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MORE ON SUBFITNESS AND FITNESS

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Abstract: The concepts of fitness and subfitness (as defined in Isbell [9]) are useful separation properties in point-free topology. The categorical behaviour of subfitness is bad and fitness is the closest modification that behaves well. The separation power of the two, however, differs very substantially and subfitness is transparent and turns out to be useful in its own right. Sort of supplementing the article [20] we present several facts on these concepts and their relation. First the "supportive" role subfitness plays when added to other properties is emphasized. In particular we prove that the numerous Dowker-Strauss type Hausdorff axioms become one for subfit frames. The aspects of fitness as a hereditary subfitness are analyzed, and a simple proof of coreflectivity of fitness is presented. Further, another property, prefitness, is shown to also produce fitness by heredity, in this case in a way usable for classical spaces, which results in a transparent characteristics of fit spaces. Finally, the properties are proved to be independent.

Keywords: Frame, locale, sublocale lattice, fit frame, subfit frame, prefit frame, Hausdorff property, nearness, quasi-nearness.

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1. Introduction

In his celebrated paper [9], Isbell introduced the concepts of fitness and subfitness. The subfitness was (with some regret) soon dismissed, after an application in compact context, because of its bad categorical behaviour: it was not generally inherited by subobjects and by products, while fit frames constituted a coreflective subcategory of the category of frames. Let us mention right away, though, that subfitness can be translated into a transparent property of a separation nature while fitness has not been given for a while any intuitive geometrical interpretation (even the topological description given later in [7] was not very satisfactory; the first one that is really

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simple may be that given below in 4.5.2). Somewhat unexpectedly, subfitness occurred in [7] as the necessary and sufficient condition for admitting a nearness (not only the regular one) extending the general concept defined for spaces by Herrlich in [6]. This prompted Simmons (who had introduced subfitness in [19] – independently and for other purposes – as *conjunctivity*) to write his very interesting paper [20] discussing several aspects of the mentioned properties.

The present article can be viewed as a supplement to [19]. In Section 3 we discuss the role of subfitness as a supporting separation property. Besides mentioning the well-known associations of subfitness with T_D and with normality we prove that numerous Hausdorff axioms of Dowker-Strauss type merge when this condition is added. In Section 4 we discuss some aspects of fitness as a hereditary subfitness and present a simple proof of coreflectivity of fitness based on this and on localic techniques. Then, we introduce another concept, prefitness, which also has the property that it produces fitness if modified by heredity. Unlike subfitness, it suffices to assume it for closed sublocales, so that it can be applied for subspaces and yield a simple and transparent characteristics of fit spaces. Further we show that subfitness, obviously inherited by closed sublocales, is inherited by all complemented ones. In the last section we confront the discussed properties and show their independence.

2. Preliminaries

2.1. Recall that a *frame* resp. *co-frame* is a complete lattice L satisfying the distributive law

$$a \wedge (\bigvee B) = \bigvee\{a \wedge b \mid b \in B\} \quad \text{resp. } a \vee (\bigwedge B) = \bigwedge\{a \vee b \mid b \in B\}$$

for all $a \in L$ and $B \subseteq L$. A typical frame is the lattice

$$\Omega(X)$$

of all open sets of a topological space X. A *frame homomorphism* $h\colon L \to M$ preserves all joins and finite meets; if $f\colon X \to Y$ is a continuous map we have a frame homomorphism $\Omega(f)\colon \Omega(Y) \to \Omega(X)$ defined by $\Omega(f)(U) = f^{-1}[U]$.

2.2. If L is a frame, the mapping $(x \mapsto x \wedge a)\colon L \to L$ preserves suprema and hence it is a left Galois adjoint; thus we have the (uniquely defined) *Heyting* *operation* $x \to y$ satisfying

$$a \wedge b \le c \quad \text{iff} \quad a \le b \to c.$$

We will use some standard facts like $1 \to a = a$, $b \le a \to b$, $a \le b$ iff $a \to b = 1$, $a \to (b \to c) = (a \wedge b) \to c = b \to (a \to c)$ or $a \to (\bigwedge b_i) = \bigwedge (a \to b_i)$.

2.2.1. For each a in a frame we have the pseudocomplement

$$a^* = \bigvee \{x \mid x \wedge a = 0\} = a \to 0.$$

We will use the standard facts like $a \le b \Rightarrow b^* \le a^*$, $a \le a^{**}$ or $a^* = a^{***}$.

We set $a \prec b$ for $a^* \vee b = 1$; note that in $\Omega(X)$, $U \prec V$ says that $\overline{U} \subseteq V$.

2.3. Here are some separation axioms used for frames:

(sfit): $a \not\le b \Rightarrow \exists c,\ a \vee c = 1 \ne b \vee c.$
(fit): $a \not\le b \Rightarrow \exists c,\ a \vee c = 1\ \&\ c \to b \ne b.$
(reg): $\forall a,\ a = \bigvee \{x \mid x \prec a\}.$
(norm): $a \vee b = 1 \Rightarrow \exists u, v$ such that $u \wedge v = 0$ and $a \vee v = 1 = u \vee b.$

One speaks on *subfit*, *fit*, *regular* and *normal* frames, in this order. The subfitness is relaxed to weak subfitness ([7]; cf. property Π_0 in [22])

(wsfit): $a \not\le 0 \Rightarrow \exists c \ne 1, a \vee c = 1.$

$\Omega(X)$ is regular resp. normal iff X is regular or normal in the classical sense.

2.4. One thinks of a frame L as of a generalized space. One of several representations of a (generalized) subspace of L is that of a *sublocale*. It is a subset $S \subseteq L$ such that

(S1) $M \subseteq S \Rightarrow \bigwedge M \in S$, and
(S2) $\forall x \in L, \forall s \in S, x \to s \in S$.

S is a frame in the order of L and inherits its Heyting structure; the left adjoint

$$\nu_S \colon L \to S \tag{2.4.1}$$

to the embedding map $j = j_S \colon S \subseteq L$ is a surjective frame homomorphism given by $\nu_S(x) = \bigwedge \{s \in S \mid s \ge x\}$. The system of all sublocales constitutes a co-frame

$$\mathcal{S}\ell(L)$$

with the order given by inclusion, meet coinciding with the intersection and the join given by

$$\bigvee S_i = \{\bigwedge M \mid M \subseteq \bigcup S_i\};$$

the top is L and the bottom is the set $\mathsf{O} = \{1\}$.

2.4.1. A sublocale S is complemented if there is a sublocale T such that $S \vee T = L$ and $S \cap T = \mathsf{O}$. An important property of complemented S is that for any system T_i, $i \in I$, of sublocales one has

$$S \cap \bigvee T_i = \bigvee (S \cap T_i)$$

(note that this is exceptional: $\mathcal{S}\ell(L)$ is a co-frame, not a frame; in fact this law characterizes complementarity – see [14, VI.4.4.3]).

2.4.2. Open resp. closed subspaces are represented by open resp. closed sublocales

$$\mathfrak{o}(a) = \{x \mid a \to x = x\} = \{a \to x \mid x \in L\} \quad \text{resp. } \mathfrak{c}(a) = \uparrow a = \{x \mid x \geq a\}.$$

$\mathfrak{o}(a)$ and $\mathfrak{c}(a)$ are complements of each other. Here are a few rules (see [14]):

- $\mathfrak{o}(0) = \mathsf{O}, \mathfrak{o}(1) = L, \mathfrak{o}(a \wedge b) = \mathfrak{o}(a) \cap \mathfrak{o}(b), \mathfrak{o}(\bigvee a_i) = \bigvee \mathfrak{o}(a_i)$,
- $\mathfrak{c}(0) = L, \mathfrak{c}(1) = \mathsf{O}, \mathfrak{c}(a \wedge b) = \mathfrak{c}(a) \vee \mathfrak{c}(b), \mathfrak{c}(\bigvee a_i) = \bigcap \mathfrak{c}(a_i)$,
- $\mathfrak{o}(a) \cap \mathfrak{c}(b) \neq \mathsf{O}$ iff $a \not\leq b$,
- $\mathfrak{c}(a) \subseteq \mathfrak{o}(b)$ iff $a \vee b = 1$,
- $T = S \cap \mathfrak{o}(a)$ is an open sublocale in the sublocale S. More precisely, $T = \mathfrak{o}_S(\nu_S(a))$. Similarly for closed sublocales.

Due to (S2) one has an extremely simple formula for the closure $\overline{S}$ (the smallest closed sublocale containing S):

$$\overline{S} = \uparrow \bigwedge S.$$

It is easy to see that

$$\mathfrak{o}(a) \cap S \neq \mathsf{O} \quad \text{iff} \quad \mathfrak{o}(a) \cap \overline{S} \neq \mathsf{O} \tag{2.4.2}$$

(since $\mathfrak{o}(a) \cap S = \mathsf{O}$ iff $S \subseteq \uparrow a$ iff $\overline{S} \subseteq \uparrow a$).

2.4.3. Note that the original definitions of fitness and subfitness in [9] are (in our present terminology)

(sfit): each open sublocale is a join of closed ones,
(fit): each sublocale is a meet of open ones.

Now we think of these statements rather as of characterization theorems.

For more about frames see e.g. [10, 14, 13, 17, 16].

2.5. A *cover* of a frame L is a subset $C \subseteq L$ such that $\bigvee C = 1$. For a cover C and an element $x \in L$ resp. sublocale $S \subseteq L$ set

$$Cx = \bigvee\{c \in C \mid c \wedge x \neq 0\} \quad \text{resp.} \ CS = \bigvee\{c \in C \mid \mathfrak{o}(c) \cap S \neq \mathsf{O}\}. \quad (2.5.1)$$

Note that $Cx = C\mathfrak{o}(x)$ (since $\mathfrak{o}(c) \cap \mathfrak{o}(x) \neq \mathsf{O}$ iff $\mathfrak{c}(c) \vee \mathfrak{c}(x) = {\uparrow}c \vee {\uparrow}x \neq L$ iff $c \wedge x \neq 0$).

A system $\mathscr{C}$ of covers is *admissible* resp. *quasi-admissible* if

$$\forall a \in L,\ a = \bigvee\{x \mid \exists C \in \mathscr{C}, Cx \leq a\} \quad \text{resp.} \ \mathfrak{o}(a) = \bigvee\{S \mid \exists C \in \mathscr{C}, CS \leq a\}.$$

Note that $Cx \leq a$ implies $x \prec a$, and that, by (2.4.2), $C\overline{S} = CS$. Thus (using also 2.4.3) we see that

2.5.1. *The existence of an admissible system of covers implies regularity and the existence of a quasi-admissible system of covers implies subfitness.*

2.6. We say that a cover A *refines* a cover B and write $A \leq B$ if for every $a \in A$ exists a $b \in B$ such that $a \leq b$.

A *nearness* (see e.g. [1]) on L is an admissible system of covers $\mathscr{A}$ such that

(N1) if $A \in \mathscr{A}$ and $A \leq B$ then $B \in \mathscr{A}$, and
(N2) if $A, B \in \mathscr{A}$ then $A \wedge B = \{a \wedge b \mid a \in A, b \in B\} \in \mathscr{A}$.

This extends the concept of a *regular* nearness, as defined for spaces by Herrlich [6], to the point-free context. If we wish to extend the concept of *general* nearness, we relax the admissibility to quasi-admissibility and speak of a *quasi-nearness* or *generalized nearness*.

A *basis of* (*quasi-*)*nearness* is a system of covers $\mathcal{B}$ such that $\mathcal{A} = \{C \mid C \geq B \in \mathcal{B}\}$ is a (quasi-)nearness. Note that obviously $\mathcal{A}$ is (quasi-)admissible iff $\mathcal{B}$ is.

Convention. We say that a frame L "admits a quasi-nearness" rather than that it "quasi-admits a nearness".

2.6.1. Proposition. *A frame admits a nearness* (*resp. a quasi-nearness*) *iff it is regular* (*resp. subfit*).

(The implications $\Rightarrow$ are in 2.5.1, the implication $\Leftarrow$ for nearness is almost trivial; for a quasi-nearness see e.g. [7].)

3. Subfitness as a supportive property

3.1. Subfitness added to another requirement often results in a more desirable property. First, however, let us recall the nature of plain subfitness. It can be viewed as a separation axiom slightly weaker than T_1. For spaces we have

Theorem (Simmons [20], Isbell [9])**.** *A topological space is subfit iff for every $x \in U$ open there exists a $y \in \overline{\{x\}}$ such that $\overline{\{y\}} \subseteq U$.*

Recall that a space X is T_D if for each $x \in X$ there is an open U such that $x \in U$ and $U \smallsetminus \{x\}$ is open.
From the theorem above we now easily infer

3.2. Proposition. *A topological space is T_1 iff it is T_D and subfit.*

In [2] the point-free aspects of the T_D axiom were discussed. The following property makes a spatial frame representable by a T_D-space:

$$\text{each prime element } p \text{ in } L \text{ is completely prime.} \qquad (\mathrm{pf}T_D)$$

From the facts in [2] we can infer

3.2.1. Proposition. *Let L satisfy* (pfT_D) *and* (sfit)*. Then the spectrum ΣL is T_1.*

3.3. The Hausdorff axiom is mimicked in point-free topology using a number of different requirements. The strongest is the Isbell's Hausdorff axiom requiring that the codiagonal in the coproduct $L \oplus L$ be closed. Then there is a number of variations on the Dowker-Strauss separation from [3, 4]. Let us list them:

- S_2': if $a \vee b = 1$ and $a, b \neq 1$ then there are u, v with $u \not\leq a$, $v \not\leq b$ and $u \wedge v = 0$ (the axiom from [3]).
- S_2'': if $a \not\leq b$ and $b \not\leq a$ then there are u, v with $u \not\leq a$, $v \not\leq b$ and $u \wedge v = 0$ (the standard strengthening of S_2').
- T_2': if $1 \neq a \not\leq b$ then there are u, v with $u \not\leq a$, $v \not\leq b$ and $u \wedge v = 0$ (P. Johnstone & Sun Shu Hao [11]).

T_2: if $1 \neq a \nleq b$ then there are u, v with $u \nleq a$, $v \nleq b$, $v \leq a$ and $u \wedge v = 0$ (Paseka & Šmarda [12]).

$T_<$: if $b < a \neq 1$ then there are u, v with $u \nleq a$, $v \nleq b$, $v \leq a$ and $u \wedge v = 0$ (equivalent with T_2, introduced for technical reasons).

$\overline{S_2}$: if $a \nleq b$ and $b \nleq a$ then there are u, v with $u \nleq a$, $v \nleq b$, $v \leq a$, $u \leq b$ and $u \wedge v = 0$.

The relations between them are depicted in the following diagram:

$$\begin{tikzcd} T_2 \arrow[dd, Rightarrow] & \overline{S_2} \,\&\, T_< \arrow[l, Rightarrow] \arrow[d, Rightarrow] & \\ & \overline{S_2} \arrow[d, Rightarrow] & \\ T_2' \arrow[r, Rightarrow] & S_2'' \arrow[r, Rightarrow] & S_2' \end{tikzcd} \qquad (*)$$

Further one has axioms based on the properties of meet irreducibility (see [18]), weaker than S_2'', but we are not discussing them here.

3.3.1. To obtain a property that would coincide with the Hausdorff one in the spatial case, Dowker & Strauss [4] introduced the combination

$$S_{2sf} \equiv S_2' \;\&\; (\text{sfit}).$$

Now adding the subfitness in fact identifies all the axioms of the group $(*)$. We have

Proposition. $\overline{S_2}$ & $T_<$ & (sfit) $\equiv T_2$ & (sfit) $\equiv \overline{S_2}$ & (sfit) $\equiv T_2'$ & (sfit) $\equiv S_2'$ & (sfit) $\equiv S_2''$ & (sfit).

Proof: Let L be subfit and let it satisfy S_2'. First we will show that it satisfies S_2''.

Let $a \nleq b$ and $b \nleq a$. Then there is a c such that $a \vee c = 1 \neq b \vee c$. Hence $a \nleq b \vee c$ (else $1 = a \vee c \leq b \vee c$) and $a \vee (b \vee c) = 1$ so that we have $u \nleq a$ and $v \nleq b \vee c$ (and hence $v \nleq b$) such that $u \wedge v = 0$.

Now, again, take $a \nleq b$ and $b \nleq a$. Then there are c_1, c_2 such that

$$a \vee c_1 = 1 \neq b \vee c_1 \quad \text{and} \quad a \vee c_2 \neq 1 = b \vee c_2.$$

We have

$$b \vee c_1 \nleq a \vee c_2 \quad \text{and} \quad a \vee c_2 \nleq b \vee c_1$$

(indeed, if $b \le a \vee c_2$ then $1 = b \vee c_2 \le a \vee c_2$; the other statement by symmetry). We already know that L satisfies S_2'' and hence we have u', v' such that $u' \nleq a \vee c_2$, $v' \nleq b \vee c_1$ and $u' \wedge v' = 0$. Then $u = u' \wedge b \nleq a$ since otherwise $u = u \wedge (b \vee c_2) \le a \vee (u \wedge c_2) \le a \vee c_2$, and similarly $v = v' \wedge a \nleq b$. Hence we have $\overline{S_2}$.

Finally let $b < a \neq 1$. Again take a c such that $a \vee c = 1 \neq b \vee c$. Then we have $a \vee (b \vee c) = 1$ with $a, b \vee c$ incomparable, and using $\overline{S_2}$ for this pair we obtain $T_<$. ■

3.4. Let us also recall the standard fact that normality does not imply regularity but augmented with subfitness it does.

3.4.1. In [5] the authors introduced, a.o., the concept of *almost normality*:

(a.norm): if $a \vee b = 1$ and $a = a^{**}$ then there are u, v with $a \vee v = 1 = u \vee b$ and $u \wedge v = 0$.

Note that

3.4.2. *A frame L is almost normal iff the relation $\prec$ in L interpolates.*

(Indeed: if $a \prec b$ then $a^* \vee b = 1$ and we have u, v with $u \wedge v = 0$ – and hence $v^* \ge u$ – and $a^* \vee v = 1 = u \vee b$. Then $a \prec v \prec b$. On the other hand, if $\prec$ interpolates and $a^{**} \vee b = 1$ then $a^* \prec b$ and we have $a^* \prec v \prec b$ for some v; then $a^* \vee v = 1$ and $v^* \vee b = 1$.)

3.4.3. Similarly with the implication (normal) & (subfit) $\Rightarrow$ (regular) we have

Proposition. *If L is almost normal and subfit then for every $a \in L$*

$$a^{**} = \bigvee\{x \mid x \prec a^{**}\}.$$

Proof: Suppose $a^{**} \nleq b = \bigvee\{x \mid x \prec a^{**}\}$. By (sfit) there is a c such that $a \vee c = 1 \neq c \vee b$. Then, by (a.norm) there is a u such that $a \vee u^* = 1 = u \vee c$. Hence we have $u \prec a$, and then $u \le b$ and $b \vee c = 1$, a contradiction. ■

4. Subfitness and others in sublocales

4.1. Fitness is well-known to be inherited by all sublocales and it implies subfitness. Although the following fact is standard, we will present a proof. It will be shorter than the proofs usually found in literature, but first of all, it will introduce a class of sublocales which will be of interest later.

Proposition. *If every sublocale of a locale L is subfit then L is fit.*

Proof: Suppose not. Then there are $a \nleq b$ such that $a \vee u = 1 \Rightarrow u \to b = b$. Set

$$S = \{x \mid a \vee u = 1 \Rightarrow u \to x = x\}.$$

Then S is obviously a sublocale and $b \in S$. We also have $a \in S$ since if $a \vee u = 1$ then $a = (a \vee u) \to a = (a \to a) \wedge (u \to a) = u \to a$. If S is subfit there is a $c \in S$ such that $a \vee c = 1$ and $b \vee c \neq 1$; but since $c \in S$, $a \vee c = 1$ implies $c \to c = c$ and $c \to c = 1$, a contradiction. ∎

4.1.1. Note that the sublocale S from the previous proof is in fact the intersection

$$\mathfrak{sc}(a) = \bigcap\{\mathfrak{o}(u) \mid \mathfrak{c}(a) \subseteq \mathfrak{o}(u)\}.$$

We will speak of these sublocales as of *semiclosed* ones. From proof in 4.1 we now obtain

Theorem. *The following are equivalent for any frame L:*

(1) *L is fit.*
(2) *Each semiclosed sublocale is closed.*
(3) *Each semiclosed sublocale is subfit.*

Proof: (1)⇔(2) is in the standard definition of fitness, (1)⇒(3) is trivial, and for (3)⇒(1) realize that in the proof above we have shown that in the non-fit case $\mathfrak{sc}(a)$ is not subfit. ∎

4.1.2. Obviously, if every closed sublocale of L is weakly subfit then L is subfit. Consequently, we have:

Corollary. *If every sublocale of a frame L is weakly subfit then L is fit.*

4.1.3. Note the sharp contrast between inheriting subfitness by all the subspaces of a space X and by all the sublocales of $\Omega(X)$. In the former case we will not need more than T_1 (stronger than (sfit) and subspace hereditary). In the latter one we obtain in fact a rather strong separation akin of regularity, see 4.4-4.5.2.

4.2. It is easy to see that subfitness is inherited by every closed sublocale. But we have much more. For that we need the following lemma about the operator from (2.5.1) and the map from (2.4.1).

4.2.1. Lemma. *Let S be a sublocale of L. For any cover C of L and any $a, b \in L$,*

$$C\mathfrak{c}(b) \le a \Rightarrow \nu_S[C](\mathfrak{c}(b) \cap S) \le \nu_S(a).$$

Proof: Suppose $\mathfrak{o}(\nu_S(c)) \cap \mathfrak{c}(b) \cap S \ne \mathsf{O}$. Then there is a $s \ge b$ in S such that $\nu_S(c) \to s = s$ from which it follows that $s = j_S(\nu_S(c) \to s) = c \to s$ (see the localic map formula in [14, II.2.3]), that is, $s \in \mathfrak{o}(c)$. Therefore $\mathfrak{o}(c) \cap \mathfrak{c}(b) \ne \mathsf{O}$ and consequently $c \le a$. Thus $\nu_S(c) \le \nu_S(a)$. ■

4.2.2. Proposition. *Let $\mathcal{N}$ be a quasi-nearness on L. If S is a complemented sublocale of L, then $\{\nu_S[C] \mid C \in \mathcal{N}\}$ is a basis of a quasi-nearness on S.*

Proof: It is obvious that $\mathcal{B} = \{\nu_S[C] \mid C \in \mathcal{N}\}$ is a basis of nearness on S. We have to show that it is quasi-admissible.

Let T be an open sublocale of S. Then $T = \mathfrak{o}_S(a) = S \cap \mathfrak{o}(a)$ for some $a \in S$. By the hypothesis we have

$$\mathfrak{o}(a) = \bigvee\{\mathfrak{c}(x) \mid x \in L, \exists C \in \mathcal{N}, C\mathfrak{c}(x) \le a\}$$

and so, using the lemma, we obtain

$$\begin{aligned}\mathfrak{o}_S(a) = S \cap \mathfrak{o}(a) &= S \cap (\bigvee\{\mathfrak{c}(x) \mid x \in L, \exists C \in \mathcal{N}, C\mathfrak{c}(x) \le a\})\\ &= \bigvee\{S \cap \mathfrak{c}(x) \mid x \in L, \exists C \in \mathcal{N}, C\mathfrak{c}(x) \le a\}\\ &\le \bigvee\{S \cap \mathfrak{c}(x) \mid x \in L, \exists C \in \mathcal{N}, \nu_S[C](S \cap \mathfrak{c}(x)) \le a\}\\ &\le \bigvee\{\mathfrak{c}_S(x) \mid x \in S, \exists C \in \mathcal{N}, \nu_S[C]\mathfrak{c}_S(x) \le a\}. \quad ■\end{aligned}$$

4.2.3. Since a frame is subfit iff it admits a quasi-nearness we obtain

Corollary. *Let L be subfit. Then each of its complemented sublocales is subfit.* ■

4.3. The point of view of 4.1.1 can be used for a simple proof of the coreflectivity of the category **FitFrm** of fit frames in the category **Frm** of all frames.

For that recall the localic map $f\colon M \to L$ associated with a frame homomorphism $h\colon L \to M$, that is, its right Galois adjoint ([14], II.3) and the concepts of image and coimage of a sublocale ([14], III.4 – $f[S]$ is the set theoretical image, $f_{-1}[S]$ is the set theoretical preimage slightly modified). One has $f[S] \subseteq T$ iff $S \subseteq f_{-1}[T]$ and hence $f_{-1}[-]$ preserves meets. We will use the formulas ([14], III.6.3)

$$f_{-1}[\mathfrak{c}(a)] = \mathfrak{c}(h(a)) \quad \text{and} \quad f_{-1}[\mathfrak{o}(a)] = \mathfrak{o}(h(a)). \tag{4.3}$$

4.3.1. For a frame L define

$$F_1(L) = \{a \in L \mid \mathfrak{c}(a) = \mathfrak{sc}(a)\} = \{a \in L \mid \mathfrak{c}(a) = \bigcap\{\mathfrak{o}(x) \mid \mathfrak{c}(a) \subseteq \mathfrak{o}(x)\}\}.$$

Explicitly, $a \in F_1(L)$ iff

$$(a \vee u = 1 \Rightarrow u {\to} x = x) \Rightarrow x \geq a. \tag{$*$}$$

Lemma. *$F_1(L)$ is a subframe of L, and $F_1(L) = L$ iff L is fit.*

Proof: Obviously, $0, 1 \in F_1(L)$. Let $a_i \in F_1(L)$. We will show that $\bigvee a_i$ satisfies $(*)$. Assume

$$\bigvee a_i \vee u = 1 \Rightarrow u {\to} x = x$$

and suppose that $a_i \vee u = 1$. Then $\bigvee a_i \vee u = 1$ and consequently $u {\to} x = x$. Since a_i satisfies $(*)$, we have that $x \geq a_i$ for every i and hence $x \geq \bigvee a_i$.

Finally let $a, b \in F_1(L)$. We have

$$\begin{aligned}
\mathfrak{c}(a \wedge b) = \mathfrak{c}(a) \vee \mathfrak{c}(b) &= \bigcap\{\mathfrak{o}(x) \mid \mathfrak{c}(a) \subseteq \mathfrak{o}(x)\} \vee \bigcap\{\mathfrak{o}(y) \mid \mathfrak{c}(b) \subseteq \mathfrak{o}(y)\} \\
&= \bigcap\{\mathfrak{o}(x \vee y) \mid \mathfrak{c}(a) \subseteq \mathfrak{o}(x), \mathfrak{c}(b) \subseteq \mathfrak{o}(y)\} \\
&\supseteq \bigcap\{\mathfrak{o}(u) \mid \mathfrak{c}(a) \vee \mathfrak{c}(b) \subseteq \mathfrak{o}(u)\}.
\end{aligned}$$

This shows that $F_1(L)$ is a subframe of L.

The second statement is in 4.1.1. ∎

4.3.2. For ordinals α define F_α as follows:

$$F_0(L) = L,\ F_{\alpha+1} = F_1(F_\alpha(L)) \text{ and } F_\alpha(L) = \bigcap_{\beta<\alpha} F_\beta(L) \text{ for a limit ordinal.}$$

Since $F_\alpha(L)$ decrease there is an ordinal $\gamma(L)$ such that $F_1(F_{\gamma(L)}(L)) = F_{\gamma(L)}(L)$. Set
$$F(L) = F_{\gamma(L)}(L).$$

Theorem. *F can be extended to a functor* **Frm** $\to$ **FitFrm** *and together with the inclusion homomorphisms $\iota_L\colon F(L) \to L$ it constitutes a coreflection.*

Proof: It suffices to show that for each frame homomorphism $h\colon L \to M$ one has $h[F_1(L)] \subseteq F_1(M)$. Let $a \in F_1(L)$ and consider the localic map f adjoint to h. We have
$$\mathfrak{c}(a) = \bigcap\{\mathfrak{o}(x) \mid \mathfrak{c}(a) \subseteq \mathfrak{o}(x)\}$$
and hence, by (4.3) and since $f_{-1}[-]$ preserves meets,
$$\begin{aligned}\mathfrak{c}(h(a)) = f_{-1}[\mathfrak{c}(a)] &= f_{-1}[\bigcap\{\mathfrak{o}(x) \mid \mathfrak{c}(a) \subseteq \mathfrak{o}(x)\}]\\ &= \bigcap\{f_{-1}[\mathfrak{o}(x)] \mid \mathfrak{c}(a) \subseteq \mathfrak{o}(x)\}\\ &= \bigcap\{\mathfrak{o}(h(x)) \mid \mathfrak{c}(a) \subseteq \mathfrak{o}(x)\} \supseteq \bigcap\{\mathfrak{o}(y) \mid \mathfrak{c}(h(a)) \subseteq \mathfrak{o}(y)\}. \quad \blacksquare\end{aligned}$$

4.4. A frame is said to be *prefit* if
$$a \neq 0 \quad \Rightarrow \quad \exists x \neq 0,\ x \prec a. \tag{pfit}$$

Note. In [21] the author introduced *almost regularity* for spaces as the requirement that for a regular open non-empty U (that is, $\emptyset \neq U = \operatorname{int}\overline{U}$) there be a non-empty open V such that $\overline{V} \subseteq U$. This corresponds to relaxing our (pfit) by assuming $a = a^{**}$.

4.4.1. Prefitness is in fact quite a strong property akin to regularity. Set
$$\rho(a) = \bigvee\{x \mid x \prec a\}.$$
We have

Proposition. *A frame L is prefit iff for each $a \in L$,*
$$a \leq \rho(a)^{**}.$$
In other words, if $\mathfrak{o}(a) \subseteq \overline{\mathfrak{o}(\rho(a))}$.

Proof: Suppose $\rho(a)^{**} \not\geq a$. Then $a \wedge \rho(a)^* \neq 0$ and hence there is an $x > 0$ such that $x \prec (a \wedge \rho(a)^*)$, that is,
$$x^* \vee (a \wedge \rho(a)^*) = (x^* \vee a) \wedge (x^* \vee \rho(a)^*) = 1$$

so that in particular $x \prec a$ (and hence $x \le \rho(a)$ so that further $\rho(a)^* \le x^*$), and $x^* \vee \rho(a)^* = 1$ and consequently $x^* = x^* \vee \rho(a)^* = 1$ and hence $x \le x^{**} = 0$, a contradiction. ■

Note. This is not to be confused with another relaxation of regularity, (a.norm) & (sfit) from 3.4.1. In regularity one has $\forall a$, $a = \rho(a)$; in prefit $\forall a$, $a^{**} = \rho(a)^{**}$ and in (a.norm) & (sfit) $\forall a$, $a^{**} = \rho(a^{**})$.

4.5. Note that

each fit frame is prefit.

(Indeed, if $a \not\le 0$ we have a c such that $a \vee c = 1$ and $c^* = c \to 0 \ne 0$. Set $x = c^*$.)

4.5.1. Realize that the pseudocomplement in the closed sublocale $\mathfrak{c}(b) = \uparrow b$ of L is given by the formula

$$x^{*_b} = x \to b.$$

Proposition. *A frame is fit iff each of its closed sublocales is prefit.*

Proof: I. Let each closed sublocale $\mathfrak{c}(b) = \uparrow b$ of L be prefit. Let $a \not\le b$. Then $a_1 = a \vee b > b = 0_{\mathfrak{c}(b)}$ and hence there is an $x > b$ such that

$$a_1 \vee x^{*_b} = a \vee b \vee (x \to b) = (x \to b) \vee a = 1.$$

Set $c = x \to b = x^{*_b}$. Then $a \vee c = 1$ and $c \to b = c^{*_b} = x^{*_b *_b} \ge x > b$.

II. Let L be fit and let $\mathfrak{c}(b)$ be a closed sublocale. Let $a > b = 0_{\mathfrak{c}(b)}$; then we have a c such that $a \vee c = 1$ and $c \to b > b$. Set $x = c^{*_b} = c \to b$. Then $x^{*_b} = c^{*_b *_b} \ge c$ and hence $x \prec a$ in $\mathfrak{c}(b)$. ■

4.5.2. Each closed sublocale of a topological space is induced by a closed subspace. Thus, unlike in the sublocale characterization of fitness in 4.1, we obtain here a characterization of fit spaces:

Corollary. *A topological space X is fit if and only if for each closed $Y \subseteq X$ and each open U such that $U \cap Y \ne \emptyset$ there is an open V such that $V \cap Y \ne \emptyset$ and $\overline{V} \cap Y \subseteq U \cap Y$.* ■

4.6. As we said in 4.1.3, comparing 4.5 (and 4.4.1) with 4.1 is an indication of the difference between the system of all subspaces and that of all sublocales of a space. If we require a property that in itself is weaker than T_1 to be inherited by all subspaces we do not go beyond T_1. If we require it to be inherited by all the sublocales we reach a property of regularity type! (For more about inheriting subfitness by subspaces see Theorem 3.4 in [7].)

Also note that we have in the sublocales $\mathfrak{sc}(a)$ in subfit but not fit frames a store of examples of non-spatial frames.

5. Comparing the properties

5.1. It is obvious that
$$\text{(sfit)} \;\not\Rightarrow\; \text{(pfit)} :$$
even T_1 does not imply (pfit) in spaces: see the cofinite topology.

5.2. The prefitness, however, is a fairly strong property and we will have more trouble to show it does not imply subfitness. It does not, as we will see in the following

Example. Let $\mathbb{N}$ be the set of natural numbers, $\omega \notin \mathbb{N}$. Set
$$X = (\mathbb{N} \times \{0,1\}) \cup \{\omega\}$$
and endow it with the following topology:
$$U \subseteq X \text{ is open if } \begin{cases} (\exists n,\ (n,1) \in U) \;\Rightarrow\; \omega \in U, \text{ and} \\ \omega \in U \;\Rightarrow\; \exists k\ (n \geq k \;\Rightarrow\; (n,0) \in U). \end{cases}$$
Thus in particular $U_0 = \mathbb{N} \times \{0\}$, $U_1 = (\mathbb{N} \times \{0\}) \cup \{\omega\}$ and each of $U(n) = (\mathbb{N} \times \{0\}) \cup \{\omega\} \cup \{(n,1)\}$ are open and we have $\omega \in U_1$ and $U_1 \smallsetminus \{\omega\} = U_0$, $(n,1) \in U(n)$ and $U(n) \smallsetminus \{(n,1)\} = U_1$, and finally each $\{(n,0)\}$ is open, so that X is T_D. Thus,
$$X \text{ is not subfit}$$
since otherwise, by 3.1, it would be T_1, and $\overline{\{\omega\}}$ contains $\mathbb{N} \times \{1\}$. But
$$X \text{ is prefit.}$$
Indeed, each $\{(n,0)\}$ is clopen and each non-void open set contains some of the $\{(n,0)\}$.

5.3. On the other hand prefit obviously implies (wsfit): if $a \neq 0$ then $x \prec a$ for some $x \neq 0$. Put $c = x^*$. Then $a \vee c = 1$ and $c \neq 1$ because $c = 1$ would yield $x^{**} = 0$ and hence $x = 0$.

Thus, the situation is as follows

$$\begin{tikzcd}
& (\mathrm{fit}) \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
(\mathrm{pfit}) \arrow[rr, Rightarrow, shift left, "/" marking] \arrow[dr, Rightarrow] & & (\mathrm{sfit}) \arrow[ll, Rightarrow, shift left, "/" marking] \arrow[dl, Rightarrow] \\
& (\mathrm{wsfit}) &
\end{tikzcd}$$

with none of the indicated implications reversible.

5.4. Finally we will show that

$$(\mathrm{pfit})\ \&\ (\mathrm{sfit}) \ \not\Rightarrow\ (\mathrm{fit}).$$

Example. Consider the square $X = I \times I$ where I is the standard unit interval and set $Y = \{(x, 1) \mid x \in I\}$. On X define a topology by declaring U open if

either $U \cap Y = \emptyset$ and for each $(x, y) \in U$ there is a standard ε-neighbourhood $V \subseteq U$,

or $U \cap Y \neq \emptyset$, for each $(x, y) \in U$ there is a standard ε-neighbourhood $V \subseteq U$, and $Y \smallsetminus U$ is finite.

Then X is T_1 and hence it is subfit. The space X is also prefit. Indeed, let U be non-void open. Then $U \cap (X \smallsetminus I)$ is non-void open and we can choose a non-void open $U' \subseteq U \cap (X \smallsetminus I)$ such that the standard metric closure of U' does not meet Y (it suffices to take an $(x, y) \in U \cap (X \smallsetminus I)$ and an open ε-neighbourhood of (x, y) with ε sufficiently small). Then the closure of any $V \subseteq U'$ in X coincides with the standard metric closure in $I \times I$ and the statement follows. On the other hand, X is not fit: Y is closed in X and it is obviously not prefit.

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