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FROM A-SPACES TO ARBITRARY SPACES VIA SPATIAL FIBROUS PREORDERS

NELSON MARTINS-FERREIRA

Abstract: The well known equivalence between preorders and Alexandrov spaces is extended to an equivalence between arbitrary topological spaces and spatial fibrous preorders, a new notion to be introduced.

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1. Introduction

In modern terms, the main result in [1] establishes a categorical equivalence between preorders and A-spaces. A preorder is simply a reflexive and transitive relation while an A-space is a topological space in which any intersection of open sets is open. The later trivially holds for finite topological spaces and the equivalence between finite topological spaces and finite preorders was used in [4, 5] to answer to open problems in topological descent theory. In [3], p.61, Erné writes *"Hence the question arises: How can we enlarge the category of A-spaces on the one hand and the category of quasiordered sets on the other hand, so that we still keep an equivalence between the topological and the order-theoretical structures, but many interesting 'classical' topologies are included in the extended definition?"* and proposes the two notions of B-space and C-space [3].

With a different motivation, and not being restricted to the *order-theoretical structures*, we propose a new structure, which we call fibrous preorder, and generalizes the one of a preorder. With the appropriate morphisms, called fibrous morphisms, and a suitable equivalence between them, we observe that the category Top, of topological spaces and continuous maps, is sitting in the

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middle

$$Preord \longrightarrow Top \longrightarrow FibPreord$$

of the category of preorders and the category of fibrous preorders.

The main result of this work is the description of the subcategory of fibrous preorders which is equivalent to the category of topological spaces. Inspired by what is called spatial frames in point-free-topology (see e.g. [7]), the fibrous preorders arising in this way are called spatial fibrous preorders.

A fibrous preorder is a generalization of a preorder. It was obtained while looking for a simple description of topological spaces in terms of internal categorical structures. By an internal categorical structure we mean a structure which can be defined in an arbitrary category with finite limits — as for instance the notion of internal category or internal groupoid, internal preorder or internal equivalence relation. A detailed description on these topics can be found for instance in [2].

This work is organized as follows: in the section 2 we describe the category of fibrous preorders, by defining its objects and morphisms and an equivalence relation on each hom-set of fibrous morphisms; that induces an equivalence between fibrous preorders that we make explicit; at the end we recover the classical Alexandrov theorem stating that every A-space is equivalent to a preorder. In section 3 we introduce the notion of spacial fibrous preorder and prove that (up to equivalence) it defines a subcategory of the category of fibrous preorders and moreover that it is isomorphic to the category of topological spaces. In section 4 we provide some examples to illustrate how the use of spacial fibrous preorders can be used to work with topological spaces described by systems of open neighbourhoods.

2. Fibrous preorders and fibrous morphisms

The following definition is a generalization of the notion of preorder, i.e. a reflexive and transitive relation. The word fibrous is a derivation of the word fibre and it is motivated by the presence of a morphism $p\colon A \longrightarrow B$ (see below), suggesting that A may be considered as a fibre over the base B, moreover when p is an isomorphism the classical notion of preorder is recovered.

Definition 2.1. A fibrous preorder is a sequence

$$R \xrightarrow{\partial} A \xrightarrow{p} B$$

in which A and B are sets, p and ∂ are maps, $R \subseteq A \times B$ is a binary relation (and as usual we simply write $(a,b) \in R$ as aRb) such that the following conditions hold:

(F1) $p\partial(a,b) = b$;
(F2) $aRp(a)$;
(F3) $\partial(a,b)Ry \Rightarrow aRy$;

for every $a \in A$ and $b, y \in B$ with aRb.

Definition 2.2. Let $\mathbf{A} = (R \xrightarrow{\partial} A \xrightarrow{p} B)$ and $\mathbf{A}' = (R' \xrightarrow{\partial'} A' \xrightarrow{p'} B')$ be two fibrous preorders. A fibrous morphism between $\mathbf{A}$ and $\mathbf{A}'$ is a pair (f, f^*) with $f\colon B \longrightarrow B'$ a map from B to B' and $f^*\colon A'_f \longrightarrow A$ a map from

$$A'_f = \{(a', b) \in A' \times B \mid p'(a') = f(b)\}$$

to A such that

$$pf^*(a', b) = b \tag{1}$$

and

$$f^*(a', b)Ry \Rightarrow a'R'f(y) \tag{2}$$

for all $a' \in A'$ and $b, y \in B$ with $p'(a') = f(b)$.

In other words a fibrous morphism from $\mathbf{A}$ to $\mathbf{A}'$ is a map f from B to B' together with a span

$$A' \xleftarrow{\pi_1} A'_f \xrightarrow{f^*} A$$

such that the following diagram (in which the top line is to be considered as a single composite arrow) is a pullback diagram

$$\begin{tikzcd}
A'_f \arrow[r, "f^*"] \arrow[d, "\pi_1"'] & A \arrow[r, "p"] & B \arrow[d, "f"] \\
A' \arrow[rr, "p'"] & & B'
\end{tikzcd}$$

and moreover the condition (2) is satisfied. Note that the commutativity of the previous diagram is equivalent to condition (1).

Now, if (g, g^*) is another fibrous morphism, say from $\mathbf{A}'$ to $\mathbf{A}''$, then the composition $(g, g^*) \circ (f, f^*)$ is computed as

$$(g, g^*) \circ (f, f^*) = (gf, f^* g^*_f)$$

with $f^*g_f^*(a'', b) = f^*(g^*(a'', f(b)), b)$. The following diagram illustrates how the above formula is obtained (simply complete the diagram by inserting the upper left pullback square) and it also shows that this formula is associative up to a canonical isomorphism of pullbacks.

$$\begin{tikzcd}
A''_{gf} \arrow[r, "g_f^*"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "p.b."] & A'_f \arrow[r, "f^*"] \arrow[d, "\pi_1"'] & A \arrow[r, "p"] & B \arrow[d, "f"] \\
A''_g \arrow[r, "g^*"] \arrow[d, "\pi_1"'] & A' \arrow[rr, "p'"] & & B' \arrow[d, "g"] \\
A'' \arrow[rrr, "p''"] & & & B''
\end{tikzcd}$$

It is a simple calculation to check that it is well defined, that is, condition (2) is satisfied.

We will consider the category FibPreord of fibrous preorders and fibrous morphisms with the following identification of parallel fibrous morphisms.

Definition 2.3. Two parallel fibrous morphisms (f, f^*) and (g, g^*) are said to be equivalent if and only if $f = g$.

This equivalence of morphisms immediately gives the following equivalence between two objects: we identify two fibrous preorders whenever they have the same base object and the identity map is fibrous in both directions.

Proposition 2.4. *Two fibrous preorders* (R, A, B, p, ∂) *and* $(R', A', B', p', \partial')$ *are equivalent*

$$(R, A, B, p, \partial) \sim (R', A', B', p', \partial)$$

if and only if $B = B'$ *and there exist two maps*

$$\begin{tikzcd}
A \arrow[r, shift left, "\varphi"] \arrow[d, "p"'] & A' \arrow[l, shift left, "\gamma"] \arrow[d, "p'"] \\
B \arrow[r, equal] & B'
\end{tikzcd}$$

such that $p'\varphi = p$, $p\gamma = p'$ *and*

$$\varphi(a)R'b \Rightarrow aRb \tag{3}$$

$$\gamma(a')Rb \Rightarrow a'R'b \tag{4}$$

for every $a \in A$, $a' \in A'$ *and* $b \in B$.

Proof: Straightforward. ■

A preorder $(B, \leq)$ is in particular a fibrous preorder with $A = B$, $p = 1_B$, xRy if and only if $x \leq y$ and $\partial(x, y) = y$. In fact we have more.

Proposition 2.5. *There is an embedding of the category of preorders into the category of fibrous preorders and moreover an object (R, A, B, p, ∂) is (up to equivalence) in the image of the embedding if and only if there exists a map*

$$u\colon B \longrightarrow A$$

such that $pu = 1_B$ and

$$up(a)Ry \Rightarrow aRy,$$

for every $a \in A$ and $b \in B$.

Proof: Clearly if $(B, \leq)$ is a preorder then (R, A, B, p, ∂) with $A = B$, $p = 1_B = 1_A$, xRy if and only if $x \leq y$ and $\partial(x, y) = y$ is a fibrous preorder. Also every monotone map gives a fibrous morphism.

Conversely, let (R, A, B, p, ∂) be a fibrous preorder. If there exists a map

$$u\colon B \longrightarrow A$$

such that $pu = 1_B$ and

$$up(a)Ry \Rightarrow aRy,$$

for every $a \in A$ and $b \in B$, then it is equivalent to

$$(R^\circ, B, B, 1_B, \partial^\circ)$$

with

$$xR^\circ y \Leftrightarrow u(x)Ry$$

and $\partial^\circ(x, y) = y$. First let us observe that R° is a preorder, so that $(R^\circ, B, B, 1_B, \partial^\circ)$ is well defined. Indeed $xR^\circ x \Leftrightarrow u(x)Rx \Leftrightarrow u(x)Rpu(x)$ which holds by condition $(F2)$. This proves reflexivity. For transitivity, suppose we have $xR^\circ y$ and $yR^\circ z$ that is $u(x)Ry$ and $u(y)Rz$, hence we also have $up(\partial(u(x), y))Rz$ and then it follows $\partial(u(x), y)Rz$ from which we conclude $u(x)Rz$, proving that $xR^\circ z$ as desired. It is also clear that we have maps

$$\begin{tikzcd} A \arrow[r, shift left, "p"] \arrow[d, "p"'] & B \arrow[l, shift left, "u"] \arrow[d, "1_B"] \\ B \arrow[r, equal] & B \end{tikzcd}$$

with
$$p(a)R^{\circ}b \Rightarrow up(a)Rb \Rightarrow aRb$$
and $u(x)Rb \Rightarrow xR^{\circ}b$. ∎

There is also a functor from the category of topological spaces to the one of fibrous preorders which will be used in the next section.

Proposition 2.6. *If (B,τ) is a topological space then the structure*
$$A = \{(U,x) \mid x \in U \in \tau\}$$
$$p(U,x) = x$$
$$(U,x)Ry \Leftrightarrow y \in U$$
$$\partial((U,x),y) = (U,y)$$
defines a fibrous preorder. Moreover, if $f\colon (B,\tau) \longrightarrow (B',\tau')$ is a continuous map then
$$(f,f^{*})\colon (R,A,B,p,\partial) \longrightarrow (R',A',B',p',\partial'),$$
with $f\colon B \longrightarrow B'$ the underlying map and
$$f^{*}((U',x'),y) = (f^{-1}(U'),y)$$
for all $(U',x') \in A'$ and $y \in B$ with $x' = f(y)$, is a fibrous morphism between fibrous preorders.

Proof: Straightforward verification. ∎

In the next section we describe the fibrous preorders arising from a topological space, called spatial fibrous preorders and prove that the category of topological spaces is isomorphic to the category of spatial fibrous preorders. Before that we illustrate how the classical result of [1] can be obtained via this new setting.

Proposition 2.7 (Alexandrov, [1])**.** *Let (B,τ) be a topological space and consider the fibrous preorder, say $F(B,\tau)$, described in Proposition 2.6. It is an Alexandrov space if and only if $F(B,\tau)$ is equivalent to a preorder.*

Proof: We only observe that if (B,τ) is an Alexandrov space then there exists a map
$$u\colon B \longrightarrow A$$
assigning to each point $x \in B$ the element $(\theta_x,x) \in A$ with θ_x the intersection of all open neighbourhoods of x, moreover this map satisfies the requirements of proposition 2.5. ∎

3. The main result

The so called spatial frames are the frames that are isomorphic to the topology of some space (see e.g. [7]). Here our main result is the description of the full subcategory of fibrous preorders and fibrous morphisms (with equivalent morphisms identified) which is equivalent to the category of topological spaces.

Definition 3.1. A fibrous preorder ($R \xrightarrow{\partial} A \xrightarrow{p} B$) is said to be *spatial* when there exists $s\colon B \longrightarrow A$ and $m\colon A \times_B A \longrightarrow A$ with $A \times_B A = \{(a, a') \in A \times A \mid p(a) = p(a')\}$ such that

(F4) $ps(y) = y$;
(F5) $pm(a, a') = p(a) = p(a')$;
(F6) $m(a, a')Ry \Rightarrow aRy \& a'Ry$;

for every $a, a' \in A$ and $y \in B$ with $p(a) = p(a')$.

Observe that because we are identifying fibrous morphisms as in Definition 2.3, the notion of spatial fibrous preorder is a property of a given fibrous preorder and not an extra structure. That is to say that the category of spatial fibrous preorders and (equivalent) fibrous morphisms is a full subcategory of FibPreord.

Theorem 3.2. *There are functors F and G between the category of spatial fibrous preorders and topological spaces*

$$SpFibPreord \underset{G}{\overset{F}{\rightleftarrows}} Top$$

such that $FG = 1$ and $GF \sim 1$.

Proof: The functor G is defined as in Proposition 2.6, together with $s(x) = (B, x)$ and $m((U, x), (V, x)) = (U \cap V, x)$. The functor F associates to each spatial fibrous preorder

$$R \xrightarrow{\partial} A \underset{s}{\overset{p}{\rightleftarrows}} B\ , \qquad A \times_{\langle p,p \rangle} A \xrightarrow{m} A$$

the topological space (B, τ) in which τ is defined by

$$\mathcal{O} \in \tau \Leftrightarrow \forall y \in \mathcal{O}\ \exists a \in A,\ p(a) = y,\ N(a) \subseteq \mathcal{O}, \tag{5}$$

with $N(a) = \{y \in B \mid aRy\}$. If (f, f^*) is a morphism in *SpFibPreord* then $F(f, f^*) = f$. In order to see that the functor F is well defined we observe:

(a) The empty set is in τ. Indeed it is an immediate consequence of (5).
(b) If $\mathcal{O}, \mathcal{O}' \in \tau$ then $\mathcal{O} \cap \mathcal{O}' \in \tau$. Indeed if $x \in \mathcal{O} \cap \mathcal{O}'$ then by (5) there exist $a, a' \in A$ such that $p(a) = p(a') = x$ and $N(a) \subseteq \mathcal{O}$, $N(a') \subseteq \mathcal{O}'$. Using $(F5)$ and $(F6)$ we obtain $m(a, a')$ such that $pm(a, a') = x$ and $N(m(a, a')) \subseteq N(a) \cap N(a')$, yielding the desired result that for every $x \in \mathcal{O} \cap \mathcal{O}'$ there exists $m(a, a') \in A$ such that $pm(a, a') = x$ and $N(m(a, a')) \subseteq \mathcal{O} \cap \mathcal{O}'$. This proves that $\mathcal{O} \cap \mathcal{O}'$ is in τ.
(c) The fact that p is surjective $(F4)$ implies (in fact is equivalent to the fact) that B is in τ.
(d) Again by definition of τ it is easy to see that it is closed under arbitrary unions.

Concerning morphisms we have to show that if (f, f^*) is a morphism in $SpFibPreord$ then f is a continuous map from (B, τ) to (B', τ'), assuming that τ and τ' are obtained as in (5). Suppose $\mathcal{O}' \in \tau'$, we shall prove $f^{-1}(\mathcal{O}') \in \tau$. Given any $y \in f^{-1}(\mathcal{O}')$, because $\mathcal{O}' \in \tau'$ and $f(y) \in \mathcal{O}'$, by (5) there exists $a' \in A'$ such that $p(a') = f(y)$ and $N'(a') \subseteq \mathcal{O}'$. Now, the very structure of fibrous morphism gives us an element $f^*(a', y) \in A$ such that (see (1) and (2))

$$pf^*(a', y) = y$$

and

$$N(f^*(a', y)) \subseteq f^{-1}(N'(a')) \subseteq f^{-1}(O'),$$

proving thus that $f^{-1}(O') \in \tau$, whenever $O' \in \tau'$. And hence f is a continuous map.

It is also immediate to observe that $FG = 1_{Top}$. Indeed

$$FG(B, \tau) = (B, \bar{\tau})$$

with

$$\mathcal{O} \in \bar{\tau} \Leftrightarrow \forall y \in \mathcal{O}, \exists u \in \tau, y \in u \subseteq \mathcal{O}$$

and it is clear that $\bar{\tau} = \tau$.

It remains to prove $GF \simeq 1$. To do that we will show the existence of two maps

$$\begin{tikzcd} A \arrow[r, shift left, "\varphi"] \arrow[d, "p"'] & GF(A) = \bar{A} \arrow[l, shift left, "\gamma"] \arrow[d, "\bar{p}"] \\ B \arrow[r, equal] & B \end{tikzcd}$$

such that the diagram above commutes and for every $a \in A$, $(u,x) \in \bar{A}$ and $b \in B$,

$$\varphi(a)\bar{R}b \Rightarrow aRb$$

$$\gamma(u,x)Rb \Rightarrow (u,x)\bar{R}b.$$

The map φ is given by

$$\varphi(a) = (N(a), p(a))$$

while (assuming the axiom of choice)

$$\gamma(u,x) = a_{u,x} = a_x$$

where $a_x \in A$ is any element of A such that $p(a_x) = x$ and $N(a_{u,x}) \subseteq u$, which exists by definition of $(u,x) \in \bar{A}$. Recall that $(u,x) \in \bar{A}$ if and only if $x \in u$ and $u \subseteq B$ is such that

$$\forall y \in u, \exists a \in A, \quad p(a) = y, N(a) \subseteq u.$$

In order to prove that $\varphi(a)$ is well defined we observe: by $(F2)$, $aRp(a)$, and so $p(a) \in N(a)$; now suppose $y \in N(a)$, this means aRy, and if we put $a' = \partial(a,y)$ then by $(F1)$ $p(a') = y$ and by $(F3)$ we know that

$$N(a') \subseteq N(a),$$

showing that φ is well defined.

Finally we observe that

$$(N(a), p(a))\bar{R}b \Leftrightarrow b \in N(a) \Leftrightarrow aRb$$

and

$$a_{u,x}Rb \Leftrightarrow b \in N(a_{u,x}) \subseteq u \Rightarrow (u,x)\bar{R}b,$$

which concludes the proof. ■

4. Some examples

We conclude with a list of examples to illustrate how this notion of spatial fibrous preorder can be used to work with arbitrary topological spaces described by basic neighbourhood relations.

Normed vector spaces. Let $B = (B, +, 0)$ be an abelian group and $I \subseteq B$ a subset of B together with a map $h\colon I \longrightarrow \mathbb{N}$ such that:

(1) $0 \in I$;
(2) it $nn'a \in I$ then $na, n'a \in I$;
(3) if $na \in I$ and $h(a)a' \in I$ then $n(a + a') \in I$.

We construct a spatial fibrous preorder as follows:

$$A = \mathbb{N} \times B$$
$$p(n, x) = x, \quad s(x) = (1, x), \quad m(n, n', x) = (nn', x)$$
$$(n, x)Ry \Leftrightarrow n(x - y) \in I$$
$$\partial((n, x), y) = \begin{cases} (h(x - y), y) \text{ if } x \neq y \\ (n, x) \text{ if } x = y \end{cases}$$

A concrete example is the case when B is a normed vector space with $I = \{x \in B \mid \mathsf{norm}(x) < 1\}$ and $h(x) \in \mathbb{N}$ is such that $\frac{1}{h(x)} < \frac{1}{k} - \mathsf{norm}(x)$ with k the unique natural number such that

$$\frac{1}{k+1} \leq \mathsf{norm}(x) < \frac{1}{k}$$

if $\mathsf{norm}(x) \neq 0$.

Metric Spaces. The intuitive idea of working with metric spaces with open balls of radius $\frac{1}{n}$ for some natural number n may be formalized in terms of spatial fibrous preorders in the following way. Let (B, d) be a metric space and consider

$$A = \mathbb{N} \times B$$
$$p(n, x) = x, \quad s(x) = (1, x)$$
$$m((n, x), (n', x)) = (nn', x)$$
$$(n, x)Ry \Leftrightarrow d(x, y) < \frac{1}{n}$$
$$\partial((n, x), y) = (k, y)$$

with $k \in \mathbb{N}$ any number greater than $\frac{n}{1 - nd(x,y)}$.

This shows the existence of a functor (actually and embedding) from the category of metric spaces and continuous maps into the category of spatial fibrous preorders and fibrous morphisms. If $f\colon (B, d) \longrightarrow (B', d')$ is a continuous map we may define $f^*((n, f(y), y))$ as the pair (k, y) with k any natural number such that for every $z \in Z$

$$d(z,y) < \frac{1}{k} \Rightarrow d'(f(z), f(y)) < \frac{1}{n}.$$

We may now ask for a characterization of those spatial fibrous preorders which arise from a metric space in the same way as above. As it is well-known in point-set topology there is not a simple answer to that question. Nevertheless the notion of a *natural* space is a good substitute to the one of a metric space.

***Natural* spaces.** Generalizing the construction for metric spaces from above we observe that more in general, every map $N\colon \mathbb{N} \times B \longrightarrow P(B)$ such that

(1) $\forall n \in \mathbb{N}, \forall x \in B, \quad x \in N(n,x)$
(2) $N(nn', x) \subseteq N(n,x) \cap N(n',x)$
(3) $N(n,x) \subseteq \{y \in B \mid \exists n' \in \mathbb{N}, \quad N(n',y) \subseteq N(n,x)\}$

gives a spatial fibrous preorder as follows: $A = \mathbb{N} \times B$, $p(n,x) = x$, $s(x) = (1,x)$, $m(n,n',x) = (nn',x)$,

$$(n,x)Ry \Leftrightarrow y \in N(n,x),$$

and $\partial(n,x,y) = n'$ where n' is such that $N(n',y) \subseteq N(n,x)$ which exists by definition of N.

For the purpose of this note a topological space is said to be *natural* if it admits a base of neighbourhoods of the form above. It is clear from the above that every metrizable space is natural.

In this case, a morphism $f\colon B \longrightarrow B'$ is continuous if for every $n \in \mathbb{N}$ and $y \in B$ there exists $k \in \mathbb{N}$ such that $f(N(k,y)) \subseteq N'(n, f(y))$.

The particular the case of metric spaces is recaptured by letting $N(n,x) = \{y \in B \mid d(x,y) < \frac{1}{n}\}$.

However, as it is well known, not every natural space is metrizable.

Tangent disk topology. An example of a well known non-metrizable space which is natural in the sense above is the so-called tangent disk topology. In this case $B = \{(x,y) \in \mathbb{R}^2 \mid y \geq 0\}$ and

$$N(n,(x,y)) = \begin{cases} \{(x_1,x_2) \mid d((x_1,x_2),(x,y)) < \frac{1}{n}\} & \text{if} \quad y > 0 \\ \{(x_1,x_2) \mid d((x_1,x_2),(x,y)) < \frac{1}{n}\} \cup \{(x,y)\} & \text{if} \quad y = 0. \end{cases}$$

The Cantor set. The Cantor Set is another well known concrete example that fits in the setting of natural spaces: in this case we let

$$B = \{u \mid u \colon \mathbb{N} \longrightarrow \{0, 2\}\}$$

and

$$N(n, u) = \{w \in B \mid w(i) = u(i), \quad i \leq n\}.$$

More generally we may consider as B any set of the form $\{u \mid u \colon \mathbb{N} \longrightarrow X\}$ with X an arbitrary set.

The p-adic topology. The p-adic topology on the set of integers is obtained as

$$N(n, x) = \{z \in \mathbb{Z} \mid z = x + kp^n, \quad k \in \mathbb{Z}\}$$

with $B = \mathbb{Z}$.

If instead of a map $N \colon \mathbb{N} \times B \longrightarrow P(B)$ we consider a family of binary relations R_n over B, then we have examples of the following type, with $I = \mathbb{N}$.

Indexed families of preorders. A more general example is obtained as follows. Let I be a monoid, B a set, $(R_i)_{i \in I}$ a family of binary relations $R_i \subseteq B \times B$, and $(\partial_i \colon R_i \longrightarrow I)$ a family of maps such that:

(1) xR_ix
(2) $xR_{ij}y \Rightarrow xR_iy \& xR_jy$
(3) $xR_ib \& bR_{\partial_i(x,b)}y \Rightarrow xR_iy$

for all $i, j \in I$ and $x, y, b \in B$.

In this case we construct a fibrous preorder as follows: $A = I \times B$, $p(i, x) = x$, $s(x) = (1, x)$, $m(i, j, x) = (ij, x)$,

$$(i, x)Ry \Leftrightarrow xR_iy$$

and $\partial(i, x, y) = (\partial_i(x, y), y)$ if xR_iy.

For morphisms (from $(\partial_i \colon R_i \longrightarrow I)_i \in I$ to $(\partial'_{i'} \colon R'_{i'} \longrightarrow I')_{i'} \in I'$) we have a map $f \colon B \longrightarrow B'$ and a family of maps $f_j \colon B \longrightarrow I'{}_j \in I'$ such that

$$xR_{f_j(x)}y \Rightarrow f(x)R'_jf(y).$$

5. Conclusion

In this note we introduced the notions of (spatial) fibrous preorder and fibrous morphism, showing that the category of topological spaces is the quotient category of the category of spatial fibrous preorders, obtained by identifying two fibrous morphisms whenever they have the same underlying map. The examples show that this notion provides a convenient setting to work with the intuitive notion of base of open neighbourhoods. However, as explained in the introduction, the main motivation that leads to the definition of fibrous preorder was the purpose of finding a purely categorical definition of topological space. Future work ([6]) will specify the internal version of a fibrous preorder, by replacing the relation $R \subseteq A \times B$ with a jointly monomorphic pair of morphisms and by giving the appropriate translation of axioms (F1)-(F3) and (F4)-(F6). In particular, the additional structure of spatial fibrous preorder is nothing but a comonoid structure in the monoidal category of fibrous preorders and fibrous morphisms, with an appropriate tensor product. Further studies will then take place in $FibPreOrd(\mathcal{C})$ and $SpFibPreord(\mathcal{C})$ for an arbitrary category $\mathcal{C}$ with finite limits. For instance if $\mathcal{C}$ is the category of finite sets then $Preord(\mathcal{C}) \simeq SpFibPreord(\mathcal{C}) \simeq FibPreord(\mathcal{C})$, as easily follows from Proposition 2.5.

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Nelson Martins-Ferreira
Departamento de Matemática, Escola Superior de Tecnologia e Gestão, Centro para o Desenvolvimento Rápido e Sustentado do Produto, Instituto Politécnico de Leiria, Leiria, Portugal

E-mail address: martins.ferreira@ipleiria.pt