Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 13–47

# SEMIDIRECT PRODUCTS OF (TOPOLOGICAL) SEMI-ABELIAN ALGEBRAS

MARIA MANUEL CLEMENTINO, ANDREA MONTOLI AND LURDES SOUSA

Abstract: We give an explicit description of the semidirect products in any semi-abelian variety. Moreover, we use this description to characterize the topology of the semidirect products in the topological models of any semi-abelian theory.

## 1. Introduction

The semidirect product is a classical construction in group theory, which is used to obtain an equivalence between group actions and split extensions. D. Bourn and G. Janelidze gave in [5] a categorical definition of semidirect products, and proved that it still gives an equivalence between split extensions and internal actions in the context of semi-abelian categories, i.e. pointed Barr-exact protomodular categories with finite colimits.

In the category of groups, the categorical semidirect product coincides with the classical one. Moreover, it is known that the semidirect product of two groups, with respect to a given action, is, as a set, the cartesian product of the two groups. This is not true in all semi-abelian varieties. E.B. Inyangala proved in [8] that it is true in varieties of right $\Omega$-loops, showing that, given two right $\Omega$-loops $X$ and $B$ and an action $\xi$ of $B$ on $X$ there exist bijections $\varphi$ and $\psi$ making the diagram

$$\begin{tikzcd}
X \arrow[r, "\langle 1,0\rangle"] \arrow[d, equal] & X \times B \arrow[r, shift right, "\pi_B"'] \arrow[d, shift right, "\varphi"'] & B \arrow[l, shift right, "\langle 0,1\rangle"'] \arrow[d, equal] \\
X \arrow[r, "k"'] & X \rtimes_\xi B \arrow[r, shift right, "f"'] \arrow[u, shift right, "\psi"'] & B \arrow[l, shift right, "s"']
\end{tikzcd}$$

Received October 28, 2013.
This work was partially supported by the Centro de Matemática da Universidade de Coimbra (CMUC), funded by the European Regional Development Fund through the program COMPETE and by the Portuguese Government through the FCT - Fundação para a Ciência e a Tecnologia under the project PEst-C/MAT/UI0324/2013 and grants number PTDC/MAT/120222/2010 and SFRH/BPD/69661/2010.

commutative, where the bottom row is the split extension corresponding to $\xi$. J.R.A. Gray and N. Martins-Ferreira showed in [7] that right $\Omega$-loops are the unique varieties where this property is valid.

On the other hand, F. Borceux and M.M. Clementino proved in [2] that the equivalence between internal actions and split extensions, obtained via the categorical semidirect product, holds in the categories of topological models of any semi-abelian algebraic theory.

In the present paper, we give an explicit description of the semidirect products in any semi-abelian variety, showing that the semidirect product corresponding to an internal action of an object $B$ on an object $X$ can be described as a subset of the cartesian product of $B$ and a suitable number of copies of $X$, extending the results of [8]. Moreover, we use this fact to prove that, in the case of topological models of a semi-abelian theory, the semidirect product is always a retract of the topological product of $B$ and some copies of $X$.

The paper is organized as follows: in Section 2 we recall the categorical definition of semidirect product. In Section 3 we describe semidirect products in the context of semi-abelian varieties, showing that a semidirect product can be always seen as a subset of a cartesian product. In Section 4 we characterize the semi-abelian varieties in which the inclusion of the semidirect product into the corresponding cartesian product is a bijection, showing that any semi-abelian variety such that all semidirect products of objects $B$ and $X$ are in bijections with cartesian products of the form $X^n \times B$ is a variety of right $\Omega$-loops; in this way we generalize Inyangala's results. Moreover, we study an example of a variety that can be described as semi-abelian using two different sets of operations, with different cardinality, showing that they give rise to different inclusions of the semidirect products into the corresponding cartesian products. In Section 5 we study explicitly other concrete examples. In Section 6 we consider the case of topological models of semi-abelian theories.

**Acknowledgement.** We are very grateful to Francis Borceux for suggesting us the study of semidirect products in topological semi-abelian algebras.

## 2. The categorical notion of semidirect product

In this section we recall from [5] the categorical notion of semidirect product.

Let $\mathbb{C}$ be a finitely complete category. For any morphism $p\colon E \to B$ in $\mathbb{C}$, we can define the pullback functor

$$p^*\colon \mathrm{Pt}(B) \to \mathrm{Pt}(E),$$

where the category $\mathrm{Pt}(B)$, called the category of *points* over $B$, is the category of points of the comma category $\mathbb{C}$ over $B$, i.e. the cocomma category $1_B$ over $\mathbb{C}/B$. This amounts to the category whose objects are the split epimorphisms with codomain $B$. In fact a morphism from the terminal object $1_B\colon B \to B$ to an object $f\colon A \to B$ is precisely an arrow $s\colon B \to A$ such that $fs = 1_B$.

**Definition 2.1.** A finitely complete category $\mathbb{C}$ is said to be a *category with semidirect products* if, for any arrow $p\colon E \to\ B$ in $\mathbb{C}$, the pullback functor $p^*$ (has a left adjoint and) is monadic.

In this case, denoting by $T^p$ the monad defined by this adjunction, given a $T^p$-algebra $(D, \xi)$ the semidirect product $(D, \xi) \rtimes (B, p)$ is an object in $\mathrm{Pt}(B)$ corresponding to $(D, \xi)$ via the canonical equivalence $K$:

$$\begin{tikzcd}
 & {[\mathrm{Pt}(E)]^{T^p}} \arrow[d, shift right, ""'] \arrow[d, phantom, "\vdash"] \\
\mathrm{Pt}(B) \arrow[ur, dashed, "K"] \arrow[r, shift right, "p^*"'] \arrow[r, phantom, "\perp"] & \mathrm{Pt}(E) \arrow[l, shift right, "p_!"'] \arrow[u, shift right]
\end{tikzcd}$$

Let us recall from [5] that, being $\mathbb{C}$ finitely complete, the pullback functors $p^*$ have left adjoints $p_!$ (for any $p$ in $\mathbb{C}$) if and only if $\mathbb{C}$ has pushouts of split monomorphisms. For Barr-exact categories [1], if, moreover, the functors $p^*$ are conservative, that is if $\mathbb{C}$ is protomodular [4], the existence of semidirect products is guaranteed. In fact:

**Theorem 2.2** ([5], Theorem 3.4)**.** *A finitely complete Barr-exact category is a category with semidirect products if and only if it is protomodular and has pushouts of split monomorphisms.*

If $\mathbb{C}$ is finitely complete, so that we can define $p^*$ for every morphism $p$, has pushouts of split monomorphisms, so that the functors $p^*$ have left adjoints $p_!$, and an initial object 0, then it is enough to consider the functors ${i_B}^*$ for the unique morphisms $i_B\colon 0 \to B$:

**Proposition 2.3** ([11], Corollary 3)**.** *Let $\mathbb{C}$ be a category with finite limits, pushouts of split monomorphisms and initial object. Then the following statements are equivalent:*

(i) *all pullback functors $i_B^*$ defined by the initial arrows are monadic;*
(ii) *for any morphism $p$ in $\mathbb{C}$, the pullback functor $p^*$ is monadic, i.e. $\mathbb{C}$ admits semidirect products.*

When the category $\mathbb{C}$ is pointed, the algebras for the monad $(T^{i_B}, \eta, \mu)$ are called *internal actions* in [3] and the endofunctor $T^{i_B}$ is usually denoted by $B\flat(-)$. We recall that $\eta_X$ and $\mu_X$ are the unique morphisms such that $k_0\eta_X = \iota_X$ and $k_0\mu_X = [k_0, \iota_B]k_0'$, as displayed in the diagrams

$$\begin{tikzcd} B\flat X \arrow[r, "k_0"] & X+B \\ X \arrow[u, "\eta_X"] \arrow[ur, "\iota_X"'] & \end{tikzcd} , \qquad \begin{tikzcd} B\flat(B\flat X) \arrow[r, "k_0'"] \arrow[d, "\mu_X"'] & (B\flat X)+B \arrow[d, "{[k_0,\iota_B]}"] \\ B\flat X \arrow[r, "k_0"'] & X+B, \end{tikzcd}$$

where $k_0'$ and $k_0$ denote the kernels of $[0,1]\colon (B\flat X)+B \to B$ and of $[0,1]\colon X+B \to B$, respectively.

The algebras for this monad are pairs $(X, \xi\colon B\flat X \to X)$ satisfying the usual conditions:

$$\xi\eta_X = 1_X, \quad \text{and} \quad \xi\mu_X = \xi(1\flat\xi).$$

Consequently, for $\mathbb{C}$ as above, saying that $\mathbb{C}$ has semidirect products means that, for each internal action $\xi : B\flat X \to X$, there exists (up to isomorphism) a unique split epimorphism $A \underset{f}{\overset{s}{\leftrightarrows}} B$ such that $X = \mathrm{Ker} f$ and making the following diagram commute:

$$\begin{tikzcd} B\flat X \arrow[r, "k_0"] \arrow[d, "\xi"'] & X+B \arrow[r, shift right, "{[0,1]}"'] \arrow[d, "{[k,s]}"] & B \arrow[l, shift right, "\iota_B"'] \arrow[d, equal] \\ X \arrow[r, "k"'] & A \arrow[r, shift right, "f"'] & B, \arrow[l, shift right, "s"'] \end{tikzcd} \tag{$*$}$$

Then $A \underset{f}{\overset{s}{\leftrightarrows}} B$ is the *semidirect product of $X$ and $B$ with respect to $\xi$.* Sometimes we will identify this semidirect product with the object $A$ or with the *split extension*

$$X \xrightarrow{k} A \underset{f}{\overset{s}{\leftrightarrows}} B. \tag{$**$}$$

When $\mathbb{C}$ is the category of groups, $B\flat X$ is the subgroup of the free product $X + B$ generated by the elements of the form $bxb^{-1}$, with $b \in B$ and $x \in X$. Hence an internal action $\xi$ is nothing but the realization in $X$ of the conjugation in the classical semidirect product $X \rtimes B$.

## 3. Semidirect products in semi-abelian varieties

An immediate consequence of Theorem 2.2 is that every semi-abelian category [9] has semidirect products in the categorical sense recalled above. This is the case, in particular, of semi-abelian varieties, which were characterized by D. Bourn and G. Janelidze in [6].

**Theorem 3.1.** *A variety of universal algebra is semi-abelian if and only if it has, among its operations, a unique constant $0$, $n$ binary operations $\alpha_i$, $i = 1, \dots, n$, and an $(n+1)$-ary operation $\theta$ satisfying the following equations:*

$$\alpha_i(x, x) = 0 \qquad \textit{for any } x \tag{I}$$

$$\theta(\underline{\alpha}(x, y), y) = x \qquad \textit{for any } x, y, \tag{II}$$

*where $\underline{\alpha}(x, y)$ denotes $(\alpha_1(x, y), \dots, \alpha_n(x, y))$.*

The aim of this section is to give an explicit description of the semidirect product in semi-abelian varieties.

We start by recalling a result of E.B. Inyangala [8]. Let $\mathbb{C}$ be a variety of right $\Omega$-loops, i.e. a (semi-abelian) variety which has, among its operations, a unique constant $0$, a binary $+$ and a binary $-$ satisfying the following equations:

(a) $x + 0 = x$;
(b) $0 + x = x$;
(c) $(x - y) + y = x$;
(d) $(x + y) - y = x$.

Then, given a split extension $(**)$ in $\mathbb{C}$, it is possible to define two set-theoretical maps

$$\begin{array}{rcl} \varphi\colon X\times B & \longrightarrow & A \\ (x,b) & \longmapsto & x+s(b) \end{array} \qquad \begin{array}{rcl} \psi\colon A & \longrightarrow & X\times B \\ a & \longmapsto & (a-sf(a),f(a)), \end{array}$$

treating $k$ as an inclusion.

**Proposition 3.2** ([8]). *The two maps $\varphi$ and $\psi$ are inverse to each other.*

In other terms, given any split extension $(**)$, $A$ is in bijection with the cartesian product of $B$ and $\mathrm{Ker}f$.

Moreover, E.B. Inyangala proved in [8] that, if a semi-abelian variety $\mathbb{C}$ has a binary $+$ and a binary $-$, and the maps $\varphi$ and $\psi$ are defined using $+$ and $-$ as above, then $\varphi$ and $\psi$ are bijections, whose restrictions to $X$ and $B$ are identities, if and only if the equations (a)-(d) are satisfied, i.e. if and only if $\mathbb{C}$ is a variety of right $\Omega$-loops. Later, in [7] J.R.A. Gray and N. Martins-Ferreira extended this result, showing that the existence of such $\varphi$ and $\psi$ induces binary operations $+$ and $-$ making the algebras right $\Omega$-loops. In order to achieve this result, they made a thorough study of the maps $\varphi$ and $\psi$ described above, and of their generalizations in semi-abelian varieties, seen as natural transformations between suitable functors.

We are now going to express the natural transformations studied in [7] in terms of the operations of the semi-abelian variety, in order to get an explicit description of semidirect products. This way we will obtain some results already contained in [7], but we will give different proofs, that will be useful later in the study of semidirect products of topological algebras.

Let $\mathbb{C}$ be a semi-abelian variety. For each split extension $(**)$ in $\mathbb{C}$, using the operations $\alpha_1,\dots,\alpha_n$ and $\theta$ introduced at the beginning of the section, we can define two set-theoretical maps

$$\begin{array}{rcl} \varphi\colon X^n\times B & \longrightarrow & A \\ (\underline{x},b) & \longmapsto & \theta(\underline{x},s(b)) \end{array} \qquad \begin{array}{rcl} \psi\colon A & \longrightarrow & X^n\times B \\ a & \longmapsto & (\underline{\alpha}(a,sf(a)),f(a)), \end{array}$$

where $\underline{x}$ denotes $(x_1,\dots,x_n)$, and treating again $k$ as an inclusion. To simplify our calculations, for any map $h:Z\to W$ and $\underline{z}=(z_1,\dots,z_n)\in Z^n$, $h^n(\underline{z})$ denotes $(h(z_1),\dots,h(z_n))$.

**Proposition 3.3.** *For any split epimorphism* $A \underset{f}{\overset{s}{\leftrightarrows}} B$ *in* $\mathbb{C}$*, we have that:*

1. $\varphi\psi = \mathrm{id}_A$*, and therefore* $A$ *is a retract of* $X^n \times B$*.*
2. $A$ *is in bijection with the subset*

$$Y = \{(\underline{x}, b) \in X^n \times B \,|\, \underline{\alpha}(\theta(\underline{x}, s(b)), s(b)) = \underline{x}\}.$$

*Proof*: 1. For any $a \in A$ we have:

$$\varphi\psi(a) = \varphi(\underline{\alpha}(a, sf(a)), f(a)) = \theta(\underline{\alpha}(a, sf(a)), sf(a)) = a,$$

where the last equality follows from equation (II).

2. Let us first prove that $\psi(A) \subseteq Y$: for any $a \in A$, we have $\psi(a) = (\underline{\alpha}(a, sf(a)), f(a))$, hence:

$$\underline{\alpha}(\theta(\underline{\alpha}(a, sf(a)), sf(a)), sf(a)) = \underline{\alpha}(a, sf(a)).$$

It remains to prove that $\psi\varphi_{|_Y} = \mathrm{id}_Y$. Let us observe that, for any algebra $Z$ and any $z \in Z$, we have $\theta(\underline{0}, z) = \theta(\underline{\alpha}(z, z), z) = z$. Then, for any $(\underline{x}, b) \in Y$:

$$\begin{aligned}
\psi\varphi(\underline{x}, b) &= \psi\theta(\underline{x}, s(b)) \\
&= (\underline{\alpha}(\theta(\underline{x}, s(b)), sf\theta(\underline{x}, s(b))), f\theta(\underline{x}, s(b))) \\
&= (\underline{\alpha}(\theta(\underline{x}, s(b)), \theta(s^n f^n(\underline{x}), sfs(b))), \theta(f^n(\underline{x}), fs(b))) \\
&= (\underline{\alpha}(\theta(\underline{x}, s(b)), \theta(\underline{0}, s(b))), \theta(\underline{0}, b)) \\
&= (\underline{\alpha}(\theta(\underline{x}, s(b)), s(b)), b) = (\underline{x}, b),
\end{aligned}$$

where the last equality holds because $(\underline{x}, b) \in Y$. ∎

Proposition 3.3 allows us to give an explicit description of semidirect products in any semi-abelian variety.

**Theorem 3.4.** *Given a semi-abelian variety* $\mathbb{C}$*, objects* $B, X \in \mathbb{C}$ *and an internal action* $\xi\colon B\flat X \to X$ *of* $B$ *on* $X$*, the semidirect product* $X \rtimes_\xi B$ *of* $X$ *and* $B$ *w.r.t. the action* $\xi$ *is the set* $Y$ *described in the previous proposition:*

$$Y = \{(\underline{x}, b) \in X^n \times B \,|\, \underline{\alpha}(\theta(\underline{x}, s(b)), s(b)) = \underline{x}\},$$

*where* $A \underset{f}{\overset{s}{\leftrightarrows}} B$ *is the split epimorphism in* $\mathbb{C}$ *corresponding to the action* $\xi$*, equipped with the following structure: if* $\omega$ *is an* $m$*-ary operation of the* *variety, then in $Y$ we have:*

$$\begin{aligned}&\omega_Y((\underline{x}_1,b_1),\dots,(\underline{x}_m,b_m)) \;=\\ &\qquad = \;(\xi^n\underline{\alpha}_{B\flat X}(\omega_{B\flat X}(\theta_{B\flat X}(\underline{x}_1,b_1),\dots,\theta_{B\flat X}(\underline{x}_m,b_m)),\omega_{B\flat X}(\underline{b})),\omega_B(\underline{b})).\end{aligned}$$

*Proof*: Being $A \underset{f}{\overset{s}{\leftrightarrows}} B$ the split epimorphism in $\mathbb{C}$ corresponding to the semidirect product $X \rtimes_\xi B$, diagram $(*)$ says that $\xi$ is the restriction of the morphism $[k,s]$. We know that $A$ is in bijection with $Y$ via the maps $\varphi$ and $\psi$ studied in Proposition 3.3. Given $(\underline{x}_i,b_i)\in Y$, for $i=1,\dots,m$, let

$$u=\omega_A(\theta_A(\underline{x}_1,s(b_1)),\dots,\theta_A(\underline{x}_m,s(b_m))).$$

Then

$$\begin{aligned}\omega_Y((\underline{x}_1,b_1),\dots,(\underline{x}_m,b_m)) \;&=\; \psi\omega_A(\varphi(\underline{x}_1,b_1),\dots,\varphi(\underline{x}_m,b_m))=\psi(u)\\ &=\;(\underline{\alpha}_A(u,sf(u)),f(u)).\end{aligned}$$

Since $f$ and $s$ are morphisms (and so they preserve the operations):

$$f(u)=\omega_B(\theta_B(f^n(\underline{x}_1),fs(b_1)),\dots,\theta_B(f^n(\underline{x}_m),fs(b_m))),$$

and, since $fs=\mathrm{id}_B$ and $X=\mathrm{Ker}f$,

$$f(u)\;=\;\omega_B(\theta_B(\underline{0},b_1),\dots,\theta_B(\underline{0},b_m))\;=\;\omega_B(\underline{b}).$$

Thus, $sf(u)=\omega_A(s^m(\underline{b}))$, and we obtain:

$$\begin{aligned}\omega_Y((\underline{x}_1,b_1),\dots,(\underline{x}_m,b_m)) \;&=\; (\underline{\alpha}_A(u,\omega_A(s^m(\underline{b}))),\omega_B(\underline{b}))\\ &=\;(\underline{\alpha}_A(\omega_A(\theta_A(\underline{x}_1,s(b_1)),\dots,\theta_A(\underline{x}_m,s(b_m))),\omega_A(s^m(\underline{b}))),\omega_B(\underline{b}))\\ &=\;([k,s]^n\underline{\alpha}_{B\flat X}(\omega_{B\flat X}(\theta_{B\flat X}(\underline{x}_1,b_1),\dots,\theta_{B\flat X}(\underline{x}_m,b_m)),\omega_{B\flat X}(\underline{b})),\omega_B(\underline{b})),\end{aligned}$$

and, since $\xi$ is the restriction of $[k,s]$, we finally obtain the claimed equality. ∎

## 4. A detailed description of $\varphi$ and $\psi$

The aim of this section is to make explicit the relationships between the properties of the maps $\varphi$ and $\psi$ defined in Section 3 and the equations of the variety.

Let $\mathbb{C}$ be a variety which has, among its operations, a unique constant 0, $n$ binary operations $\alpha_i$, $i=1,\dots,n$, satisfying the equations (I), and an $(n+1)$-ary operation $\theta$. Given a split epimorphism with kernel $X$ as in $(**)$, let $\varphi\colon X^n\times B\to A$ and $\psi\colon A\to X^n\times B$ be the maps defined in Section 3.

Observe that the map $\psi$ is well-defined (i.e. it takes value in $X^n \times B$) thanks to the equations (I). We have therefore the following diagram:

$$\begin{tikzcd} X^n \arrow[r, "\langle 1,0\rangle"] \arrow[d, shift right, "\varphi_X"'] & X^n \times B \arrow[r, shift left, "\pi_B"'] \arrow[d, shift right, "\varphi"'] & B \arrow[l, shift left, "\langle 0,1\rangle"'] \arrow[d, shift right, "\varphi_B"'] \\ X \arrow[r, "k"'] \arrow[u, shift right, "\psi_X"'] & A \arrow[r, shift right, "f"'] \arrow[u, shift right, "\psi"'] & B \arrow[l, shift right, "s"'] \arrow[u, shift right, "\psi_B"'] \end{tikzcd} \qquad (***)$$

where $\varphi_X, \psi_X$ are obtained via the universal property of kernels, and

$$\varphi_B(b) = f(\varphi(\underline{0}, b)) = f\theta(\underline{0}, s(b)) = \theta(\underline{0}, b), \qquad \psi_B(b) = \pi_B(\psi(s(b))) = \mathrm{id}_B,$$

so that $f\varphi = \varphi_B \pi_B$ and $\psi s = \langle 0, 1\rangle \psi_B$.

The following proposition is a reformulation of some results in [7].

**Proposition 4.1.** *We have that:*

1. $\varphi_B = \mathrm{id}_B$ *for any split epimorphism* $(**)$ *if and only if the following equation is satisfied in the variety:*
$$\theta(\underline{0}, x) = x \quad \textit{for all } x; \tag{III}$$
2. $\varphi\psi = \mathrm{id}_A$ *for any split epimorphism* $(**)$ *if and only if equation* (II) *is satisfied in the variety;*
3. *if equation* (III) *holds in the variety, then* $\psi\varphi = \mathrm{id}_{X^n \times B}$ *for any split epimorphism* $(**)$ *if and only if the equation*
$$\underline{\alpha}(\theta(\underline{x}, y), y) = \underline{x} \quad \textit{for all } x, y \tag{IV}$$
*is satisfied in the variety.*

*Proof*: 1. Suppose that equation (III) holds. Then $\varphi_B(b) = \theta(\underline{0}, b) = b$, for all $b \in B$. Conversely, suppose that $\varphi_B = \mathrm{id}_B$ for any split epimorphism $(**)$. Applying this fact to the split epimorphism $A \underset{1_A}{\overset{1_A}{\rightleftarrows}} A$ for any algebra $A$, we easily get equation (III).

2. The fact that equation (II) implies that $\varphi\psi = \mathrm{id}_A$ was already proved (see Proposition 3.3). Conversely, suppose that $\varphi\psi = \mathrm{id}_A$ for any split epimorphism of the form $A \times A \underset{\pi_2}{\overset{\langle 1,1\rangle}{\rightleftarrows}} A$. Then, for any $x, y \in A$, we have

$$(x, y) = \theta(\underline{\alpha}((x, y), (y, y)), (y, y)) = (\theta(\underline{\alpha}(x, y), y), \theta(\underline{\alpha}(y, y), y)),$$

and hence $\theta(\underline{\alpha}(x,y),y)=x$ for all $x,y$.

3. We have that

$$\begin{aligned}\psi\varphi(\underline{x},b) &= \psi(\theta(\underline{x},s(b))=(\underline{\alpha}(\theta(\underline{x},s(b)),sf\theta(\underline{x},s(b))),f\theta(\underline{x},s(b)))\\ &= (\underline{\alpha}(\theta(\underline{x},s(b)),\theta(s^nf^n(\underline{x}),sfs(b))),\theta(f^n(\underline{x}),fs(b)))\\ &= (\underline{\alpha}(\theta(\underline{x},s(b)),\theta(\underline{0},s(b))),\theta(\underline{0},b))\\ &= (\underline{\alpha}(\theta(\underline{x},s(b)),s(b)),b).\end{aligned}$$

Hence, if equation (IV) holds, we get $\psi\varphi=\mathrm{id}_{X^n\times B}$. Conversely, suppose that $\psi\varphi=\mathrm{id}$ for any split epimorphism of the form $A\times A \underset{\pi_2}{\overset{\langle 1,1\rangle}{\leftrightarrows}} A$. Then, for any $\underline{x},y$, we have

$$(\underline{x},y)=(\underline{\alpha}(\theta(\underline{x},y),y),y),$$

and the first component of this equality gives equation (IV). ■

**Theorem 4.2.** 1. *For each semi-abelian variety $\mathbb{C}$, and each $n$ binary operations $\alpha_i$ and $(n+1)$-ary operation $\theta$ satisfying equations* (I)-(II), *for any split epimorphism $A \underset{f}{\overset{s}{\leftrightarrows}} B$ the maps $\varphi$ and $\psi$ are bijections between $A$ and the cartesian product $X^n\times B$ if and only if equation* (IV) *is satisfied in $\mathbb{C}$.*

2. *If a semi-abelian variety $\mathbb{C}$ satisfies equation* (IV), *then it is possible to define in $\mathbb{C}$ binary operations $+$ and $-$ satisfying the conditions for right $\Omega$-loops.*

*Proof*: 1. Thanks to the previous proposition, it suffices to observe that equations (I) and (II) imply equation (III); indeed:

$$\theta(\underline{0},x)=\theta(\underline{\alpha}(x,x),x)=x.$$

2. The operations $+$ and $-$ can be defined in the following way:

$$x+y=\theta(\underline{\alpha}(x,0),y),\qquad x-y=\theta(\underline{\alpha}(x,y),0).$$

They satisfy the equations of a right $\Omega$-loop:

(a) $x+0=\theta(\underline{\alpha}(x,0),0)=x$, by equation (II);

(b) $0+x=\theta(\underline{\alpha}(0,0),x)=\theta(\underline{0},x)=x$;

(c)

$$\begin{aligned}(x-y)+y &= \theta(\underline{\alpha}(x,y),0)+y = \theta(\underline{\alpha}(\theta(\underline{\alpha}(x,y),0),0),y) && \text{(by (IV))}\\ &= \theta(\underline{\alpha}(x,y),y) = x && \text{(by (II));}\end{aligned}$$

(d)

$$\begin{aligned}(x+y)-y &= \theta(\underline{\alpha}(x,0),y)-y = \theta(\underline{\alpha}(\theta(\underline{\alpha}(x,0),y),y),0) && \text{(by (IV))}\\ &= \theta(\underline{\alpha}(x,0),0) = x && \text{(by (II)).}\end{aligned}$$

■

Let us observe that J.R.A. Gray and N. Martins-Ferreira proved in [7] that the existence of suitable maps $\varphi$ and $\psi$ for any split epimorphism in a variety (giving natural transformations of suitable functors between categories of points) allows to define operations $\theta$ and $\alpha_i$ satisfying equations (I). This means that, in a semi-abelian variety, for any set of operations $(\alpha_i, \theta)$ there exists exactly one pair of maps $(\varphi, \psi)$ for any split epimorphism. Theorem 4.2 extends then the results of [8] and [7], giving a complete characterization of those semi-abelian varieties in which the semidirect product of two objects $X$ and $B$ (w.r.t. an action of $B$ on $X$) naturally underlies the cartesian product of $B$ and a certain number of copies of $X$, and, moreover, we show that one single copy of $X$ suffices.

Let us observe, moreover, that, if a semi-abelian variety $\mathbb{C}$ satisfies the conditions of Theorem 4.2, then the maps $\varphi$ and $\psi$ induce bijections between $X$ and $X^n$ for any split epimorphism $(**)$. If $n \geq 2$, this implies that all the algebras of the variety, except the trivial one, are infinite. The following is an example of a variety, with $n \geq 2$, which satisfies equation (IV).

**Example 4.3.** Let $\mathbb{C}$ be the variety defined by the following operations: a unique constant 0, two binary operations $\alpha_1$ and $\alpha_2$ and a ternary operation $\theta$ satisfying the equations (I), (II) and (IV). A concrete example of an algebra belonging to this variety is given by the set $\mathbb{R}^{\mathbb{N}}$ of real sequences (but $\mathbb{R}$ can be replaced by any non-trivial, not necessarily infinite, right $\Omega$-loop) equipped with the operations defined by

$$\alpha_1(x,y) = (x_{2n-1} - y_{2n-1})_{n\in\mathbb{N}} = (x_1 - y_1, x_3 - y_3, \ldots),$$

$$\alpha_2(x,y) = (x_{2n} - y_{2n})_{n\in\mathbb{N}} = (x_2 - y_2, x_4 - y_4, \ldots),$$

$$\theta(x,y,z) = (x_1 + z_1, y_1 + z_2, x_2 + z_3, y_2 + z_4, \ldots),$$

for any $x = (x_n)_{n\in\mathbb{N}}$, $y = (y_n)_{n\in\mathbb{N}}$ and $z = (z_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^{\mathbb{N}}$. It is immediate to see that the equations (I) are satisfied. Concerning equation (II) we have:

$$\begin{aligned}\theta(\alpha_1(x,y),\alpha_2(x,y),y) &= \theta((x_{2n-1}-y_{2n-1})_{n\in\mathbb{N}},(x_{2n}-y_{2n})_{n\in\mathbb{N}},y)\\ &= (x_1-y_1+y_1,x_2-y_2+y_2,x_3-y_3+y_3,x_4-y_4+y_4,\ldots)\\ &= x.\end{aligned}$$

Finally, concerning equation (IV), we have:

$$\begin{aligned}\alpha_1(\theta(x,y,z),z) &= \alpha_1((x_1+z_1,y_1+z_2,x_2+z_3,y_2+z_4,\ldots),z)\\ &= (x_1+z_1-z_1,x_2+z_3-z_3,\ldots)=x,\end{aligned}$$

and

$$\begin{aligned}\alpha_2(\theta(x,y,z),z) &= \alpha_2((x_1+z_1,y_1+z_2,x_2+z_3,y_2+z_4,\ldots),z)\\ &= (y_1+z_2-z_2,y_2+z_4-z_4,\ldots)=y.\end{aligned}$$

Then $\mathbb{C}$ satisfies the conditions of Theorem 4.2, and hence, for any split extension $(**)$ in $\mathbb{C}$, we have that $A$ is in bijection with the cartesian product $X^2\times B$. This example can be easily generalized to the case of any $n\geq 2$. Observe that, in this case, the operations $+$ and $-$ which give the structure of right $\Omega$-loop are the usual sum and subtraction of sequences, respectively, i.e.:

$$x+y=(x_n+y_n)_{n\in\mathbb{N}},$$

$$x-y=(x_n-y_n)_{n\in\mathbb{N}}.$$

Then $\mathbb{C}$ can be seen as a semi-abelian variety using both sets of operations $\{0,+,-\}$ and $\{0,\alpha_1,\alpha_2,\theta\}$. Using the first one, we have that, for any split extension $(**)$, the object $A$ is in bijection with $X\times B$, while, using the second one, $A$ is in bijection with $X^2\times B$.

The previous example shows that, if a variety can be described as semi-abelian using two different sets of operations, then the two corresponding descriptions of the semidirect products may be different.

## 5. Examples

In this section we present examples that illustrate the results of Section 3 in the absence of equation (IV). Given a split epimorphism $A \underset{f}{\overset{s}{\leftrightarrows}} B$

in a semi-abelian variety $\mathbb{C}$ with binary operations $\alpha_i$, $i = 1, \dots, n$, and an $(n+1)$-operation $\theta$ satisfying equations (I) and (II), diagram $(***)$:

$$\begin{tikzcd}
X^n \arrow[r, "\langle 1,0\rangle"] \arrow[d, shift right, "\varphi_X"'] & X^n \times B \arrow[r, shift right, "\pi_B"'] \arrow[d, shift right, "\varphi"'] & B \arrow[l, shift right, "\langle 0,1\rangle"'] \arrow[d, shift right, "\varphi_B"'] \\
X \arrow[r, "k"'] \arrow[u, shift right, "\psi_X"'] & A \arrow[r, shift right, "f"'] \arrow[u, shift right, "\psi"'] & B \arrow[l, shift right, "s"'] \arrow[u, shift right, "\psi_B"']
\end{tikzcd}$$

gives an inclusion $\langle 1,0\rangle\psi_X$ of $X$ into the cartesian product $X^n \times B$. In the following examples we show that this inclusion can be of different forms.

**Example 5.1.** Let $\mathbb{C}$ be the variety of Heyting semilattices, which is defined by a constant $\top$ and two binary operations $\wedge$ and $\Rightarrow$ satisfying the following equations:

$$\top \wedge x = x, \qquad x \wedge x = x, \qquad x \wedge y = y \wedge x,$$
$$x \wedge (y \wedge z) = (x \wedge y) \wedge z, \qquad (x \Rightarrow x) = \top,$$
$$x \wedge (x \Rightarrow y) = x \wedge y, \qquad y \wedge (x \Rightarrow y) = y,$$
$$x \Rightarrow (y \wedge z) = (x \Rightarrow y) \wedge (x \Rightarrow z).$$

P.T. Johnstone proved in [10] that the variety of Heyting semilattices is semi-abelian, with the following operations:

$$\alpha_1(x,y) = (x \Rightarrow y), \qquad \alpha_2(x,y) = (((x \Rightarrow y) \Rightarrow y) \Rightarrow x),$$
$$\theta(x,y,z) = (x \Rightarrow z) \wedge y.$$

In this variety, given a split extension $(**)$, we have, for $x \in X$:

$$\psi(x) = (\alpha_1(x, sf(x)), \alpha_2(x, sf(x)), f(x)) = (\alpha_1(x,\top), \alpha_2(x,\top), \top) =$$
$$= ((x \Rightarrow \top), (((x \Rightarrow \top) \Rightarrow \top) \Rightarrow x), \top) = (\top, x, \top),$$

hence the inclusion of $X$ into $X \times X \times B$ is given by the second inclusion of $X$ into the product:

$$X \xrightarrow{\langle 0,1,0\rangle} X \times X \times B.$$

**Example 5.2.** Let $\mathbb{C}$ be the variety defined by the following operations: a unique constant $0$, a binary subtraction $\alpha$ (which means a binary operation $\alpha$ such that $\alpha(x,x) = 0$ and $\alpha(x,0) = x$ for any $x$) and a ternary operation $\theta$ satisfying the following equation:

$$\theta(\alpha(x,y), \alpha(x,y), y) = x.$$

It is a semi-abelian variety, with $n = 2$ and $\alpha_1 = \alpha_2 = \alpha$. A concrete example of this situation is given by the divisible abelian group $(\mathbb{Q}, +)$ with $\alpha$ and $\theta$ given by:

$$\alpha(x, y) = x - y, \qquad \theta(x, y, z) = \frac{x + y + 2z}{2}.$$

In this variety, given a split extension $(**)$, we have, for $x \in X$:

$$\psi(x) = (\alpha(x, sf(x)), \alpha(x, sf(x)), f(x)) = (\alpha(x, 0), \alpha(x, 0), 0) = (x, x, 0),$$

hence the inclusion of $X$ into $X \times X \times B$ is given by the diagonal of $X$:

$$X \xrightarrow{\langle 1,1,0\rangle} X \times X \times B.$$

**Example 5.3.** Let $\mathbb{C}$ be the semi-abelian variety having a unique constant 0, a binary subtraction $\alpha$, a binary sum $\rho$ and a ternary operation $\theta$, such that $\rho$ and $\alpha$ satisfy the usual group equations, and, moreover, the following equation is satisfied:

$$\theta(\alpha(x, y), \alpha(x, y), y) = x.$$

A concrete example of this situation is the divisible abelian group $(\mathbb{Q}, +)$ considered in the example above, with $\rho$, $\alpha$ and $\theta$ given by:

$$\rho(x, y) = x + y, \qquad \alpha(x, y) = x - y, \qquad \theta(x, y, z) = \frac{x + y + 2z}{2}.$$

There are two ways of describing $\mathbb{C}$ as a semi-abelian variety. One is with $n = 1$, using the group operations $\rho$ and $\alpha$. From this point of view, given a split extension $(**)$, we have that $A$ is in bijection with the cartesian product $X \times B$. The second way is with $n = 2$, using $\alpha_1 = \alpha_2 = \alpha$ and the ternary operation $\theta$. From this second point of view, given a split extension $(**)$, we have that $A$ is in bijection with a subset of the cartesian product $X \times X \times B$. The two points of view are not in contradiction, because, as observed in the previous example, the fact that $\alpha$ is a subtraction implies that the inclusion of $X$ into $X \times X \times B$ is given by the diagonal of $X$.

## 6. The semidirect product of topological algebras

F. Borceux and M.M. Clementino proved in [2] that, given a semi-abelian theory $\mathbb{T}$, the category $Top(\mathbb{T})$ of its topological models has semidirect products in the categorical sense, although it is not a semi-abelian category, because it fails to be Barr-exact in general. The results presented in Section 3 allow us to give an explicit description of the semidirect products in $Top(\mathbb{T})$, as we are going to show.

Let $\mathbb{T}$ be a semi-abelian theory determined by a constant 0, $n$ binary operations $\alpha_i$ and an $(n+1)$-ary operation $\theta$ satisfying equations (I) and (II), and let $\mathcal{E}$ be a finitely complete category. We will denote by $\mathcal{E}(\mathbb{T})$ the category of models of $\mathbb{T}$ in $\mathcal{E}$. When $\mathcal{E} = Set$, $\mathbb{C}$ is the semi-abelian variety corresponding to the theory $\mathbb{T}$. Let

$$U : \mathcal{E}(\mathbb{T}) \to \mathcal{E}$$

be the (forgetful) functor which forgets the algebraic structure of any object in $\mathcal{E}(\mathbb{T})$. Given a split epimorphism $A \underset{f}{\overset{s}{\leftrightarrows}} B$ in $\mathcal{E}(\mathbb{T})$, we can repeat internally in $\mathcal{E}$ the construction of the maps $\varphi$ and $\psi$ studied in Proposition 3.3, obtaining thus two morphisms in $\mathcal{E}$:

$$\varphi : U(X)^n \times U(B) \to U(A), \qquad \psi : U(A) \to U(X)^n \times U(B),$$

where $X$ is the kernel of $f$. The proof of Proposition 3.3 uses only finite limits, hence it is Yoneda-invariant. This means that, for any finitely complete category $\mathcal{E}$ and for any split extension $(**)$ in $\mathcal{E}(\mathbb{T})$, $U(A)$ is a subobject of $U(X)^n \times U(B)$.

In particular, when $\mathcal{E} = Top$, for any split extension $(**)$ in $Top(\mathbb{T})$ we have that, as a topological space, $A$ is a subspace, actually a retract, of the topological product $X^n \times B$. More explicitly:

**Proposition 6.1.** *Given a semi-abelian theory $\mathbb{T}$ and a split extension $(**)$ in $Top(\mathbb{T})$, the maps*

$$\varphi : X^n \times B \to A \qquad \text{and} \qquad \psi : A \to X^n \times B,$$

*of Proposition 3.3 are continuous.*

*Proof*: $\varphi$ and $\psi$ are defined using only the operations $\theta$ and $\alpha_i$, that are continuous because they are morphisms in $Top$, and the canonical morphisms induced by the products in $Top$, hence they are continuous. ■

From Proposition 6.1 and Theorem 3.4 it follows that:

**Theorem 6.2.** *Let $\mathbb{T}$ be an algebraic theory. Given objects $B, X$ and an internal action $\xi : B\flat X \to X$ on $X$ in $Top(\mathbb{T})$, the semidirect product $X \rtimes_\xi B$ of $X$ and $B$ w.r.t the action $\xi$ is the algebra $Y \subseteq X^n \times B$ described in Theorem 3.4 equipped with the product topology. In particular, if the theory $\mathbb{T}$ defines* *a variety of right $\Omega$-loops, then $X \rtimes_\xi B$ is isomorphic, as a topological space, to the topological product $X \times B$.*

The results above can be applied not only for $\mathcal{E} = Top$, but for any subcategory of $Top$ which is closed under finite products, like the subcategories of compact, Hausdorff, connected, and totally disconnected spaces. Moreover, we immediately get that, given a split extension $(**)$ in $Top(\mathbb{T})$, if both $X$ and $B$ are compact, then $A$ also is, and the same holds for the properties of being Hausdorff, connected and totally disconnected.

We conclude by observing that the last statement of Theorem 6.2 actually holds if we replace $Top$ with any finitely complete category $\mathcal{E}$: if the algebraic theory $\mathbb{T}$ defines a variety of right $\Omega$-loops, let us consider the category $\mathcal{E}(\mathbb{T})$ of its models in $\mathcal{E}$. Given a split extension $(**)$ in $\mathcal{E}(\mathbb{T})$, we have that $U(A)$ is isomorphic to $U(X) \times U(B)$, where $U$ is, as above, the forgetful functor $U\colon \mathcal{E}(\mathbb{T}) \to \mathcal{E}$. If the category $\mathcal{E}(\mathbb{T})$ has pushouts of split monomorphisms, this fact can be use to prove that it has semidirect products in the categorical sense recalled in Section 2, using the same argument that was used in [11] for the case of internal groups and internal groupoids with a fixed object of objects.

## References

[1] M. Barr, *Exact categories*, in: Lecture Notes in Mathematics, vol. 236 (1971), Springer-Verlag, 1-120.

[2] F. Borceux, M.M. Clementino, *Topological semi-abelian algebras*, Adv. Math. 190 (2005), no. 2, 425-453.

[3] F. Borceux, G. Janelidze, G.M. Kelly, *Internal object actions*, Commentationes Mathematicae Universitatis Carolinae 46 (2005), no. 2, 235-255.

[4] D. Bourn, *Normalization equivalence, kernel equivalence and affine categories*, in Lecture Notes in Mathematics, vol. 1488 (1991), Springer-Verlag, 43-62.

[5] D. Bourn, G. Janelidze, *Protomodularity, descent, and semidirect products*, Theory Appl. Categories 4 (1998), 37-46.

[6] D. Bourn, G. Janelidze, *Characterization of protomodular varieties of universal algebra*, Theory Appl. Categories 11 (2003), 143-147.

[7] J.R.A. Gray, N. Martins-Ferreira, *On algebraic and more general categories whose split epimorphisms have underlying product projections*, Appl. Categorical Structures, to appear, arxiv:1208.2032v1.

[8] E.B. Inyangala, *Semidirect products and crossed modules in varieties of right $\Omega$-loops*, Theory Appl. Categories 25 (2011), 426-435.

[9] G. Janelidze, L. Márki, W. Tholen, *Semi-abelian categories*, J. Pure Appl. Algebra 168 (2002), no. 2-3, 367-386.

[10] P.T. Johnstone, *A note on the semiabelian variety of Heyting semilattices*, Fields Institute Communications 43 (2004), 317-318.

[11] G. Metere, A. Montoli, *Semidirect products of internal groupoids*, J. Pure Appl. Algebra 214 (2010), 1854-1861.

Maria Manuel Clementino
CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal
*E-mail address*: mmc@mat.uc.pt

Andrea Montoli
CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal
*E-mail address*: montoli@mat.uc.pt

Lurdes Sousa
Polytechnic Institute of Viseu, Portugal & CMUC, Centre for Mathematics of the University of Coimbra, Portugal
*E-mail address*: sousa@estv.ipv.pt