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FROM HYPERSYMPLECTIC STRUCTURES TO COMPATIBLE PAIRS OF TENSORS ON A LIE ALGEBROID

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ABSTRACT: A hypersymplectic structure on a Lie algebroid determines several Poisson-Nijenhuis, ΩN and $P\Omega$ structures on that Lie algebroid. We show that these Poisson-Nijenhuis (respectively, ΩN, $P\Omega$) structures on the Lie algebroid, are pairwise compatible.

Introduction

Pairs of tensor fields on manifolds, which are compatible in a certain sense, were studied by Magri and Morosi [7], in view of their application to integrable hamiltonian systems. These pairs determine Poisson-Nijenhuis (PN for short), ΩN and $P\Omega$ structures on the manifold where they are defined. The extension of these concepts to the Lie algebroid framework was done in [1] and [6]. Other related structures on Lie algebroids are the complementary forms [8] and the Hitchin pairs [4]. In [2], we studied the compatibility of pairs of these structures, where being compatible means that the sum of two structures of a certain type is still a structure of the same type.

Hypersymplectic structures on Lie algebroids were introduced in [1]. They are triples of symplectic forms on the Lie algebroid, satisfying some conditions. Taking the inverse of each symplectic form we get three Poisson bivectors and, if we consider the composition of the Poisson bivectors with the symplectic forms, we obtain $(1,1)$-tensors on the Lie algebroid, that turn to be Nijenhuis. In [1] it is proved that these symplectic forms, Poisson bivectors and Nijenhuis tensors, when considered in pairs, define PN, ΩN and $P\Omega$ structures on the Lie algebroid and that the symplectic forms ω_i are complementary forms. The aim of this Note is to show that the structures that we can derive from a given hypersymplectic structure on a Lie algebroid are, in most cases, pairwise compatible.

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1. Pairs of tensors on Lie algebroids

Let $(A, [\cdot,\cdot], \rho)$ be a Lie algebroid over a manifold M. Frequently, we shall denote this Lie algebroid simply by A. Given a bivector π and a 2-form ω on A, we consider the usual bundle maps $\pi^{\#} : A^* \to A$ and $\omega^{\flat} : A \to A^*$ and the induced morphisms on sections, denoted by the same symbols, which are defined, for all $\alpha, \beta \in \Gamma(A^*)$ and $X, Y \in \Gamma(A)$, respectively by

$$\langle \beta, \pi^{\#}(\alpha)\rangle = \pi(\alpha, \beta) \quad \text{and} \quad \langle Y, \omega^{\flat}(X)\rangle = \omega(X, Y).$$

A bivector π on a Lie algebroid $(A, [\cdot,\cdot], \rho)$ is a *Poisson* bivector if $[\pi, \pi] = 0$. Every bivector π on a Lie algebroid A defines a bracket on $\Gamma(A^*)$:

$$[\alpha, \beta]_\pi = \mathcal{L}_{\pi^{\#}(\alpha)}\beta - \mathcal{L}_{\pi^{\#}(\beta)}\alpha - \mathrm{d}(\pi(\alpha, \beta)), \tag{1}$$

where d is the differential of the Lie algebroid A and $\mathcal{L}$ is the Lie derivative determined by d. In the case where π is a Poisson bivector on A, the bracket $[\cdot,\cdot]_\pi$, given by (1), is a Lie bracket and $(A^*, [\cdot,\cdot]_\pi, \rho \circ \pi^{\#})$ is a Lie algebroid.

Let π be a Poisson bivector on $(A, [\cdot,\cdot], \rho)$. A 2-form ω on A is said to be a *complementary form* of π if $[\omega, \omega]_\pi = 0$ ([8]). In other words, ω is a Poisson bivector on the Lie algebroid $(A^*, [\cdot,\cdot]_\pi, \rho \circ \pi^{\#})$.

Let N be a $(1,1)$-tensor on $(A, [\cdot,\cdot], \rho)$ and consider the *deformed* bracket $[\cdot,\cdot]_N$ on $\Gamma(A)$,

$$[X, Y]_N = [NX, Y] + [X, NY] - N[X, Y]. \tag{2}$$

The Nijenhuis torsion of N is the $(1,2)$-tensor $\mathcal{T}N$ given by

$$\mathcal{T}N(X, Y) = [NX, NY] - N[X, Y]_N, \;\; X, Y \in \Gamma(A).$$

When the Nijenhuis torsion vanishes, N is said to be a *Nijenhuis* tensor on A.

Let π be a bivector on A and N a $(1,1)$-tensor, seen as a vector bundle map $N : A \to A$. If $N \circ \pi^{\#} = \pi^{\#} \circ N^*$, where N^* denotes the transpose of N, π_N given by $\pi_N(\alpha, \beta) = \pi(N^*\alpha, \beta)$, $\alpha, \beta \in \Gamma(A^*)$, is a bivector on A. Analogously, if ω is a 2-form on A such that $\omega^{\flat} \circ N = N^* \circ \omega^{\flat}$, then ω_N given by $\omega_N(X, Y) = \omega(NX, Y)$, $X, Y \in \Gamma(A)$, is a 2-form on A.

Let π be a bivector and N a $(1,1)$-tensor on a Lie algebroid A such that $N \circ \pi^{\#} = \pi^{\#} \circ N^*$. Recall that the *Magri-Morosi concomitant* $\mathcal{C}(\pi, N)$ of π and N is the $(2,1)$-tensor on A defined by [5, 7]:

$$\mathcal{C}(\pi, N)(\alpha, \beta) = ([\alpha, \beta]_\pi)_{N^*} - [\alpha, \beta]_{\pi_N}, \quad \alpha, \beta \in \Gamma(A^*), \tag{3}$$

where $[\cdot,\cdot]_{\pi_N}$ is the bracket (1) determined by the bivector π_N and $([\cdot,\cdot]_\pi)_{N^*}$ denotes the bracket $[\cdot,\cdot]_\pi$ deformed by N^*. From (1), (2) and (3), it is obvious that

$$\mathcal{C}(\pi+\pi', N+N') = \mathcal{C}(\pi,N) + \mathcal{C}(\pi,N') + \mathcal{C}(\pi',N) + \mathcal{C}(\pi',N'), \tag{4}$$

for all bivectors π, π' and $(1,1)$-tensors N, N' on A.

Let $(A,[\cdot,\cdot],\rho)$ be a Lie algebroid, π a bivector, ω a 2-form and N a $(1,1)$-tensor on A. The tensor fields N, π and ω, when taken in pairs, define well known structures on the Lie algebroid A. Next, we recall some of these structures.

A pair (π,N) is a *Poisson-Nijenhuis structure* (*PN structure*, for short) on $(A,[\cdot,\cdot],\rho)$ if π is Poisson, N is Nijenhuis, $N\circ\pi^\# = \pi^\#\circ N^*$ and $\mathcal{C}(\pi,N)=0$.

A pair (ω,N) is an *ΩN structure* on $(A,[\cdot,\cdot],\rho)$ if ω is closed, N is Nijenhuis, $\omega^\flat\circ N = N^*\circ\omega^\flat$ and ω_N is closed. If ω is symplectic, $\omega^\flat\circ N = N^*\circ\omega^\flat$ and ω_N is closed, the pair (ω,N) is a *Hitchin pair* [4].

A pair (π,ω) is a *$P\Omega$ structure* on $(A,[\cdot,\cdot],\rho)$ if π is Poisson, ω is closed and ω_N is closed, where N is the $(1,1)$-tensor on A defined by $N=\pi^\#\circ\omega^\flat$.

A $P\Omega$ structure on a Lie algebroid determines a PN and an ΩN structure on this Lie algebroid and is equivalent to a closed complementary form. More precisely, we have:

Proposition 1.1. [3, 6] *Let π and ω be, respectively, a Poisson bivector and a 2-form on a Lie algebroid $(A,[\cdot,\cdot],\rho)$ and consider the $(1,1)$-tensor $N=\pi^\#\circ\omega^\flat$.*

(i) *If (π,ω) is a $P\Omega$ structure on A, then (π,N) and (ω,N) are, respectively, a PN and an ΩN structure on A.*
(ii) *The pair (π,ω) is a $P\Omega$ structure on A if and only if ω is a closed complementary form of π.*

Let N and N' be two $(1,1)$-tensors on a Lie algebroid $(A,[\cdot,\cdot],\rho)$ and $[N,N']_{FN}$ their Frölicher-Nijenhuis bracket which is defined, for all sections X and Y of A, by

$$\begin{aligned}[N,N']_{FN}(X,Y) &= [NX,N'Y] - N[X,N'Y] - N'[NX,Y] + NN'[X,Y] \\ &\quad + [N'X,NY] - N'[X,NY] - N[N'X,Y] + N'N[X,Y]. \end{aligned} \tag{5}$$

Notice that $[N,N']_{FN} = [N',N]_{FN}$ and if $N = N'$, then $[N,N']_{FN} = 2\mathcal{T}N$. So, if N and N' are Nijenhuis tensors on A, we get

$$\mathcal{T}(N+N') = 0 \Leftrightarrow [N,N']_{FN} = 0. \tag{6}$$

When the Nijenhuis tensors N and N' satisfy (6), they are said to be *compatible.*

2. Hypersymplectic structures on Lie algebroids

In this section we recall, from [1] and [3], the notion of hypersymplectic structure on a Lie algebroid as well as their main properties, needed in the sequel.

Let $(A,[\cdot,\cdot],\rho)$ be a Lie algebroid and consider 3 symplectic forms ω_1, ω_2 and ω_3 on A with inverse Poisson bivectors π_1, π_2 and π_3, respectively. Then, for all $i \in \{1,2,3\}$, we have $\omega_i^\flat \circ \pi_i^\# = \mathrm{Id}_{A^*}$ and $\pi_i^\# \circ \omega_i^\flat = \mathrm{Id}_A$. These ω_i and π_i determine the *transition* $(1,1)$-tensors N_1, N_2 and N_3 on A, defined by

$$N_i = \pi_{i-1}^\# \circ \omega_{i+1}^\flat, \tag{7}$$

where the indices of $\pi^\#$ and $\omega^\flat$ are considered as elements of $\mathbb{Z}_3$.

Definition 2.1. A triple $(\omega_1,\omega_2,\omega_3)$ of symplectic structures is an $\boldsymbol{\varepsilon}$-*hypersymplectic structure* on a Lie algebroid $(A,[\cdot,\cdot],\rho)$ if the transition $(1,1)$-tensors N_i, $i=1,2,3$, given by (7), satisfy $N_i^2 = \varepsilon_i \mathrm{Id}_A$, where the parameters $\varepsilon_i = \pm 1$ form the triple $\boldsymbol{\varepsilon}= (\varepsilon_1,\varepsilon_2,\varepsilon_3)$.

Notice that $N_i^{-1} = \varepsilon_i N_i$, for all $i \in \{1,2,3\}$ and that

$$\pi_{i-1}^\# \circ \omega_i^\flat = \varepsilon_{i+1} N_{i+1}, \tag{8}$$

with the indices in $\mathbb{Z}_3$.

Proposition 2.2. *Let $(\omega_1,\omega_2,\omega_3)$ be an $\boldsymbol{\varepsilon}$-hypersymplectic structure on a Lie algebroid $(A,[\cdot,\cdot],\rho)$. Then, $[N_i,N_j]_{FN} = 0$ for all $i,j \in \{1,2,3\}$. In particular, $\mathcal{T}N_i = 0$, $i=1,2,3$.*

Recall that two Poisson bivectors π and π' on a Lie algebroid A are *compatible* if $\pi+\pi'$ is a Poisson bivector on A or, equivalently, if $[\pi,\pi'] = 0$. The Poisson bivectors π_i, $i=1,2,3$, of an $\boldsymbol{\varepsilon}$-hypersymplectic structure are pairwise compatible, i.e.,

$$[\pi_i,\pi_j] = 0, \quad i,j \in \{1,2,3\}. \tag{9}$$

Having an $\boldsymbol{\varepsilon}$-hypersymplectic structure $(\omega_1,\omega_2,\omega_3)$ on a Lie algebroid A, we may define $g \in \bigotimes^2 A^*$, through the vector bundle map $g^\flat$, by setting

$$g^\flat = \varepsilon_{i-1}\varepsilon_{i+1}\,\omega_{i-1}^\flat \circ \pi_i^\# \circ \omega_{i+1}^\flat, \tag{10}$$

where the indices are taken in $\mathbb{Z}_3$. Moreover, $(g^\flat)^* = -\varepsilon_1\varepsilon_2\varepsilon_3\, g^\flat$, which means that, in the case where $\varepsilon_1\varepsilon_2\varepsilon_3 = 1$, g is a 2-form on A and, consequently, g^{-1} is a bivector on A. Notice that

$$(g^{-1})^\# = \varepsilon_{i-1}\varepsilon_{i+1}\,\pi_{i+1}^\# \circ \omega_i^\flat \circ \pi_{i-1}^\#, \tag{11}$$

with the indices taken in $\mathbb{Z}_3$.

Proposition 2.3. *Let $(\omega_1,\omega_2,\omega_3)$ be an $\boldsymbol{\varepsilon}$-hypersymplectic structure on a Lie algebroid $(A,[\cdot,\cdot],\rho)$, with $\varepsilon_1\varepsilon_2\varepsilon_3 = 1$. Then,*

i) *g is a symplectic form on A;*
ii) *g^{-1} is a Poisson bivector on A;*
iii) *g^{-1} is compatible with π_i, i.e., $[\pi_i, g^{-1}] = 0$, $i = 1,2,3$.*

From an $\boldsymbol{\varepsilon}$-hypersymplectic structure on a Lie algebroid, we may derive several PN, ΩN and $P\Omega$ structures on that Lie algebroid, according to the next theorem.

Theorem 2.4. *Let $(\omega_1,\omega_2,\omega_3)$ be an $\boldsymbol{\varepsilon}$-hypersymplectic structure on a Lie algebroid $(A,[\cdot,\cdot],\rho)$. Then, for all $i,j \in \{1,2,3\}$,*

i) *the pairs (π_i, N_j), with $i \neq j$, are PN structures on A;*
ii) *the pairs (ω_i, N_j), with $i \neq j$, are ΩN structures on A;*
iii) *the pairs (π_i, ω_j), are $P\Omega$ structures on A.*

Moreover, when $\varepsilon_1\varepsilon_2\varepsilon_3 = 1$,

iv) *the pairs (π_i, N_i) and (g^{-1}, N_i), are PN structures on A;*
v) *the pairs (ω_i, N_i) and (g, N_i), are ΩN structures on A;*
vi) *the pairs (π_i, g) and (g^{-1}, ω_i), are $P\Omega$ structures on A.*

From Proposition 1.1 ii) we immediately deduce:

Corollary 2.5. *If $(\omega_1,\omega_2,\omega_3)$ is an $\boldsymbol{\varepsilon}$-hypersymplectic structure on $(A,[\cdot,\cdot],\rho)$, then ω_i is a complementary form of π_j, for all $i,j \in \{1,2,3\}$. In the case where $\varepsilon_1\varepsilon_2\varepsilon_3 = 1$, g is a complementary form of π_i and each ω_i is a complementary form of g^{-1}, $i = 1,2,3$.*

Remark 2.6. Notice that the 2-forms of the ΩN structures in Theorem 2.4 ii) and v) are symplectic. So, besides being ΩN structures, the pairs in the statements ii) and v) of that theorem, are also Hitchin pairs.

3. Compatibility of structures

The compatibility of structures that are defined by pairs of tensors on a Lie algebroid was studied in [2].

Definition 3.1. Two PN (respectively, ΩN, $P\Omega$) structures on a Lie algebroid A are said to be *compatible* if their sum is still a PN (respectively, ΩN, $P\Omega$) structure on A. Two complementary forms of a given Poisson bivector on A are *compatible* if their sum is still a complementary form of the same Poisson bivector.

In what follows, we show that the PN, ΩN and $P\Omega$ structures determined by an $\boldsymbol{\varepsilon}$-hypersymplectic structure on a Lie algebroid provide many examples of compatibility. In some cases, we also have the compatibility of complementary forms.

3.1. The general case. Let $(\omega_1, \omega_2, \omega_3)$ be an $\boldsymbol{\varepsilon}$-hypersymplectic structure on a Lie algebroid $(A, [\cdot,\cdot], \rho)$. Set, as before, $\pi_i^\# = (\omega_i^\flat)^{-1}$ and $N_i = \pi_{i-1}^\# \circ \omega_{i+1}^\flat$.

We start with the general case, i.e., the product $\varepsilon_1\varepsilon_2\varepsilon_3$ can be either 1 or -1. In this case we know, from Theorem 2.4, that for $i, j \in \{1, 2, 3\}$, with $i \neq j$, the pairs (π_i, N_j), (ω_i, N_j) and (π_i, ω_j) are, respectively, PN, ΩN and $P\Omega$ structures on A. So, we have six structures of each type determined by the $\boldsymbol{\varepsilon}$-hypersymplectic structure. Notice that the pairs (π_i, ω_i), $i = 1, 2, 3$, are also $P\Omega$ structures, but they are *trivial* in the sense that $\pi_i^\# \circ \omega_i^\flat = \mathrm{Id}_A$, so that they are not interesting from our point of view.

Proposition 3.2. *With the indices taken in $\mathbb{Z}_3$, we have:*

i) *the PN structures (π_i, N_{i-1}) and (π_i, N_{i+1}) are compatible;*
ii) *the PN structures (π_{i-1}, N_i) and (π_{i+1}, N_i) are compatible.*

Proof: i) From Proposition 2.2 and equation (6), we have that $N_{i-1} + N_{i+1}$ is a Nijenhuis tensor on A. Since the pairs (π_i, N_{i-1}) and (π_i, N_{i+1}) are PN structures on A, it is obvious that the equality $(N_{i-1} + N_{i+1}) \circ \pi_i^\# = \pi_i^\# \circ (N_{i-1} + N_{i+1})^*$ holds and also that $\mathcal{C}(\pi_i, N_{i-1} + N_{i+1}) = 0$, see (4). Thus, i) is proved.

ii) We use (9) to get that $\pi_{i-1} + \pi_{i+1}$ is a Poisson bivector. Since the conditions $N_i \circ (\pi_{i-1} + \pi_{i+1})^{\#} = (\pi_{i-1} + \pi_{i+1})^{\#} \circ N_i^*$ and $\mathcal{C}(\pi_{i-1} + \pi_{i+1}, N_i) = 0$ hold because (π_{i-1}, N_i) and (π_{i+1}, N_i) are PN structures on A, the proof is complete. ■

Proposition 3.3. *With the indices taken in $\mathbb{Z}_3$, we have:*

i) *the ΩN structures (ω_i, N_{i-1}) and (ω_i, N_{i+1}) are compatible;*
ii) *the ΩN structures (ω_{i-1}, N_i) and (ω_{i+1}, N_i) are compatible.*

Proof: As we already observed in the proof of Proposition 3.2 i), $N_{i-1} + N_{i+1}$ is a Nijenhuis tensor on A. We have $\omega_i^\flat \circ (N_{i-1} + N_{i+1}) = (N_{i-1} + N_{i+1})^* \circ \omega_i^\flat$, because $\omega_i^\flat \circ N_{i-1} = N_{i-1}^* \circ \omega_i^\flat$ and $\omega_i^\flat \circ N_{i+1} = N_{i+1}^* \circ \omega_i^\flat$. Since $(\omega_i)_{N_{i-1}+N_{i+1}} = (\omega_i)_{N_{i-1}} + (\omega_i)_{N_{i+1}}$ and the 2-forms $(\omega_i)_{N_{i-1}}$ and $(\omega_i)_{N_{i+1}}$ are closed, the 2-form $(\omega_i)_{N_{i-1}+N_{i+1}}$ is also closed. This completes the proof of i). The proof of ii) is similar. ■

According to the observation in Remark 2.6, all the pairs in Proposition 3.3 are Hitchin pairs. Moreover, from i), we have that the Hitchin pairs (ω_i, N_{i-1}) and (ω_i, N_{i+1}) are compatible in the sense that $(\omega_i, N_{i-1} + N_{i+1})$ is still a Hitchin pair.

Proposition 3.4. *The $P\Omega$ structures (π_{i-1}, ω_i) and (π_{i+1}, ω_i) are compatible, where the indices are taken in $\mathbb{Z}_3$.*

Proof: The sum $\pi_{i-1} + \pi_{i+1}$ is a Poisson bivector on A (see (9)). From (7) and (8), we have $(\pi_{i+1} + \pi_{i-1})^{\#} \circ (\omega_i)^\flat = N_{i-1} + \varepsilon_{i+1} N_{i+1}$. It remains to prove that $(\omega_i)_{N_{i-1}+\varepsilon_{i+1}N_{i+1}}$ is closed. Since, from Theorem 2.4 ii), the pairs (ω_i, N_{i-1}) and (ω_i, N_{i+1}) are ΩN structures, the 2-forms $(\omega_i)_{N_{i-1}}$ and $(\omega_i)_{N_{i+1}}$ are closed. Thus, $(\omega_i)_{N_{i-1}+\varepsilon_{i+1}N_{i+1}} = (\omega_i)_{N_{i-1}} + \varepsilon_{i+1}(\omega_i)_{N_{i+1}}$ is closed. ■

3.2. The case $\varepsilon_1\varepsilon_2\varepsilon_3 = 1$. Now, we consider the case where we have an $\boldsymbol{\varepsilon}$-hypersymplectic structure on a Lie algebroid $(A, [\cdot,\cdot], \rho)$, with $\varepsilon_1\varepsilon_2\varepsilon_3 = 1$. In this case, we know from Theorem 2.4 iv) that the pairs (π_i, N_i) and (g^{-1}, N_i), $i = 1, 2, 3$, are PN structures on $(A, [\cdot,\cdot], \rho)$ and so the total number of PN structures determined by the $\boldsymbol{\varepsilon}$-hypersymplectic structure is twelve.

We may improve Proposition 3.2.

Proposition 3.5. *For all indices $i, j, k, l \in \{1, 2, 3\}$,*

i) *the PN structures (π_i, N_j) and (π_k, N_l) are compatible;*
ii) *the PN structures (g^{-1}, N_i) and (g^{-1}, N_j) are compatible;*

iii) *the PN structures* (g^{-1}, N_i) *and* (π_j, N_k) *are compatible.*

Proof: i) As before, $\pi_i + \pi_k$ is a Poisson bivector and $N_j + N_l$ is a Nijenhuis tensor. Since $N_i \circ \pi_j^{\#} = \pi_j^{\#} \circ N_i^*$, the equality $(N_j + N_l) \circ (\pi_i + \pi_k)^{\#} = (\pi_i + \pi_k)^{\#} \circ (N_j + N_l)^*$ holds for all $i, j, k, l \in \{1, 2, 3\}$. Finally, from (4), we get $\mathcal{C}(\pi_i + \pi_k, N_j + N_l) = 0$ and the proof is complete.

ii) The proof is similar to Proposition 3.2 i).

iii) From Proposition 2.3, $g^{-1} + \pi_j$ is a Poisson bivector on A. The rest of the proof is similar to case i). ■

Summarizing, we have:

Theorem 3.6. *Let* $(\omega_1, \omega_2, \omega_3)$ *be an* $\boldsymbol{\varepsilon}$*-hypersymplectic structure on a Lie algebroid* $(A, [\cdot, \cdot], \rho)$, *with* $\varepsilon_1\varepsilon_2\varepsilon_3 = 1$. *Then, the twelve PN structures* (π_i, N_j), (g^{-1}, N_k), $i, j, k \in \{1, 2, 3\}$, *on A, are pairwise compatible.*

Concerning the ΩN structures obtained from an $\boldsymbol{\varepsilon}$-hypersymplectic structure, with $\varepsilon_1\varepsilon_2\varepsilon_3 = 1$, we know, from Theorem 2.4 v), that the pairs (ω_i, N_i) and (g, N_i), $i = 1, 2, 3$, are ΩN structures. Thus, the total number of ΩN structures is twelve and we have:

Proposition 3.7. *For all indices* $i, j, k, l \in \{1, 2, 3\}$,

i) *the* ΩN *structures* (ω_i, N_j) *and* (ω_k, N_l) *are compatible;*
ii) *the* ΩN *structures* (g, N_i) *and* (g, N_j) *are compatible;*
iii) *the* ΩN *structures* (g, N_i) *and* (ω_j, N_k) *are compatible.*

Thus, we get the following:

Theorem 3.8. *Let* $(\omega_1, \omega_2, \omega_3)$ *be an* $\boldsymbol{\varepsilon}$*-hypersymplectic structure on a Lie algebroid* $(A, [\cdot, \cdot], \rho)$, *with* $\varepsilon_1\varepsilon_2\varepsilon_3 = 1$. *Then, the twelve* ΩN *structures on A,* (ω_i, N_j), (g, N_k), $i, j, k \in \{1, 2, 3\}$, *are pairwise compatible.*

Lastly we discussed the $P\Omega$ structures. It is known that, in the case of an $\boldsymbol{\varepsilon}$-hypersymplectic structure with $\varepsilon_1\varepsilon_2\varepsilon_3 = 1$, the pairs (π_i, g) and (g^{-1}, ω_i) are $P\Omega$ structures, for all $i \in \{1, 2, 3\}$ (see Theorem 2.4 vi)). Thus, the total number of *nontrivial* $P\Omega$ structures is twelve. Notice that the pair (g^{-1}, g) is a trivial $P\Omega$ structure.

Proposition 3.9. *For all indices* $i, j, k, l \in \{1, 2, 3\}$,

i) *the* $P\Omega$ *structures* (π_i, g) *and* (π_j, g) *are compatible;*
ii) *the* $P\Omega$ *structures* (π_i, g) *and* (g^{-1}, ω_j) *are compatible;*

iii) *the $P\Omega$ structures (π_i, g) and (π_j, ω_k) are compatible;*
iv) *the $P\Omega$ structures (g^{-1}, ω_i) and (g^{-1}, ω_j) are compatible;*
v) *the $P\Omega$ structures (g^{-1}, ω_i) and (π_j, ω_k) are compatible;*
vi) *the $P\Omega$ structures (π_i, ω_j) and (π_k, ω_l) are compatible;*
vii) *the trivial $P\Omega$ structure (g^{-1}, g) is compatible with all the other $P\Omega$ structures.*

Proof: i) From equation (9), $\pi_i + \pi_j$ is a Poisson bivector on A. We have $\pi_i^{\#} \circ g^{\flat} = \varepsilon_{i-1}\varepsilon_{i+1}N_{i+1} \circ \pi_i^{\#} \circ \omega_{i+1}^{\flat} = \varepsilon_{i-1}\pi_{i-1}^{\#} \circ \omega_{i+1}^{\flat} = \varepsilon_{i-1}N_i$. The 2-forms $g_{\pi_i^{\#}\circ g^{\flat}} = \varepsilon_{i-1}g_{N_i}$ being closed, it is obvious that $g_{(\pi_i^{\#}+\pi_j^{\#})\circ g^{\flat}} = \varepsilon_{i-1}g_{N_i} + \varepsilon_{j-1}g_{N_j}$ is also closed.

ii) From Proposition 2.3, $\pi_i + g^{-1}$ is a Poisson bivector on A. From (11), (7) and (8) we get $(g^{-1})^{\#} \circ \omega_j^{\flat} = \varepsilon_{j-1}N_{j-1} \circ N_{j+1}$, which implies $N_j \circ (g^{-1})^{\#} \circ \omega_j^{\flat} = \varepsilon_{j-1}\mathrm{Id}_A$, because $N_j \circ N_{j-1} \circ N_{j+1} = \mathrm{Id}_A$. But, since $N_j^{-1} = \varepsilon_j N_j$, we get $(g^{-1})^{\#} \circ \omega_j^{\flat} = \varepsilon_{j-1}\varepsilon_j N_j$. The pairs $(g, \varepsilon_{i-1}N_i)$ and $(\omega_j, \varepsilon_j N_j)$ being compatible ΩN structures (see Proposition 3.7), the 2-form $(g + \omega_j)_{\pi_i^{\#}\circ g^{\flat} + (g^{-1})^{\#}\circ \omega_j^{\flat}} = (g + \omega_j)_{\varepsilon_{i-1}N_i + \varepsilon_{j-1}\varepsilon_j N_j}$ is closed and the proof of ii) is complete.

iii) As in i), we have that $\pi_i + \pi_j$ is a Poisson bivector on A. For the rest of the proof, just notice that $\pi_j^{\#} \circ \omega_k$ is equal to Id_A if $j = k$, or is equal to N_{j-1}, if $k = j + 1$, or $\varepsilon_{j+1}N_{j+1}$, if $k = j - 1$. We also have $\pi_i^{\#} \circ g^{\flat} = \varepsilon_{i-1}N_i$. In any case, the 2-form $(\omega_k + g)_{\pi_i^{\#}\circ g^{\flat} + \pi_j^{\#}\circ\omega_k^{\flat}}$ is closed because the pairs (g, N_i) and (ω_j, N_k) are compatible ΩN structures.

The proofs of iv) – vi) are similar. ■

The Proposition 3.9 can be summarized as follows:

Theorem 3.10. *Let $(\omega_1, \omega_2, \omega_3)$ be an $\boldsymbol{\varepsilon}$-hypersymplectic structure on a Lie algebroid $(A, [\cdot,\cdot], \rho)$, with $\varepsilon_1\varepsilon_2\varepsilon_3 = 1$. Then, the sixteen $P\Omega$ structures (π_i, ω_j), (π_k, g), (g^{-1}, ω_l) and (g^{-1}, g), $i, j, k, l \in \{1, 2, 3\}$, on A, are pairwise compatible.*

From Proposition 1.1, we get the following:

Corollary 3.11. *Let $(\omega_1, \omega_2, \omega_3)$ be an $\boldsymbol{\varepsilon}$-hypersymplectic structure on a Lie algebroid $(A, [\cdot,\cdot], \rho)$, with $\varepsilon_1\varepsilon_2\varepsilon_3 = 1$. Then, the complementary forms g, ω_1, ω_2 and ω_3 of π_1, (respectively of π_2, of π_3 and of g^{-1}) are pairwise compatible.*

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