Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 13–21

CHARACTERIZATION THEOREM FOR LAGUERRE-HAHN ORTHOGONAL POLYNOMIALS ON NON-UNIFORM LATTICES

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Abstract: It is stated and proved a characterization theorem for Laguerre-Hahn orthogonal polynomials on non-uniform lattices. This theorem proves the equivalence between the Riccati equation for the formal Stieltjes function, linear first-order difference relations for the orthogonal polynomials as well as for the associated polynomials of the first kind, and linear first-order difference relations for the functions of the second kind.

Keywords: Laguerre-Hahn orthogonal polynomials; Divided difference operator; non-uniform lattices; Riccati difference equation; Structure relations.
AMS Subject Classification (2000): 33C45; 33C47; 33D45.

1. Introduction

The present paper concerns orthogonal polynomials of a discrete variable on non-uniform lattices (commonly denoted by snul). These lattices are associated with divided differences operators such as the Wilson or Askey-Wilson operator ([2, Section 5], and [3, 12, 17, 18]). Specifically, we focus our attention on the so-called Laguerre-Hahn orthogonal polynomials. The Laguerre-Hahn orthogonal polynomials on non-uniform lattices were introduced by A. Magnus in [14], as the ones for which the formal Stieltjes function satisfies a Riccati difference equation with polynomial coefficients, with the difference operator taken as a general divided difference operator given by [14, Eq. (1.1)] (see Section 2 of the present paper for the precise definitions and main properties). In this pioneering work, Magnus establishes difference relations as well as representations for the Laguerre-Hahn orthogonal polynomials and he proves that, under certain restrictions on the degrees of the coefficient of the Riccati difference equation, the Laguerre-Hahn orthogonal polynomials are the associated Askey-Wilson polynomials [1, 2].

Received May 31, 2013.
This work was partially supported by Centro de Matemática da Universidade de Coimbra (CMUC), funded by the European Regional Development Fund through the program COMPETE and by the Portuguese Government through the FCT - Fundação para a Ciência e a Tecnologia under the project PEst-C/MAT/UI0324/2011.

As it is well known from the setting of continuous orthogonality, Laguerre-Hahn orthogonal polynomials inherit many properties from the classical and semi-classical families [5, 7, 13, 16]. Indeed, one of the research topics within the Laguerre-Hahn theory of a discrete variable is the so-called structure relations, that is, linear difference relations involving the orthogonal polynomials (see [4, 8, 10, 11] and their lists of references). In the semi-classical case, it was proven in [15] the characterization of semi-classical orthogonal polynomials on non-uniform lattices in terms of structure relations. A more recent contribution, [9], proves the characterization of classical polynomials on non-uniform lattices in terms of two types of structure relations, using the so-called functional approach.

In the present paper we show a characterization theorem for Laguerre-Hahn orthogonal polynomial on arbitrary non-uniform lattices. Our main result is given in Theorem 2, where it is shown the equivalence between:

(a) the Riccati difference equation for the formal Stieltjes function, S;
(b) linear first-order difference relations for orthogonal polynomials related to S, as well as for the associated polynomials of the first kind;
(c) linear first-order difference relations for the functions of the second kind related to S.

The difference relations contained in Theorem 2 for Laguerre-Hahn families extend some of the difference relations for the classical families given in [9, 15, 22].

This paper is organized as follows. In Section 2 we give the definitions and state the basic results which will be used in the forthcoming sections. In Section 3 we show the main results of the paper, namely, the equivalence between the above referred conditions (a), (b) and (c), stated in Theorem 2 Section 4 is devoted to the proof of Theorem 2.

2. Preliminary results

2.1. The operators $\mathbb{D}, \mathbb{E}_j, \mathbb{M}$ and the related non-uniform lattices. We consider the divided difference operator $\mathbb{D}$ given in [14], involving the values of a function at two points, with the fundamental property that $\mathbb{D}$ leaves a polynomial of degree $n-1$ when applied to a polynomial of degree n. The operator $\mathbb{D}$, defined on the space of arbitrary functions, is given by

$$\mathbb{D}f(x) = \frac{f(y_2(x)) - f(y_1(x))}{y_2(x) - y_1(x)}, \tag{1}$$

where, at this stage, y_1 and y_2 are still unknown functions. To define them, one starts by using the property that $\mathbb{D}f$ is a polynomial of degree $n-1$ whenever f is a polynomial of degree n. Then, applying $\mathbb{D}$ to $f(x)=x^2$ and $f(x)=x^3$, one obtains, respectively,

$$y_1(x)+y_2(x) = \text{polynomial of degree } 1\,, \tag{2}$$

$$(y_1(x))^2+y_1(x)y_2(x)+(y_2(x))^2 = \text{polynomial of degree } 2\,, \tag{3}$$

the later condition being equivalent to $y_1(x)y_2(x)=$ polynomial of degree less or equal than 2. The conditions (2) and (3) define y_1 and y_2 as the two roots of a quadratic equation

$$\hat{a}y^2+2\hat{b}xy+\hat{c}x^2+2\hat{d}y+2\hat{e}x+\hat{f}=0\,, \quad \hat{a}\neq 0\,. \tag{4}$$

Some identities involving y_1 and y_2, following from the fact that y_1, y_2 are the roots of (4):

$$y_1(x)+y_2(x) = -2(\hat{b}x+\hat{d})/\hat{a}\,,$$
$$y_1(x)y_2(x) = (\hat{c}x^2+2\hat{e}x+\hat{f})/\hat{a}\,.$$

There are four primary classes of lattices and related divided difference operators (1):

(i) the linear lattice, related to the forward difference operator [19, Chapter 2, Section 12] ;
(ii) the q-linear lattice, related to the q-difference operator [12] ;
(iii) the quadratic lattice, related to the Wilson operator [2] ;
(iv) the q-quadratic lattice, related to the Askey-Wilson operator [2].

This classification of lattices is done according to the two parameters $\lambda=\hat{b}^2-\hat{a}\hat{c}$ and $\tau=\left((\hat{b}^2-\hat{a}\hat{c})(\hat{d}^2-\hat{a}\hat{f})-(\hat{b}\hat{d}-\hat{a}\hat{e})^2\right)/\hat{a}$, assuming $\hat{a}\hat{c}\neq 0$: $\lambda=\tau=0$ in case (i); $\lambda>0, \tau=0$ in case (ii); $\lambda=0, \tau<0$ in case (iii); $\lambda\tau<0$ in case (iv).

We would like to remark [15, Section 2], where it is given a geometric interpretation of the lattices. For the quadratic class of lattices (the so-called snul), it is possible to have a parametric representation of the conic (4), say $\{x(s),y(s)\}$, such that $y_1(x(s))=y(s)=x(s-1/2)$ and $y_2(x(s))=y(s+1)=x(s+1/2)$, ading to [3, 17, 18]

$$\begin{cases} x(s)=c_4s^2+c_5s+c_6\,, & \text{if } \lambda=0, \tau<0\,,\\ x(s)=c_1q^s+c_2q^{-s}+c_3\,, & \text{if } \lambda\tau<0\,, \quad q+q^{-1}=4\hat{b}^2/(\hat{a}\hat{c})-2\,. \end{cases}$$

Note that each of the operators in (i)-(iv) is an extension of the preceding one, which is recovered as a particular case or as a limit case, up to a linear transformation of the variable.

In the present paper we shall operate with the divided difference operator given in its general form (1). By defining the operators $\mathbb{E}_1$ and $\mathbb{E}_2$, acting on arbitrary functions f, as

$$\mathbb{E}_1 f(x) = f(y_1(x)) \, , \;\; \mathbb{E}_2 f(x) = f(y_2(x)) \, ,$$

(1) is given by

$$\mathbb{D} f(x) = \frac{\mathbb{E}_2 f(x) - \mathbb{E}_1 f(x)}{y_2(x) - y_1(x)} \, .$$

We define the companion operator of $\mathbb{D}$ as

$$\mathbb{M} f(x) = \frac{\mathbb{E}_1 f(x) + \mathbb{E}_1 f(x)}{2} \, . \tag{5}$$

Some useful identities involving $\mathbb{D}, \mathbb{M}$ and $\mathbb{E}_1, \mathbb{E}_2$ are listed below:

$$\mathbb{D}(gf) = \mathbb{D}g \, \mathbb{M}f + \mathbb{M}g \, \mathbb{D}f \, , \tag{6}$$

$$\mathbb{D}(g/f) = \frac{\mathbb{D}g \, \mathbb{M}f - \mathbb{D}f \, \mathbb{M}g}{\mathbb{E}_1 f \, \mathbb{E}_2 f} \, , \tag{7}$$

$$\mathbb{D}(1/f) = \frac{-\mathbb{D}f}{\mathbb{E}_1 f \, \mathbb{E}_2 f} \, , \tag{8}$$

$$\mathbb{M}(gf) = \mathbb{M}g \, \mathbb{M}f + \mathbb{D}g \, \mathbb{D}f \frac{(y_1 - y_2)^2}{4} \, ,$$

$$\mathbb{M}(g/f) = \frac{\mathbb{E}_1 g \, \mathbb{E}_2 f + \mathbb{E}_2 g \, \mathbb{E}_1 f}{2\mathbb{E}_1 f \, \mathbb{E}_2 f} \, , \tag{9}$$

$$\mathbb{M}(1/f) = \frac{\mathbb{M}f}{\mathbb{E}_1 f \, \mathbb{E}_2 f} \, . \tag{10}$$

Eq. (6) has the equivalent forms:

$$\mathbb{D}(gf) = \mathbb{D}g \, \mathbb{E}_1 f + \mathbb{D}f \, \mathbb{E}_2 g \, , \tag{11}$$

$$\mathbb{D}(gf) = \mathbb{D}g \, \mathbb{E}_2 f + \mathbb{D}f \, \mathbb{E}_1 g \, .$$

Also, one has two equivalent forms for (7):

$$\mathbb{D}(g/f) = \frac{\mathbb{D}g\,\mathbb{E}_1 f - \mathbb{D}f\,\mathbb{E}_1 g}{\mathbb{E}_1 f\,\mathbb{E}_2 f}\,, \tag{12}$$

$$\mathbb{D}(g/f) = \frac{\mathbb{D}g\,\mathbb{E}_2 f - \mathbb{D}f\,\mathbb{E}_2 g}{\mathbb{E}_1 f\,\mathbb{E}_2 f}\,. \tag{13}$$

2.2. Laguerre-Hahn orthogonal polynomials and auxiliary results. We shall consider formal orthogonal polynomials related to a (formal) Stieltjes function defined by

$$S(x) = \sum_{n=0}^{+\infty} u_n x^{-n-1} \tag{14}$$

where (u_n), the sequence of moments, is such that $\det\left[u_{i+j}\right]_{i,j=0}^{n} \neq 0$, $n \geq 0$, $u_0 = 1$. The orthogonal polynomials related to S, $P_n, n \geq 0$, are the diagonal Padé denominators of (14), thus the numerator polynomial (of degree $n-1$), henceforth denoted by $P_{n-1}^{(1)}$, and the denominator P_n (of degree n) are determined through

$$S(x) - P_{n-1}^{(1)}(x)/P_n(x) = \mathcal{O}(x^{-2n-1})\,, \quad x \to \infty\,. \tag{15}$$

Throughout the paper we consider each P_n monic, and we will denote the sequence of monic polynomials $\{P_n\}_{n\geq 0}$ by SMOP.

Monic orthogonal polynomials satisfy a three term recurrence relation [20]

$$P_{n+1}(x) = (x - \beta_n)P_n(x) - \gamma_n P_{n-1}(x)\,, \quad n = 0, 1, 2, \ldots\,, \tag{16}$$

with $P_{-1}(x) = 0,\ P_0(x) = 1$, and $\gamma_n \neq 0,\ n \geq 1,\ \gamma_0 = u_0 = 1$.

The sequence $\{P_n^{(1)}\}_{n\geq 0}$, also known as the sequence of associated polynomials of the first kind, satisfies the three term recurrence relation

$$P_n^{(1)}(x) = (x - \beta_n)P_{n-1}^{(1)}(x) - \gamma_n P_{n-2}^{(1)}(x)\,, \quad n = 1, 2, \ldots$$

with $P_{-1}^{(1)}(x) = 0,\ P_0^{(1)}(x) = 1$.

An equivalent form of (15), often encountered in the literature of orthogonal polynomials (see, for example, [21] and its list of references), is given by

$$q_n = P_n S - P_{n-1}^{(1)}\,, \quad n \geq 1\,, \ q_0 = S\,, \tag{17}$$

where $q_n, n \geq 0$, are the so-called functions of the second kind corresponding to $\{P_n\}_{n\geq 0}$. The sequence $\{q_n\}_{n\geq 0}$ also satisfies a three term recurrence relation,

$$q_{n+1}(x) = (x-\beta_n)q_n(x) - \gamma_n q_{n-1}(x)\,, \quad n = 0,1,2,... \tag{18}$$

with initial conditions $q_{-1} = 1,\ q_0(x) = S(x)$.

We will make use of the following result (see [6]).

Lemma 1. *Let $\{P_n\}_{n\geq 0}$ be a SMOP and let $\{P_n^{(1)}\}_{n\geq 0}$ be the sequence of associated polynomials of the first kind. The following holds:*

$$\mathbb{E}_j P_n^{(1)}\, \mathbb{E}_j P_n - \mathbb{E}_j P_{n+1}\, \mathbb{E}_j P_{n-1}^{(1)} = \prod_{k=0}^{n} \gamma_k\,, \quad j = 1,2\,, \quad n \geq 0\,. \tag{19}$$

Therefore, for each $j = 1,2$, $\mathbb{E}_j P_n^{(1)}$ and $\mathbb{E}_j P_{n+1}$ do not share zeroes.

Proof: Eq. (19) follows from the application of the operator $\mathbb{E}_j$, $j = 1,2$, to the identity

$$P_n^{(1)} P_n - P_{n+1} P_{n-1}^{(1)} = \prod_{k=0}^{n} \gamma_k\,, \quad n \geq 0\,.$$

From (19) there follows the statement concerning the zeros. ∎

Definition 1. A SMOP $\{P_n\}_{n\geq 0}$ related to a Stieltjes function S (14) is said to be *Laguerre-Hahn* if S satisfies a Riccati equation

$$A(x)\, \mathbb{D}S(x) = B(x)\, \mathbb{E}_1 S(x)\, \mathbb{E}_2 S(x) + C(x)\, \mathbb{M}S(x) + D(x)\,, \tag{20}$$

where A, B, C, D are polynomials in x, $A \neq 0$.
If $B \equiv 0$, then $\{P_n\}_{n\geq 0}$ is said to be *semi-classical.*

We will make use of the Theorem that follows.

Theorem 1. *Let $\{f_n\}$ be a sequence of functions satisfying a three term recurrence relation*

$$f_{n+1}(x) = (x-\beta_n) f_n(x) - \gamma_n f_{n-1}(x)\,, \;\; \gamma_n \neq 0,\; n \geq 0\,. \tag{21}$$

Let $g_n = f_{n+1}/f_n$ satisfy for all $n \geq 0$

$$A_n(x)\, \mathbb{D}g_n(x) = B_n(x)\, \mathbb{E}_1 g_n(x)\, \mathbb{E}_2 g_n(x) + C_n\, \mathbb{M}g_n(x) + D_n(x)\,, \tag{22}$$

with $\mathbb{D},\mathbb{M}$ the operators defined in (1) and (5), and A_n, B_n, C_n, D_n bounded degree polynomials. Then, for all $n \geq 0$, the following relations hold:

$$A_{n+1} = A_n - \frac{(y_1-y_2)^2}{2}\frac{D_n}{\gamma_{n+1}}\,, \tag{23}$$

$$B_{n+1} = \frac{D_n}{\gamma_{n+1}}\,, \tag{24}$$

$$C_{n+1} = -C_n - 2\mathbb{M}(x-\beta_{n+1})\frac{D_n}{\gamma_{n+1}}\,, \tag{25}$$

$$D_{n+1} = A_n + \gamma_{n+1}B_n + \mathbb{M}(x-\beta_{n+1})C_n + (y_1-\beta_{n+1})(y_2-\beta_{n+1})\frac{D_n}{\gamma_{n+1}}. \tag{26}$$

Proof: From (21) we get $g_n = (x-\beta_n) - \gamma_n/g_{n-1}$, thus, writing the above equation to $n+1$,

$$g_{n+1} = (x-\beta_{n+1}) - \gamma_{n+1}/g_n\,. \tag{27}$$

Applying $\mathbb{D}$ to (27) and using $\mathbb{D}(1/g_n) = -\mathbb{D}g_n/(\mathbb{E}_1 g_n \mathbb{E}_2 g_n)$ (cf. (8)) we get

$$\mathbb{D}g_{n+1} = 1 + \gamma_{n+1}\frac{\mathbb{D}g_n}{\mathbb{E}_1 g_n\,\mathbb{E}_2 g_n}\,.$$

Now we multiply the above equation by A_n and use (22), as well as $\mathbb{M}(1/g_n) = \mathbb{M}g_n/(\mathbb{E}_1 g_n\,\mathbb{E}_2 g_n)$ (cf. (10)), thus obtaining

$$A_n\,\mathbb{D}g_{n+1} = A_n + \gamma_{n+1}B_n + \gamma_{n+1}C_n\,\mathbb{M}(1/g_n) + \frac{\gamma_{n+1}D_n}{\mathbb{E}_1 g_n\,\mathbb{E}_2 g_n}\,. \tag{28}$$

Note that from (27) we have

$$\mathbb{M}(1/g_n) = \frac{\mathbb{M}(x-\beta_{n+1})}{\gamma_{n+1}} - \frac{\mathbb{M}g_{n+1}}{\gamma_{n+1}}\,. \tag{29}$$

Also,

$$\frac{\gamma_{n+1}D_n}{\mathbb{E}_1 g_n \mathbb{E}_2 g_n} = \frac{D_n}{\gamma_{n+1}}\left(y_1-\beta_{n+1}-\mathbb{E}_1 g_{n+1}\right)\left(y_2-\beta_{n+1}-\mathbb{E}_2 g_{n+1}\right)\,,$$

and some computations yield

$$\begin{aligned}\frac{\gamma_{n+1}D_n}{\mathbb{E}_1 g_n\,\mathbb{E}_2 g_n} = \frac{D_n}{\gamma_{n+1}}\Big((y_1-\beta_{n+1})(y_2-\beta_{n+1}) + (y_1-y_2)^2/2\,\mathbb{D}g_{n+1} \\ -2\mathbb{M}(x-\beta_{n+1})\mathbb{M}g_{n+1} + \mathbb{E}_1 g_{n+1}\mathbb{E}_2 g_{n+1}\Big)\,. \end{aligned} \tag{30}$$

The substitution of (29) and (30) into (28) yields

$$\left(A_n - 2\frac{(y_1-y_2)^2}{4}\frac{D_n}{\gamma_{n+1}}\right)\mathbb{D}g_{n+1} = \frac{D_n}{\gamma_{n+1}}\mathbb{E}_1 g_{n+1}\mathbb{E}_2 g_{n+1}$$
$$+\left(-C_n - 2\mathbb{M}(x-\beta_{n+1})\frac{D_n}{\gamma_{n+1}}\right)\mathbb{M}g_{n+1}$$
$$+\left(A_n + \gamma_{n+1}B_n + \mathbb{M}(x-\beta_{n+1})C_n + (y_1-\beta_{n+1})(y_2-\beta_{n+1})\frac{D_n}{\gamma_{n+1}}\right).$$

The comparison between the above equation and (22) written to $n+1$ gives us (23)-(26). ■

3. Characterization theorem

Theorem 2. *Let S be a Stieltjes function, let $\{P_n\}_{n\geq 0}$ be the corresponding SMOP, and let $\{P_n^{(1)}\}_{n\geq 0}$, $\{q_n\}_{n\geq 0}$ be the sequence of associated polynomials of the first kind and the sequence of functions of the second kind, respectively. The following statements are equivalent:*

(a) *S satisfies the Riccati equation (20),*
$$A\,\mathbb{D}S = B\,\mathbb{E}_1 S\,\mathbb{E}_2 S + C\,\mathbb{M}S + D\,,$$
where A, B, C, D are polynomials;

(b) *P_n and $P_n^{(1)}$ satisfy the difference relations for all $n \geq 1$,*
$$\begin{cases} A\,\mathbb{D}P_n = l_{n-1}\mathbb{E}_1 P_n - C/2\,\mathbb{E}_2 P_n - B\,\mathbb{E}_2 P^{(1)}_{n-1} + \Theta_{n-1}\mathbb{E}_1 P_{n-1}\,, \\ A\,\mathbb{D}P^{(1)}_{n-1} = l_{n-1}\mathbb{E}_1 P^{(1)}_{n-1} + C/2\,\mathbb{E}_2 P^{(1)}_{n-1} + D\,\mathbb{E}_2 P_n + \Theta_{n-1}\mathbb{E}_1 P^{(1)}_{n-2}\,; \end{cases} \tag{31}$$

(c) *q_n satisfies for all $n \geq 0$,*
$$A\,\mathbb{D}q_n = l_{n-1}\,\mathbb{E}_1 q_n + (B\,\mathbb{E}_1 S + C/2)\,\mathbb{E}_2 q_n + \Theta_{n-1}\,\mathbb{E}_1 q_{n-1}\,, \tag{32}$$
where l_n, Θ_n are polynomials of uniformly bounded degrees satisfying the initial conditions $l_{-1} = C/2$, $\Theta_{-1} = D$.

The proof of Theorem 2 will be given at the next section.

Remark . The characterizations stated in Theorem 1 are not uniquely represented. One can also deduce that the following statements (a), (b), (c) are equivalent:

(a) S satisfies the Riccati equation (20),
$$A\mathbb{D}S = B\,\mathbb{E}_1 S\,\mathbb{E}_2 S + C\,\mathbb{M}S + D\,;$$

(b) P_n and $P_n^{(1)}$ satisfy the difference relations for all $n \geq 1$,

$$\begin{cases} A\,\mathbb{D}P_n = l_{n-1}\mathbb{E}_2 P_n - C/2\,\mathbb{E}_1 P_n - B\,\mathbb{E}_1 P_{n-1}^{(1)} + \Theta_{n-1}\mathbb{E}_2 P_{n-1}\,, \\ A\,\mathbb{D}P_{n-1}^{(1)} = l_{n-1}\mathbb{E}_2 P_{n-1}^{(1)} + C/2\,\mathbb{E}_1 P_{n-1}^{(1)} + D\,\mathbb{E}_1 P_n + \Theta_{n-1}\mathbb{E}_2 P_{n-2}^{(1)}\,; \end{cases} \tag{33}$$

(c) q_n satisfies for all $n \geq 0$,

$$A\,\mathbb{D}q_n = l_{n-1}\,\mathbb{E}_2 q_n + (B\,\mathbb{E}_2 S + C/2)\,\mathbb{E}_1 q_n + \Theta_{n-1}\,\mathbb{E}_2 q_{n-1}\,. \tag{34}$$

Therefore, we deduce the result that follows.

Theorem 3. *Let S be a Stieltjes function satisfying the Riccati equation*

$$A\,\mathbb{D}S = B\,\mathbb{E}_1 S\,\mathbb{E}_2 S + C\,\mathbb{M}S + D\,,$$

where A, B, C, D are polynomials. Let $\{P_n\}_{n\geq 0}$ be the SMOP related to S, and let $\{P_n^{(1)}\}_{n\geq 0}$, $\{q_n\}_{n\geq 0}$ be the sequence of associated polynomials of the first kind and the sequence of functions of the second kind, respectively. The following relations hold, for all $n \geq 0$:

$$A\,\mathbb{D}P_{n+1} = (l_n - C/2)\mathbb{M}P_{n+1} - B\,\mathbb{M}P_n^{(1)} + \Theta_n\mathbb{M}P_n\,, \tag{35}$$

$$A\,\mathbb{D}P_n^{(1)} = (l_n + C/2)\mathbb{M}P_n^{(1)} + D\,\mathbb{M}P_{n+1} + \Theta_n\mathbb{M}P_{n-1}^{(1)}\,, \tag{36}$$

$$A\,\mathbb{D}q_n = (l_{n-1} + C/2)\mathbb{M}q_n + B\,(\mathbb{M}S\,\mathbb{M}q_n - \mathbb{M}(Sq_n)) + \Theta_{n-1}\mathbb{M}q_{n-1}\,. \tag{37}$$

Proof: Sum (31) and (33) to get (35) and (36). Following the same idea, sum (32) and (34) to get (37). ■

Remark . The equations (35)-(37) extend the ones given in [22] for the semi-classical case.

Corollary 1. *The polynomials l_n, Θ_n of Theorems 2, 3 satisfy, for all $n \geq 0$,*

$$l_{n+1} + l_n + \mathbb{M}(x - \beta_{n+1})\frac{\Theta_n}{\gamma_{n+1}} = 0\,, \tag{38}$$

$$\begin{aligned} \Theta_{n+1} = A + (y_1 - \beta_{n+1})(y_2 - \beta_{n+1})\frac{\Theta_n}{\gamma_{n+1}} \\ + \left(\gamma_{n+1} - \frac{(y_1 - y_2)^2}{4}\right)\frac{\Theta_{n-1}}{\gamma_n} + 2\mathbb{M}(x - \beta_{n+1})l_n\,, \end{aligned} \tag{39}$$

with initial conditions $l_{-1} = C/2, \Theta_{-1} = D$.

Proof: Multiply (32), written to $n+1$, by $\mathbb{E}_2 q_n$ and subtract to (32) multiplied by $\mathbb{E}_2 q_{n+1}$. Then, multiply the resulting equation by $1/(\mathbb{E}_1 q_n \mathbb{E}_2 q_n)$, thus obtaining

$$A\,\mathbb{D}\left(\frac{q_{n+1}}{q_n}\right) = l_n \mathbb{E}_1\left(\frac{q_{n+1}}{q_n}\right) - l_{n-1}\mathbb{E}_2\left(\frac{q_{n+1}}{q_n}\right) + \Theta_n - \Theta_{n-1}\mathbb{E}_1\left(\frac{q_{n-1}}{q_n}\right)\mathbb{E}_2\left(\frac{q_{n+1}}{q_n}\right), \quad (40)$$

where we used the property (13). From the recurrence relation for q_n there holds

$$\frac{q_{n-1}}{q_n} = \frac{(x-\beta_n)}{\gamma_n} - \frac{1}{\gamma_n}\frac{q_{n+1}}{q_n},$$

thus

$$\mathbb{E}_1\left(\frac{q_{n-1}}{q_n}\right) = \frac{1}{\gamma_n}\mathbb{E}_1(x-\beta_n) - \frac{1}{\gamma_n}\mathbb{E}_1\left(\frac{q_{n+1}}{q_n}\right). \quad (41)$$

The substitution of (41) in (40) yields

$$A\,\mathbb{D}\left(\frac{q_{n+1}}{q_n}\right) = l_n \mathbb{E}_1\left(\frac{q_{n+1}}{q_n}\right) - l_{n-1}\mathbb{E}_2\left(\frac{q_{n+1}}{q_n}\right) + \Theta_n - \frac{\Theta_{n-1}}{\gamma_n}\mathbb{E}_1(x-\beta_n)\mathbb{E}_2\left(\frac{q_{n+1}}{q_n}\right) + \frac{\Theta_{n-1}}{\gamma_n}\mathbb{E}_1\left(\frac{q_{n+1}}{q_n}\right)\mathbb{E}_2\left(\frac{q_{n+1}}{q_n}\right). \quad (42)$$

On the other hand, if we proceed as above, but starting with the eq. (32) and using the property (12), we obtain

$$A\,\mathbb{D}\left(\frac{q_{n+1}}{q_n}\right) = l_n \mathbb{E}_2\left(\frac{q_{n+1}}{q_n}\right) - l_{n-1}\mathbb{E}_1\left(\frac{q_{n+1}}{q_n}\right) + \Theta_n - \frac{\Theta_{n-1}}{\gamma_n}\mathbb{E}_2(x-\beta_n)\mathbb{E}_1\left(\frac{q_{n+1}}{q_n}\right) + \frac{\Theta_{n-1}}{\gamma_n}\mathbb{E}_1\left(\frac{q_{n+1}}{q_n}\right)\mathbb{E}_2\left(\frac{q_{n+1}}{q_n}\right). \quad (43)$$

From the sum of (42) with (43) there follows

$$A\,\mathbb{D}\left(\frac{q_{n+1}}{q_n}\right) = (l_n - l_{n-1})\mathbb{M}\left(\frac{q_{n+1}}{q_n}\right) + \Theta_n + \frac{\Theta_{n-1}}{\gamma_n}\mathbb{E}_1\left(\frac{q_{n+1}}{q_n}\right)\mathbb{E}_2\left(\frac{q_{n+1}}{q_n}\right) - \frac{\Theta_{n-1}}{2\gamma_n}\left(\mathbb{E}_1(x-\beta_n)\mathbb{E}_2\left(\frac{q_{n+1}}{q_n}\right) + \mathbb{E}_2(x-\beta_n)\mathbb{E}_1\left(\frac{q_{n+1}}{q_n}\right)\right). \quad (44)$$

The use of

$$\mathbb{E}_1(x-\beta_n)\mathbb{E}_2\left(\frac{q_{n+1}}{q_n}\right)+\mathbb{E}_2(x-\beta_n)\mathbb{E}_1\left(\frac{q_{n+1}}{q_n}\right) = 2\mathbb{M}(x-\beta_n)\mathbb{M}\left(\frac{q_{n+1}}{q_n}\right)-2\frac{(y_1-y_2)^2}{4}\mathbb{D}\left(\frac{q_{n+1}}{q_n}\right)$$

in (44) gives us the Riccati equation for $g_n = q_{n+1}/q_n$,

$$A_n\,\mathbb{D}g_n = B_n\,\mathbb{E}_1 g_n\,\mathbb{E}_2 g_n + C_n\,\mathbb{M}g_n + D_n$$

with

$$A_n = A - \frac{(y_1-y_2)^2}{4}\frac{\Theta_{n-1}}{\gamma_n}\,,$$
$$B_n = \frac{\Theta_{n-1}}{\gamma_n}\,,$$
$$C_n = l_n - l_{n-1} - \mathbb{M}(x-\beta_n)\frac{\Theta_{n-1}}{\gamma_n}\,,$$
$$D_n = \Theta_n\,.$$

Now we use Theorem 1. Taking into account the relations (23)-(26) for A_n, B_n, C_n, D_n, there follows, for all $n \geq 0$,

$$l_{n+1} - l_{n-1} - \mathbb{M}(x-\beta_n)\frac{\Theta_{n-1}}{\gamma_n} + \mathbb{M}(x-\beta_{n+1})\frac{\Theta_n}{\gamma_{n+1}} = 0\,, \tag{45}$$

$$\Theta_{n+1} = A + (y_1-\beta_{n+1})(y_2-\beta_{n+1})\frac{\Theta_n}{\gamma_{n+1}} + \left(\gamma_{n+1} - \frac{(y_1-y_2)^2}{4} - \mathbb{M}(x-\beta_n)\mathbb{M}(x-\beta_{n+1})\right)\frac{\Theta_{n-1}}{\gamma_n} + \mathbb{M}(x-\beta_{n+1})(l_n - l_{n-1})\,. \tag{46}$$

To deduce (38) we write (45) in the equivalent form

$$M_{n+1} = M_n\,,\quad n\geq 0 \quad \text{and} \quad M_{n+1} = l_{n+1} + l_n + \mathbb{M}(x-\beta_{n+1})\frac{\Theta_n}{\gamma_{n+1}}\,,$$

from which there follows $M_{n+1} = M_0$, $n \geq 0$. The use of the initial conditions $l_0 + l_{-1} + \mathbb{M}(x-\beta_0)\frac{\Theta_{-1}}{\gamma_0} = 0$ yield (38).

Using (38) written to $n-1$ in (46) we obtain (39). ■

4. Proof of Theorem 2

Proof of $(a) \Rightarrow (b)$.

If we use $S = \dfrac{q_n}{P_n} + \dfrac{P^{(1)}_{n-1}}{P_n}$, $n \geq 1$ (cf. (17)), then (20) yields

$$M_n = -A\mathbb{D}\left(\frac{P^{(1)}_{n-1}}{P_n}\right) + B\mathbb{E}_1\left(\frac{P^{(1)}_{n-1}}{P_n}\right)\mathbb{E}_2\left(\frac{P^{(1)}_{n-1}}{P_n}\right) + C\mathbb{M}\left(\frac{P^{(1)}_{n-1}}{P_n}\right) + D\,, \quad (47)$$

where

$$M_n = A\mathbb{D}\left(\frac{q_n}{P_n}\right) - B\left[\mathbb{E}_1\left(\frac{q_n}{P_n}\right)\mathbb{E}_2\left(\frac{q_n}{P_n}\right) + \mathbb{E}_1\left(\frac{q_n}{P_n}\right)\mathbb{E}_2\left(\frac{P^{(1)}_{n-1}}{P_n}\right) + \mathbb{E}_1\left(\frac{P^{(1)}_{n-1}}{P_n}\right)\mathbb{E}_2\left(\frac{q_n}{P_n}\right)\right] - C\mathbb{M}\left(\frac{q_n}{P_n}\right).$$

By multiplying both hand sides of (47) by $\mathbb{E}_1 P_n \mathbb{E}_2 P_n$ and using the properties (12) and (9), we obtain

$$\begin{aligned} M_n\,\mathbb{E}_1P_n\,\mathbb{E}_2P_n = {}& -A\,\mathbb{D}P^{(1)}_{n-1}\,\mathbb{E}_1P_n + A\,\mathbb{D}P_n\,\mathbb{E}_1P^{(1)}_{n-1} + B\,\mathbb{E}_1P^{(1)}_{n-1}\,\mathbb{E}_2P^{(1)}_{n-1} \\ & + \frac{C}{2}\left(\mathbb{E}_1P^{(1)}_{n-1}\,\mathbb{E}_2P_n + \mathbb{E}_1P_n\,\mathbb{E}_2P^{(1)}_{n-1}\right) + D\mathbb{E}_1P_n\,\mathbb{E}_2P_n\,. \end{aligned}$$

Now let us write

$$\begin{aligned} & -A\,\mathbb{D}P^{(1)}_{n-1}\,\mathbb{E}_1P_n + A\,\mathbb{D}P_n\,\mathbb{E}_1P^{(1)}_{n-1} + B\,\mathbb{E}_1P^{(1)}_{n-1}\,\mathbb{E}_2P^{(1)}_{n-1} \\ & \quad + \frac{C}{2}\left(\mathbb{E}_1P^{(1)}_{n-1}\,\mathbb{E}_2P_n + \mathbb{E}_1P_n\,\mathbb{E}_2P^{(1)}_{n-1}\right) + D\,\mathbb{E}_1P_n\,\mathbb{E}_2P_n = \hat{\Theta}_{n-1}\,, \end{aligned} \quad (48)$$

where $\hat{\Theta}_{n-1}$ is a bounded degree polynomial, as $q_n(x) = \mathcal{O}(x^{-2n-1})$. One has $\deg(\hat{\Theta}_{n-1}) = \max\{\deg(A)-2, \deg(B)-2, \deg(C)-1\}$.

Taking into account $\mathbb{E}_1(P^{(1)}_{n-1})\mathbb{E}_1(P_{n-1}) - \mathbb{E}_1(P_n)\mathbb{E}_1(P^{(1)}_{n-2}) = \prod_{k=0}^{n-1}\gamma_k$, $n \geq 1$, (cf. (19)), then (48) can be written as

$$\begin{aligned} & -A\,\mathbb{D}P^{(1)}_{n-1}\,\mathbb{E}_1P_n + A\,\mathbb{D}P_n\,\mathbb{E}_1P^{(1)}_{n-1} + B\,\mathbb{E}_1P^{(1)}_{n-1}\,\mathbb{E}_2P^{(1)}_{n-1} \\ & \quad + \frac{C}{2}\left(\mathbb{E}_1P^{(1)}_{n-1}\,\mathbb{E}_2P_n + \mathbb{E}_1P_n\,\mathbb{E}_2P^{(1)}_{n-1}\right) + D\mathbb{E}_1P_n\,\mathbb{E}_2P_n \\ & \quad = \Theta_{n-1}\left(\mathbb{E}_1P^{(1)}_{n-1}\,\mathbb{E}_1P_{n-1} - \mathbb{E}_1P_n\,\mathbb{E}_1P^{(1)}_{n-2}\right)\,, \end{aligned} \quad (49)$$

where $\Theta_{n-1} = \hat{\Theta}_{n-1} / \prod_{k=0}^{n-1} \gamma_k$.

Now, let us write (49) as

$$\left\{ A\,\mathbb{D}P_n + C/2\,\mathbb{E}_2 P_n + B\,\mathbb{E}_2 P_{n-1}^{(1)} - \Theta_{n-1}\mathbb{E}_1 P_{n-1} \right\} \mathbb{E}_1 P_{n-1}^{(1)}$$
$$= \left\{ A\,\mathbb{D}P_{n-1}^{(1)} - C/2\,\mathbb{E}_2 P_{n-1}^{(1)} - D\,\mathbb{E}_2 P_n - \Theta_{n-1}\mathbb{E}_1 P_{n-2}^{(1)} \right\} \mathbb{E}_1 P_n\,. \quad (50)$$

Since $\mathbb{E}_1 P_{n-1}^{(1)}$ and $\mathbb{E}_1 P_n$ do not have common zeroes, for all $n \geq 1$, then there exists a polynomial, l_{n-1}, such that

$$\begin{cases} A\,\mathbb{D}P_n + C/2\,\mathbb{E}_2 P_n + B\,\mathbb{E}_2 P_{n-1}^{(1)} - \Theta_{n-1}\mathbb{E}_1 P_{n-1} = l_{n-1}\mathbb{E}_1 P_n\,, \\ A\,\mathbb{D}P_{n-1}^{(1)} - C/2\,\mathbb{E}_2 P_{n-1}^{(1)} - D\,\mathbb{E}_2 P_n - \Theta_{n-1}\mathbb{E}_1 P_{n-2}^{(1)} = l_{n-1}\mathbb{E}_1 P_{n-1}^{(1)}\,, \end{cases}$$

that is, we get (31).

Proof of $(b) \Rightarrow (a)$**.**

Let us define $\psi_n = \begin{bmatrix} P_{n+1} \\ P_n^{(1)} \end{bmatrix}$. From the three term recurrence relation for $\{P_n\}$ and $\{P_n^{(1)}\}$, there follows that ψ_n satisfies

$$\psi_n = (x - \beta_n)\psi_{n-1} - \gamma_n \psi_{n-2}\,, \quad n \geq 1\,, \;\; \psi_{-1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \;\; \psi_0 = \begin{bmatrix} x - \beta_0 \\ 1 \end{bmatrix}. \quad (51)$$

With the notation $\mathbb{D}\psi_n = \begin{bmatrix} \mathbb{D}P_{n+1} \\ \mathbb{D}P_n^{(1)} \end{bmatrix}$, $\mathbb{E}_j \psi_n = \begin{bmatrix} \mathbb{E}_j P_{n+1} \\ \mathbb{E}_j P_n^{(1)} \end{bmatrix}$, $j = 1, 2$, (31) reads as

$$A\,\mathbb{D}\psi_{n-1} = l_{n-1}\mathbb{E}_1 \psi_{n-1} + \mathcal{C}\,\mathbb{E}_2 \psi_{n-1} + \Theta_{n-1}\mathbb{E}_1 \psi_{n-2}\,, \quad (52)$$

where $\mathcal{C} = \begin{bmatrix} -C/2 & -B \\ D & C/2 \end{bmatrix}$.

In turn, (52) reads as

$$A \left(\frac{\psi_{n-1}(y_2) - \psi_{n-1}(y_1)}{y_2 - y_1} \right) = l_{n-1}\psi_{n-1}(y_1) + \mathcal{C}\psi_{n-1}(y_2) + \Theta_{n-1}\psi_{n-2}(y_1)\,,$$

that is,

$$\mathcal{A}_n \psi_{n-1}(y_1) + \mathcal{B}\psi_{n-1}(y_2) = \mathcal{C}_n \psi_{n-2}(y_1) \quad (53)$$

with

$$\mathcal{A}_n = (-\frac{A}{y_2 - y_1} - l_{n-1})I, \;\; \mathcal{B} = \frac{A}{y_2 - y_1}I - \mathcal{C}, \;\; \mathcal{C}_n = \Theta_{n-1} I,$$

and I denoting the identity matrix of order 2.

Taking $n+1$ in (53) and using the recurrence relation (51) we get

$$\tilde{\mathcal{A}}_n\psi_{n-1}(y_1)+\tilde{\mathcal{B}}_n\psi_{n-1}(y_2)=\tilde{\mathcal{C}}_n\psi_{n-2}(y_1)+\tilde{\mathcal{D}}_n\psi_{n-2}(y_2)\,, \tag{54}$$

with

$$\tilde{\mathcal{A}}_n=(y_1-\beta_n)\mathcal{A}_{n+1}-\mathcal{C}_{n+1},\ \tilde{\mathcal{B}}_n=(y_2-\beta_n)\mathcal{B},\ \tilde{\mathcal{C}}_n=\gamma_n\mathcal{A}_{n+1},\ \tilde{\mathcal{D}}_n=\gamma_n\mathcal{B}\,.$$

Now, we gather (53) and (54) in the system

$$\mathcal{E}_n\begin{bmatrix}\psi_{n-1}(y_1)\\ \psi_{n-1}(y_2)\end{bmatrix}=\mathcal{F}_n\begin{bmatrix}\psi_{n-2}(y_1)\\ \psi_{n-2}(y_2)\end{bmatrix}\,, \tag{55}$$

where $\mathcal{E}_n$ and $\mathcal{F}_n$ are the block matrices

$$\mathcal{E}_n=\begin{bmatrix}\mathcal{A}_n & \mathcal{B}_n\\ \tilde{\mathcal{A}}_n & \tilde{\mathcal{B}}_n\end{bmatrix}\,,\quad \mathcal{F}_n=\begin{bmatrix}\mathcal{C}_n & 0_{2\times 2}\\ \tilde{\mathcal{C}}_n & \tilde{\mathcal{D}}_n\end{bmatrix}\,.$$

Note that $\mathcal{E}_n$ is invertible,

$$\mathcal{E}_n^{-1}=\frac{\gamma_{n-1}}{\gamma_n\Theta_{n-2}}\begin{bmatrix}(y_2-\beta_n)I & -I\\ -\mathcal{B}^{-1}\tilde{\mathcal{A}}_n & \mathcal{B}^{-1}\mathcal{A}_n\end{bmatrix}\,. \tag{56}$$

From (55) there follows

$$\begin{bmatrix}\psi_{n-1}(y_1)\\ \psi_{n-1}(y_2)\end{bmatrix}=\mathcal{G}_n\begin{bmatrix}\psi_{n-2}(y_1)\\ \psi_{n-2}(y_2)\end{bmatrix}\,,\quad \mathcal{G}_n=\mathcal{E}_n^{-1}\mathcal{F}_n\,, \tag{57}$$

being $\mathcal{G}_n$ an invertible matrix as is a product of invertible matrices.

Take $n+1$ in (57). On the one hand we have

$$\begin{bmatrix}\psi_{n}(y_1)\\ \psi_{n}(y_2)\end{bmatrix}=\mathcal{G}_{n+1}\begin{bmatrix}\psi_{n-1}(y_1)\\ \psi_{n-1}(y_2)\end{bmatrix} \tag{58}$$

and, on the other hand, using the three term recurrence relation (51), we have

$$\begin{bmatrix}\psi_{n}(y_1)\\ \psi_{n}(y_2)\end{bmatrix}=\begin{bmatrix}(y_1-\beta_n)I & 0\\ 0 & (y_2-\beta_n)I\end{bmatrix}\begin{bmatrix}\psi_{n-1}(y_1)\\ \psi_{n-1}(y_2)\end{bmatrix}-\gamma_n\begin{bmatrix}\psi_{n-2}(y_1)\\ \psi_{n-2}(y_2)\end{bmatrix}\,,$$

thus,

$$\begin{bmatrix}\psi_{n}(y_1)\\ \psi_{n}(y_2)\end{bmatrix}=\left(\begin{bmatrix}(y_1-\beta_n)I & 0\\ 0 & (y_2-\beta_n)I\end{bmatrix}-\gamma_n\mathcal{G}_n^{-1}\right)\begin{bmatrix}\psi_{n-1}(y_1)\\ \psi_{n-1}(y_2)\end{bmatrix}\,. \tag{59}$$

Consequently, (58) and (59) yield

$$\mathcal{G}_{n+1}=\begin{bmatrix}(y_1-\beta_n)I & 0\\ 0 & (y_2-\beta_n)I\end{bmatrix}-\gamma_n\mathcal{G}_n^{-1}\,. \tag{60}$$

Let us compute $\mathcal{G}_n^{-1}$. Taking into account (56), we obtain

$$\mathcal{G}_n = \begin{bmatrix} X_n I & Y_n \mathcal{B} \\ U_n \mathcal{B}^{-1} & V_n I \end{bmatrix}, \tag{61}$$

where X_n, Y_n, U_n, V_n are the functions given by

$$X_n = \alpha_n \left((y_2 - \beta_n)\Theta_{n-1} + \gamma_n (\frac{A}{y_2 - y_1} + l_n) \right),$$
$$Y_n = -\gamma_n \alpha_n ,$$
$$U_n = \alpha_n \left((\frac{A}{y_2 - y_1} + l_n) \left[\gamma_n (\frac{A}{y_2 - y_1} + l_{n-1}) + (y_1 - \beta_n)\Theta_{n-1} \right] + \Theta_n \Theta_{n-1} \right),$$
$$V_n = -\gamma_n \alpha_n (\frac{A}{y_2 - y_1} + l_{n-1}),$$

with $\alpha_n = \dfrac{\gamma_{n-1}}{\gamma_n \Theta_{n-2}}$. Therefore, it turns out that

$$\mathcal{G}_n^{-1} = \frac{1}{\delta_n} \begin{bmatrix} V_n I & -Y_n \mathcal{B} \\ -U_n \mathcal{B}^{-1} & X_n I \end{bmatrix}, \tag{62}$$

where δ_n is the function given by $\delta_n = X_n V_n - Y_n U_n$.

Taking into account (61) and (62), (60) reads

$$X_{n+1} = (y_1 - \beta_n) - \gamma_n V_n / \delta_n , \tag{63}$$
$$Y_{n+1} = \gamma_n Y_n / \delta_n , \tag{64}$$
$$U_{n+1} = \gamma_n U_n / \delta_n \tag{65}$$
$$V_{n+1} = (y_2 - \beta_n) - \gamma_n X_n / \delta_n . \tag{66}$$

From (63)-(66) there follows that $\delta_{n+1} = X_{n+1} V_{n+1} - Y_{n+1} U_{n+1}$ is given by

$$\delta_{n+1} = (y_1 - \beta_n)(y_2 - \beta_n) - \gamma_n \left((y_1 - \beta_n) X_n + (y_2 - \beta_n) V_n \right) \frac{1}{\delta_n} + \frac{\gamma_n^2}{\delta_n}.$$

Now we proceed analogously with Magnus [14, 15]. Write $\delta_n = \mu_n / \mu_{n-1}$. Then, we obtain

$$\mu_n X_{n+1} = (y_1 - \beta_n)\mu_n - \gamma_n \mu_{n-1} V_n ,$$
$$\mu_n V_{n+1} = (y_2 - \beta_n)\mu_n - \gamma_n \mu_{n-1} X_n ,$$
$$\mu_{n+1} = (y_1 - \beta_n)(y_2 - \beta_n)\mu_n - \gamma_n \mu_{n-1}((y_1 - \beta_n) X_n + (y_2 - \beta_n) V_n) + \gamma_n^2 \mu_{n-1}.$$

The change of variables

$$\hat{X}_{n+1} = \mu_n X_{n+1}, \; \hat{V}_{n+1} = \mu_n V_{n+1}$$

yields the relations

$$\hat{X}_{n+1} = (y_1 - \beta_n)\mu_n - \gamma_n \hat{V}_n \,,$$
$$\hat{V}_{n+1} = (y_2 - \beta_n)\mu_n - \gamma_n \hat{X}_n \,,$$
$$\mu_{n+1} = (y_1 - \beta_n)(y_2 - \beta_n)\mu_n - \gamma_n \left((y_1 - \beta_n)\hat{X}_n + (y_2 - \beta_n)\hat{V}_n \right) + \gamma_n^2 \mu_{n-1} \,.$$

Remark that the above recurrence relations for $\hat{X}_n, \hat{V}_n$ and μ_n are precisely the recurrence relations satisfied by the products of solutions of the three term recurrence relation (16) at y_1 and y_2. Indeed, if

$$\xi_{n+1} = (y_1 - \beta_n)\xi_n - \gamma_n \xi_{n-1} \,, \quad \eta_{n+1} = (y_2 - \beta_n)\eta_n - \gamma_n \eta_{n-1} \,,$$

then the above recurrence relation for $\hat{X}_n, \hat{V}_n, \mu_n$ is precisely the relation for $\xi_n \eta_{n-1}$, $\xi_{n-1}\eta_n, \xi_n \eta_n$, respectively. Taking into account that a basis of the three term recurrence relation $\tau_{n+1} = (x - \beta_n)\tau_n - \gamma_n \tau_{n-1}$ is constituted by $\{P_n\}$ and $\{q_n\}$, (cf. (16) and (18)), the following must hold: ξ_n must be a combination of $P_n(y_1)$ and $q_n(y_1)$, and η_n must be a combination of $P_n(y_2)$ and $q_n(y_2)$. Thus, there are four choices to be considered:

$$(i) \; \xi_n = P_n(y_1), \; \eta_n = P_n(y_2),$$
$$(ii) \; \xi_n = P_n(y_1), \; \eta_n = q_n(y_2) \,,$$
$$(iii) \; \xi_n = q_n(y_1), \; \eta_n = P_n(y_2) \,,$$
$$(iv) \; \xi_n = q_n(y_1), \; \eta_n = q_n(y_2).$$

Therefore, we obtain

$$\mu_n = \alpha P_n(y_1)P_n(y_2) + \beta P_n(y_1)q_n(y_2) + \gamma q_n(y_1)P_n(y_2) + \delta q_n(y_1)q_n(y_2) \,. \quad (67)$$

Taking $n = 0$ in (67) we obtain

$$\mu_0 = \alpha + \beta q_0(y_2) + \gamma q_0(y_1) + \delta q_0(y_1)q_0(y_1) \,,$$

and such a relation is $A\mathbb{D}S = B\mathbb{E}_1 S\, \mathbb{E}_2 S + C\mathbb{M}S + D$, with

$$A = \frac{(\gamma - \beta)}{2}(y_2 - y_1), \; B = \delta, \; C = \gamma + \beta, \; D = \alpha - \mu_0 \,.$$

Proof of $(a) \Rightarrow (c)$**.**
Note that $A\,\mathbb{D}S = B\,\mathbb{E}_1 S\,\mathbb{E}_2 S + C\,\mathbb{M}S + D$ is

$$A\,\mathbb{D}q_n = l_{n-1}\,\mathbb{E}_1 q_n + (B\,\mathbb{E}_1 S + C/2)\,\mathbb{E}_2 q_n + \Theta_{n-1}\,\mathbb{E}_1 q_{n-1}$$

with $n = 0$, since $q_{-1} = 1, q_0 = S, l_{-1} = C/2, \Theta_{-1} = D$.

Let us now deduce the above difference equation for $n \geq 1$.

Applying $A\mathbb{D}$ to $q_n = P_n S - P^{(1)}_{n-1}$, $n \geq 1$ (cf. (17)), and using the property (11) we obtain

$$A\,\mathbb{D}q_n = A\,\mathbb{D}P_n\,\mathbb{E}_1 S + A\,\mathbb{D}S\,\mathbb{E}_2 P_n - A\,\mathbb{D}P^{(1)}_{n-1}\,.$$

Using the equations (31) as well as (20) in the above equation, we obtain

$$A\,\mathbb{D}q_n = l_{n-1}\mathbb{E}_1(P_n S - P^{(1)}_{n-1}) + (B\,\mathbb{E}_1 S + C/2\,)\,\mathbb{E}_2(P_n S - P^{(1)}_{n-1}) \\ + \Theta_{n-1}\mathbb{E}_1(P_{n-1}S - P^{(1)}_{n-2})\,,$$

thus (32) follows.

Proof of $(c) \Rightarrow (a)$**.**

Take $n = 0$ in (32).

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