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LAPLACE TRANSFORM - FINITE ELEMENT METHOD FOR NON FICKIAN FLOWS

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ABSTRACT: In this paper we consider numerical methods for integro-differential problems based on time discretization via Laplace transformation. We focus our attention in models arising in the context of non Fickian solute transport phenomena in porous media. The mathematical models which describe the evolution of the solute concentrations are characterized by Volterra equations. We present and analyze an hybrid method which combines the Laplace transformation with respect to the time variable with the finite element discretization in the spatial variables. Numerical results illustrate the performance of the method.

1. Introduction

In this paper we consider the following Volterra equation

$$\frac{\partial c}{\partial t}(t) + Ac(t) + \int_0^t k(t-s)Bc(s)\,ds = f(t)\,,\ t>0, \tag{1}$$

with

$$Ac(t) = -\nabla.\Big(A_{22}\nabla c(t)\Big) + \nabla.(A_2 c(t)) + A_1 c(t),$$

$$Bc(t) = -\nabla.\Big(B_{22}\nabla c(t)\Big) + \nabla.(B_2 c(t)) + B_1 c(t),$$

where k denotes the kernel, and $A_{22}, B_{22}, A_2, B_2, A_1$ and B_1 represent functions dependent on (x,y), being $A_{22}=[a_{ij}]$ and $B_{22}=[b_{ij}]$ 2 by 2 symmetric matrix functions, $A_2=[a_i]$ and $B_2=[b_i]$ vectorial functions and A_1 and B_1 scalar functions.

Solute transports in porous media are commonly characterized by the convection-diffusion equation

$$\frac{\partial c}{\partial t} + \nabla.(vc) = \nabla.(D\nabla c) + f \text{ in}\,\Omega\times(0,T], \tag{2}$$

where c denotes the solute concentration, D denotes the dispersion tensor (which can be c dependent) and v represents the fluid velocity. Ω is the spatial domain with boundary $\partial\Omega$. Equation (2) is established using the so

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called Fick's law for the mass flux due to molecular diffusion, J_d, and the convective flux, J_c, given by, respectively,

$$J_d = -D\nabla c, \tag{3}$$

and

$$J_c = vc\,. \tag{4}$$

Then the mass flux $J = J_d + J_c$ is given by

$$J = vc - D\nabla c\,, \tag{5}$$

which combined with the mass conservation equation

$$\frac{\partial c}{\partial t} + \nabla \cdot J = f\,, \tag{6}$$

leads to the convection-diffusion equation (2).

When the memory effect of the fluid flow has an important role in the solute transport, equation (2) should be modified in order to incorporate such effect. One possible approach to accomplish this is to assume that the mass flux J_d is given by

$$J_d(t) = -\frac{1}{\tau}\int_0^t e^{-\frac{t-s}{\tau}} D\nabla c(s)\, ds + J_d(0), \tag{7}$$

where τ is a relaxation parameter. Combining now $J = J_c + J_d$ with J_d given by (7) with the mass conservation equation (6) we arrive to

$$\frac{\partial c}{\partial t} + \nabla \cdot (vc) = \frac{1}{\tau}\int_0^t e^{-\frac{t-s}{\tau}} \nabla \cdot (D\nabla c(s)) ds + f - \nabla J_d(0) \text{ in}\,\Omega \times (0,T], \tag{8}$$

that replaces (2) for non Fickian flows.

Equation (8) is a particular case of an equation of type (1), which is the model that we are going to study in this article. This type of equations have been proposed in the literature to describe non Fickian diffusion processes as for instance in [11], [22], [23], [26], [28]. The development of efficient and accuracy numerical methods to solve the initial boundary value problem (IBVP) defined by (1) has attracted the attention of several researchers during the last two decades. A significative number of contributions can be found in the literature. Without be exhaustive we mention [24], [25], [38], [40] for the study of finite element semi-discrete approximations, [31] for the study of semi-discrete lumped mass approximations, [16], [17] and [34] for the study of finite volume semi-discrete approximations, [2], [4], [5], [6], [8],

[18], [20] and [21] for finite difference methods presenting the same qualitative behavior of the integro-differential problem.

Integro-differential equations (1) can be rewritten as equivalent linear differential systems: a partial differential equation involving only a time derivative and an integro-differential equation presenting only partial derivatives with respect to the space variables. This approach was used, for instance, in [18] and recently in [35] where mixed finite element methods were used for the disretization. Systems of differential equations that are equivalent to nonlinear versions of equation (1) were considered in [7] and [32].

In what follows we will consider that $\Omega \subset \mathbb{R}^2$ is bounded polygonal domain. We will introduce an hybrid method for the IBVP defined by (1), with the Dirichlet boundary condition

$$c(t) = \psi(t) \text{ on } \partial\Omega \times \mathbb{R}^+, \tag{9}$$

and with the initial condition

$$c(0) = c_0 \text{ in } \Omega. \tag{10}$$

The method is based on the use of Laplace transform to the IBVP (1), (9), (10) which converts the IBVP in an elliptic boundary value problem that depends on the Laplace parameter. The elliptic problem is solved by using finite element methods for the spatial variables, for a choice of a finite set of quadrature points in the Laplace domain. This set of elliptic equations can be solved in parallel. Finally the numerical approximation for the solution on the physical time space domain is obtained by using numerical inverse Laplace transforms. This type of approach was considered as for instance in [3], [15], [19], [33], [36], [37]. The convergence analysis of methods designed using this procedure were presented *e.g.* in [9], [27] and [39] (see also the references cited in these two last papers). The present paper presents error bounds with respect H^1-norm which are based on the Paley–Wiener Theorem and the generalization of the classical arguments of the finite element analysis to complex Sobolev spaces ([9]). This type of approach allow us to consider more general differential and integro-differential operators compared with those studied in [27].

The paper is organized as follows. In Section 2 we introduce the variational formulation of the IBVP (1), (9), (10) and its finite element formulation. The weak variational problem in the Laplace space is introduced in Section 3 and the existence and uniqueness of the solution of this problem are also studied in this section. In Section 4 we describe the finite element approximation of the variational problem introduced in the previous section and an error estimate for such approximation is established. Using the Paley–Wiener Theorem we return to the initial variables and we estimate the error for Laplace inverse of the finite element solution in the Laplace space. Finally some numerical experiments illustrating the convergence results are also included.

2. Weak solution and its Ritz-Galerkin approximation

Let $L^2(\Omega), H^1(\Omega)$ be the usual Sobolev spaces endowed, respectively, with the usual inner products $(.,.), (.,.)_{H^1(\Omega)}$ and norms $\|.\|_{L^2(\Omega)}, \|.\|_{H^1(\Omega)}$. The space of functions $v \in H^1(\Omega)$ such that $v = 0$ on $\partial\Omega$, is denoted by $H_0^1(\Omega)$. By $L^2(\mathbb{R}^+, H^1(\Omega))$ we denote the space of functions $v : \mathbb{R}^+ \to H^1(\Omega)$ such that

$$\int_{\mathbb{R}^+} \|v(s)\|^2_{H^1(\Omega)}\, ds < \infty \tag{11}$$

and by $H^1(\mathbb{R}^+, L^2(\Omega))$ we denote the space of functions $v : \mathbb{R}^+ \to H^1(\Omega)$ such that

$$\int_{\mathbb{R}^+} \Big(\|v(s)\|^2_{L^2(\Omega)} + \|\frac{dv}{dt}(s)\|^2_{L^2(\Omega)}\Big) ds < \infty. \tag{12}$$

In (12), the time derivative is in the weak sense.

The weak solution for the IBVP (1), (9), (10) is obtained solving the following problem: find $c \in L^2(\mathbb{R}^+, H^1(\Omega)) \bigcap H^1(\mathbb{R}^+, L^2(\Omega))$ such that $c(t) = \psi(t)$ on $\partial\Omega$ and, for any $T > 0$,

$$\begin{cases} (\dfrac{\partial c}{\partial t}(t), v) + ((A_{22}\nabla c(t), \nabla v)) - ((c(t)A_2, \nabla v)) + (A_1 c(t), v) \\ \qquad\qquad + \displaystyle\int_0^t K(t-s)\Big(((B_{22}\nabla c(s), \nabla v)) - ((B_2 c(s), \nabla v)) \\ \qquad\qquad + (B_1 c(s), v)\Big)\, ds = (f(t), v) \ \text{ a.e. in } (0,T),\ \forall v \in H_0^1(\Omega), \\ c(0) = c_0. \end{cases} \tag{13}$$

We remark that we use the notation

$$((u,v)) = \sum_{i=1}^{2} (u_i, v_i)$$

for $u = (u_1, u_2), v = (v_1, v_2), u_i, v_i \in L^2(\Omega), i = 1, 2$.

To compute the the semi-discrete Ritz-Galerkin approximation c_H for the weak solution c defined by (13), we introduce in $\overline{\Omega}$ an admissible triangulation

$\mathcal{T}_H$ and the corresponding finite dimension space

$$V_{H,m} = \{u \in C^0(\overline{\Omega}) : u(x) = P_m(x),\, x \in \Delta, \Delta \in \mathcal{T}_H\},$$

where $P_m(x)$ denotes a polynomial in space variables with degree $\leq m$.

Then, given $c_{0,H} \in V_{H,m}$, c_H is obtained solving the following problem: find $c_H \in L^2(\mathbb{R}^+, H^1_0(\Omega)) \bigcap H^1(\mathbb{R}^+, L^2(\Omega))$ such that $c_H(t) = \psi(t)$ on $\partial\Omega$ and, for any $T > 0$,

$$\begin{cases} (\dfrac{\partial c_H}{\partial t}(t), v_H) + ((A_{22}\nabla c_H(t), \nabla v_H)) - ((c_H(t)A_2, \nabla v_H)) + (A_1 c_H(t), v_H) \\ \qquad\qquad + \displaystyle\int_0^t K(t-s)\Big(((B_{22}\nabla c_H(s), \nabla v_H)) - ((c_H(s)B_2, \nabla v_H)) \\ \qquad\qquad + (B_1 c_H(s), v_H)\Big)\, ds = (f(t), v_H) \ \text{ a.e. in } (0,T),\ \forall v_H \in V_{H,m}, \\ c_H(0) = c_{0,H}. \end{cases} \tag{14}$$

In what follows we present an approach that allows us to compute an approximation for the weak solution of the IBVP (1), (9), (10) avoiding the computation of the solution of the integro-differential problem (14). We will also derive error estimates for the numerical solution.

3. Weak solution in Laplace space

In what follows we replace the IBVP (1), (9), (10) by the corresponding problem obtained applying the Laplace transform $\mathcal{L}$.

Applying Laplace transform to (1) we obtain

$$\left(I_d + \frac{1}{p}A + \frac{\tilde{k}}{p}B\right)\tilde{c} = \frac{1}{p}\big(c_0 + \tilde{f}\big) \ \text{ in } \Omega, \tag{15}$$

whereI_d is the identity operator, $\tilde{k}, \tilde{f}$ denote the Laplace transforms of k and f, respectively, and $\tilde{c}$ is the Laplace transform of c. Equation (15) is complemented with the boundary condition

$$\tilde{c} = \tilde{\psi} \ \text{ on } \partial\Omega, \tag{16}$$

where $\tilde{\psi}$ represents the Laplace transform of ψ.

In order to define the weak solution for the boundary value problem (15), (16) we introduce now the set of functional spaces need to this definition. We denoted by $\operatorname{Re} z$ the real part of $z \in \mathbb{C}$. Let $H^1(\Omega, \mathbb{C}^+_\sigma)$ and $L^2(\Omega, \mathbb{C}^+_\sigma)$ be the Sobolev spaces of functions that depend on the complex number $p \in$

$\mathbb{C}_\sigma^+ = \{p \in \mathbb{C} : \mathrm{Re}\, p \geq \sigma > 0\}$ where they are analytic. In $L^2(\Omega, \mathbb{C}_\sigma^+)$ we consider the inner product

$$(\tilde{u}, \tilde{v}) = \int_\Omega \tilde{u}\overline{\tilde{v}}\, dx,\ \tilde{u}, \tilde{v} \in L^2(\Omega, \mathbb{C}_\sigma^+), \tag{17}$$

and the corresponding norm

$$\|\tilde{u}\|_{L^2(\Omega,\mathbb{C}_\sigma^+)} = (\tilde{u}, \tilde{u})^{1/2},\ \ \tilde{u} \in L^2(\Omega, \mathbb{C}_\sigma^+).$$

The inner product (17) allows us to introduce in $L^2(\Omega, \mathbb{C}_\sigma^+) \times L^2(\Omega, \mathbb{C}_\sigma^+)$ the following inner product

$$(((\tilde{u}_1, \tilde{u}_2), (\tilde{v}_1, \tilde{v}_2))) = \sum_{i=1}^{2} (\tilde{u}_i, \tilde{v}_i),\ \ (\tilde{u}_1, \tilde{u}_2), (\tilde{v}_1, \tilde{v}_2) \in L^2(\Omega, \mathbb{C}_\sigma^+) \times L^2(\Omega, \mathbb{C}_\sigma^+).$$

The space $H^1(\Omega, \mathbb{C}_\sigma^+)$ is endowed with the inner product

$$(\tilde{u}, \tilde{v})_{H^1(\Omega,\mathbb{C}_\sigma^+)} = (\tilde{u}, \tilde{v}) + ((\nabla\tilde{u}, \nabla\tilde{v}))\, ,\ \tilde{u}, \tilde{v} \in H^1(\Omega, \mathbb{C}_\sigma^+), \tag{18}$$

which induces the following norm

$$\|\tilde{u}\|_{H^1(\Omega,\mathbb{C}_\sigma^+)} = (\tilde{u}, \tilde{u})^{1/2}_{H^1(\Omega,\mathbb{C}_\sigma^+)},\ \tilde{u} \in H^1(\Omega, \mathbb{C}_\sigma^+). \tag{19}$$

By $|.|_{H^1(\Omega,\mathbb{C}_\sigma^+)}$ we denote the following semi-norm in $H^1(\Omega, \mathbb{C}_\sigma^+)$

$$|\tilde{u}|_{H^1(\Omega,\mathbb{C}_\sigma^+)} = ((\nabla\tilde{u}, \nabla\tilde{u}))^{1/2},\ \ \tilde{u} \in H^1(\Omega, \mathbb{C}_\sigma^+).$$

The subspace of $H^1(\Omega, \mathbb{C}_\sigma^+)$ composed by the functions vanishing on $\partial\Omega$ is represented by $H_0^1(\Omega, \mathbb{C}_\sigma^+)$.

Let $a_p(.,.) : H^1(\Omega, \mathbb{C}_\sigma^+) \times H^1(\Omega, \mathbb{C}_\sigma^+) \to \mathbb{C}$ be the sesquilinear form

$$a_p(\tilde{u}, \tilde{v}) = (\tilde{u}, \tilde{v}) + \frac{1}{p}\Big(a(\tilde{u}, \tilde{v}) + \tilde{k}b(\tilde{u}, \tilde{v})\Big) \tag{20}$$

where

$$a(\tilde{u}, \tilde{v}) = ((A_{22}\nabla\tilde{u}, \nabla\tilde{v})) - ((A_2\tilde{u}, \nabla\tilde{v})) + (A_1\tilde{u}, \tilde{v}) \tag{21}$$

and

$$b(\tilde{u}, \tilde{v}) = ((B_{22}\nabla\tilde{u}, \nabla\tilde{v})) - ((B_2\tilde{u}, \nabla\tilde{v})) + (B_1\tilde{u}, \tilde{v}) \tag{22}$$

for $\tilde{u}, \tilde{v} \in H^1(\Omega, \mathbb{C}_\sigma^+)$. By $\ell : H^1(\Omega, \mathbb{C}_\sigma^+) \to \mathbb{C}$ we denote the following functional

$$\ell(\tilde{v}) = \frac{1}{p}(c_0 + \tilde{f}, \tilde{v}). \tag{23}$$

We associate with the sesquilinear form $a_p(.,.)$ the following operator $L : H^1_0(\Omega, \mathbb{C}^+_\sigma) \to H^1_0(\Omega, \mathbb{C}^+_\sigma)'$,

$$L\tilde{u}(\tilde{v}) = a_p(\tilde{u}, \tilde{v}),$$

where $H^1_0(\Omega, \mathbb{C}^+_\sigma)'$ denotes the dual space of $H^1_0(\Omega, \mathbb{C}^+_\sigma)$.

The existence and uniqueness of the solution of the variational problem: find $\tilde{c} \in H^1(\Omega, \mathbb{C}^+_\sigma)$ such that $\tilde{c} = \tilde{\psi}$ on $\partial\Omega$ and

$$a_p(\tilde{c}, \tilde{v}) = \ell(\tilde{v}), \, \forall \tilde{v} \in H^1_0(\Omega, \mathbb{C}^+_\sigma), \tag{24}$$

is established in the next result. To simplify the proof we consider homogeneous boundary conditions.

Theorem 1. *Let $f \in L^2(\mathbb{R}^+, L^2(\Omega)), c_0 \in L^2(\Omega)$ and $a_{ij}, b_{i,j}, a_i, b_i, A_1, B_1 \in L^\infty(\Omega), i, j = 1, 2$. If there exists $\sigma \in \mathbb{R}^+$ and $e : \mathbb{C}^+_\sigma \to \mathbb{R}^+$ such that, for $p \in \mathbb{C}^+_\sigma$, holds the following*

$$Re\, a_p(\tilde{u}, \tilde{u}) \geq e(p)\|\tilde{u}\|^2_{H^1(\Omega,\mathbb{C}^+_\sigma)}, \, \forall \tilde{u} \in H^1_0(\Omega, \mathbb{C}^+_\sigma), \tag{25}$$

then the variational problem (24) with $\psi = 0$ has only one solution $\tilde{c} \in H^1_0(\Omega, \mathbb{C}^+_\sigma)$.

Proof: In the proof of this result we use the Lax-Milgram Theorem. We start by noticing that if $f \in L^2(\mathbb{R}^+, L^2(\Omega)), c_0 \in L^2(\Omega)$ then the linear functional $\ell : H^1(\Omega, \mathbb{C}^+_\sigma) \to \mathbb{C}$ defined by (23) belongs to $H^1_0(\Omega, \mathbb{C}^+_\sigma)'$. In what follows we prove that the sesquilinear form $a_p(.,.) : H^1_0(\Omega, \mathbb{C}^+_\sigma) \times H^1_0(\Omega, \mathbb{C}^+_\sigma) \to \mathbb{C}$ defined by (20) is elliptic, that is, there exist positive constants $a_{c,p}$ and $a_{e,p}$ such that

$$|a_p(\tilde{u}, \tilde{v})| \leq a_{c,p}\|\tilde{u}\|_{H^1(\Omega,\mathbb{C}^+_\sigma)}\|\tilde{v}\|_{H^1(\Omega,\mathbb{C}^+_\sigma)}, \;\; \forall \tilde{u}, \tilde{v} \in H^1_0(\Omega, \mathbb{C}^+_\sigma), \tag{26}$$

and

$$|a_p(\tilde{u}, \tilde{u})| \geq a_{e,p}\|\tilde{u}\|^2_{H^1(\Omega,\mathbb{C}^+_\sigma)}, \;\; \forall \tilde{u} \in H^1_0(\Omega, \mathbb{C}^+_\sigma). \tag{27}$$

To establish (26) we need only to use the fact that the coefficients functions are assumed to be in $L^\infty(\Omega)$. The condition (27) is a trivial consequence of the assumption (25).

□

In the next results we specify necessary conditions that guarantee that the assumption (25) holds true. We start associating with $a(.,.)$ defined by (21) the following sesquilinear forms

$$a_I(\tilde{u}, \tilde{v}) = ((A_{22}\nabla\tilde{u}, \nabla\tilde{v})), \tilde{u}, \tilde{v} \in H^1(\Omega, \mathbb{C}^+_\sigma), \tag{28}$$

$$a_{II}(\tilde{u},\tilde{v}) = -((A_2\tilde{u},\nabla\tilde{v})) + (A_1\tilde{u},\tilde{v}),\ \tilde{u},\tilde{v}\in H^1(\Omega,\mathbb{C}_\sigma^+). \tag{29}$$

Analogously, we associate with the sesquilinear $b(.,.)$ defined by (22) the sesquilinear forms $b_I(.,.)$ and $b_{II}(.,.)$

$$b_I(\tilde{u},\tilde{v}) = ((B_{22}\nabla\tilde{u},\nabla\tilde{v})), \tilde{u},\tilde{v}\in H^1(\Omega,\mathbb{C}_\sigma^+), \tag{30}$$

$$b_{II}(\tilde{u},\tilde{v}) = -((B_2\tilde{u},\nabla\tilde{v})) + (B_1\tilde{u},\tilde{v}),\ \tilde{u},\tilde{v}\in H^1(\Omega,\mathbb{C}_\sigma^+). \tag{31}$$

We remark that, for $\tilde{u}\in H^1(\Omega,\mathbb{C}_\sigma^+)$,

$$a_I(\tilde{u},\tilde{u})\in\mathbb{R},\quad b_I(\tilde{u},\tilde{u})\in\mathbb{R}.$$

Lemma 1. *Let a_e and C_b be positive constants such that*

$$a_I(\tilde{u},\tilde{u}) \geq a_e|\tilde{u}|^2_{H^1(\Omega,\mathbb{C}_\sigma^+)},\ \tilde{u}\in H^1(\Omega,\mathbb{C}_\sigma^+), \tag{32}$$

$$|b_I(\tilde{u},\tilde{u})| \leq C_b|\tilde{u}|^2_{H^1(\Omega,\mathbb{C}_\sigma^+)},\ \tilde{u}\in H^1(\Omega,\mathbb{C}_\sigma^+), \tag{33}$$

If $Re\,p\geq\sigma>0$ and $|\overline{p}\tilde{k}|\leq\tilde{C}$, then there exists a positive constant $e(p)$ such that (25) holds.

Proof: From (32) and (33) we find

$$\mathrm{Re}\,a_p(\tilde{u},\tilde{u})\ \geq \|\tilde{u}\|^2_{L^2(\Omega,\mathbb{C}_\sigma^+)} + \frac{1}{|p|^2}\big(a_e\mathrm{Re}\,p - C_b\tilde{C}\big)|\tilde{u}|^2_{H^1(\Omega,\mathbb{C}_\sigma^+)}.$$

To have (25) it is sufficient to choose σ such that

$$\sigma > \frac{C_b\tilde{C}}{a_e}. \tag{34}$$

□

As an example we notice that, for the kernel $k(t) = \dfrac{1}{\tau}e^{-\frac{t}{\tau}}, \tau>0$, which was introduced in equation (8), we have $\overline{p}\tilde{k} = \dfrac{\overline{p}}{1+\tau p}$ and then $|\overline{p}\tilde{k}|\leq\frac{1}{\tau}$ for $\mathrm{Re}\,p\geq 0$.

For more general sesquilinear forms $a_p(.,.)$ we have the following sufficient conditions:

Lemma 2. *Let us suppose that $a_I(.,.)$ defined by (28) satisfies (32). If*

$$|b(\tilde{u},\tilde{u})| \leq C_b\|\tilde{u}\|^2_{H^1(\Omega,\mathbb{C}_\sigma^+)}, \tilde{u}\in H^1(\Omega,\mathbb{C}_\sigma^+), \tag{35}$$

and

$$|\tilde{k}| = O(|p|^{-1}) \tag{36}$$

then there exists $\sigma>0$ such that for $p\in\mathbb{C}_\sigma^+$, $a_p(.,.)$ defined by (20) satisfies (25).

Proof: Using convenient algebraic manipulations, we can show that

$$|\frac{1}{p}a_{II}(\tilde{u},\tilde{u})| \leq \frac{1}{|p|^2}\frac{1}{4\epsilon^2}\|A_2\|_\infty^2|\tilde{u}|^2_{H^1(\Omega,\mathbb{C}_\sigma^+)} + \Big(2\epsilon^2 + \frac{1}{|p|}\|A_1\|_\infty\Big)\|\tilde{u}\|^2_{L^2(\Omega,\mathbb{C}_\sigma^+)}, \quad (37)$$

$\forall \tilde{u} \in H^1(\Omega, \mathbb{C}_\sigma^+)$, for all $\epsilon \neq 0$. From (28) and (35) we get the estimate

$$\mathrm{Re}\, a_p(\tilde{u},\tilde{u}) \geq e_1(p)|\tilde{u}|^2_{H^1(\Omega,\mathbb{C}_\sigma^+)} + e_0(p)\|\tilde{u}\|^2_{L^2(\Omega,\mathbb{C}_\sigma^+)},\ \forall \tilde{u} \in H^1_0(\Omega,\mathbb{C}_\sigma^+), \quad (38)$$

with $e_1(p)$ and $e_0(p)$ defined respectively by

$$e_1(p) = \frac{1}{|p|^2}\Big(a_e \mathrm{Re}\, p - \frac{1}{4\epsilon^2}\|A_2\|_\infty^2 - C_b|\tilde{k}||p|\Big) \quad (39)$$

and

$$e_0(p) = 1 - 2\epsilon^2 - \frac{1}{|p|}\|A_1\|_\infty - C_b\frac{|\tilde{k}|}{|p|}. \quad (40)$$

Now we use (36) and we conclude that there exists $\epsilon \neq 0$ and $\sigma \in \mathbb{R}^+$ such that $e_1(p)$ and $e_0(p)$ satisfy

$$e_i(p) > 0,\ \forall p \in \mathbb{C}_\sigma^+,\ i = 0, 1. \quad (41)$$

□

Lemma 3. *Let us suppose that $a_I(.,.)$ defined by (28) satisfies (32). If the sesquilinear form $b_I(.,.)$ satisfies*

$$b_I(\tilde{u},\tilde{u}) \geq b_e|\tilde{u}|^2_{H^1(\Omega,\mathbb{C}_\sigma^+)},\ \tilde{u} \in H^1(\Omega,\mathbb{C}_\sigma^+), \quad (42)$$

$$Re\frac{\tilde{k}}{p} > 0, \quad (43)$$

and

$$|\tilde{k}| = O(1), \quad (44)$$

then there exists $\sigma > 0$ such that for $p \in \mathbb{C}_\sigma^+$, $a_p(.,.)$ defined by (20) satisfies (25).

Proof: Let us suppose now that the sesquilinear form $b_I(.,.)$ satisfies (42). For $b_{II}(.,.)$ we can prove that

$$|\frac{1}{p}\tilde{k}b_{II}(\tilde{u},\tilde{u})| \leq \frac{|\tilde{k}|^2}{|p|^2}\frac{1}{4\eta^2}\|B_2\|_\infty^2|\tilde{u}|^2_{H^1(\Omega,\mathbb{C}_\sigma^+)} + \Big(2\eta^2 + \frac{|\tilde{k}|}{|p|}\|B_1\|_\infty\Big)\|\tilde{u}\|^2_{L^2(\Omega,\mathbb{C}_\sigma^+)}, \quad (45)$$

$\forall \tilde{u} \in H^1(\Omega, \mathbb{C}_\sigma^+)$, for $\eta \neq 0$. As (43) holds, from (42) and (45), we conclude that

$$\mathrm{Re}\, a_p(\tilde{u}, \tilde{u}) \geq e_1(p)|\tilde{u}|^2_{H^1(\Omega,\mathbb{C}_\sigma^+)} + e_0(p)\|\tilde{u}\|^2_{L^2(\Omega,\mathbb{C}_\sigma^+)}, \ \forall \tilde{u} \in H_0^1(\Omega, \mathbb{C}_\sigma^+), \tag{46}$$

with $e_1(p)$ and $e_0(p)$ defined respectively by

$$e_1(p) = \frac{1}{|p|^2}\Big(a_e \mathrm{Re}\, p + b_e \mathrm{Re}\,(\tilde{k}\overline{p}) - \frac{1}{4\epsilon^2}\|A_2\|_\infty^2 - \frac{1}{4\eta^2}|\tilde{k}|^2\|B_2\|_\infty^2\Big) \tag{47}$$

and

$$e_0(p) = 1 - 2\epsilon^2 - 2\eta^2 - \frac{1}{|p|}\|A_1\|_\infty - \frac{|\tilde{k}|}{|p|}\|B_1\|_\infty, \tag{48}$$

for all $\epsilon, \eta \neq 0$. Using now condition (44) we guarantee that there exist $\epsilon, \eta \neq 0$ and $\sigma \in \mathbb{R}^+$ such that $e_1(p)$ and $e_0(p)$ satisfy (41).

□

It is clear that if $a_i = b_i = 0, i = 1, 2$, $A_1 = B_1 = 0$ and A_{22}, B_{22} are diagonal matrices such that

$$a_{ii} \geq \alpha_e > 0 \text{ in } \overline{\Omega},$$

and

$$b_{ii} \geq \beta_e > 0 \text{ in } \overline{\Omega},$$

then (32), (42) and (35), respectively, hold with $a_e = \alpha_e, b_e = \beta_e$ and $C_b = \|B_{22}\|_\infty$.

If we consider the kernel $k(t) = \dfrac{1}{\tau}e^{-\frac{t}{\tau}}, \tau > 0$, introduced in (8), then $\tilde{k} = \dfrac{1}{1+\tau p}$ satisfies (36).

4. Discretization in the Laplace space

4.1. The finite element solution.

In order to simplify the presentation, in what follows we consider homogeneous Dirichlet boundary conditions, that is, $\psi = 0$. By $C^0(\overline{\Omega}, \mathbb{C}_\sigma^+)$ we represent the space of functions $\tilde{u} : \overline{\Omega} \times \mathbb{C}_\sigma^+ \to \mathbb{C}$ depending on x, p, continuous in $\overline{\Omega}$ and analytic in $\mathbb{C}_\sigma^+$.

By $P_m(x, p)$ we denote a polynomial in space variables of degree $\leq m$ with coefficients depending on p analytic in $\mathbb{C}_\sigma^+$. We consider a sequence of triangulations $\mathcal{T}_H$, with diameter $H = \max\limits_{\Delta \in \mathcal{T}_H} diam(\Delta)$, obtained by regular refinement (see [30]). By Λ we denote the sequence of diameters of the sequence of triangulations. Let $V_{H,m}(\mathbb{C}_\sigma^+), H \in \Lambda$, be the corresponding sequence of finite element spaces:

$$\begin{aligned} V_{H,m}(\mathbb{C}_\sigma^+) &= \{\tilde{u} \in C^0(\overline{\Omega}, \mathbb{C}_\sigma^+) : \tilde{u} = 0 \text{ on } \partial\Omega, \\ &\quad \tilde{u}(x,p) = P_m(x,p),\, x \in \Delta, \Delta \in \mathcal{T}_H, p \in \mathbb{C}_\sigma^+\}. \end{aligned} \tag{49}$$

We denote by $V_{H,m}(\mathbb{C}_\sigma^+)'$ the dual space of $V_{H,m}(\mathbb{C}_\sigma^+)$.

We remark that $V_{H,m}(\mathbb{C}_\sigma^+) \subset H_0^1(\Omega, \mathbb{C}_\sigma^+)$. Let $\{\phi, i = 1, \ldots, n_H\}$ be a finite element basis of $V_{H,m}(\mathbb{C}_\sigma^+)$, where ϕ_i depends only on x. The Ritz-Galerkin approximation for the solution of (24) is a function $\tilde{c}_H \in V_{H,m}(\mathbb{C}_\sigma^+)$ such that

$$a_p(\tilde{c}_H, \tilde{v}_H) = \ell_H(\tilde{v}_H),\ \forall \tilde{v}_H \in V_{H,m}(\mathbb{C}_\sigma^+), \tag{50}$$

where $a_p(.,.)$ is defined by (20) and $\ell_H : V_{H,m}(\mathbb{C}_\sigma^+) \to \mathbb{C}$,

$$\ell_H(\tilde{v}_H) = \ell(\tilde{v}_H),\ \tilde{v}_H \in V_{H,m}(\mathbb{C}_\sigma^+),$$

with ℓ defined by (23).

The existence and uniqueness of the previous finite element solution is consequence of the ellipticity of the bilinear form $a_p(.,.)$.

Theorem 2. *If $f \in L^2(\mathbb{R}^+, L^2(\Omega)), c_0 \in L^2(\Omega)$ and under the assumption of Theorem 1, there exists a positive σ such that, for each $p \in \mathbb{C}_\sigma^+$, the problem (50) has a unique solution $\tilde{c}_H \in V_{H,m}(\mathbb{C}_\sigma^+)$.*

The finite element solution $\tilde{c}_H \in V_{H,m}(\mathbb{C}_\sigma^+)$, $\tilde{c}_H = \displaystyle\sum_{i=1}^{n_H} a_i\phi_i$, where a_i depends on p, is obtained solving the linear system

$$[a_p(\phi_i, \phi_j)][a_i] = [\ell(\phi_j)]. \tag{51}$$

We remark that the variational equation (50) is equivalent to the following problem: find $\tilde{c}_H \in V_{H,m}(\mathbb{C}_\sigma^+)$ such that

$$L_H\tilde{c}_H = \ell_H \text{ in } V_{H,m}(\mathbb{C}_\sigma^+), \tag{52}$$

with $L_H : V_{H,m}(\mathbb{C}_\sigma^+) \to V_{H,m}(\mathbb{C}_\sigma^+)'$, defined by

$$L_H\tilde{u}_H(\tilde{v}_H) = a_p(\tilde{u}_H, \tilde{v}_H),\ \tilde{u}_H, \tilde{v}_H \in V_{H,m}(\mathbb{C}_\sigma^+).$$

Theorem 2 establishes a sufficient condition for the existence of a unique solution of the equation (52), $\tilde{c}_H = L_H^{-1}\ell_H$.

4.2. Error estimates for the finite element solution.

Let $\Pi_H : H_0^1(\Omega, \mathbb{C}_\sigma^+) \to V_{H,m}(\mathbb{C}_\sigma^+)$ be the finite element projection operator. Under the assumption of Theorem 1, there exists a unique solution $\tilde{c} \in H_0^1(\Omega, \mathbb{C}_\sigma^+)$ of (24) and a unique solution $\tilde{c}_H \in V_{H,m}(\mathbb{C}_\sigma^+)$ of (50). As $H_0^1(\Omega, \mathbb{C}_\sigma^+)$ and $V_{H,m}(\mathbb{C}_\sigma^+)$ are Hilbert spaces we consider $\Pi_H : H_0^1(\Omega, \mathbb{C}_\sigma^+)' \to V_{H,m}(\mathbb{C}_\sigma^+)'$. Let $S_H : H_0^1(\Omega, \mathbb{C}_\sigma^+) \to V_{H,m}$ be defined by

$$S_H = L_H^{-1} \Pi_H L.$$

This operator satisfies

$$S_H = I_d \text{ in } V_{H,m}. \tag{53}$$

In the next theorem we establish the error estimate for $\tilde{c}_H$.

Theorem 3. *Let us suppose that the finite element spaces $V_{H,m}(\mathbb{C}_\sigma^+)$, for $H \in \Lambda$, are constructed using a sequence of triangulations $\mathcal{T}_H$, with diameter $H \in \Lambda$, obtained by regular refinement. Under the assumption of Theorem 1, there exists a unique solution $\tilde{c} \in H_0^1(\Omega, \mathbb{C}_\sigma^+)$ of (24), a unique solution $\tilde{c}_H \in V_{H,m}(\mathbb{C}_\sigma^+)$ of (50) and a positive constant C, independent of $\tilde{c}$, H and p, such that, for $H \in \Lambda$ small enough, we have*

$$\|\tilde{c} - \tilde{c}_H\|_{H^1(\Omega, \mathbb{C}_\sigma^+)} \leq C H^m \|\tilde{c}\|_{H^{m+1}(\Omega, \mathbb{C}_\sigma^+)} \tag{54}$$

provided that $\tilde{c} \in H^{m+1}(\Omega, \mathbb{C}_\sigma^+)$.

Proof: Following the proof of Theorems 3 of [9] we start by proving that

$$\|\tilde{c} - \tilde{c}_H\|_{H^1(\Omega, \mathbb{C}_\sigma^+)} \leq C \|\tilde{c} - \tilde{v}_H\|_{H^1(\Omega, \mathbb{C}_\sigma^+)}, \ \forall \tilde{v}_H \in V_{H,m}(\mathbb{C}_\sigma^+). \tag{55}$$

As by using (53) we have

$$\tilde{v}_H = S_H \tilde{v}_H,$$

then

$$\begin{aligned} \|\tilde{c} - \tilde{c}_H\|_{H^1(\Omega, \mathbb{C}_\sigma^+)} &\leq \|\tilde{c} - \tilde{v}_H\|_{H^1(\Omega, \mathbb{C}_\sigma^+)} + \|\tilde{c_H} - \tilde{v}_H\|_{H^1(\Omega, \mathbb{C}_\sigma^+)} \\ &= \|\tilde{c} - \tilde{v}_H\|_{H^1(\Omega, \mathbb{C}_\sigma^+)} + \|S_H(\tilde{c} - \tilde{v}_H)\|_{H^1(\Omega, \mathbb{C}_\sigma^+)} \end{aligned}$$

and we conclude that

$$\|\tilde{c} - \tilde{c}_H\|_{H^1(\Omega, \mathbb{C}_\sigma^+)} \leq \big(1 + \|S_H\|\big) \|\tilde{c} - \tilde{v}_H\|_{H^1(\Omega, \mathbb{C}_\sigma^+)},$$

for $\tilde{v}_H \in V_{H,m}(\mathbb{C}_\sigma^+)$, where $\|S_H\|$, for $H \in \Lambda$ with H small enough, has a bound independent of $p \in \mathbb{C}_\sigma^+$.

Moreover Theorem 4 of [9] allows us to conclude that there exists a positive constant C independent on $\tilde{c}$, H and p such that, for H small enough, we have

$$\|\tilde{c}-\Pi_H\tilde{c}\|_{H^1(\Omega,\mathbb{C}_\sigma^+)} \leq CH^m\|\tilde{c}\|_{H^{m+1}(\Omega,\mathbb{C}_\sigma^+)}. \tag{56}$$

From (55) and (56) we finally obtain (54).

□

5. Returning to the initial variables

To return to the initial variables we need to apply the Laplace inverse to both members of an inequality of type (56) with convenient norms.

An essential tool to recover the initial variables is the Paley-Wiener Theorem. To present such lemma we introduce the space $L^2(\mathbb{R}^+, H^1(\Omega),\sigma)$ as the space of functions $v:\mathbb{R}^+\to H^1(\Omega)$ such that

$$\|v\|_{L^2(\mathbb{R}^+,H^1(\Omega),\sigma)} = \Big(\int_{\mathbb{R}^+} e^{-2\sigma t}\|v(t)\|^2_{H^1(\Omega)}\,dt\Big)^{1/2} \tag{57}$$

is finite. In $L^2(\mathbb{R}^+, H^1(\Omega),\sigma)$ we consider the inner product

$$(u,v)_{L^2(\mathbb{R}^+,H^1(\Omega),\sigma)} = \int_{\mathbb{R}^+}(u(t),v(t))_{H^1(\Omega)}e^{-2\sigma t}\,dt\,,\; u,v\in L^2(\mathbb{R}^+,H^1(\Omega),\sigma) \tag{58}$$

which induces the norm defined by (57). We also consider the Hardy space $\mathcal{H}^2(\mathbb{C}_\sigma^+, H^{m+1}(\Omega))$ of holomorphic functions $\tilde{f}:\mathbb{C}_\sigma^+\to H^{m+1}(\Omega)$ such that

$$\|\tilde{f}\|_{\mathcal{H}^2(\mathbb{C}_\sigma^+,H^{m+1}(\Omega))} = \Big(\sup_{p_1>\sigma}\int_{\mathbb{R}}\|\tilde{f}(p_1+ip_2)\|^2_{H^{m+1}(\Omega)}\,dp_2\Big)^{1/2} < \infty.$$

Lemma 4. *[Paley-Wiener Theorem] The Laplace transform* $\mathcal{L}: L^2(\mathbb{R}^+, H^{m+1}(\Omega),\sigma)\to\mathcal{H}^2(\mathbb{C}_\sigma^+, H^{m+1}(\Omega))$ *is an isometric isomorphism.*

Inequality (54) allows us to write

$$\|\tilde{c}-\tilde{c}_H\|_{\mathcal{H}^2(\mathbb{C}_\sigma^+,H^1(\Omega))} \leq CH^m\|\tilde{c}\|_{\mathcal{H}^2(\mathbb{C}_\sigma^+,H^{m+1}(\Omega))}. \tag{59}$$

Applying Paley-Wiener Theorem we get the main result of this paper:

Theorem 4. *Let us suppose that the finite element spaces* $V_{H,m}(\mathbb{C}_\sigma^+)$*, for* $H\in\Lambda$*, are constructed using a sequence of triangulations* $\mathcal{T}_H$*, with diameter* $H\in\Lambda$*, obtained by regular refinement. Under the assumption of Theorem 1 there exist a unique solution* $\tilde{c}\in H_0^1(\Omega,\mathbb{C}_\sigma^+)$ *of (24), a unique solution*

$\tilde{c}_H \in V_{H,m}(\mathbb{C}_\sigma^+)$ *of (50) and a positive constant* C, *independent of* $\tilde{c}$, H *and* p, *such that, for* $H \in \Lambda$ *small enough,* $c = \mathcal{L}^{-1}\tilde{c}$, $c_H = \mathcal{L}^{-1}\tilde{c}_H$ *satisfy*

$$\|c - c_H\|_{L^2(\mathbb{R}^+, H^1(\Omega), \sigma)} \leq CH^m \|c\|_{L^2(\mathbb{R}^+, H^{m+1}(\Omega), \sigma)}, \tag{60}$$

provided that $\tilde{c} \in L^2(\mathbb{R}^+, H^{m+1}(\Omega), \sigma)$.

□

6. Numerical simulation

In this section we give one example of application of the method based on the Laplace transform described in Section 4 combined with the algorithm developed in [1] for the inverse Laplace transform.

We consider the integro-differential equation (1) with $\Omega = (0,1) \times (0,1)$, $A = B = -\Delta$, where Δ denotes the Laplace operator and $K(s) = \dfrac{1}{\tau} e^{-\frac{s}{\tau}}$. The function f, the initial and boundary conditions are such that the IBVP has the following solution

$$u(x,t) = \cos(t) x_1 x_2 (1 - x_1)(1 - x_2), (x_1, x_2) \in \overline{\Omega}, t \in \mathbb{R}_0^+.$$

In Ω we introduce a triangulation $\mathcal{T}_H$ induced by a uniform rectangular grid defined considering, in $[0,1] \times [0,1]$, $(N+1) \times (N+1)$ equally spaced points. For each time t, the Laplace inverse of the finite element solution is computed using the algorithm developed in [1] with the following parameters, according the notation used in the mentioned paper: $\alpha = 0, T = 0.8t, E_r = 10^{-8}, \gamma = -\frac{\ln(E_R)}{1.6T}$, $M = 50$ and $Tol = \dfrac{T}{e^{2\gamma T} N^2}$.

The objective of this section is to illustrate the convergence behavior of the method studied in this work. We consider $m = 1, 2$, that is, we use linear and quadratic finite elements. Assuming that $\|c(t) - c_H(t)\|_{H^1(\Omega)} \simeq CH^q$, we show that $q \simeq m$. The convergence rate $Rate(t)$ is computed using

$$\text{Rate}(t) = \frac{\ln\left(\frac{\|c(t) - c_{H_1}(t)\|_{H^1}}{\|c(t) - c_{H_2}(t)\|_{H^1}}\right)}{\ln\left(\frac{H_1}{H_2}\right)},$$

where H_1 and H_2 are the diameters of two consecutive triangulations.

We remark that we observe the bound

$$\|c(t) - c_H(t)\|_{H^1(\Omega)} \leq CH^m, \tag{61}$$

N	Error(0.1)	Rate(0.1)	Error(1)	Rate(1)	Error(10)	Rate(10)
10	0.0241023	1.00	0.0130785	0.99	0.0203103	0.99
20	0.0120958	1.00	0.00656704	1.00	0.0101984	1.00
30	0.00806943	1.00	0.00438147	1.00	0.00680427	1.00
40	0.00605353	1.00	0.00328701	1.00	0.00510461	1.00
50	0.00484337	1.00	0.00262995	1.00	0.00408422	1.00
60	0.00403639	1.00	0.00219179	1.00	0.00340375	1.00
70	0.00345989	1.00	0.00187876	1.00	0.00291763	1.00
80	0.00302749	-	0.00164398	-	0.00255301	-

TABLE 1. Errors and rates obtained for linear elements at $t = 0.1, 1, 10$, computed with the norm $\|.\|_{H^1}$.

N	Error(0.1)	Rate(0.1)	Error(1)	Rate(1)	Error(10)	Rate(10)
10	0.00134738	1.99	0.000731614	1.99	0.00113616	1.99
20	0.000338144	1.99	0.000183679	1.98	0.00028516	1.99
30	0.000150626	1.97	8.21509e-05	1.89	0.000127113	1.97
40	8.54939e-005	1.87	4.76761e-05	1.63	7.21529e-05	1.84
50	5.63455e-005	1.61	3.31572e-05	1.01	4.78133e-05	1.49
60	4.20235e-005	-	2.75704e-05	-	3.64293e-05	-

TABLE 2. Errors and rates obtained for quadratic elements at $t = 0.1, 1, 10$, computed with the norm $\|.\|_{H^1}$.

which is a stronger estimate when compared with the result in Theorem 4,

$$\int_{\mathbb{R}^+} e^{-2\sigma t}\|c(t) - c_H(t)\|^2_{H^1(\Omega)} ds \leq CH^{2m}\|c\|^2_{L^2(\mathbb{R}^+, H^{m+1}(\Omega),\sigma)}.$$

In Table 1 we present the numerical error Error(t) and Rate(t) for $t = 0.1$, $t = 1$ and $t = 10$ computed using linear elements. The numerical results show that the convergence rate is in fact 1 when linear elements are used.

The numerical errors Error(t) and Rate(t) for $t = 0.1$, $t = 1$ and $t = 10$, for quadratic elements, are presented in Table 2. The numerical results show that the convergence rate is 2 when quadratic elements are used. When N increases we observe a deterioration of the convergence rates. This behavior was expected since, for large values of N, the error of the spatial discretization is very small and the error $\|c(t) - c_{H_2}(t)\|_{H^1}$ is dominated by the error induced by numerical Laplace inversion.

N	Error(0.1)	Rate(0.1)	Error(1)	Rate(1)	Error(10)	Rate(10)
10	0.000923419	1.99	0.000516432	1.99	0.000788718	1.99
20	0.000232886	2.00	0.000130441	2.00	0.000199163	2.00
30	0.000103637	2.00	5.80592e-05	2.00	8.86341e-05	2.00
40	5.83169e-005	2.00	3.26734e-05	2.00	4.98842e-05	2.00
50	3.7338e-005	2.00	2.09267e-05	2.00	3.19352e-05	2.00
60	2.59387e-005	2.00	1.45307e-05	2.00	2.21752e-05	2.00
70	1.90528e-005	2.00	1.06773e-05	2.00	1.62915e-05	2.00
80	1.45901e-005	-	8.17607e-06	-	1.24722e-05	-

TABLE 3. Errors and rates obtained for linear elements at $t = 0.1, 1, 10$, computed with the norm $\|.\|_{L^2}$.

N	Error(0.1)	Rate(0.1)	Error(1)	Rate(1)	Error(10)	Rate(10)
10	1.43765e-005	3.00	7.81467e-06	3.01	1.21519e-05	3.01
20	1.78621e-006	2.99	9.69927e-07	2.97	1.50822e-06	2.99
30	5.30795e-007	2.91	2.91427e-07	2.71	4.48644e-07	2.92
40	2.29777e-007	2.64	1.33718e-07	2.07	1.93656e-07	2.55
50	1.27406e-007	2.17	8.41604e-08	1.36	1.09575e-07	1.94
60	8.57563e-008	-	6.57133e-08	-	7.68772e-08	-

TABLE 4. Errors and rates obtained for quadratic elements at $t = 0.1, 1, 10$, computed with the norm $\|.\|_{L^2}$.

Finally we present some results obtained considering the L^2 norm in the measurement of the error. In Table 3 we present the results obtained with linear elements that show a second order convergence rate. The errors and rates obtained with quadratic elements are presented in Table 4. These results show a third order convergence rate. However, when N increases we observe, as before, a deterioration of this rate because the error $\|c(t) - c_H(t)\|_{L^2}$ is dominated by the error of the numerical Laplace inversion.

Also, we can only expect that the numerical Laplace inverse is computed with a high degree of accuracy for moderate values of t. In fact, for the example considered in the experiments, when we consider large values of t (*e.g.* $t = 100$) we don't observe a good agreement between the exact and numerical solution due to the limitations of the algorithm for the numerical Laplace inversion.

7. Conclusions

In this paper we consider a hybrid numerical method for the IBVP (1), (9), (10). The method is composed by three steps: in the first step, applying Laplace transforms, the given initial boundary value problem is replaced by an elliptic boundary value problem that depends on the Laplace parameter; in the second step the solution of this boundary value problem is approximated using the finite element method, for a choice of a finite set of quadrature points in the Laplace domain; finally, in the third stage, the numerical solution on the physical time space domain is obtained using numerical inverse Laplace transforms. The main result of this paper, Theorem 4, shows the theoretical error estimates for $c - c_H$. Although the norm used in the estimate (60) doens't give information for the error at a specific value of t, since it involves an integration over the time and a negative exponential in the variable t, the numerical results illustrate, for moderate values of t, that the method proposed has similar convergence behavior when compared to the results known for the standard finite element method for elliptic or parabolic problems.

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