V_1 \underset{u}{\overset{v}{\rightleftarrows}} V = \pi_0 \underline{Y}_1$), but also the two groupoids $\underline{X}_1$ and $\underline{Y}_1$ determine the same element in the cohomology group $H^2_{\mathbb{C}\downarrow V} V_1$.*

The groupoid $\underline{D}_1\phi$ appears then as an element of the cohomology group $H^2_{\mathbb{C}\downarrow Y}\underline{E}_1\phi$, where $\underline{E}_1\phi$ is its global direction. We can now state our result, which transfers to action accessible categories the "obstruction part" of the classical Schreier-Mac Lane theorem on extensions with non-abelian kernel.

Theorem 4.2. *Let $\mathbb{C}$ be a Barr-exact action accessible category. Given any faithful groupoid*

$$K \longrightarrow T_1 \underset{\tau_0}{\overset{\tau_1}{\rightrightarrows}} T_0$$

and any morphism $\phi \colon Y \to Q$, where Q is the coequalizer of τ_0 and τ_1, the set $\mathrm{Ext}_{(\underline{T}_1, \phi)}(Y, K)$ is not empty if and only if the groupoid $\underline{D}_1\phi$, seen as an element of the cohomology group $H^2_{\mathbb{C}\downarrow Y}\underline{E}_1\phi$, is 0.

Proof: The following proof is inspired by the one given in [10] for action representative categories. Suppose first that there exists an extension f of Y by K with abstract direction $(\underline{T}_1, \phi)$. Then we know that with f it is associated a pretorsor (f_ϕ, f), whose domain is $\underline{D}_1\phi$ and whose codomain is its global direction $\underline{E}_1\phi$, as proved in Proposition 2.4. Moreover, thanks to Proposition 4.1, we know that $\underline{D}_1\phi$ and $\underline{E}_1\phi$ determine the same element in the cohomology group $H^2_{\mathbb{C}\downarrow Y}\underline{E}_1\phi$. Clearly $\underline{E}_1\phi$ represents the zero element of this group, and then the thesis follows.

Conversely, suppose that $\underline{D}_1\phi$ is the zero element in $H^2_{\mathbb{C}\downarrow Y}\underline{E}_1\phi$. According to Theorem 12 in [8], in a Barr-exact category $\mathbb{E}$ an aspherical groupoid $\underline{Z}_1$ with direction A is zero in $H^2_{\mathbb{E}}A$ if and only if there is an object H with global support and a $()_0$-cartesian functor from $\nabla H \times K_1(A)$ to $\underline{Z}_1$. This is the same thing as the existence of a functor from ∇H to $\underline{Z}_1$. In our context, where $\mathbb{E} = \mathbb{C} \downarrow Y$, an aspherical groupoid amounts to a groupoid $H_1 \underset{c}{\overset{d}{\rightrightarrows}} H$ in $\mathbb{C}$, together with a regular epimorphism $h \colon H \to Y$, such that $hd = hc$ and

$\langle d, c\rangle \colon H_1 \to R[h]$ is a regular epimorphism. The $()_0$-cartesian functor above is a functor $\underline{l}_1 \colon R[h] \to \underline{D}_1\phi$ between groupoids in $\mathbb{C} \downarrow Y$.

Since the category $\mathbb{C} \downarrow Y$ is Barr-exact, we can construct, according to Theorem 4 in [5], a factorization

$$\underline{l}_1 = \underline{m}_1\underline{n}_1 \colon R[h] \to \underline{X}_1 \to \underline{D}_1\phi$$

such that $\underline{m}_1$ is a discrete fibration and $\underline{n}_1$ is a final functor. Since $R[h]$ is an equivalence relation, the groupoid $\underline{X}_1$ is actually an equivalence relation S on $X = X_0$ (see Proposition 1.4 in [10]). Moreover, since $\underline{n}_1$ is final, the quotients of $R[h]$ and S are isomorphic. Accordingly, we get $S = R[f]$ for some regular epimorphism f such that $fn = h$. Consider now the following diagram:

$$\begin{tikzcd}
R[h] \arrow[r, "n_1"] \arrow[d, shift right, "h_0"'] \arrow[d, shift left, "h_1"] & R[f] \arrow[r, two heads, "m_1"] \arrow[d, shift right, "f_0"'] \arrow[d, shift left, "f_1"] & D_1\phi \arrow[r, two heads, "d_{1\phi}"] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & T_1 \arrow[d, shift right, "\tau_0"'] \arrow[d, shift left, "\tau_1"] \\
H \arrow[r, "n"] \arrow[drr, "h"'] & X \arrow[r, two heads, "m"] \arrow[dr, "f"] & D\phi \arrow[r, two heads, "d_\phi"] \arrow[d, "q_\phi"] & T_0 \arrow[d, "q"] \\
 & & Y \arrow[r, two heads, "\phi"'] & Q
\end{tikzcd}$$

Since the functors $\underline{m}_1$ and $\underline{d}_{1\phi}$ are discrete fibration, also their composition is. Moreover, $d_\phi m$ and $d_{1\phi}m_1$ are regular epimorphisms. This means that $\underline{T}_1$ is the canonical faithful groupoid associated with $R[f]$. Hence we have an extension:

$$0 \longrightarrow K \xrightarrow{k} X \xrightarrow{f} Y \longrightarrow 0$$

whose abstract direction is $(\underline{T}_1, \phi)$, since the morphism $d_\phi m$ clearly induces the factorization $\phi \colon Y \to Q$. ■

5. Obstruction to extensions for Leibniz algebras

In this section, as an example of action accessible category which is not action representative, we consider the case of Leibniz algebras, introduced by Loday in [21]. In this setting we develop obstruction theory by means of Loday-Pirashvili cohomology (see [23]). We are led to Theorem 5.10, that turns out to be an instance of the more general Theorem 4.2.

5.1. Preliminaries. We will refer to the category of right k-Leibniz algebras (k-Leib from now on), which are vector spaces on a fixed field k, endowed with a bilinear operation $[-,-]$ satisfying the following identity (Leibniz identity):

$$[[x, y], z] = [[x, z], y] + [x, [y, z]]$$

We recall here from [23] only the necessary tools in order to deal with obstruction theory.

Observe that the notion of action of a Leibniz algebra Y on another Leibniz algebra K is a special case of those introduced by Orzech [28], in the context of categories of interest, under the name of "derived actions", i.e. actions induced by split extensions:

$$0 \longrightarrow K \longrightarrow X \rightleftarrows Y \longrightarrow 0 \ .$$

At a purely categorical level, via the semi-direct product construction which is available in any semi-abelian category, they correspond to internal object actions in the sense of Borceux, Janelidze and Kelly [4]. The notion of crossed module of Leibniz algebras is again a special case of a more general one, introduced by Porter [30] in the context of groups with operations; in the same article, the author proved the equivalence between crossed modules and internal categories in any category of groups with operations. Actually, in [20] Janelidze introduced a categorical notion of crossed module, based on internal actions, and proved that the same equivalence holds in the context of semi-abelian categories. So, since the category k-Leib is semi-abelian, Definitions 5.3 and 5.4 (borrowed from [23]) are in fact instances of the corresponding internal ones.

In [23], Loday and Pirashvili defined cohomology groups for Leibniz algebras over a commutative ring k. They proved that, given an abelian Leibniz algebra A and another Leibniz algebra Y with a fixed action on A, their second cohomology group $HL^2(Y, A)$ is isomorphic to the abelian group of (isomorphism classes of) extensions of Y by A which are split extensions of k-modules and induce the given action of Y on A. On the other hand, Bourn first cohomology group $H^1(Y, A)$ is isomorphic to the abelian group of (isomorphism classes of) extensions of Y by A inducing the given action, which are not necessarily split as k-linear maps. In the case where k is a field, as in the present paper, the two cohomology groups are obviously isomorphic. Moreover, the results of this section will show that also $HL^3(Y, ZK)$ and