Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 13–14

HAGEMANN'S THEOREM FOR REGULAR CATEGORIES

ZURAB JANELIDZE, DIANA RODELO AND TIM VAN DER LINDEN

Dedicated to Hvedri Inassaridze on the occasion of his eightieth birthday

Abstract: In this paper, we extend the characterisation of n-permutable varieties of universal algebras due to J. Hagemann to regular categories. In particular, we show that a regular category has n-permutable congruences if and only if every internal reflexive relation R in it satisfies $R^\circ \leqslant R^{n-1}$, and if and only if every internal reflexive relation R in it satisfies $R^n \leqslant R^{n-1}$. In the case when $n = 2$ this result is well known.

1. Introduction

The notion of congruence n-permutability for varieties of universal algebras, which is well known in universal algebra, was extended to regular categories in [3]. More precisely, a regular category (in the sense of [1]) is said to have n-permutable congruences when for any two (internal) congruences (i.e. effective equivalence relations) C_1 and C_2 on an object W in the category, we have

$$(C_1, C_2)_n = (C_2, C_1)_n,$$

where, following the notation used in [3], for any two relations $S\colon X \to Y$ and $R\colon Y \to X$,

$$(S, R)_n = \underbrace{SRSRS\dots}_{n} = \begin{cases} (SR)^{\frac{n}{2}}, & \text{when } n \text{ is even,} \\ (SR)^{\frac{n-1}{2}}S, & \text{when } n \text{ is odd.} \end{cases}$$

In this paper we prove the following result, which is well known in the case when $n = 2$ (see [6, 4]):

Received May 6, 2013.

The first author's research was partially supported by South African National Research Foundation.

The second author's research was supported by CMUC (funded by the European Regional Development Fund through the program COMPETE and by the Portuguese Government through the FCT-Fundação para a Ciência e a Tecnologia under the project PEst-C/MAT/UI0324/2011) and the FCT Grant PTDC/MAT/120222/2010.

The third author works as *chargé de recherches* for Fonds de la Recherche Scientifique–FNRS.

The second and third author wish to thank the University of Cape Town and Stellenbosch University for their kind hospitality during their stay in South Africa.

Theorem 1.1. *For a regular category $\mathbb{C}$, and a natural number $n \geqslant 1$, the following conditions are equivalent:*

(i) *$\mathbb{C}$ has n-permutable congruences.*
(ii) *$R^\circ \leqslant R^{n-1}$ for any internal reflexive relation R in $\mathbb{C}$.*
(iii) *$R^n \leqslant R^{n-1}$ for any internal reflexive relation R in $\mathbb{C}$.*

This result was announced for varieties in [7], which refers to a work of J. Hagemann in preparation. As the third author learned from L. Márki, this work was never completed to the point of publication. However, as noted in [10], it is not difficult to prove this result for varieties via standard arguments involving free algebras. For $n = 2$ it gives a result due to H. Werner [13], which, by the way, could have inspired the discovery of the result for general n (see the Remark in [7]). The authors of [3] seem to have missed the above result, obtaining instead a different generalization of the same result for $n = 2$:

Theorem 1.2 ((ii)⇔(v)⇔(vi) of Theorem 3.5 in [3])**.** *For a regular category $\mathbb{C}$, and a natural number $n \geqslant 2$, the following conditions are equivalent:*

(i) *$\mathbb{C}$ is n-permutable.*
(ii) *$(R, R^\circ)_{n-1} = (R^\circ, R)_{n-1}$ for any internal reflexive relation R in $\mathbb{C}$.*
(iii) *$(R, R^\circ)_{n-1}$ is transitive for any internal reflexive relation R in $\mathbb{C}$.*

2. Proof of Theorem 1.1

For a background on regular categories we refer the reader to the first few pages of [3], which gives a sufficiently detailed overview of regular categories from the point of view of their use for universal algebraic considerations relevant to the present paper. Apart from the definition and basic properties of a regular category, we use the calculus of relations and the language of generalised elements for regular categories, as described in [3].

Note that the definition of $(S, R)_n$ recalled in the Introduction makes sense for all $n \geqslant 0$. Indeed, for $n = 0$ we have

$$(S, R)_n = (SR)^{\frac{0}{2}} = 1_Y$$

where 1_Y denotes the identity morphism $1_Y \colon Y \to Y$ regarded as a relation (the internal equality relation at Y) and for $n = 1$ we have

$$(S, R)_n = (SR)^0 S = S.$$

However, the property of n-permutability of congruences is "interesting" only for $n \geqslant 2$. Recall that 2-permutable categories are also called Mal'tsev categories [4] while 3-permutable categories are known as Goursat categories [4, 3].

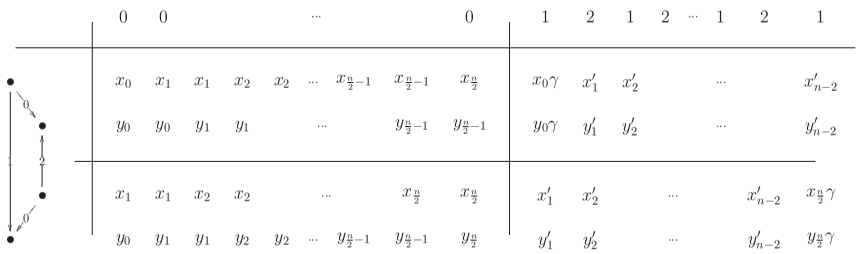


FIGURE 2.5. Matrix sketch of the proof of 2.4(iv)⇒2.4(i) for an even $n \geqslant 2$.

Lemma 2.1. *In a regular category $\mathbb{C}$, consider a span*

$$X \xleftarrow{f_1} W \xrightarrow{f_2} Y$$

and the following relations associated to it:

- *the relation $S = f_2 f_1^\circ$ defined as the composite of the opposite relation of the morphism f_1 regarded as a relation, and the morphism f_2 regarded as a relation;*
- *the kernel congruences $C_1 = f_1^\circ f_1$ and $C_2 = f_2^\circ f_2$, of f_1 and f_2 respectively.*

Then for all $n \geqslant 0$ we have:

$$\begin{aligned}(C_2, C_1)_{n+1} &= f_2^\circ (S, S^\circ)_n f_{\frac{3}{2}+\frac{(-1)^n}{2}} \\ f_2(C_2, C_1)_{n+2} f^\circ_{\frac{3}{2}-\frac{(-1)^n}{2}} &= f_2(C_1, C_2)_n f^\circ_{\frac{3}{2}-\frac{(-1)^n}{2}} \\ f_2(C_1, C_2)_n f^\circ_{\frac{3}{2}-\frac{(-1)^n}{2}} &= (S, S^\circ)_{n+1}\end{aligned}$$

Proof: Suppose first that n is even. Then:

$$\begin{aligned}(C_2, C_1)_{n+1} &= (C_2 C_1)^{\frac{n}{2}} C_2 = (f_2^\circ f_2 f_1^\circ f_1)^{\frac{n}{2}} f_2^\circ f_2 \\ &= f_2^\circ (f_2 f_1^\circ f_1 f_2^\circ)^{\frac{n}{2}} f_2 = f_2^\circ (S S^\circ)^{\frac{n}{2}} f_2 = f_2^\circ (S, S^\circ)_n f_2\end{aligned}$$

$$\begin{aligned}f_2(C_2, C_1)_{n+2} f_1^\circ &= f_2 (C_2 C_1)^{\frac{n+2}{2}} f_1^\circ = f_2 (f_2^\circ f_2 f_1^\circ f_1)^{\frac{n+2}{2}} f_1^\circ \\ &= f_2 f_2^\circ f_2 (f_1^\circ f_1 f_2^\circ f_2)^{\frac{n}{2}} f_1^\circ f_1 f_1^\circ = f_2 (f_1^\circ f_1 f_2^\circ f_2)^{\frac{n}{2}} f_1^\circ \\ &= f_2 (C_1, C_2)_n f_1^\circ\end{aligned}$$

$$\begin{aligned}f_2(C_1, C_2)_n f_1^\circ &= f_2 (f_1^\circ f_1 f_2^\circ f_2)^{\frac{n}{2}} f_1^\circ = (f_2 f_1^\circ f_1 f_2^\circ)^{\frac{n}{2}} f_2 f_1^\circ \\ &= (f_2 f_1^\circ, f_1 f_2^\circ)_{n+1} = (S, S^\circ)_{n+1}\end{aligned}$$

If n is odd, then:

$$\begin{aligned}(C_2,C_1)_{n+1} &= (C_2C_1)^{\frac{n+1}{2}} = (f_2^\circ f_2 f_1^\circ f_1)^{\frac{n-1}{2}} f_2^\circ f_2 f_1^\circ f_1\\ &= f_2^\circ (f_2 f_1^\circ f_1 f_2^\circ)^{\frac{n-1}{2}} f_2 f_1^\circ f_1\\ &= f_2^\circ (SS^\circ)^{\frac{n-1}{2}} S f_1 = f_2^\circ (S,S^\circ)_n f_1\end{aligned}$$

$$\begin{aligned}f_2(C_2,C_1)_{n+2}f_2^\circ &= f_2(C_2C_1)^{\frac{n+1}{2}} C_2 f_2^\circ = f_2(f_2^\circ f_2 f_1^\circ f_1)^{\frac{n+1}{2}} f_2^\circ f_2 f_2^\circ\\ &= f_2(f_2^\circ f_2 f_1^\circ f_1)^{\frac{n+1}{2}} f_2^\circ = f_2(f_2^\circ f_2 f_1^\circ f_1)^{\frac{n-1}{2}} f_2^\circ f_2 f_1^\circ f_1 f_2^\circ\\ &= f_2 f_2^\circ f_2 (f_1^\circ f_1 f_2^\circ f_2)^{\frac{n-1}{2}} f_1^\circ f_1 f_2^\circ\\ &= f_2(f_1^\circ f_1 f_2^\circ f_2)^{\frac{n-1}{2}} f_1^\circ f_1 f_2^\circ = f_2(C_1,C_2)_n f_2^\circ\end{aligned}$$

$$\begin{aligned}f_2(C_1,C_2)_n f_2^\circ &= f_2(f_1^\circ f_1 f_2^\circ f_2)^{\frac{n-1}{2}} f_1^\circ f_1 f_2^\circ = (f_2 f_1^\circ f_1 f_2^\circ)^{\frac{n-1}{2}} f_2 f_1^\circ f_1 f_2^\circ\\ &= (f_2 f_1^\circ, f_1 f_2^\circ)_{n+1} = (S,S^\circ)_{n+1}\end{aligned}$$

■

Proposition 2.2. *In the setting of Lemma 2.1, for any natural number $n \geqslant 2$ we always have $(S,S^\circ)_{n-1} \leqslant (S,S^\circ)_{n+1}$, whereas for each $n \geqslant 1$, the inclusion $(S,S^\circ)_{n+1} \leqslant (S,S^\circ)_{n-1}$ holds if and only if $(C_1,C_2)_n \leqslant (C_2,C_1)_n$.*

Proof: The fact that the inclusion $(S,S^\circ)_{n-1} \leqslant (S,S^\circ)_{n+1}$ holds for all $n \geqslant 2$ can be deduced from the fact that it holds for $n = 2$, in which case it becomes

$$S \leqslant SS^\circ S.$$

Next we show

$$(S,S^\circ)_{n+1} \leqslant (S,S^\circ)_{n-1} \quad \Leftrightarrow \quad (C_1,C_2)_n \leqslant (C_2,C_1)_n$$

Suppose first $(S,S^\circ)_{n+1} \leqslant (S,S^\circ)_{n-1}$. Notice that $(C_1,C_2)_n \leqslant (C_2,C_1)_{n+2}$. Then, applying Lemma 2.1 twice we get:

$$\begin{aligned}(C_1,C_2)_n &\leqslant (C_2,C_1)_{n+2} = f_2^\circ (S,S^\circ)_{n+1} f_{\frac{3}{2}+\frac{(-1)^{n+1}}{2}}\\ &\leqslant f_2^\circ (S,S^\circ)_{n-1} f_{\frac{3}{2}+\frac{(-1)^{n-1}}{2}} = (C_2,C_1)_n\end{aligned}$$

Conversely, assuming $(C_1,C_2)_n \leqslant (C_2,C_1)_n$ we get from Lemma 2.1 that for each $n \geqslant 2$,

$$\begin{aligned}(S,S^\circ)_{n+1} &= f_2(C_1,C_2)_n f^\circ_{\frac{3}{2}-\frac{(-1)^n}{2}} \leqslant f_2(C_2,C_1)_n f^\circ_{\frac{3}{2}-\frac{(-1)^n}{2}}\\ &= f_2(C_1,C_2)_{n-2} f^\circ_{\frac{3}{2}-\frac{(-1)^n}{2}} = (S,S^\circ)_{n-1}\end{aligned}$$

For $n = 1$ it is easy to see directly that we still get $(S,S^\circ)_{n+1} \leqslant (S,S^\circ)_{n-1}$. ■

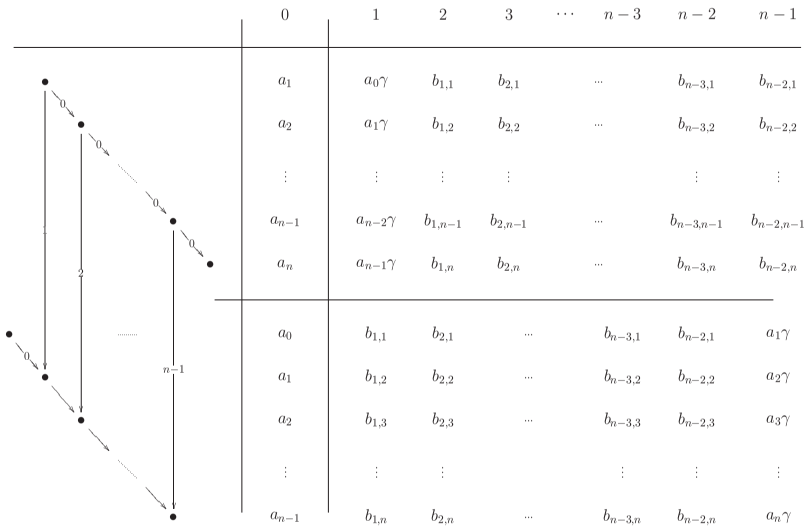


FIGURE 2.3. Matrix sketch of the proof of 2.4(iii)⇒2.4(iv) for $n \geqslant 2$.

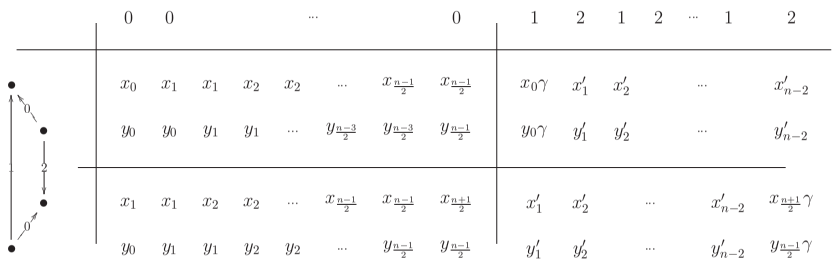


FIGURE 2.4. Matrix sketch of the proof of 2.4(iv)⇒2.4(i) for an odd $n \geqslant 3$.

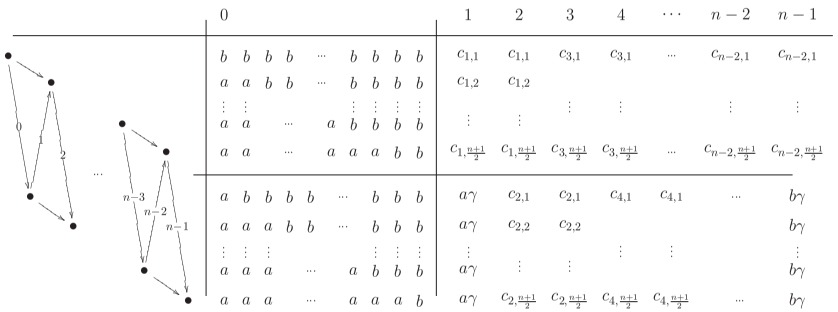


FIGURE 2.1. Matrix sketch of the proof of 2.4(ii)⇒2.4(iii) for an odd $n \geqslant 3$.

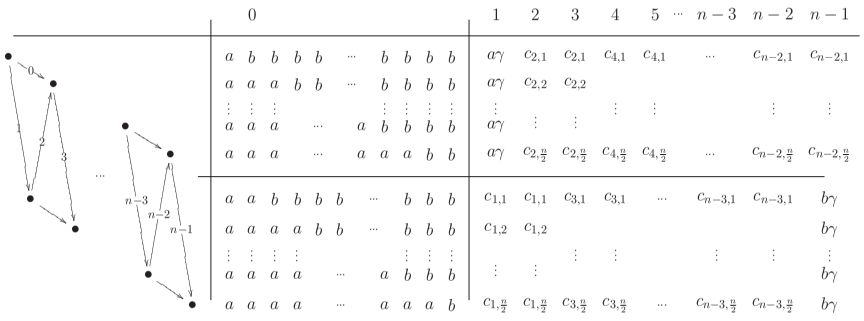


FIGURE 2.2. Matrix sketch of the proof of 2.4(ii)⇒2.4(iii) for an even $n \geqslant 2$.

The proposition above immediately gives the following (which in the case when $n \geqslant 2$ becomes (ii)⇔(iii) of Theorem 3.5 in [3]):

Corollary 2.3. *For any natural number $n \geqslant 1$, a regular category has n-permutable congruences if and only if $(S, S^\circ)_{n+1} \leqslant (S, S^\circ)_{n-1}$ for any internal relation S in it.*

This and the following proposition together imply Theorem 1.1.

Proposition 2.4. *In a regular category, the following conditions are equivalent for any natural number $n \geqslant 1$:*

(i) $(S, S^\circ)_{n+1} \leqslant (S, S^\circ)_{n-1}$ *for any internal relation S.*
(ii) $(S, S^\circ)_{n+1} \leqslant (S, S^\circ)_{n-1}$ *for any internal endorelation S.*
(iii) $R^\circ \leqslant R^{n-1}$ *for any internal reflexive relation R.*
(iv) $R^n \leqslant R^{n-1}$ *for any internal reflexive relation R.*

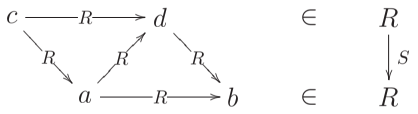
Proof: Let us immediately note that the implication (i)⇒(ii) is trivial. We will show (ii)⇒(iii), (iii)⇒(iv), and (iv)⇒(i).

The case when $n = 1$ is quite straightforward. Indeed, in this case (i) states that for any relation S from an object X to an object Y the composite SS° is a subrelation of the equality relation on Y, (ii) states the same for $X = Y$, (iii) states that the dual of any reflexive relation on an object A coincides with the equality relation on A, while (iv) states that any reflexive relation on an object A coincides with the equality relation on A. Thus, (iii) and (iv) are trivially equivalent. To get (ii)⇒(iv), let R be a reflexive relation on A and use (ii) for $S = R$ after noting that since R is reflexive, we have $R \leqslant RR^\circ$. Finally, for (iv)⇒(i) consider an internal relation $(s_1, s_2) \colon S \to X \times Y$ and note that (iv) will imply that the kernel congruence of s_1 is the equality relation on S. Hence we get the following, where the first equality comes from Lemma 2.1:

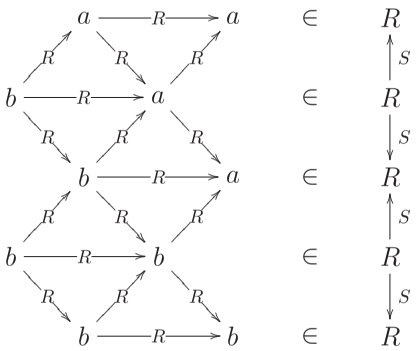
$$(S, S^\circ)_2 = s_2 1_S s_2^\circ \leqslant s_2 s_2^\circ \leqslant 1_S = (S, S^\circ)_0.$$

In the case when $n = 2$, (i) states that any internal relation is difunctional, (ii) states that any internal endorelation is difunctional, (iii) states that any internal reflexive relation is symmetric, while (iv) states that any internal reflexive relation is transitive. The equivalence of these conditions for any finitely complete category was first obtained by A. Carboni, M. C. Pedicchio and N. Pirovano in [5]. Our argument for $n \geqslant 2$ can be seen as a generalisation of their argument, despite of the fact that the context of finitely complete categories is more general than the context of regular categories (see Remark 3.2). To see how to get the generalisation, it will be useful to treat the case $n = 3$ separately.

(ii)⇒(iii) for $n = 3$: Given an internal reflexive relation $R \to A \times A$, we construct a relation $S \to R \times R$ by taking S to be the subobject of $R \times R$ whose generalised elements are given by the displays

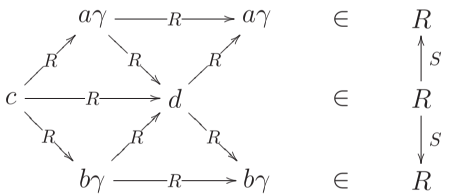


To have $R^\circ \leqslant R^2$ we must show that for any two generalised elements $a\colon V \to A$ and $b\colon V \to A$ of A such that bRa, there exists a regular epimorphism $\gamma\colon U \to V$ and a generalised element $d\colon U \to A$ such that $a\gamma RdRb\gamma$. Now, using the reflexivity of R we have



(1)

The inequality $SS^\circ SS^\circ \leqslant SS^\circ$ applied to the above situation gives that there exists a regular epimorphism $\gamma\colon U \to V$ and there exist generalised elements c, $d\colon U \to A$ with



(2)

from which we get $a\gamma RdRb\gamma$.

[4] A. Carboni, J. Lambek, and M. C. Pedicchio, *Diagram chasing in Mal'cev categories*, J. Pure Appl. Algebra **69** (1991), 271–284.
[5] A. Carboni, M. C. Pedicchio, and N. Pirovano, *Internal graphs and internal groupoids in Mal'cev categories*, Proceedings of Conf. Category Theory 1991, Montreal, Am. Math. Soc. for the Canad. Math. Soc., Providence, 1992, pp. 97–109.
[6] T. Fay, *On categorical conditions for congruences to commute*, Algebra Universalis **8** (1978), 173–179.
[7] J. Hagemann and A. Mitschke, *On n-permutable congruences*, Algebra Universalis **3** (1973), 8–12.
[8] Z. Janelidze, *Closedness properties of internal relations V: Linear Mal'tsev conditions*, Algebra Universalis **58** (2008), 105–117.
[9] A. I. Mal'cev, *On the general theory of algebraic systems*, Mat. Sbornik N. S. **35** (1954), no. 6, 3–20.
[10] N. Martins-Ferreira and T. Van der Linden, *Categories vs. groupoids via generalised Mal'tsev properties*, preprint `arXiv:1206.2745v1`, 2012.
[11] D. Rodelo and T. Van der Linden, *Approximate Hagemann–Mitschke co-operations*, Appl. Categ. Structures, accepted for publication, 2013.
[12] J. D. H. Smith, *Mal'cev varieties*, Lecture Notes in Math., vol. 554, Springer, 1976.
[13] H. Werner, *A Mal'cev condition for admissible relations*, Algebra Universalis **3** (1973), 263.

Zurab Janelidze
Department of Mathematical Sciences, Stellenbosch University, Private Bag X1, Matieland 7602, South Africa
E-mail address: `zurab@sun.ac.za`

Diana Rodelo
CMUC, Universidade de Coimbra, 3001–454 Coimbra, Portugal
Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade do Algarve, Campus de Gambelas, 8005–139 Faro, Portugal
E-mail address: `drodelo@ualg.pt`

Tim Van der Linden
Institut de Recherche en Mathématique et Physique, Université catholique de Louvain, chemin du cyclotron 2 bte L7.01.02, 1348 Louvain-la-Neuve, Belgium
E-mail address: `tim.vanderlinden@uclouvain.be`

algebras other than (quasi)varieties. In fact, already noting that in Proposition 3.1 it is sufficient to require that $\mathbb{C}$ is closed in $\mathbb{X}$ under powers and equalisers, gives some new examples. Indeed, in any variety $\mathcal{V}$ take any family $(\mathcal{V}_i)_{i\in I}$ of subvarieties; then, their join $\mathcal{U} = \bigvee_{i\in I}\mathcal{V}_i$ gives a class of algebras in $\mathcal{V}$ which is closed under all powers and subalgebras, and this would give that the conditions of Proposition 3.1 for $\mathbb{C} = \mathcal{U}$ and $\mathbb{X} = \mathcal{V}$ are equivalent. In fact, they hold if and only if each $\mathcal{V}_i$ has n-permutable congruences. There are also examples which are not of this type; for instance, consider groups among all monoids... in each case, we could also restrict to, say, finite universal algebras.

Finally, we would like to note that the arguments we used to prove Proposition 2.4 have many variations which could allow to obtain other similar characterisations of n-permutability. For instance, the matrix display in Figure 2.3, which can be used to prove 2.4(iii)⇒2.4(iv) for $n \geqslant 2$, can be used similarly to prove that 2.4(iii) implies the following condition:

$(*)$ For any internal relation R on an object X in $\mathbb{C}$,

$$R \leqslant (1_X \wedge R)R \quad \Rightarrow \quad R^n \leqslant R^{n-1}.$$

Notice that 2.4(iii)⇒$(*)$ still holds for $n = 1$, since 1-permutability, as mentioned earlier, gives that $\mathbb{C}$ is a preorder, while in any preorder any relation R on X is contained in the equality relation 1_X. Next, $(*)$ implies 2.4(iv) since when R is reflexive, $R = (1_X \wedge R)R$. So $(*)$ also characterises n-permutability. One can also get this way conditions involving more than one relation. Here is one example:

$(**)$ For any two internal relations S, T in $\mathbb{C}$,

$$S \leqslant T \quad \Rightarrow \quad (S^\circ T)^n \leqslant (S^\circ T)^{n-1}.$$

This obviously implies 2.4(iv) since, if we take T above to be a relation $T = R$ on X and $S = 1_X$, then we get exactly the statement in 2.4(iv). At the same time, $(**)$ can be deduced from $(*)$ by setting $R = S^\circ T$, since then $S \leqslant T$ implies $R \leqslant (1_X \wedge R)R$. So $(**)$ also characterises n-permutability.

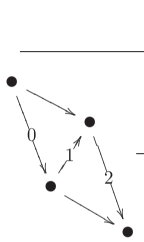
References

[1] M. Barr, P. A. Grillet, and D. H. van Osdol, *Exact categories and categories of sheaves*, Lecture Notes in Math., vol. 236, Springer, 1971.

[2] D. Bourn and Z. Janelidze, *Approximate Mal'tsev operations*, Theory Appl. Categ. **21** (2008), no. 8, 152–171.

[3] A. Carboni, G. M. Kelly, and M. C. Pedicchio, *Some remarks on Maltsev and Goursat categories*, Appl. Categ. Structures **1** (1993), 385–421.

This argument can be exhibited by the following matrix display:

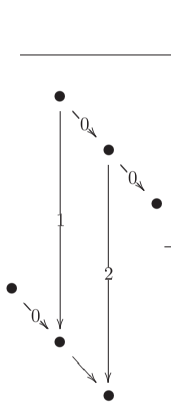


0				1	2
b	b	b	b	c	c
a	a	b	b	d	d
a	b	b	b	$a\gamma$	$b\gamma$
a	a	a	b	$a\gamma$	$b\gamma$

Here:

- the diagram of arrows indicates how S is constructed;
- columns represent generalised elements of S used in the argument. The columns on the left side translate the display (1). In particular, the first column indicates that $(b, a)S(a, a)$, i.e. that $(a, a)S^\circ(b, a)$, which comes from the top part of the relations displayed in (1). The label 0 of the first column of variables indicates which pair of variables from that column was assumed to belong to R in the beginning of the argument — it is the pair that corresponds to the arrow with the same label 0;
- in a similar way, the columns on the right side sketch the display (2), with the numerical labels giving the required conclusion of the argument.

(iii)⇒(iv) for $n = 3$: In this case, the required argument can be summarised using a similar matrix display:



0	1	2
a_1	$a_0\gamma$	b_1
a_2	$a_1\gamma$	b_2
a_3	$a_2\gamma$	b_3
a_0	b_1	$a_1\gamma$
a_1	b_2	$a_2\gamma$
a_2	b_3	$a_3\gamma$

In detail, given an internal reflexive relation $R \to A \times A$, construct another one $S \to X \times X$ as follows: generalised elements of X are triples (a, b, c) of generalised elements of A such that $aRbRc$, and S is defined by

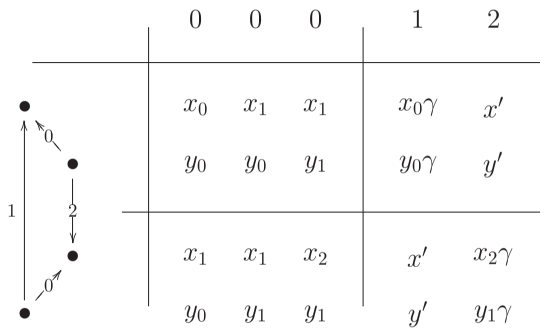
$$(a, b, c)S(a', b', c') \qquad \Leftrightarrow \qquad aRb' \text{ and } bRc'.$$

It is easy to see that S is a reflexive relation. Now, using $S^\circ \leqslant SS$ we want to show that $RRR \leqslant RR$. Assume $a_0Ra_1Ra_2Ra_3$ where a_0, a_1, a_2, a_3 are generalised elements of A. Then, $(a_1, a_2, a_3)S(a_0, a_1, a_2)$ which gives

$$(a_0\gamma, a_1\gamma, a_2\gamma)S(b_1, b_2, b_3)S(a_1\gamma, a_2\gamma, a_3\gamma)$$

for some regular epimorphism γ and generalised elements b_1, b_2, b_3 of A (having the same domain as γ). By the definition of S we get $a_0\gamma Rb_2Ra_3\gamma$.

(iv)⇒(i) for $n = 3$: This proof is sketched by the following matrix display:



This proves Proposition 2.4 for $n = 3$. The proof for a general $n \geqslant 2$ is sketched by the matrices displayed in Figures 2.1–2.5. ∎

3. Final remarks

While working on [10] with N. Martins-Ferreira, the third author noticed that Theorem 1.1 above, announced for varieties in [7], did not appear in [3]. In [10] the varietal proof was sketched, essentially based on the syntactical characterisation of n-permutability obtained in [7]. In [11] the second and the third authors looked at extending this proof to regular categories with sums, based on the technique of approximate operations introduced by the first author and D. Bourn in [2]. The problem of obtaining Theorem 1.1 for regular categories free of sums remained open. We then looked at transforming the varietal proof into a categorical one, based on an insight suggested by the

algorithm for translating linear Mal'tsev conditions into categorical conditions described by the first author in [8].

In the literature, when one speaks of n-permutability one usually assumes $n \geqslant 2$. It is easy to see that in the case when $n = 1$, this property becomes equivalent to the regular category being a preorder, while for $n = 0$ it holds trivially. The cases when $n = 2$ and $n = 3$ are the best known ones. Regular categories with 2-permutable congruences were called Mal'tsev categories in [4], following J. D. H. Smith who called varieties with 2-permutable congruences Mal'tsev varieties in [12], after A. I. Mal'tsev who studied them in [9]. Since [5], the term "Mal'tsev category" refers to a finitely complete category where every internal reflexive relation is an equivalence relation (or equivalently, where every internal relation is difunctional). Thus, in modern terminology, regular categories with 2-permutable congruences are the same as regular Mal'tsev categories. Regular categories with 3-permutable congruences are known as Goursat categories since [4] and [3].

We would like to point out that there are several obvious ways in which one could extend the results we obtain in this paper, to more general axiomatic contexts than that of regular categories. For example, the proof of Proposition 2.4 can be easily adapted to prove the following:

Proposition 3.1. *For any full subcategory $\mathbb{C}$ of a regular category $\mathbb{X}$, closed under finite limits in $\mathbb{X}$, the following conditions are equivalent for any natural number $n \geqslant 1$:*

(i) $(S, S^\circ)_{n+1} \leqslant (S, S^\circ)_{n-1}$ *for any internal relation $S \to X \times Y$ in $\mathbb{X}$ where $S, X, Y \in \mathbb{C}$.*

(ii) $(S, S^\circ)_{n+1} \leqslant (S, S^\circ)_{n-1}$ *for any internal relation S as above with $X = Y$.*

(iii) $R^\circ \leqslant R^{n-1}$ *for any internal reflexive relation $R \to A \times A$ in $\mathbb{X}$ where $R, A \in \mathbb{C}$.*

(iv) $R^n \leqslant R^{n-1}$ *for any internal reflexive relation R as above.*

Remark 3.2. In the above proposition, setting $n = 2$ and $\mathbb{X} = \mathbf{Set}^{\mathbb{C}^{\mathrm{op}}}$, and regarding $\mathbb{C}$ as a full subcategory of $\mathbb{X}$ via the Yoneda embedding, we get Proposition 1.2 in [5], which establishes the equivalence between difunctionality of internal relations, and symmetry/transitivity of internal reflexive relations in any finitely complete category.

A natural question to ask is whether such generalisations would allow to obtain a characterisation of n-permutability in interesting classes of universal