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INVERSE SPECTRAL PROBLEMS FOR STRUCTURED PSEUDO-SYMMETRIC MATRICES

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ABSTRACT: Inverse spectral problems for Jacobi and periodic Jacobi matrices with certain sign patterns are investigated. Necessary and sufficient conditions under which the problems are solvable are presented. Uniqueness results are also discussed. Algorithms to construct the solutions from the spectral data are provided and illustrative examples are given.

KEYWORDS: Inverse eigenvalue problem, periodic pseudo-Jacobi matrix, pseudo-Jacobi matrix, pseudo-symmetric matrix, eigenvalues.
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1. Introduction

In the sequel we consider $\mathbb{C}^n$ endowed with the indefinite inner product defined by $[x, y] = y^* Hx$, for any $x, y \in \mathbb{C}^n$, where $H = \operatorname{diag}(\delta_1, \delta_2, \cdots, \delta_n)$ is a selfadjoint involution, i.e., $H = H^*$ and $H^2 = I$. A real matrix A is H-*symmetric* or *pseudo-symmetric*, if $HA^T H = A$, being $HA^T H$ the H-*adjoint* of A, usually denoted by $A^\#$.

A *periodic pseudo-Jacobi* matrix is one of the form

$$J_n = \begin{bmatrix} a_1 & \epsilon_1 b_1 & 0 & \cdots & 0 & b_n \\ b_1 & a_2 & \epsilon_2 b_2 & \cdots & 0 & 0 \\ 0 & b_2 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & \epsilon_{n-1} b_{n-1} \\ \epsilon_1 \cdots \epsilon_{n-1} b_n & 0 & 0 & \cdots & b_{n-1} & a_n \end{bmatrix}, \tag{1}$$

where all entries are real, $b_i \geq 0$ and $\epsilon_i = \pm 1$. The matrix J_n is H-symmetric for $H = \operatorname{diag}(1, \epsilon_1, \epsilon_1\epsilon_2, \cdots, \epsilon_1 \cdots \epsilon_{n-1})$. If $b_n = 0$, the tridiagonal matrix so obtained is called a *pseudo-Jacobi matrix*, which is said *unreduced* if $b_i > 0$, $i = 1, \ldots, n-1$. In the sequel, we shall be concerned with this case. If H is the identity, matrices of the type (1) are termed *periodic Jacobi*

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matrices. These matrices deserved the attention of many authors as they appear in different subjects of pure and applied mathematics, see [10, 11] and the references therein. For example, they arise in the discretization of the one-dimensional Schrödinger equation with periodic boundary conditions [3], or in connection with small vibrations of a nonhomogeneous ring [8]. Jacobi matrices have motivated the interest of researchers due to its applications in many areas (cf. e.g. [2, 7, 9] and the above cited monographs).

Following [12], the matrix J_n can be represented as

$$J_n = \begin{bmatrix} J_{n-1} & y \\ z^T & a_n \end{bmatrix}, \tag{2}$$

where J_{n-1} is the matrix obtained from J_n by deleting the last row and column and

$$y = (b_n, 0, \cdots, 0, \epsilon_{n-1} b_{n-1})^T \in \mathrm{R}^{n-1}, \; z^T = (\epsilon_1 \cdots \epsilon_{n-1} b_n, 0, \cdots, 0, b_{n-1}) \in \mathrm{R}^{n-1}.$$

The matrix J_{n-1} is pseudo-symmetric, so its spectrum is symmetric relatively to the real axis. We shall assume that J_{n-1} has real and simple eigenvalues, this condition ensuring that J_{n-1} has a corresponding set of real H-orthonormal eigenvectors.

Our main aim is the investigation of the following inverse spectral problem, throughout referred as IPPJ:

Given the data $\{\lambda, \mu, \beta, \delta\}$ satisfying the following conditions:

(i) $\lambda_1, \ldots, \lambda_n$ are complex and pairwise distinct numbers, closed under complex conjugation;
(ii) $\mu_1, \ldots, \mu_{n-1}$ are real and pairwise distinct numbers;
(iii) β is a positive number;
(iv) $\delta_1 = 1$, and $\delta_j = \pm 1$, $j = 2, \cdots, n-1$;

determine a necessary and sufficient condition for the existence of a matrix J_n of the form (1), H-symmetric for $H = \mathrm{diag}(\delta_1, \ldots, \delta_n)$, such that $\sigma(J_n) = \{\lambda_1, \ldots, \lambda_n\}$, $\sigma(J_{n-1}) = \{\mu_1, \ldots, \mu_{n-1}\}$, where $\sigma(X)$ denotes the spectrum of the matrix X, and $\prod_{i=1}^n b_i = \beta$.

This note is organized as follows. In Section 2 a modified Lanczos algorithm, which constructs a pseudo-Jacobi matrix from prescribed spectral data, is presented. In Section 3, IPPJ is investigated. In Section 4, an algorithm to construct its solutions is proposed and an illustrative example is given. The obtained results are parallel to the analogous ones for classical Jacobi and periodic Jacobi matrices, from which the here considered matrices differ in sign patterns. The approaches of Ferguson's in [8] and of Xu and Jiang in [12] have been followed with convenient adaptations.

2. An inverse problem for pseudo-Jacobi matrices

In this section we show that the Lanczos algorithm can be used to recover the entries of a pseudo-Jacobi matrix from its real, simple eigenvalues $\mu_1, \ldots, \mu_{n-1}$, the first entries of the corresponding eigenvectors $v_1, \ldots, v_{n-1}$ and the respective norms $\delta_1, \ldots, \delta_{n-1}$.

For convenience, we shall use capital letters to denote the entries of vectors and matrices obtained by an algorithm. As usual, δ_{jk} denotes the Kronecker symbol (δ_{jk} is 0 if $j \neq k$ and 1 if $j = k$). A real matrix V is called *H-orthogonal* if $VV^{\#} = I$.

The next statement is a sort of uniqueness theorem and the proof provides a reconstruction algorithm.

Theorem 2.1. *Let J_{n-1} be H-symmetric for*

$$H = \mathrm{diag}(1, \epsilon_1 \ldots, \epsilon_1 \cdots \epsilon_{n-2}) = \mathrm{diag}(\delta_1, \ldots, \delta_{n-1}).$$

Let $\sigma(J_{n-1}) = \{\mu_1, \ldots, \mu_{n-1}\}$, $v_{11}, \ldots, v_{n-1,1}$ be the first entries of the corresponding eigenvectors $v_1, \ldots, v_{n-1}$, and $\delta_1 = [v_1, v_1], \ldots, \delta_{n-1} = [v_{n-1}, v_{n-1}]$. Then, J_{n-1} can be constructed by the following:

Algorithm 1.

1. *Set* $b_0 = 1$;
2. *For* $j = 1, \ldots, n-1$, *set* $Y_{0j} = 0$;
3. *For* $j = 1, \ldots, n-1$, *set* $Y_{1j} = v_{1j}$;
4. *Iterate for* $i = 1, \ldots, n-1$;
5. $a_i = \delta_i \sum_{k=1}^{n-1} \delta_k \mu_k Y_{ik}^2$;
6. $b_i = \sqrt{\mathcal{D}_i}, \qquad \mathcal{D}_i := \delta_{i+1} \sum_{k=1}^{n-1} \delta_k ((\mu_k - a_i) Y_{ik} - b_{i-1} Y_{i-1,k})^2$;
7. $Y_{i+1,j} = \dfrac{(\mu_j - a_i) Y_{ij} - b_{i-1} Y_{i-1,j}}{\epsilon_i b_i}$ *for* $j = 1, \ldots, n-1$;
8. *Take next* i;
9. $a_{n-1} = \delta_{n-1} \sum_{k=1}^{n-1} \delta_k \mu_k Y_{n-1,k}^2$.

Proof: The theorem is a consequence of the following three Lemmas. ■

Lemma 2.1. *Under the conditions of Theorem 2.1, we have*

$$a_j = \delta_j \sum_{k=1}^{n-1} \delta_k \mu_k v_{jk}^2, \tag{3}$$

$$b_1^2 = \delta_2 \sum_{k=1}^{n-1} \delta_k (\mu_k - a_1) v_{1k}, \tag{4}$$

$$b_j^2 = \delta_{j+1} \sum_{k=1}^{n-1} \delta_k ((\mu_k - a_j) v_{jk} - b_{j-1} v_{j-1,k})^2, \quad j = 2, \dots, n-2, \tag{5}$$

$$v_{2j} = \epsilon_1 b_1^{-1} (v_{1j}\mu_j - a_1 v_{1j}), \tag{6}$$

$$v_{k+1,j} = \epsilon_k b_k^{-1} (v_{kj}\mu_j - (b_{k-1} v_{k-1,j} + a_k v_{kj})), \quad k = 2, \dots, n-2. \tag{7}$$

Proof: Consider the matrix

$$J_{n-1} = \begin{bmatrix} a_1 & \epsilon_1 b_1 & 0 & \cdots & 0 & 0 \\ b_1 & a_2 & \epsilon_2 b_2 & \cdots & 0 & 0 \\ 0 & b_2 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-2} & \epsilon_{n-2} b_{n-2} \\ 0 & 0 & 0 & \cdots & b_{n-2} & a_{n-1} \end{bmatrix}, \quad b_1, \dots, b_{n-1} > 0, \tag{8}$$

which is H-symmetric for $H = \operatorname{diag}(1, \epsilon_1, \epsilon_1\epsilon_2, \dots, \epsilon_1 \cdots \epsilon_{n-2}) = \operatorname{diag}(\delta_1, \cdots, \delta_{n-1})$. The real matrix V whose columns are, respectively, $v_1, \cdots, v_{n-1}$, satisfy $J_{n-1} v_j = \mu_j v_j$ and $v_j^T J_{n-1} v_k = \delta_k \delta_{jk}$, $j, k = 1, \dots, n-1$. Thus, $VV^\# = I_{n-1}$ and $J_{n-1} = V \operatorname{diag}(\mu_1, \dots, \mu_{n-1})\, V^\#$. Now, we prove (3). From

$$J_{n-1} V = VD, \quad D = \operatorname{diag}(\mu_1, \cdots, \mu_{n-1}), \tag{9}$$

it follows that

$$a_1 v_{1j} + \epsilon_1 b_1 v_{2j} = v_{1j} \mu_j, \tag{10}$$

$$b_{k-1} v_{k-1,j} + a_k v_{kj} + \epsilon_k b_k v_{k+1,j} = v_{kj} \mu_j, \quad k = 2, \dots, n-2. \tag{11}$$

Multiplying these relations by $v_{lj}\delta_j$, summing over j and having in mind that the v_j are H-orthogonal, (2) follows. Next, we prove (4), (5). Clearly, from (10), (11), we get

$$\epsilon_2 b_2 v_{2,k} = v_{1k} (\mu_k - a_1),$$

$$\epsilon_j b_j v_{j+1,k} = v_{jk} (\mu_k - a_j) - v_{j-1,k} b_{j-1}, \quad j = 2, \dots, n-2.$$

If these relations are squared and the result summed over k after multiplication by δ_k, (4) (5) are obtained.

Finally we observe that (6), (7), follow from (10), (11). ■

Lemma 2.2. *Under the conditions of Theorem 2.1, we have*

$$v_{1j}v_{n-1,j}\chi'(\mu_j) = \delta_1\delta_j b_1 \dots b_{n-2}, \quad \text{for} \quad j = 1, \dots, n-1,$$

where $\chi(z) = \det(J_{n-1} - zI_{n-1})$. *In particular,* $v_{1j} \neq 0$, $v_{n-1,j} \neq 0$, *for* $j = 1, \dots, n-1$.

Proof: Clearly,

$$\chi(z)^{-1}\mathrm{Adj}(J_{n-1} - zI_{n-1}) = (J_{n-1} - zI_{n-1})^{-1} = V(D - zI_{n-1})^{-1}HV^T H.$$

Considering the $(1, n-1)$th entry of this relation, we find that

$$b_1 \cdots b_{n-2} = \delta_{n-1}\sum_{j=1}^{n-1}\frac{\chi(z)}{\mu_j - z}v_{1,j}v_{n-1,j}\delta_j. \tag{12}$$

Taking the limit as z approaches μ_j and observing that

$$\chi'(\mu_j) = \prod_{k\neq j}(\mu_k - \mu_j),$$

the result follows. ■

Lemma 2.3. *The pseudo-Jacobi matrix* J_{n-1} *constructed by the modified Lanczos algorithm is unique.*

Proof: Let $\hat{J}_{n-1}$ be a pseudo-Jacobi matrix constructed by the modified Lanczos algorithm from the data $\{\mu, y, \delta\}$. Then, $y_{11}, \dots, y_{1,n-1}$ are the first components of a set $\hat{Y}_1, \dots, \hat{Y}_{n-1}$ of real, pseudo-orthogonal eigenvectors of $\hat{J}_{n-1}$, with pseudo-norms $\delta_1, \dots, \delta_{n-1}$, corresponding to the eigenvalues $\mu_1, \dots, \mu_{n-1}$. If $\hat{Y}$ denotes the matrix whose jth column is $\hat{Y}_j$, then $\hat{Y}$ is H-orthogonal for $H = \mathrm{diag}(\delta_1, \dots, \delta_{n-1})$ and

$$\hat{J}_{n-1}\hat{Y} = D\hat{Y}, \quad D = \mathrm{diag}(\mu_1, \dots, \mu_{n-1}).$$

We will now prove that the entries $(\hat{a}, \hat{b}, \delta)$ of $\hat{J}_{n-1}$ are identical to the entries (a, b, δ) of the matrix J_{n-1} computed by the modified Lanczos algorithm. The entries $\hat{Y}_{jk}$ of $\hat{Y}$ satisfy the H-orthogonality relations

$$\sum_{k=1}^{n-1}\hat{Y}_{jk}\hat{Y}_{ik}\delta_k = \delta_i\delta_{ij},$$

and also satisfy the recurrence relations

$$\hat{a}_1\hat{Y}_{1j} + \epsilon_1\hat{b}_1\hat{Y}_{2j} = \hat{Y}_{1j}\mu_j,$$

$$\hat{b}_{k-1}\hat{Y}_{k-1,j} + \hat{a}_k\hat{Y}_{kj} + \epsilon_k\hat{b}_k\hat{Y}_{k+1,j} = \hat{Y}_{kj}\mu_j, \quad k = 2,\dots,n-2,$$

which follow from $\hat{J}_{n-1}\hat{Y} = D\hat{Y}$. If we multiply the recurrence relations by $\hat{Y}_{lj}\delta_j$ and sum over j the result so obtained, we find that

$$a_j = \delta_j \sum_{k=1}^{n-1} \delta_k\mu_k\hat{Y}_{jk}^2.$$

The recurrence relations also imply that

$$\epsilon_2\hat{b}_2\hat{Y}_{2,k} = \hat{Y}_{1k}(\mu_k - \hat{a}_1),$$

$$\epsilon_j\hat{b}_j\hat{Y}_{j+1,k} = \hat{Y}_{jk}(\mu_k - \hat{a}_j) - \hat{Y}_{j-1,k}\hat{b}_{j-1}, \quad j = 2,\dots,n-2.$$

If these relations are squared, and then summed over k after multiplication by δ_k, we get

$$\hat{b}_i = \sqrt{\delta_{i+1}\sum_{k=1}^{n-1}\delta_k((\mu_k - \hat{a}_i)\hat{Y}_{ik} - \hat{b}_{i-1}\hat{Y}_{i-1,k})^2}.$$

Since $\hat{Y}_{1j} = y_{1j}$, we easily prove, following the sequence of computations in the modified Lanczos algorithm, that

$$\hat{a}_1 = a_1,\ \hat{b}_1 = b_1,\ \hat{Y}_{2j} = \hat{Y}_{2j},$$

$$\hat{a}_2 = a_2,\ \hat{b}_2 = b_2,\ \hat{Y}_{3j} = \hat{Y}_{3j},$$

$$\dots$$

$$\hat{a}_{n-2} = a_{n-2},\ \hat{b}_{n-2} = b_{n-2},\ \hat{Y}_{n-1,j} = \hat{Y}_{n-1j},$$

$$\hat{a}_{n-1} = a_{n-1}.$$

∎

We observe that if $\mathcal{D}_i \leq 0$ at step 6 of the Algorithm 1, then its execution is interrupted. The next result gives sufficient conditions for the algorithm to work.

Theorem 2.2. *Let the data* $\{\mu, y, \delta\}$ *be given and satisfy:*

(i) $\mu_1,\dots,\mu_{n-1}$ *are real and pairwise distinct numbers;*
(ii) $y_{11},\dots,y_{n-1,1}$ *are real, nonzero numbers;*
(iii) $\delta_1 = 1$, $\delta_j = \pm1$, $j = 2,\cdots,n-1$, *and* $\sum_{k=1}^{n-1}\delta_k y_k^2 = \delta_1$*;*

(iv) $\mathcal{D}_i > 0$, $i = 1, \ldots, n-2$.

Then the Algorithm 1 constructs a H-Jacobi matrix J_{n-1},

$$H = \operatorname{diag}(\delta_1, \ldots, \delta_{n-1}),$$

such that $\sigma(J_{n-1}) = \{\mu_1, \ldots, \mu_{n-1}\}$, $y_{11}, \ldots, y_{n-1,1}$ *are the first entries of the corresponding eigenvectors* $y_1, \ldots, y_{n-1}$, *and* $\delta_1 = [y_1, y_1], \ldots, \delta_{n-1} = [y_{n-1}, y_{n-1}]$.

Proof: Assume that $\mathcal{D}_i > 0$, $i = 1, \ldots, n-2$. We show that the rows of the matrix Y constructed by the algorithm satisfy the pseudo-orthogonality relations

$$\sum_{k=1}^{n-1} Y_{ik} Y_{jk} \delta_k = \delta_i \delta_{ij}, \quad j = 1, \cdots, i, \tag{13}$$

and $i = 1, \ldots, n-2$. By the hypothesis, $\sum_{j=1}^{n-1} \delta_j y_{j1}^2 = \delta_1$. If (13) holds for $i = 1, \cdots, l$, we will show that it also holds for $i = l+1$. Consider $Y_{l+1,k}$ according to step 7,

$$\sum_{k=1}^{n-1} Y_{l+1,k} Y_{jk} \delta_k = \frac{1}{\epsilon_l b_l} \sum_{k=1}^{n-1} ((\mu_k - a_l) Y_{lk} - b_{l-1} Y_{l-1,k}) Y_{jk} \delta_k$$

$$= \frac{1}{\epsilon_l b_l} \left(\sum_{k=1}^{n-1} \mu_k Y_{lk} Y_{jk} \delta_k - a_l \delta_{lj} \delta_j - b_{l-1} \delta_{l-1,j} \delta_j \right).$$

Thus, taking $j = l$ and using step 5, we obtain

$$\sum_{k=1}^{n-1} Y_{l+1,k} Y_{lk} \delta_k = \frac{1}{\epsilon_l b_l} \left(\sum_{k=1}^{n-1} \mu_k Y_{lk} Y_{lk} \delta_k - a_l \delta_l \right) = 0.$$

For $j < l$ we have

$$\sum_{k=1}^{n-1} Y_{l+1,k} Y_{jk} \delta_k = \frac{1}{\epsilon_l b_l} \left(\sum_{k=1}^{n-1} \mu_k Y_{lk} Y_{jk} \delta_k - b_{l-1} \delta_{l-1,j} \delta_j \right).$$

We consider now $Y_{j+1,k}$ given by step 7, for $j < l$, and observe that

$$\sum_{k=1}^{n-1} Y_{j+1,k} Y_{lk} \delta_k = \delta_{j+1,l}\delta_l = \frac{1}{\epsilon_j b_j} \sum_{k=1}^{n-1} ((\mu_k - a_j) Y_{jk} Y_{lk} \delta_k - b_{j-1} Y_{j-1,k} Y_{lk} \delta_k)$$
$$= \frac{1}{\epsilon_j b_j} \left(\sum_{k=1}^{n-1} \mu_k Y_{jk} Y_{lk} \delta_k - b_{j-1} \delta_{j-1,l} \delta_l \right),$$

so that

$$\sum_{k=1}^{n-1} \mu_k Y_{jk} Y_{lk} \delta_k = \delta_{j+1,l} \epsilon_j b_j \delta_l + b_{j-1} \delta_{j-1,l} \delta_l.$$

Thus, since $j < l$,

$$\sum_{k=1}^{n-1} Y_{l+1,k} Y_{jk} \delta_k = \frac{1}{\epsilon_l b_l} \left(\sum_{k=1}^{n-1} \mu_k Y_{lk} Y_{jk} \delta_k - b_{l-1} \delta_{l-1,j} \delta_j \right)$$
$$= \frac{1}{\epsilon_l b_l} (\delta_{j+1,l} \epsilon_j b_j \delta_l + b_{j-1} \delta_{j-1,l} \delta_l - b_{l-1} \delta_{l-1,j} \delta_j)$$
$$= \epsilon_l \delta_{j,l+1} \delta_l = \delta_{j,l+1} \delta_{l+1}.$$

This completes the proof that the rows of the real matrix Y, whose jth column is the vector Y_j, are H-orthogonal, that is, $YHY^T = H$ holds for $H = \mathrm{diag}(1, \epsilon_1, \cdots, \epsilon_1 \cdots \epsilon_{n-2})$. This implies that the vectors $Y_1, \ldots, Y_{n-1}$ are such that $Y_j^T H Y_k = \delta_{jk} \delta_k$. Moreover, we have $J_{n-1} Y_j = \mu_j Y_j$. Thus, $\sigma(J_{n-1}) = \{\mu_1, \ldots, \mu_{n-1}\}$ and $y_{11}, \ldots, y_{n-1,1}$ are the first entries of the corresponding eigenvectors $Y_1, \ldots, Y_{n-1}$. ∎

Example 2.1. *Consider the data*

$$(\mu_1, \mu_2, \mu_3, y_{11}, y_{21}, y_{31}, \delta_1, \delta_2, \delta_3)$$
$$= \left(-\sqrt{2}, \sqrt{2}, 0, \frac{3 - 2\sqrt{2}}{2\sqrt{3 - 2\sqrt{2}}}, \frac{-2 + \sqrt{2}}{2\sqrt{3 - 2\sqrt{2}}}, \frac{1}{2\sqrt{3 - 2\sqrt{2}}}, 1, -1, 1 \right).$$

From these data, the modified Lanczos algorithm determines the matrix

$$J_3 = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}.$$

Now, consider the data

$$(\mu_1,\ \mu_2,\ \mu_3,\ y_{11},\ y_{21},\ y_{31},\ \delta_1,\ \delta_2,\ \delta_3) = \left(-\sqrt{2},\ \sqrt{2},\ 0,\ \sqrt{4-\sqrt{2}}, \sqrt{7-2\sqrt{2}},\ 2,\ 1,\ -1,\ 1\right).$$

These data lead to $a_1 = \mu_1 y_{11}^2 - \mu_2 y_{21}^2 + \mu_3 y_{31}^2 = -2$. *However the algorithm has been interrupted at step 6, since* $\mathcal{D}_1 = ((\mu_1 - a_1)y_{11})^2 - ((\mu_2 - a_1)y_{21})^2 + ((\mu_3 - a_1)y_{31})^2 = -12 + 2\sqrt{2} < 0$. *Thus, a matrix* J_3 *corresponding to these data does not exist.*

3. An inverse problem for periodic pseudo-Jacobi matrices

The main result of this section is the solution of IPPJ (cf. [4] for a particular case).

Theorem 3.1. *Let the data* $\{\lambda, \mu, \beta, \delta\}$ *be given and satisfy (i), (ii), (iii), (iv). There exists a periodic pseudo-Jacobi matrix* J_n *of the form (1), H-symmetric for* $H = \mathrm{diag}(\delta_1, \cdots, \delta_n)$, *such that*

$$\sigma(J_n) = \{\lambda_1, \dots, \lambda_n\},\ \sigma(J_{n-1}) = \{\mu_1, \dots, \mu_{n-1}\},\ \textit{and } \beta = \prod_{i=1}^{n} b_i$$

if and only if the following conditions hold for j=1,...,n-1:

1) $-\delta_n\delta_j x_j \geq 0$, *where* $x_j = \prod_{i=1}^{n}(\lambda_i - \mu_j)\prod_{\substack{i=1\\ i\neq j}}^{n-1}(\mu_i - \mu_j)^{-1}$ *for* $\mu_j \in \{\lambda_1, \dots, \lambda_n\}$, *and* $x_j = 0$ *otherwise;*

2) $\prod_{i=1}^{n}|\lambda_i - \mu_j| - 4\delta_n\delta_j\beta\ \mathrm{sign}\chi'(\mu_j) \geqslant 0$, *where* $\chi'(\mu_j) = \prod_{j\neq k=1}^{n-1}(\mu_k - \mu_j)$;

3) For a selected sign $\pm$,

$$\sum_{j=1}^{n-1}\delta_j\left(\sqrt{-\delta_n\delta_j x_j} \pm \sqrt{-\delta_n\delta_j x_j - \delta_n\delta_j\frac{4\beta}{\chi'(\mu_j)}}\right)^2 > 0\,;$$

4) $\mathcal{D}_j > 0$ *(cf. Algorithm 1) for* $j = 1, \dots, n-2$.
Furthermore, there are at most 2^{n-1} *different solutions* J_n.

Next, we present four auxiliary lemmas needed for the proof of Theorem 3.1.

Lemma 3.1. *Let $\mu_1, \ldots, \mu_{n-1}$ be the eigenvalues of J_{n-1} and*

$$v_i = (v_{i1}, \ldots, v_{i,n-1})^T \in \mathrm{R}^{n-1}, \quad i = 1, \ldots, n-1,$$

be the corresponding eigenvectors, satisfying $[v_i, v_i] = \delta_i$. Then μ_i is an eigenvalue of J_n if and only if $b_n v_{i1} + \delta_n b_{n-1} v_{i,n-1} = 0$.

Proof: Let V be the matrix whose ith column is the eigenvector v_i associated with μ_i such that $[v_i, v_i] = \delta_i$, $i = 1, \cdots, n-1$. It is easy to confirm that the matrix

$$U = \begin{bmatrix} V & 0 \\ 0^T & 1 \end{bmatrix},$$

satisfies $U^{\#} U = I_n$, since $V V^{\#} = I_{n-1}$. Furthermore,

$$U^{\#} J_n U = \begin{bmatrix} V^{\#} J_{n-1} V & V^{\#} y \\ z^T V & a_n \end{bmatrix} = \begin{bmatrix} \mathrm{diag}(\mu_1, \cdots, \mu_{n-1}) & d \\ f^T & a_n \end{bmatrix},$$

where

$$\begin{aligned} d &= (d_1, \cdots, d_{n-1})^T = V^{\#} y && (14) \\ &= (\delta_1(b_n v_{11} + b_{n-1} v_{1,n-1} \delta_n), \cdots, \delta_{n-1}(b_n v_{n-1,1} + b_{n-1} v_{n-1,n-1} \delta_n))^T, \\ f^T &= (f_1, \cdots, f_{n-1}) = z^T V && (15) \\ &= \delta_n (b_n v_{11} + b_{n-1} v_{1,n-1} \delta_n, \cdots, b_n v_{n-1,1} + b_{n-1} v_{n-1,n-1} \delta_n). \end{aligned}$$

A simple computation yields

$$\det (U^{\#} J_n U - z I_n) = (a_n - z) \prod_{j=1}^{n-1} (\mu_j - z) - \sum_{i=1}^{n-1} d_i f_i \prod_{\substack{j=1 \\ j \neq i}}^{n-1} (\mu_j - z).$$

Henceforth, μ_ℓ is an eigenvalue of J_n if and only if $\det (J_n - \mu_\ell I) = 0$, that is, $d_\ell f_\ell \prod_{\substack{i=1 \\ i \neq \ell}}^{n-1} (\mu_i - \mu_\ell) = 0$. ■

Next, we shall assume that J_n and J_{n-1} do not have common eigenvalues.

Lemma 3.2. *If $b_n v_{i1} + \delta_n b_{n-1} v_{i,n-1} \neq 0$, for $i = 1, \ldots, n$, then the eigenvalues of J_n are the n zeros of the function*

$$f(z) = a_n - z - \sum_{i=1}^{n-1} \frac{\delta_n \delta_i (\delta_1 b_n v_{i1} + \delta_n b_{n-1} v_{i,n-1})^2}{\mu_i - z}. \quad (16)$$

Proof: The lemma is an easy consequence of the following observation

$$\det\left(J_n - zI_n\right) = (\mu_1 - z)\cdots(\mu_{n-1} - z)\left(a_n - z - \sum_{j=1}^{n-1}\frac{f_j d_j}{\mu_j - z}\right),$$

keeping in mind expressions (14) and (15). ∎

In the following, we consider that J_n and J_{n-1} have common eigenvalues.

Lemma 3.3. *Let S be a subset of $\{1,\ldots,n-1\}$ with s elements such that $\delta_1 b_n v_{i1} + \delta_n b_{n-1} v_{i,n-1} = 0$ for $i \in S$ and $\delta_1 b_n v_{i1} + \delta_n b_{n-1} v_{i,n-1} \neq 0$ for $i \notin S$. Then, μ_i is an eigenvalue of J_n for $i \in S$. The remaining eigenvalues of J_n are the $n-s$ zeros of the function defined by*

$$f(z) = a_n - z - \sum_{\substack{i=1\\ i\notin S}}^{n-1}\frac{\delta_n\delta_i(\delta_1 b_n v_{i1} + \delta_n b_{n-1} v_{i,n-1})^2}{\mu_i - z}.$$

Proof: By Lemma 3.1 μ_i is an eigenvalue of J_n if $i \in S$, being the remaining eigenvalues of J_n the zeros of the polynomial

$$p(z) = \frac{\det\left(U^{\#}J_nU - zI_n\right)}{\prod_{j\in S}(\mu_j - z)} = \left(a_n - z - \sum_{i=1}^{n-1}\frac{f_i d_i}{\mu_i - z}\right)\prod_{\substack{j=1\\ j\notin S}}^{n-1}(\mu_j - z).$$

Thus, $p(z) = \det\left(U^{\#}J_nU - zI_n\right)\prod_{j\in S}(\mu_j - z)^{-1} = 0$ if and only if

$$f(z) = \left(a_n - z - \sum_{i=1}^{n-1}\frac{d_i f_i}{\mu_i - z}\right) = \left(a_n - z - \sum_{i=1, i\notin S}^{n-1}\frac{d_i f_i}{\mu_i - z}\right) = 0.$$

The polynomial $p(z)$ has degree $n-s$, and by construction, it has $n-s$ distinct zeros. The result follows by Lemma 3.2. ∎

The next lemma follows [12, Theorem 4].

Lemma 3.4. *For λ, μ satisfying (i), (ii) and $\mu_j \notin \{\lambda_1,\ldots,\lambda_n\}$, we have*

$$\frac{x_1}{\mu_1 - \lambda_i} + \frac{x_2}{\mu_2 - \lambda_i} + \cdots + \frac{x_{n-1}}{\mu_{n-1} - \lambda_i} + a_n - \lambda_i = 0, \quad \textit{for} \quad i = 1,\ldots,n, \tag{17}$$

if and only if $x_j = \prod_{i=1}^{n}(\lambda_i - \mu_j)\prod_{i=1,\ i\neq j}^{n-1}(\mu_i - \mu_j)^{-1}$, for $j = 1,\ldots,n-1$, and $a_n = \sum_{i=1}^{n}\lambda_i - \sum_{i=1}^{n-1}\mu_i$.

Proof: ($\Rightarrow$) Let

$$q(z) = a_n - z + \sum_{i=1}^{n-1} \frac{x_i}{\mu_i - z}\,.$$

By easy computations we get

$$q(z) = \frac{(a_n - z)\prod_{i=1}^{n-1}(\mu_i - z) + x_1\prod_{i=2}^{n-1}(\mu_i - z) + \cdots + x_{n-1}\prod_{i=1}^{n-2}(\mu_i - z)}{\prod_{i=1}^{n-1}(\mu_i - z)},$$

and so $q(z) = Q(z)\prod_{i=1}^{n-1}(\mu_i - z)^{-1}$, where $Q(z)$ is a monic polynomial in $-z$ of degree n. By the hypothesis, $q(\lambda_1) = \cdots = q(\lambda_n) = 0$, so $Q(z) = \prod_{i=1}^{n}(\lambda_i - z)$. Hence $q(z) = \prod_{i=1}^{n}(\lambda_i - z)\prod_{i=1}^{n-1}(\mu_i - z)^{-1}$. For each i, the residue of $q(z)$ at μ_i is given by $\mathrm{Res}_{z=\mu_i} q(z) = -\prod_{j=1}^{n}(\lambda_j - \mu_i)\prod_{\substack{i=1\\ i\neq j}}^{n-1}(\mu_j - \mu_i)^{-1}$, and yields the value of x_i. Having in mind that $\lim_{z\to\infty}(q(z) + z) = \sum_{j=1}^{n}\lambda_j - \sum_{j=1}^{n-1}\mu_j$, the direct implication follows. The converse holds by analogous arguments. ■

Proof of Theorem 3.1

($\Rightarrow$) Assume that there exists a periodic pseudo-Jacobi matrix J_n of the form (1) such that $\sigma(J_n) = \{\lambda_1, \ldots, \lambda_n\}$ and $\sigma(J_{n-1}) = \{\mu_1, \ldots, \mu_{n-1}\}$. If $\mu_j \notin \sigma(J_n)$, by Lemma 3.1 we have $(b_n v_{j1} + \delta_1\delta_{n-1}b_{n-1}v_{j,n-1}) \neq 0$ and, bearing in mind Lemma 3.2 and Lemma 3.4, $x_j = \prod_{i=1}^{n}(\lambda_i - \mu_j)\prod_{\substack{i=1\\ i\neq j}}^{n-1}(\mu_i - \mu_j)^{-1}$ $= -\delta_n\delta_j(\delta_1 b_n v_{j1} + \delta_n b_{n-1}v_{j,n-1})^2$. If $\mu_j \in \sigma(J_n)$, by Lemma 3.1 we have $x_j = -\delta_n\delta_j(\delta_1 b_n v_{j1} + \delta_n b_{n-1}v_{j,n-1})^2 = 0$.

Henceforth, the condition 1) holds.

Next, we show that the condition 2) is satisfied. From Lemma 3.3 we get $\delta_1 b_n v_{j1} + \delta_n b_{n-1}v_{j,n-1} = \pm\sqrt{-\delta_n\delta_j x_j}$, and from Lemma 2.2 $(v_{j1}b_n)(v_{j,n-1}b_{n-1}) =$

$\delta_1\delta_j\frac{\beta}{\chi'(\mu_j)}$, for $j=1,\ldots,n-1$, where $\prod_{i=1}^n b_i=\beta$. Thus

$$v_{j1}b_n=\frac{1}{2}\left(\pm\sqrt{-\delta_n\delta_j x_j}\pm\sqrt{-\delta_n\delta_j x_j-4\delta_n\delta_j\beta/\chi'(\mu_j)}\right),\quad j=1,\ldots,n-1\,. \tag{18}$$

Let

$$\begin{aligned}\Delta_j &= -\delta_n\delta_j x_j-\frac{4\beta\delta_1\delta_j}{\chi'(\mu_j)}=-\delta_n\delta_j\frac{\prod_{i=1}^{n}(\lambda_i-\mu_j)}{\prod_{\substack{i=1\\ i\neq j}}^{n-1}(\mu_i-\mu_j)}-\frac{4\beta\delta_n\delta_j}{\prod_{\substack{i=1\\ i\neq j}}^{n-1}(\mu_i-\mu_j)}\,,\\ &=\frac{|\prod_{i=1}^{n}(\lambda_i-\mu_j)|}{|\prod_{\substack{i=1\\ i\neq j}}^{n-1}(\mu_i-\mu_j)|}-\frac{4\beta\delta_n\delta_j\mathrm{sign}\prod_{\substack{\mathrm{i=1}\\ \mathrm{i\neq j}}}^{\mathrm{n-1}}(\mu_\mathrm{i}-\mu_\mathrm{j})}{|\prod_{\substack{i=1\\ i\neq j}}^{n-1}(\mu_i-\mu_j)|}\,,\quad \text{for } j=s+1,\ldots,n-1\,.\end{aligned}$$

Since v_{j1} and b_n are real, we necessarily have $\Delta_j\geq 0$ or, equivalently,

$$\prod_{i=1}^{n}|\lambda_i-\mu_j|-4\beta\delta_1\delta_j\mathrm{sign}\prod_{\substack{\mathrm{i=1}\\ \mathrm{i\neq j}}}^{\mathrm{n-1}}(\mu_\mathrm{i}-\mu_\mathrm{j})\geq 0\,,\quad \text{for } j=1,\ldots,n-1.$$

Thus, the condition 2) follows.

By (18) and having in mind that $\sum_{j=1}^{n-1}\delta_j v_{1j}^2=\delta_1$, we infer that the condition 3) is satisfied. Finally, 4) clearly follows because $\mathcal{D}_i=b_i^2>0$.

Next, we prove the converse. Assume that 1) 2) 3) and 4) hold. Consider $x_j=\prod_{i=1}^n(\lambda_i-\mu_j)\prod_{\substack{i=1\\ i\neq j}}^{n-1}(\mu_i-\mu_j)^{-1}$ if $\mu_j\notin\{\lambda_1,\ldots,\lambda_n\}$ and $x_j=0$ otherwise. Let define

$$b_n=\frac{1}{2}\sqrt{\sum_{j=1}^{n-1}\delta_j\left(\sqrt{-\delta_n\delta_j x_j}\pm\sqrt{-\delta_n\delta_j x_j-\delta_n\delta_j\frac{4\beta}{\chi'(\mu_j)}}\right)^2}\,. \tag{19}$$

We show that $b_n>0$. In fact, condition 1) ensures that $-\delta_n\delta_j x_j\geq 0$, and condition 2) guarantes that $-\delta_n\delta_j x_j-\delta_n\delta_j\frac{4\beta}{\chi'(\mu_j)}\geq 0$. On the other hand, condition 3) ensures that a selected sign $\pm$ may be chosen such that

$\sum_{j=1}^{n-1} \delta_j \left(\sqrt{-\delta_n\delta_j x_j} \pm \sqrt{-\delta_n\delta_j x_j - \delta_n\delta_j 4\beta/\chi'(\mu_j)}\right)^2 > 0$. Hence, $b_n > 0$. Considering the selected sign $\pm$ we find

$$v_{j1} = \frac{\sqrt{-\delta_n\delta_j x_j} \pm \sqrt{-\delta_n\delta_j x_j - \delta_n\delta_j 4\beta/\chi'(\mu_j)}}{2\, b_n}, \quad j = 1, \ldots, n-1,$$

being the real numbers $v_{11}, \ldots, v_{n-1,1}$ clearly nonzero. So, applying Theorem 2.2 to the data $\{\mu_1, \ldots, \mu_{n-1}\}$, $\{v_{11}, \ldots, v_{n-1,1}\}$, $\{\delta_1, \ldots, \delta_{n-1}\}$ and recalling that $\mathcal{D}_j > 0$ by condition 4), a unique pseudo-Jacobi matrix J_{n-1} can be constructed such that $\sigma(J_{n-1}) = \{\mu_1, \ldots, \mu_{n-1}\}$, $v_{11}, \ldots, v_{1n-1}$ are the first entries of the corresponding eigenvectors $v_1, \ldots, v_{n-1}$ and $\delta_1 = [v_1, v_1], \ldots, \delta_{n-1} = [v_{n-1}, v_{n-1}]$. Next, we may determine $b_{n-1} = \frac{\beta}{b_1 \cdots b_{n-2} b_n}$, and so $a_n = \sum_{j=1}^{n} \lambda_j - \sum_{j=1}^{n-1} \mu_j$. Lemma 3.2 ensures that $\sigma(J_n) = \{\lambda_1, \ldots, \lambda_n\}$.

We finish the proof, by observing that if strict inequality $\Delta_j > 0$ occurs for each j, there are two possible choices for the signs $\pm$ of v_{j1}, $j = 1, \ldots, n-1$. Thus, there are at most 2^{n-1} different solutions. ∎

4. An algorithm and an example

Several algorithms for the construction of periodic Jacobi matrices with given spectral data are known [1, 5, 6, 13]. The solutions of IPPJ can be constructed from the given data $\{\lambda, \mu, \beta, \delta\}$, according to the following algorithm.

Algorithm 2.

Step 1 *For* $j = 1, \ldots, n-1$, *set*

$$\chi'(\mu_j) = \prod_{\substack{i=1 \\ i \neq j}}^{n-1} (\mu_i - \mu_j).$$

Step 2 *Set*

$$x_j = \prod_{i=1}^{n} (\lambda_i - \mu_j) \prod_{\substack{i=1 \\ i \neq j}}^{n-1} (\mu_i - \mu_j)^{-1}, \quad if \quad \mu_j \notin \sigma(J_n),$$

$$x_j = 0 \quad if \quad \mu_j \in \sigma(J_n).$$

Step 3 *Choose* δ_n, δ_j *such that* $-\delta_n\delta_j x_j \geq 0$.

Step 4 *Check that* $\Delta_j = -\delta_n x_j \delta_j - \delta_n\delta_j \frac{4\beta}{\chi'(\mu_j)} \geq 0$.

Step 5 *Compute the quantities*

$$W_j = \frac{1}{2}\left(\sqrt{-\delta_n x_j \delta_j} \pm \sqrt{-\delta_n x_j \delta_j - \delta_n \delta_j \frac{4\beta}{\chi'(\mu_j)}}\right)$$

and select the $\pm$ *sign ensuring that* $\displaystyle\sum_{j=1}^{n-1} W_j^2 \delta_j > 0.$

Step 6 *For the selected* $\pm$ *sign, let*

$$b_n = \frac{1}{2}\sqrt{\sum_{j=1}^{n-1} \delta_j \left(\sqrt{-\delta_n \delta_j x_j} \pm \sqrt{-\delta_n \delta_j x_j - \delta_n \delta_j 4\beta/\chi'(\mu_j)}\right)^2}.$$

Step 7 *For* $j = 1, \dots, n-1$ *and for the selected* $\pm$ *sign, set*

$$v_{j1} = \frac{\sqrt{-\delta_n \delta_j x_j} \pm \sqrt{-\delta_n \delta_j x_j - \delta_n \delta_j 4\beta/\chi'(\mu_j)}}{2\, b_n}.$$

Step 8 *Apply the modified Lanczos algorithm to the sets* $\{\mu_1, \dots, \mu_{n-1}\}$, $\{v_{11}, \dots, v_{n-1,1}\}$ *and* $\{\delta_1, \dots, \delta_{n-1}\}$.

Step 9 *Let*

$$b_{n-1} = \frac{\beta}{b_1 \cdots b_{n-2} b_n}.$$

Step 10 *Set*

$$a_n = \sum_{j=1}^{n} \lambda_j - \sum_{j=1}^{n-1} \mu_j.$$

Example 4.1. *We illustrate the algorithm with the following example. Let us consider* $\delta = (\delta_1, \delta_2, \delta_3, \delta_4)$ *and*

$$\begin{aligned}(\lambda_1,&\lambda_2, \lambda_3, \lambda_4; \mu_1, \mu_2, \mu_3; \beta)\\ &= (2.82843, 0, 0, -2.82843; 2.90321, 0.806063, -1.70928; 1).\end{aligned}$$

We look for a 4×4 *matrix* J_4 *such that* $\sigma(J_4) = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$, $\sigma(J_3) = \{\mu_1, \mu_2, \mu_3\}$ *and* $b_1 b_2 b_3 = 1$.

Consider the function

$$f(z) = \frac{(\lambda_1 - z)(\lambda_2 - z)(\lambda_3 - z)(\lambda_3 - z)}{(\mu_1 - z)(\mu_2 - z)(\mu_3 - z)}.$$

The residues $-x_1, -x_2, -x_3$ *of* $f(z)$ *at* μ_1, μ_2, μ_3 *are:*

$$-x_1 = -0.405642, \quad -x_2 = -0.69142, \quad -x_3 = 1.09706.$$

We infer that $\delta_1\delta_4 = -1$, $\delta_2\delta_4 = -1$, $\delta_3\delta_4 = 1$. *Hence,* J_3 *and* J_4 *are, respectively,* H'*-symmetric for* $H' = \mathrm{diag}(1,1,-1)$ *and* H*-symmetric for* $H = \mathrm{diag}(1,1,-1,-1)$.

We compute $\chi'(\mu_1)$, $\chi'(\mu_2)$, $\chi'(\mu_3)$*:*

$$\chi'(\mu_1) = 13.7627, \quad \chi'(\mu_2) = -9.60928, \quad \chi'(\mu_3) = 31.8471\ .$$

Next, we analyze whether the discriminants $\Delta_1, \Delta_2, \Delta_3$ *are positive:*

$$\Delta_1 = 0.692883, \quad \Delta_2 = 0.275179, \quad \Delta_3 = 0.9710463\ .$$

Applying (18), we compute $v_{j1}b_4$. *For a particular choice of the* $\pm$ *signs, we find* $b_4 = 1$ *and*

$$v_{11}b_4 = \frac{1}{2}(-\sqrt{x_1} - \sqrt{\Delta_1}) = -0.735668$$
$$v_{21}b_4 = \frac{1}{2}(-\sqrt{x_2} - \sqrt{\Delta_2}) = -0.678046$$
$$v_{13}b_4 = \frac{1}{2}(\sqrt{-x_3} - \sqrt{\Delta_3}) = 0.0308898.$$

Applying the modified Lanczos algorithm, we get $a_1 = a_2 = -a_3 = -a_4 = 2,\ b_1 = b_2 = b_3 = 1$.

5. Final remarks

We have solved IPPJ under the restrictive condition of the μ's being real and pairwise distinct. This condition ensures that J_{n-1} is diagonalizable under a pseudo-orthogonal similarity. The existence of a real multiple μ_j may prevent this diagonalizability, and then the previous approach does not apply. The same occurs if complex μ's exist. Hence, IPPJ is an open problem in these cases, and its study appears to be of some interest.

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