R_2 of the projection disk bounded by the component of $\beta_2 - c_2$ that has no crossings with other blocks and that is the smallest that contains one of the two components of $\beta_3 - c_2$. Notice that R_2 has fewer crossings than R_1. Repeating this process we get a contradiction to the diagram being minimal or to the number of crossings being finite. ■

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Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 13–46

THE MINIMUM CROSSING NUMBER OF ESSENTIAL TANGLES

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Abstract: In this paper we compute the sharp lower bounds for the crossing number of n-string k-loop tangles. For tangles with only string components, we characterise the ones with the minimum crossing number for a given number of components, both when the tangle has knotted strings or only unknotted strings.

Keywords: Essential tangle, minimum crossing number.
AMS Subject Classification (2010): 57M25.

1. Introduction

A *n-string k-loop tangle* (B, σ), also denoted by $\mathcal{T}$, is a 3-ball B together with a compact 1-submanifold σ, with boundary, that is a disjoint collection of $n \geq 1$ arcs (called *strings*) and $k \geq 0$ simple closed curves (called *loops*). When $k = 0$ we say simply that $\mathcal{T}$ is a *n-string tangle*. In this paper we say that two tangles $\mathcal{T}_1$, $\mathcal{T}_2$ are *equivalent* if there is an homeomorphism of $(B, \partial B)$ sending $\mathcal{T}_1$ to $\mathcal{T}_2$. The classification of the tangles with the minimum crossing number up to isotopy follows from the classification up to homeomorphism described in this paper. A tangle $\mathcal{T}$ is said to be *essential* if σ is a single knotted arc* in B, or if σ has more than one component and there is no properly embedded disk in B separating the components of σ. Otherwise, we say that the tangle is *inessential*. The *crossing number* of a tangle $\mathcal{T}$, denoted by $c(\mathcal{T})$, is the minimum crossing number of the diagrams of $\mathcal{T}$.

Tangles were first introduced by Conway in [6] where he defines and classifies 2-string (*rational*) tangles as an instrument to list knots. Since then, the concept of thangle has been important in knot theory and its applications, and 3-manifold topology. For instance, in [11] Kirby and Lickorish prove that

Received October 23, 2013.

This work was partially supported by the Centro de Matemática da Universidade de Coimbra (CMUC), funded by the European Regional Development Fund through the program COMPETE and by the Portuguese Government through the FCT - Fundação para a Ciência e a Tecnologia under the project PEst-C/MAT/UI0324/2011.

*An arc of σ is *unknotted* in B if it cobounds a disk embedded in B together with an arc in ∂B, otherwise it is said to be *knotted*.

any knot is concordant to a prime knot by introducing the concept of essential tangle. Other example is the work of Lickorish in [12] and its extension by Gómez-Larrañaga in [10] where conditions for knot or link primeness are given based on tangle decompositions of a knot or link. In 3-manifold topology we can also find a pertinent use of tangles in the study of Dehn fillings and related problems as the cabling conjecture [14]. We also have the presence of tangles in applied mathematics as in the study of the DNA topology. This application was pioneered by Ernst and Sumners in [7], and in the survey paper [4] by Buck we can find a concise explanation on how tangles are used to study the DNA topology.

In this paper we compute the sharp lower bound for the crossing number of all n-string k-loop essential tangles, for each $n \geq 1$ and $k \geq 0$. In the case when the tangles have no loops, only strings, we also characterize which n-string essential tangles have the minimum crossing number possible. This question was initially motivated by an observation of Buck in [4] that 2-string essential tangles have crossing number at least five and citing the work of Bleiler in [2] for this statement. The aim of this paper is to present a sharp lower on the crossing number of n-string k-loop essential tangles. The results obtained are presented in the following theorems.

Theorem 1. *Let $\mathcal{T}$ be a n-string essential tangle. Then*

$$c(\mathcal{T}) \geq 2n + 1.$$

We have $c(\mathcal{T}) = 2n + 1$ if and only if $\mathcal{T}$ is equivalent to the tangle in Figure 1.

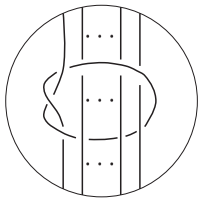

FIGURE 1. The n-string essential tangle with the minimum crossing number.

Proof of Theorem 3. Consider a partition of $\mathcal{T}$ into blocks. First note that if two blocks $\mathcal{T}_1$ and $\mathcal{T}_2$ of this partition cross at least twice, with $c(\mathcal{T}_i) \geq 2(n_i+k_i-1)$, where n_i and k_i are respectively the number of strings and loops of $\mathcal{T}_i$, $i = 1, 2$, then $\mathcal{T}_1 \cup \mathcal{T}_2$ is a subblock of $\mathcal{T}$ such that

$$c(\mathcal{T}_1 \cup \mathcal{T}_2) \geq 2(n_1+k_1-1)+2(n_2+k_2-1)+2 = 2\left((n_1+n_2)+(k_1+k_2)-1\right).$$

We can therefore replace $\mathcal{T}_1$ and $\mathcal{T}_2$ by $\mathcal{T}_1 \cup \mathcal{T}_2$ obtaining a smaller partition of $\mathcal{T}$ into blocks which have the property stated in Lemma 7.
Consider the collection of all minimal subblocks of $\mathcal{T}$ with respect to a minimal diagram of $\mathcal{T}$. Clearly this collection is a partition of $\mathcal{T}$ by blocks that verify the inequality in Lemma 7. By repeating the above process on this collection, we eventually obtain a partition of $\mathcal{T}$ such that two blocks intersect each other at most once, and that also verifies the inequality in Lemma 7. Next we prove that this partition has only one block. Being this the case, we have

$$c(\mathcal{T}) \geq 2(n + k - 1),$$

as in the statement of the theorem.
In fact, suppose that the partition has more than one block, and that each block intersects each other at most once. Then all these blocks have at least two outermost crossings.
Consider a block β_1 and an outermost crossing c_1 of β_1 (with β_2). Let R_1 be the outer region of the projection disk bounded by the component of $\beta_1 - c_1$ that has no crossings with other blocks and that is the smallest that contains one of the two components of $\beta_2 - c_1$. (See Figure 15.)

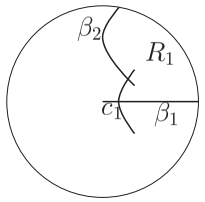

FIGURE 15. The outer region defined by an outermost crossing.

If β_2 has no outermost crossing in R_1, then we can reduce the number of crossings by an isotopy of β_1. Otherwise, consider an outermost crossing c_2 of β_2 (with β_3). Note that $\beta_3 \neq \beta_1$ as we are assuming that two blocks do not share more than one crossing. We can similarly consider the outer region

as the identity or reflection on the plane. Therefore, if two of these essential tangles are homeomorphic the knots obtained are also equivalent, which is a contradiction. Then, these three essential tangles are not equivalent. ■

5. Crossing number of essential tangles with closed strings

Let $\mathcal{T}$ be a n-string k-loop tangle with $k > 0$, and $\mathcal{D}$ be a minimal diagram of $\mathcal{T}$. A *block* β (with respect to $\mathcal{D}$) is a subtangle of $\mathcal{T}$ defined by a set of connected components of σ such that:

(1) if $l \in \beta$ is a loop, then every string or loop that crosses l is in β;
(2) if $s \in \beta$ is a string, then every loop that crosses s is in β.

The union of two blocks is a block and a block $\beta_1 \subsetneq \beta_2$ is called a *subblock* of β_2. We call a crossing of a block β_1 with another block β_2 *outermost* (in β_1) if it separates the projection of β_1 into two components such that one of them has no crossings with other blocks. Notice that a string s that doesn't cross any loop is a block, and a crossing of the string s is outermost if and only if it is outermost as a crossing of the block s.

Lemma 7. *If $\mathcal{T}$ has no subblocks, then $c(\mathcal{T}) \geq 2(n + k - 1)$.*

Proof: Let l_1 be any loop of $\mathcal{T}$ and consider the subtangle $\mathcal{T}_1$ of $\mathcal{T}$ formed by l_1 and all the n_1 strings that cross l_1. Since every string of $\mathcal{T}_1$ crosses l_1 at least twice, then $c(\mathcal{T}_1) \geq 2n_1$. If $\mathcal{T}_1 \neq \mathcal{T}$, then, since $\mathcal{T}_1$ is not a subblock of $\mathcal{T}$, there is a loop l_2 in $\mathcal{T} - \mathcal{T}_1$ that crosses some component of $\mathcal{T}_1$, at least twice. Consider the subtangle $\mathcal{T}_2$ of $\mathcal{T}$ formed by l_1, l_2 and all the n_2 strings that cross l_1 or l_2. Again, since every string of $\mathcal{T}_2$ crosses l_2 at least twice, then $c(\mathcal{T}_2) \geq 2(n_2 + 1)$. By repeating this argument, adding at stage i, one loop l_i and the new strings of $\mathcal{T}$ that cross l_i, we obtain a subtangle $\mathcal{T}_i$ of $\mathcal{T}$ such that $c(\mathcal{T}_i) \geq 2(n_i + i - 1)$. At stage k all the loops of $\mathcal{T}$ have been added, and, since $\mathcal{T}_k$ is not a subblock of $\mathcal{T}$, then $\mathcal{T}_k = \mathcal{T}$, which shows that $c(\mathcal{T}) \geq 2(n + k - 1)$. ■

We notice that this inequality is sharp, as the tangle $\mathcal{T}$ of the Figure 3, with one loop and n strings, verifies the equality $c(\mathcal{T}) = 2n$. A similar argument to that on the end of the proof of Theorem 1 shows that $\mathcal{T}$ is essential. To see that this is the only n-string 1-loop essential tangle with $c(\mathcal{T}) = 2n$, observe that every string must cross the loop twice so that there are no crossings between any two strings.

Now we prove Theorem 3.

Theorem 2. *Let $\mathcal{T}$ be a n-string essential tangle, with all strings unknotted. Then*

$$c(\mathcal{T}) \geq 2n + 2.$$

We have $c(\mathcal{T}) = 2n + 2$ if and only if T is equivalent to one of the tangles in Figure 2.

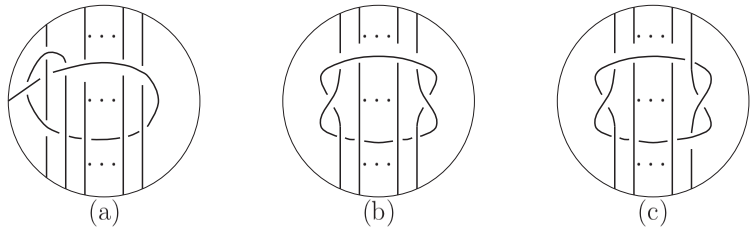

FIGURE 2. The all strings unknotted n-string essential tangles with the minimum crossing number.

Theorem 3. *Let $\mathcal{T}$ be a n-string k-loop essential tangle, with $k \geq 1$. Then*

$$c(\mathcal{T}) \geq 2(n + k) - 2.$$

If $k = 1$, then $c(\mathcal{T}) = 2n$ if and only if $\mathcal{T}$ is equivalent to the tangle in Figure 3.

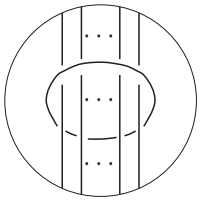

FIGURE 3. The n-string 1-loop essential tangle with the minimum crossing number.

There have been several other studies on tangles and their diagrams. For instance, in the work of Conway in [6], besides introducing the concept of tangle, the rational tangles with two strings are also classified. Also, in [9] Kanenobu,

Saito and Satoh classify up to isotopy 2-string k-loop prime tangles with up to seven crossings. In a more general approach, the paper [3] presents a study on the enumeration of k-string tangle projections, and in [8] it is presented a computerized enumeration of alternating tangles. Also, in [5] Cochran and Ruberman present invariants of 2-string tangles and use them to distinguish some knots from their mutants.

For a brief introduction to the concept of tangle, as in Conway's work, we refer to the book [1] by Colin Adams, and throughout this paper we work in the piecewise linear category. In Section 2 we introduce notation and prove some lemmas that define a base for the proofs of the main results in this paper. In sections 3, 4 and 5, we prove Theorems 1, 2 and 3, respectively, using combinatorics in the diagram projection of tangles.

2. Preliminaries

A *projection* of a tangle $\mathcal{T}$ is the image $p(\mathcal{T})$ of the tangle by an orthogonal projection p to a plane such that $p(B)$ is a disk, the preimage of each point of $p(\sigma)$ has at most two points, and there is a finite number of double points, which are called the *crossings* of the projection. A projection always exists in the piecewise linear category. The connected components of the complement of the crossings are called the *segments* of the projection. If the crossings are decorated with broken lines to show the overcrosses and undercrosses, then we get a *diagram* of $\mathcal{T}$. The *crossing number* of a projection or a diagram of $\mathcal{T}$ is its number of crossings. A *minimal diagram* of $\mathcal{T}$ is a diagram with the minimum crossing number $c(\mathcal{T})$.

There are four segments adjacent to each crossing. Two of these segments are *opposite* if their preimages by the projection are adjacent, and *consecutive* otherwise. A crossing is called *outermost* if it is the first crossing of a string for some orientation, and *inner* otherwise.

Lemma 1. *In a diagram of an essential tangle $\mathcal{T}$, each string has at least two crossings with the other strings.*

Proof: Suppose that a string s has at most one crossing with the other strings. Then there is a disk D with boundary $\partial D = s \cup \alpha$, where α is a curve in ∂B. A small neighborhood of D is then a ball that contains only the string s, which shows that $\mathcal{T}$ is not essential. ∎

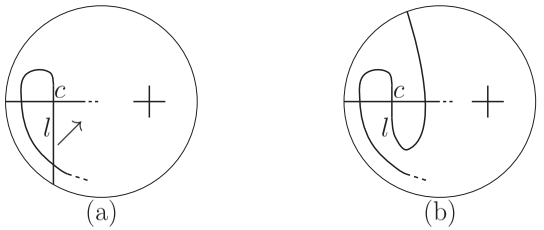

FIGURE 13. When s_1 and s_2 are on both inner crossings, we can assume that the inner crossings are as in (b).

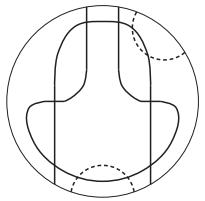

FIGURE 14. Diagram projection when s_1 and s_2 are on both inner crossings.

the case of Figure 2(b), or the granny knot, in the case of Figure 2(c). Then, if there is a properly embedded disk in B separating the strings s_1, s_2 we would have a 2-bridge decomposition of a composite knot, which is a contradiction because 2-bridge knots are prime. To prove that the n-string tangle $\mathcal{T}$ is essential in these cases we now follow a similar argument as the one used in case (i).

The three tangles depicted in Figure 2 are not equivalent. In fact, suppose there is a homeomorphism between two of these tangles, and consider the 2-string trivial tangle added to the tangles in the argument and the respective knots obtained from adding this trivial tangle. Consider B in S^3 and the disk of the diagram in $\mathbb{R}^2$. Note that the diagram of this trivial tangle is a collection of two disjoint arcs in the complement in $\mathbb{R}^2$ of the disk from the diagram of each essential tangle, with one arc at the top of this disk of the diagram and the other at the bottom. The homeomorphism between the essential tangles restricted to the boundary of the ball B can be isotoped to either the identity or reflexion of ∂B, and from here extended to the ball in the complement of B

knot 6_2 from Rolfsen's list in [13] that we denote by K. Suppose there is a disk separating s_1 and s_2. As s_2 is unknotted we could isotope it to the complement of B in S^3, and realize that the knot K is the connected sum of two unknots. This is a contradiction as K is knotted. All other strings are parallel. Then if $\mathcal{T}$ is not essential there is a disk separating s_1 and s_2 from the other strings, and it suffices to study the case when the number of strings in $\mathcal{T}$ is three. Denote the third string by s and suppose there is a disk in B separating s from $s_1 \cup s_2$. Then, from the diagram $\mathcal{D}$, the string s cobounds a disk D_1 with an arc in ∂B intersecting s_1 once, and disjoint from s_2, and cobounds a disk D_2, with another arc in ∂B, disjoint from $s_1 \cup s_2$. Therefore, the union of these two disks intersects $s_1 \cup s_2$ in a point of s_1. As s_1 is not knotted, $D_1 \cup D_2$ separates a ball from B and an unknotted arc of s_1 from $s_1 \cup s_2$ in it. Hence, this arc can be isotoped to an arc in D_1. With a proper choice of D_1, we can isotope s_1 in a way that reduces the crossing number of the tangle defined by s_1 and s_2 in B, which contradicts the crossing number minimality of the diagram with s_1 and s_2. Therefore, the tangle $\mathcal{T}$ is essential.

(ii) Suppose s_1 and s_2 are on both inner crossings. In this way, each inner crossing is adjacent to an internal segment for each string s_1 and s_2 and these segments are necessarily consecutive. For each inner crossing we can assume that the internal segments separate their opposite segments from the other inner crossings. In fact, if that is not the case, as in Figure 13(a), then at least a segment opposite to an internal segment l is lateral and in the direction of a outermost crossing adjacent to the same inner crossing c. In this case, we can isotope a string by taking the internal segment l over (or under, depending on the outermost crossing type) the segment opposite to the other internal segment adjacent to c, without increasing the crossing number.
After this isotopy, c is as claimed before: the internal segments of c separate the corresponding opposite segments and the other inner crossing. (See Figure 13(b).) Under this setting we are left with only one possible diagram projection of s_1 and s_2, as in Figure 14. There are two possible cases for the crossings of $\mathcal{D}$ in this case. Either the diagram is alternating, as in Figure 2(b), or the diagram is obtained from the alternating one by changing the crossings of a twist box, as in Figure 2(c).

To prove that these tangles are essential first note that we can add a 2-string trivial tangle to the tangle with the strings s_1 and s_2 to get the square knot, in

The next lemma is a base for the theorems in this paper and provides a common weaker version of all of them.

Lemma 2. *If $\mathcal{T}$ is a n-string k-loop tangle, then $c(\mathcal{T}) \geq 2n$.*

Proof: Consider a minimal diagram of $\mathcal{T}$. Since each string has at least two crossings with the other strings, we can associate to each end of a string an outermost crossing. If two ends were associated with the same crossing, as in Figure 4(a), then we could isotope the two ends through the region between them, as in Figure 4(b), reducing the number of crossings by one. This contradicts the diagram being minimal, and therefore this shows that the association is injective. Hence $c(\mathcal{T}) \geq 2n$. ∎

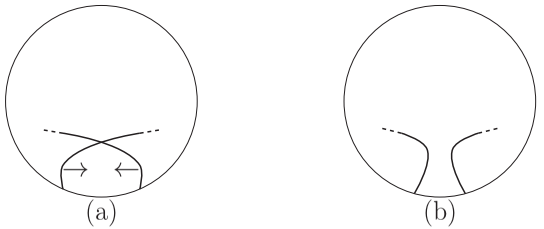

FIGURE 4. In a minimal diagram, two strings do not share an outermost crossing.

In a minimal diagram of a tangle, each outermost crossing c defines a single segment s containing an end of a string. Then s is referred to as the *external segment* of c. The opposite segment with respect to c is called the *internal segment* and the other two are the *lateral segments* of c.

Lemma 3. *In a minimal diagram of a tangle, there is no segment which is both internal and lateral.*

Proof: Suppose there is a minimal diagram of a tangle with a segment s which is both internal and lateral. Hence, as illustrated in Figure 5, we can proceed with an isotopy of s and reduce the number of crossings in $\mathcal{D}$, which contradicts its minimality. ∎

Remark 1. *Since all strings are arcs, in a diagram of a tangle there is no string containing only lateral segments. Furthermore, in a n-string tangle,*

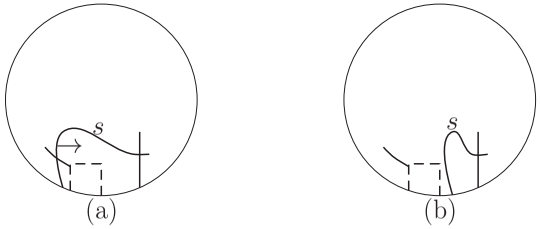

FIGURE 5. In a minimal diagram, segments are not both internal and lateral.

there is at least one inner crossing, and at least two non opposite segments adjacent to the inner crossings are lateral.

3. Crossing number of essential tangles

In this section we study the minimum crossing number of n-string essential tangles and we prove Theorem 1.

Proof of Theorem 1. Let $\mathcal{T}$ be a n-string essential tangle and $\mathcal{D}$ a minimal diagram of $\mathcal{T}$. By the proof of Lemma 2, $\mathcal{D}$ has $2n$ outermost crossings and by Remark 1, there is at least one inner crossing. Therefore $c(\mathcal{T}) \geq 2n + 1$.

Now suppose that $c(\mathcal{T}) = 2n + 1$. Then there is exactly one inner crossing c. Of the four segments adjacent to c, two are lateral and the other two are internal. Again by Remark 1, the lateral segments cannot be opposite in c. Therefore the projection of $\mathcal{T}$ is as depicted in Figure 6(a). In each *dotted region* there is a collection (possibly empty) of unknotted arcs that go around the curve that contains the inner crossing (with one overcrossing and one undercrossing), as in Figure 6(b). Each of these regions can slide along the curve and merge with the other regions, so that we could depict Figure 6(a) with a single such region.

If the string that contains the inner crossing is not alternating, then the tangle can be isotoped to a tangle with two less crossings. Therefore this string is knotted (by identifying its ends along ∂B we obtain a trefoil) and all other strings are unknotted, and $\mathcal{T}$ is equivalent to the tangle represented in Figure 1.

The tangle $\mathcal{T}$ is essential, since we can add a trivial tangle to $\mathcal{T}$ to obtain a link L with n components, one of which is a trefoil and the other $n-1$ components

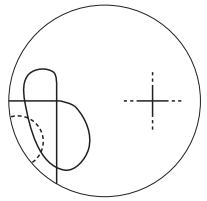

FIGURE 11. Only one string on an inner crossing and other strings on the other crossing.

(i) Suppose s_2 is only on the inner crossing c_b. Then in the projection corresponding to $\mathcal{D}$, s_2 contains two internal segments adjacent to c_b. One internal segment of s_1 is adjacent to c_b, and the opposite segment to this is adjacent to both c_a and c_b, since $\mathcal{D}$ is minimal. One segment consecutive to the latter in c_a is internal, and the remaining two segments adjacent to c_a are lateral. (See Figure 12.)

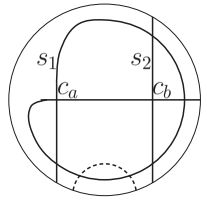

FIGURE 12. Diagram when s_2 is only on the inner crossing c_b.

As $\mathcal{D}$ is a minimal diagram of $\mathcal{T}$, following the string s_2 the crossings in the diagram are alternating. As s_1 is unknotted, following the string, two of its self-crossings are of the same type, and are in the opposite sides of the inner crossing of s_1 with s_2, otherwise we can reduce the crossing number of $\mathcal{D}$. Then, from these two statements we can assume that $\mathcal{D}$ is alternating. There are two possible alternating diagrams but they correspond to homeomorphic tangles, which are depicted in Figure 2(a).

The tangle defined by s_1 and s_2 in B is essential. In fact, we can add a 2-string trivial tangle to the tangle with the strings s_1 and s_2 to obtain the alternating

to the inner crossings and outermost crossings are uniquely determined at this point. For this case we obtain the diagram shadow as in Figure 10.

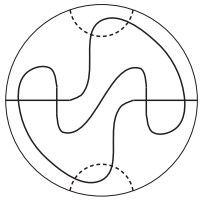

FIGURE 10. Only one string on the inner crossings with the two internal segments on distinct inner crossings.

In this section all strings of the tangles are unknotted. Consequently, in this two cases, illustrated in Figures 9 and 10, the string on the inner crossings is also unknotted. This necessarily means that two consecutive self-crossings are not alternating. Considering all consecutive self-crossings possibilities for the two cases we can obtain a diagram with a smaller crossing number than the one of $\mathcal{D}$, which contradicts $\mathcal{D}$ being a minimal crossing number diagram. ■

Lemma 6. *If on an inner crossing there is only one string s, then s is on the other inner crossing.*

Proof: Suppose that on an inner crossing there is only one string s and that s is not on the other inner crossing. Then, the two internal segments of s are adjacent to the same inner crossing, and the respective opposite segments are lateral. Hence, not to be in contradiction with Lemma 3, we have a diagram projection of $\mathcal{D}$ as in Figure 11, which is a contradiction to $\mathcal{T}$ being essential. ■

Proof of Theorem 2. As before in this section, let $\mathcal{T}$ be a n-string essential tangle with all strings unknotted, and $\mathcal{D}$ a minimal diagram of $\mathcal{T}$. In Lemma 4 we proved that $c(\mathcal{T}) \geq 2n+2$. To prove the second part of the theorem suppose that $c(\mathcal{T}) = 2n+2$. As observed before, $\mathcal{D}$ has $2n$ outermost crossings and 2 inner crossings.
From Lemma 5 exactly two strings are on the inner crossings of $\mathcal{D}$. We denote the two strings on the inner crossings by s_1 and s_2. From Lemma 6, without loss of generality we can assume that s_1 is on both inner crossings.

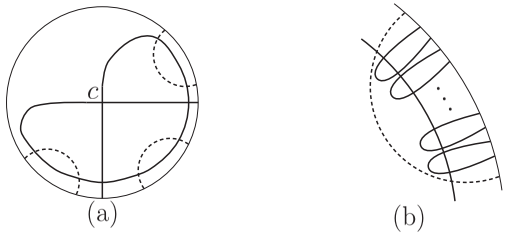

FIGURE 6. (a) A projection of a n-string essential tangle with $c(\mathcal{T}) = 2n+1$; (b) A projection of a dotted region.

are trivial and parallel[†]. With a suitable orientation, the linking number of L is $n-1$. If $\mathcal{T}$ is not essential, then there is a disk in B separating the trefoil string from the trivial strings. Then these strings could be isotoped outside of B and L would be splittable. Therefore the linking number of L would be 0, which is a contradiction to the linking number of L being $n-1$. ■

4. Crossing number of essential tangles with all strings unknotted

This section is devoted to the proof of Theorem 2. We study the minimum crossing number of n-string essential tangles with all strings unknotted, and identify the respective tangles with this crossing number.

Lemma 4. *If $\mathcal{T}$ is a n-string essential tangle with all strings unknotted then $c(\mathcal{T}) \geq 2n+2$.*

Proof: Let $\mathcal{T}$ be a n-string essential tangle with all strings unknotted. By Theorem 1, $c(\mathcal{T}) \geq 2n+1$ and, if $c(\mathcal{T}) = 2n+1$ then at least one string is knotted. Therefore $c(\mathcal{T}) \geq 2n+2$. ■

For the following lemmas we let $\mathcal{T}$ be a n-string essential tangle with all strings unknotted and $c(\mathcal{T}) = 2n+2$, and $\mathcal{D}$ be a minimal diagram of $\mathcal{T}$. Then, from the proof of Lemma 2, there are $2n$ outermost crossings and 2 inner crossings in $\mathcal{D}$, that we denote by c_a and c_b.

Lemma 5. *There are exactly two strings on the inner crossings of $\mathcal{D}$.*

[†]Two strings of $\mathcal{T}$ are *parallel* if they cobound a disk together with two disjoint arcs in ∂B.

Proof: As $\mathcal{D}$ only has two inner crossings then there are at most eight segments adjacent to the inner crossings. From Remark 1, at least two segments adjacent to the inner crossings are lateral. Then, in $\mathcal{D}$ at most six segments adjacent to inner crossings can be internal. Hence, there are at most three strings on the inner crossings of $\mathcal{D}$.

Suppose there are three strings on the inner crossings. Then, from Lemma 3, there are six internal segments adjacent to the inner crossings. From Remark 1, two segments adjacent to inner crossings are lateral. Therefore, as there are only two segments adjacent to inner crossings that are not internal, one component s of σ is on all outermost crossings containing all lateral segments but the two adjacent to the inner crossings. (See Figure 7.)

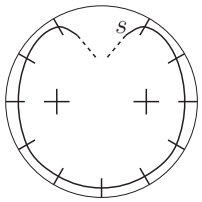

FIGURE 7. If three strings are on the inner crossings, there is a string containing all lateral segments.

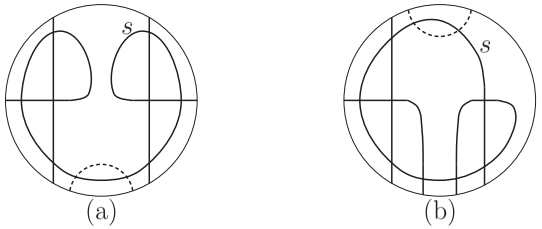

FIGURE 8. If three strings are on the inner crossings, there is always an isotopy reducing the number of crossings.

Either s is on both inner crossings, as in Figure 8(a), and each inner crossing is adjacent to three internal segments, or s is on a single inner crossing and

the other inner crossing has all ends adjacent to internal segments, as in Figure 8(b). In the case of Figure 8(a) we can reduce the number of crossings in the diagram by an isotopy of one of the strings on the inner crossings. And in the case of Figure 8(b), as s is unknotted, two of its three self-crossings in the diagram are not alternating, which allows us to isotope σ reducing the crossing number of the diagram. In both cases we have a contradiction with $\mathcal{D}$ being a minimal crossing number diagram of $\mathcal{T}$. Consequently, the number of strings on the inner crossings of $\mathcal{D}$ is at most two.

Assume now that there is only one string on the inner crossings. Then exactly two segments adjacent to the inner crossings are internal. Suppose these segments are adjacent to the same inner crossing, say c_a. It follows they have to be consecutive, otherwise more than one string would be on the inner crossings or we would have a contradiction to Lemma 3. Then c_b cannot be adjacent to three or four lateral segments, because in this case we would be in contradiction to Lemma 3. Hence, two consecutive segments adjacent to c_b are lateral and the other two segments are adjacent to the two inner crossings. In Figure 9 we have an illustration of the projection of this case.

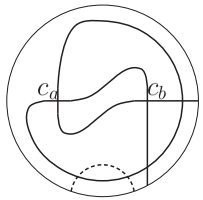

FIGURE 9. Only one string on the inner crossings with the two internal segments on the same inner crossing.

Suppose now each inner crossing is adjacent to exactly one internal segment. The ends of the inner crossings opposite to the internal segments cannot be adjacent to the other inner crossings or to a lateral segment in the direction of the outermost crossing adjacent to the internal segment it opposes, otherwise we would have a contradiction to only one string being on the inner crossings or to Lemma 3. Therefore, the ends of the inner crossings opposite to the internal segments are connected to lateral segments in the direction of the outermost crossing adjacent to the other internal segment. The other segments adjacent