Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 13–06

A NOTE ON WEIGHTED SUMS OF ASSOCIATED RANDOM VARIABLES

TONGUÇ ÇAĞIN AND PAULO EDUARDO OLIVEIRA

Abstract: We prove the convergence of weighted sums of associated random variables normalized by $n^{1/p}$. The usual assumptions include finiteness of moments somewhat larger than p. We assume the existence of moments of order closer to p than previous results, thus getting nearer to the assumption used for the case of constant weights. Besides moment conditions we assume a convenient behaviour either on truncated covariances or on joint tail probabilities. Our results extend analogous characterizations known for sums of independent or negatively dependent random variables.

Keywords: weighted sums, associated random variables, almost sure convergence, Marcinkiewicz-Zygmund strong law of large numbers.
AMS Subject Classification (2010): 60F15.

1. Introduction

Sums of random variables have always attracted a lot of interest as their asymptotic behaviour raises relevant theoretical challenges. Moreover, many statistical procedures depend on such sums. Thus, there is a natural interest in considering $T_n = \sum_{i=1}^n a_{n,i}X_i$, where the variables X_i are centered, both from a theoretical and practical point of view. For constant weights and independent and identically distributed Baum and Katz [3] proved the Marcinkiewicz-Zygmund strong law of large numbers, that is, that $n^{-1/p}T_n \longrightarrow 0$ almost surely, $p \in [1,2)$, if and only if $\mathrm{E}(|X_1|^p) < \infty$. Chow [6] and Cuzick [7] considered variables $\mathrm{E}(|X_1|^\beta) < \infty$ and weights satisfying $\sup n^{-1}\sum_{i=1}^n a_{n,i}^\alpha < \infty$ where $\alpha^{-1} + \beta^{-1} = 1$, to prove the Marcinkiewicz-Zygmund law with $p = 1$. This was extended by Cheng [5] and Bai and Cheng [2] to other values of $p \in (1,2)$. The same problem with negatively dependent random variables was considered by Ko and Kim [8], Baek, Park, Chung and Seo [1], Cai [4], Qiu and Chen [14] or Shen, Wang, Yang

Received March 20, 2013.
This work was partially supported by the Centro de Matemática da Universidade de Coimbra (CMUC), funded by the European Regional Development Fund through the program COMPETE and by the Portuguese Government through the FCT - Fundação para a Ciência e a Tecnologia under the project PEst-C/MAT/UI0324/2011.

and Hu [17]. Positively associated random variables were considered by Louhichi [9] for constant weights and Oliveira [11] for more general weights. In this paper we extend the results in [11], relaxing the moment assumption on the random variables, approaching the p-th order moment assumption used by Louhichi [9] to prove the convergence for constant weights, while strengthening the assumption on the decay rate of the covariances. We also consider the Marcinkiewicz-Zygmund law with assumptions on the 2-dimensional analogue of tail probabilities of the random variables relaxing in this case the assumption on the decay rate on the covariances, but strengthening the moment condition.

2. Framework and preliminaries

Let X_n, $n \geq 1$, be a sequence of random variables and define partial sums $S_n = \sum_{i=1}^n X_i$ and weighted partial sums $T_n = \sum_{i=1}^n a_{n,i} X_i$, where $a_{n,i} \geq 0$, $i \leq n$, $n \geq 1$. The variables X_n, $n \geq 1$, are assumed to be associated, that is, for any $m \geq 1$ and any two real-valued coordinatewise nondecreasing functions f and g,

$$\mathrm{Cov}\Big(f\left(X_1, \ldots, X_m\right), \, g\left(X_1, \ldots, X_m\right)\Big) \geq 0,$$

whenever this covariance exists. It is well known that the covariance structure of associated random variables characterizes their asymptotics, so it is natural to seek assumptions on the covariances.

In this paper we will be interested in the case where second order moments do not exist, so we will avoid using covariances directly, using them only through truncation. For this later argument, define, for each $M > 0$, the nondecreasing function $g_M(u) = \max(\min(u, M), -M)$, which performs the truncation at level M, and introduce, for each $n \geq 1$, the random variables $\bar{X}_n = g_M(X_n)$ and $\tilde{X}_n = X_n - \bar{X}_n$. It is easily checked that both these families of random variables are associated, as they are nondecreasing transformations of the original ones. Define next the weighted sums of the truncated variables: for each $n \geq 1$, $\bar{T}_n = \sum_{i=1}^n a_{n,i}(\bar{X}_i - \mathrm{E}\bar{X}_i)$ and $\tilde{T}_n = \sum_{i=1}^n a_{n,i}(\tilde{X}_i - \mathrm{E}\tilde{X}_i)$, and the maxima $T_n^* = \max_{k \leq n} |T_k|$ and $\bar{T}_n^* = \max_{k \leq n} \left|\bar{T}_k\right|$. To handle covariances define, for each $i, j \geq 1$, $\Delta_{i,j}(x, y) = \mathrm{P}(X_i \geq x, X_j \geq y) - \mathrm{P}(X_i \geq x)\mathrm{P}(X_j \geq y)$. Of course, $\mathrm{Cov}(X_i, X_j) = \int_{\mathbb{R}^2} \Delta_{i,j}(x, y)\, dxdy$. Moreover,

$$G_{i,j}(M) = \mathrm{Cov}(\bar{X}_i, \bar{X}_j) = \int_{[-M,M]^2} \Delta_{i,j}(x, y)\, dxdy. \tag{1}$$

The control of moments of maxima of partial sums is a crucial argument throughout. For nonweighted sums it was proved by Newman and Wright [10] that $\mathrm{E}\left(\max_{k\leq n} S_k^2\right) \leq \mathrm{E}S_n^2$. This maximal inequality is one of the key ingredients used by Louhichi [9] to control tail probabilities of maxima of sums of associated random variables and then prove that $n^{-1/p}S_n \longrightarrow 0$ a.s., where $p \in [1,2)$ when one only has p-th order moments. For weighted sums, the following extension of this maximal inequality was proved by Oliveira [11].

Lemma 2.1. *Let X_n, $n \geq 1$, be centered and associated random variables. Assume the coefficients are such that*

$$a_{n,i} \geq 0, \qquad \text{and} \qquad a_{n,i} \geq a_{n-1,i}, \quad i \leq n, \ n \geq 1. \tag{2}$$

Then $\mathrm{E}\left(\max_{k\leq n} T_k^2\right) \leq \mathrm{E}(T_n^2)$.

We will need some more assumptions on the weights. Define, for each $\alpha > 0$, $A_{n,\alpha} = \frac{1}{n^{1/\alpha}}\left(\sum_{i=1}^n |a_{ni}|^\alpha\right)^{1/\alpha}$. These coefficients are considered in [1, 2, 4, 7, 8, 14, 18], assuming them to be either bounded or convergent.

Finally, we recall the following extension of Lemma 1 in Louhichi [9] proved by Oliveira [11].

Lemma 2.2. *Let X_n, $n \geq 1$, be centered and identically distributed associated random variables and assume the weights satisfy (2). Then, for every $\alpha > 1$, $x \in \mathbb{R}$ and $M > 0$,*

$$\begin{aligned}\mathrm{P}(T_n^* > x) \ \leq\ & \frac{8}{x^2}n^{1+2/\alpha}A_{n,\alpha}^2\mathrm{E}\left(X_1^2\mathbb{I}_{|X_1|\leq M}\right) + \frac{8}{x^2}n^{1+2/\alpha}A_{n,\alpha}^2M^2\mathrm{P}(|X_1| > M) \\ & + \frac{16}{x^2}n^{2/\alpha}A_{n,\alpha}^2\sum_{1\leq i<j\leq n}G_{i,j}(M) + \frac{4}{x}nA_{n,\alpha}\mathrm{E}\left(|X_1|\,\mathbb{I}_{|X_1|>M}\right).\end{aligned} \tag{3}$$

3. Some Marcinkiewicz-Zygmund strong laws

We now prove the almost sure convergence of $n^{-1/p}T_n$ based on the Borel-Cantelli Lemma. Instead of considering T_n directly, we replace it by the larger T_n^*, which is an increasing sequence. For this increasing sequence T_n^*, the use of the Borel-Cantelli Lemma may be reduced to proving $\sum_n n^{-1}\mathrm{P}(T_n^* > \varepsilon n^{1/p}) < \infty$.

Theorem 3.1. *Let X_n, $n \geq 1$, be centered and identically distributed associated random variables. Assume the weights satisfy (2) and* $\sup_{n\geq 1} A_{n,\alpha} < \infty$.

Further, assume that $p \in (1,2)$, $\alpha > \frac{2p}{2-p}$ *and* $\mathrm{E}\left(|X_1|^{p\frac{\alpha-2}{\alpha-2p}}\right) < \infty$. *If*

$$\sum_{1 \le i < j < \infty} \int_{j^{(\alpha-2p)/(\alpha p)}}^{\infty} v^{-2\frac{\alpha-p}{\alpha-2p}-1} G_{i,j}(v)\, dv < \infty, \tag{4}$$

then $n^{-1/p} T_n \longrightarrow 0$ *almost surely.*

Proof: The proof follows similar arguments as in Theorem 4.1 in Oliveira [11]. Taking into account (3), with $M = n^{1/q}$, where q is to be specified later, we find that

$$\begin{aligned}
\frac{1}{n}\mathrm{P}(T_n^* > \varepsilon n^{1/p}) \;\le\; & \frac{8n^{2/\alpha-2/p}}{\varepsilon^2} A_{n,\alpha}^2 \mathrm{E}\left(X_1^2 \mathbb{I}_{|X_1| \le n^{1/q}}\right) \\
& + \frac{8n^{2/\alpha-2/p+2/q}}{\varepsilon^2} A_{n,\alpha}^2 \mathrm{P}(|X_1| > n^{1/q}) \\
& + \frac{16n^{2/\alpha-2/p-1}}{\varepsilon^2} A_{n,\alpha}^2 \sum_{1 \le i < j \le n} G_{i,j}(n^{1/q}) \\
& + \tfrac{4n^{-1/p}}{\varepsilon} A_{n,\alpha} \mathrm{E}\left(|X_1|\, \mathbb{I}_{|X_1| > n^{1/q}}\right).
\end{aligned}$$

The remaining argument is to prove that this upper bound defines a convergent series. Taking into account that $A_{n,\alpha}$ is bounded, we may drop these terms. Notice that $\alpha > \frac{2p}{2-p}$ is equivalent to $\frac{2}{\alpha} - \frac{2}{p} < -1$. Using Fubini's Theorem we easily find that:

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{2/\alpha-2/p} \mathrm{E}\left(X_1^2 \mathbb{I}_{|X_1|^p \le n}\right) = \mathrm{E}\left(X^2 \sum_{n=|X|^q}^{\infty} n^{2/\alpha-2/p}\right) \le c_1 \mathrm{E}\left(|X_1|^{q(1+2/\alpha-2/p)+2}\right), \\
&\sum_{n=1}^{\infty} n^{2/\alpha-2/p+2/q} \mathrm{E}\left(\mathbb{I}_{|X_1|^q > n}\right) = \mathrm{E}\left(\sum_{n=1}^{|X|^q} n^{2/\alpha-2/p+2/q}\right) \le c_2 \mathrm{E}\left(|X_1|^{q(1+2/\alpha-2/p)+2}\right), \\
&\sum_{n=1}^{\infty} n^{-1/p} \mathrm{E}\left(|X_1|\, \mathbb{I}_{|X_1|^q > n}\right) \le c_3 \mathrm{E}\left(|X_1|^{q(1-1/p)+1}\right).
\end{aligned} \tag{5}$$

The constants c_1, c_2 and c_3 used above only depend on p, q and α. As $1+\frac{2}{\alpha}-\frac{2}{p}<0$, in order to consider the lowest moment assumption possible on the variables, the first two terms above imply that we want to choose q as large as possible. On the other hand, as $1-\frac{1}{p}>0$, the last term implies that we should choose q as small as possible. It is clear that for small values of q we have $q\left(1-\frac{1}{p}\right)+1<q\left(1+\frac{2}{\alpha}-\frac{2}{p}\right)+2$, so we choose q such that these two expressions coincide, that is, $q=\frac{\alpha p}{\alpha-2p}$. Notice that $\alpha>\frac{2p}{2-p}$, with $p\in(1,2)$, implies that $\alpha>2p$, so the above choice for q is positive. It is now straightforward to verify that the moments considered above are of order $p\frac{\alpha-2}{\alpha-2p}$, thus finite.

Finally we control the term depending on the covariances. Again, using Fubini's Theorem we may write

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{2/\alpha-2/p-1} \sum_{1\leq i<j\leq n} G_{i,j}(n^{1/q}) \\
&\quad= \sum_{1\leq i<j<\infty} \iint \sum_{n>j} n^{2/\alpha-2/p-1}\, \mathbb{I}_{n>\max(|x|^q,|y|^q,j)}\, \Delta_{i,j}(x,y)\,dxdy \\
&\quad\leq c_4 \sum_{1\leq i<j<\infty} \iint \Big(\max(|x|^q,|y|^q,j)\Big)^{2/\alpha-2/p} \Delta_{i,j}(x,y)\,dxdy \\
&\quad= c_4 \sum_{1\leq i<j<\infty} \iint \int_0^{j^{2/\alpha-2/p}} \mathbb{I}_{|x|\leq u^{-\frac{\alpha p}{2q(\alpha-p)}}}\, \mathbb{I}_{|y|\leq u^{-\frac{\alpha p}{2q(\alpha-p)}}}\, du\, \Delta_{i,j}(x,y)\,dxdy \\
&\quad= \frac{2q(\alpha-p)c_4}{\alpha p} \sum_{1\leq i<j<\infty} \int_{j^{1/q}}^{\infty} v^{-2q\frac{\alpha-p}{\alpha p}-1}\, G_{i,j}(v)\,dv<\infty, \qquad (6)
\end{aligned}$$

taking into account (4), where c_4 depends only on p and α, so the proof is concluded. ■

Remark 3.2. *Notice that $\alpha>\frac{2p}{2-p}$, as assumed in Theorem 3.1, implies that $p\frac{\alpha-2}{\alpha-2p}<2$, thus we are still not assuming second order moments.*

Remark 3.3. *In Theorem 4.1 in Oliveira* [11] *the moment considered was $p\frac{\alpha+2}{\alpha}$. It is easily seen that $\alpha>\frac{2p}{2-p}$ implies that $p\frac{\alpha+2}{\alpha}>p\frac{\alpha-2}{\alpha-2p}$, thus we are improving somewhat the moment assumption. As what regards the integrability assumption (4), in* [11] *the exponent of the polynom in the integrand was $-3+2\frac{p}{\alpha}>-2\frac{\alpha-p}{\alpha-2p}-1$, thus the present integrability is a little stronger. The* *difference between these exponents is equal to* $4p\frac{p-\alpha}{\alpha(\alpha-2p)}$*. So, when* $\alpha \longrightarrow +\infty$ *this difference converges to 0.*

Remark 3.4. *To compare this result with Louhichi's* [9] *conditions for non-weighted sums, notice that allowing* $\alpha \longrightarrow \infty$ *in the assumptions of Theorem 3.1 we are lead to assume the existence of* p*-th order moments and the exponent in the integrability condition converges to* -3*, that is, we find the assumptions of Theorem 1 in* [9].

The above statement assumes a moment condition and adjusts the integrability condition on the truncated covariances to get the convergence. One may be interested in doing the opposite, that is, assume an integrability condition on the truncated variables and describe which moments should be required. Assume that for some $\beta > 0$ and a suitable $q > 0$ we have

$$\sum_{1\leq i<j<\infty} \int_{j^{1/q}}^{\infty} v^{-\beta}\, G_{i,j}(v)\, dv < \infty. \tag{7}$$

We now choose q conveniently. Comparing with (6) we need that $2q\frac{\alpha-p}{\alpha p}+1 \geq \beta$ or, equivalently, $q \geq \frac{p\alpha(\beta-1)}{2(\alpha-p)}$. Assume that $\alpha > \frac{2p}{2-p}$, which is equivalent to $\frac{2}{\alpha}-\frac{2}{p} < -1$ and implies that $\alpha > 2p$. So, if $\beta \in [0,1]$ the above condition is verified for every choice of $q > 0$, thus, as seen in the proof of Theorem 3.1, the choice $q^* = \frac{\alpha p}{\alpha-2p}$ optimizes the moment assumption, requiring the existence of the absolute moment of order $p^* = p\frac{\alpha-2}{\alpha-2p}$. Because of the integration region in (7) we need to assume that $q \geq q^*$. If $\beta > 1$, we look at $\frac{\alpha p}{\alpha-2p} - \frac{p\alpha(\beta-1)}{2(\alpha-p)}$. As we assumed that $\alpha > 2p$ it is easily seen that the sign of this difference is equal to the sign of $(3-\beta)\alpha - 2p(2-\beta)$. If $\beta \in (1,2]$ this means that the sign is positive if $\alpha > 2p\frac{2-\beta}{3-\beta} = 2p\left(1-\frac{1}{3-\beta}\right)$ which always holds. Thus the optimization of the moments is achieved by the choice $q^* = \frac{\alpha p}{\alpha-2p}$. If $\beta \in (2,3]$ the above difference is always nonnegative, so we choose again $q^* = \frac{\alpha p}{\alpha-2p}$. Now, if $\beta > 3$, $\frac{\alpha p}{\alpha-2p} - \frac{p\alpha(\beta-1)}{2(\alpha-p)} \geq 0$ is equivalent to $\alpha \leq 2p\frac{\beta-2}{\beta-3} = 2p\left(1+\frac{1}{\beta-3}\right)$. So, when $\beta > 3$, if $2p < \alpha \leq 2p\left(1+\frac{1}{\beta-3}\right)$ we should also choose $q^* = \frac{\alpha p}{\alpha-2p}$. Finally, when $\beta > 3$ and $\alpha > 2p\left(1+\frac{1}{\beta-3}\right)$ we must assume the finiteness of the largest of the moments appearing in (5), where q^* is taken to be $\frac{p\alpha(\beta-1)}{2(\alpha-p)}$. Thus we have proved the following statement.

Theorem 3.5. *Let X_n, $n \geq 1$, be centered and identically distributed associated random variables. Assume the weights satisfy (2) and* $\sup_{n\geq 1} A_{n,\alpha} < \infty$. *Further, assume that $p \in (1,2)$ and $\alpha > \frac{2p}{2-p}$ are satisfied. Define q^* and p^* as*

- *if $\beta \leq 3$ or if $\beta > 3$ and $\alpha \in \left(2p,\, 2p\left(1+\frac{1}{\beta-3}\right)\right]$, $q^* = \frac{\alpha p}{\alpha-2p}$ and $p^* = p\frac{\alpha-2}{\alpha-2p}$,*
- *if $\beta > 3$ and $\alpha > 2p\left(1+\frac{1}{\beta-3}\right)$, $q^* = \frac{p\alpha(\beta-1)}{2(\alpha-p)}$ and $p^* = 1 + \frac{\alpha(\beta-1)(p-1)}{2(\alpha-p)}$.*

If (7) is satisfied with $q \geq q^*$ and $\mathrm{E}\left|X_1\right|^{p^*} < \infty$ then $n^{-1/p}T_n \longrightarrow 0$ almost surely.

We will now look for assumptions on the functions $\Delta_{i,j}$ rather than on the truncated covariances. Remark that the $\Delta_{i,j}$ may also be interpreted as covariances: $\Delta_{i,j} = \mathrm{Cov}\left(\mathbb{I}_{[x,+\infty)}(X_i), \mathbb{I}_{[x,+\infty)}(X_j)\right)$. It follows from Sadikova [15] that, if the random variables have bounded density and covariances do exist that $\Delta_{i,j}(x,y) \leq c\mathrm{Cov}^{1/3}(X_i,X_j)$, where $c > 0$ is a constant depending only on the density function. This made natural to seek for assumptions on the $\Delta_{i,j}$ while studying the asymptotics of empirical processes based on associated random variables, as in Yu [19], Shao and Yu [16] or Oliveira and Suquet [12, 13]. Moreover, the $\Delta_{i,j}(x,y)$ play, in dimension two, the role of the tail probabilities usually considered in the one dimensional framework. So, we will now consider the following assumption on the limit behaviour of $\Delta_{i,j}$:

$$\sup_{i,j\geq 1} \Delta_{i,j}(x,y) = O\left(\max(|x|,|y|)^{-a}\right), \qquad \text{as } \max(|x|,|y|) \longrightarrow +\infty. \tag{8}$$

Thus, outside of some $[-j_0, j_0]^2$ we may assume that all the $\Delta_{i,j}$ are, up to the multiplication by some constant c_0, that does not depend on i or j, bounded above by $\max(|x|,|y|)^{-a}$. Thus

$$G_{i,j}(v) \leq 4j_0^2 + 4c_0 \int_{j^{1/q}}^{\infty}\int_{-x}^{x} x^{-a}\,dx = 4j_0^2 + \frac{4c_0}{2-a}\left(v^{2-a} - j_0^{2-a}\right). \tag{9}$$

Remember that $\mathrm{Cov}(X_i,X_j) = G_{i,j}(+\infty)$. Looking at the expression above, if we allow $v \longrightarrow +\infty$ we have convergence to a finite limit whenever $a \geq 2$. Thus, the most interesting case for us corresponds to $0 < a < 2$, so that we do not have finite covariances between the random variables.

Theorem 3.6. *Let X_n, $n \geq 1$, be centered and identically distributed associated random variables. Assume the weights satisfy (2) and* $\sup_{n\geq 1} A_{n,\alpha} < \infty$. *Let $p \in (1,2)$ and $\alpha > \frac{2p}{2-p}$. Assume that (8) is satisfied for some $a \in (0,2)$ and (7) holds for some $q > 0$ and $\beta > 3-a+2q$. If $\mathrm{E}\,|X_1|^{p^*} < \infty$, where $p^* = \max\left(q\left(1+\frac{2}{\alpha}-\frac{2}{p}\right)+2, q\left(1-\frac{1}{p}\right)+1\right)$ then $n^{-1/p}T_n \longrightarrow 0$ almost surely.*

Proof: Using (9) to compute the integral in (7), one easily finds that, as $\beta > 3-a+2q > 3-a$,

$$\int_{j^{1/q}}^{\infty} v^{-\beta} G_{i,j}(v)\,dv \leq c_0' j^{(1-\beta)/q} + j^{3-(\beta+a)}q,$$

where c_0' does not depend on i or j. Thus inserting this upper bound in (7) and taking into account the summation, we have a convergent series if both $1+\frac{1-\beta}{q} < -1$ and $1+\frac{3-(\beta+a)}{q} < -1$. But these two inequalities follow from $\beta > 3-a+2q$. As the summations in (5) are finite due to our moment assumptions, the proof is concluded. ∎

Remark 3.7. *The above statement allows to consider $\beta < 3$ in (7). This was out of reach in Theorem 3.1. However, the moment assumed to be finite is of order $p^* = \max\left(q\left(1+\frac{2}{\alpha}-\frac{2}{p}\right)+2, q\left(1-\frac{1}{p}\right)+1\right)$. It is easily seen that if $q > \frac{\alpha p}{\alpha-2p}$, then $p^* = q\left(1-\frac{1}{p}\right)+1$. The difference between this order and the one considered in Theorem 3.1 has the same sign as $(\alpha-2p)q-p\alpha \geq 0$ for the range of values for q where this applies. Likewise, if $q < \frac{\alpha p}{\alpha-2p}$ then the difference of order moments has the same sign as $(p-2)\alpha^2+2p\alpha(3-p)-2p^2 > 0$ for the range of values for q considered. Thus, the moment condition assumed in Theorem 3.6 is always stronger than the one in Theorem 3.1.*

References

[1] Baek, J., Park, S., Chung S., Seo, H., On the almost sure convergence of weighted sums of negatively associated random variables, Commun. Korean Math. Soc. 20 (2005), 539–546.

[2] Bai, Z., Cheng, P., Marcinkiewicz strong laws for linear statistics, Statist. Probab. Lett. 46 (2000), 105–112.

[3] Baum, L., Katz, M., Convergence rates in the law of large numbers, Trans. Amer. Math. Soc. 120 (1965), 108–123.

[4] Cai, G., Strong laws for weighted sums of NA random variables, Metrika 68 (2008), 323-331.

[5] Cheng, P., A note on strong convergence rates in nonparametric regression, Statist. Probab. Lett. 24 (1995), 357–364.

[6] Chow, Y., Some convergence theorems for independent random variables, Ann. Math. Statist. 37 (1966), 1482–1493.
[7] Cuzick, J., A Strong of Large Numbers for Weighted Sums of I.I.D. Random Variables, J. Theoret. Probab. 8 (1995), 625–641.
[8] Ko, M., Kim, T., Almost sure convergence for weighted sums of negatively orthant dependent random variables, J. Korean Math. Soc. 42 (2005), 949–957.
[9] Louhichi, S., Convergence rates in the strong law for associated random variables, Probab. Math. Statist. 20 (2000), 203-214.
[10] Newman, C., Wright, A., An invariance principle for certain dependent sequences, Ann. Probab. 9 (1981), 671–675.
[11] Oliveira, P.E., Weighted sums of associated variables, J. Kor. Statist. Soc. 41 (2012), 537-542.
[12] Oliveira, P., Suquet, C., $L^2[0,1]$ weak convergence of the empirical process for dependent variables, In: A. Antoniadis, G. Oppenheim (eds.) Actes des XVèmes Rencontres Franco-Belges de Statisticiens (Ondelettes et Statistique), *Lecture Notes in Statistics* 103 (1995), 331–344, Springer.
[13] Oliveira, P., Suquet, C., Weak convergence in $L^p(0,1)$ of the uniform empirical process under dependence, Statist. Probab. Letters 39 (1998), 363–370.
[14] Qui, D., Chen, P., Limiting Behavior of Weighted Sums of NOD Random Variables, J. Math. Res. Exposition 31 (2011), 1081-1091.
[15] Sadikova, S., Two dimensional analogues of an inequality of esséen with applications to the central limit theorem, Theory Probab. Appl. 11 (1966), 325–335.
[16] Shao, Q., Yu, H., Weak convergence for weighted empirical processes of dependent sequences, Ann. Probab. 24 (1996), 2098–2127.
[17] Shen, Y., Wang, X., Yang, W., Hu, S., Almost sure convergence theorem and strong stability for weighted sums of NSD random variables, Acta Mathematica Sinica, English Series, http://dx.doi.org/10.1007/s10114-012-1723-6.
[18] Sung, S., Strong laws for weighted sums of i.i.d. random variables, Statist. Probab. Lett. 52 (2001), 413–419.
[19] Yu, H., A Glivenko-Cantelli Lemma and weak convergence for empirical processes of associated sequences, Probab. Theory Relat. Fields 95 (1993), 357–370.

Tonguç Çağin
CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal
E-mail address: tonguc@mat.uc.pt

Paulo Eduardo Oliveira
CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal
E-mail address: paulo@mat.uc.pt