Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 13–52

KAN INJECTIVITY IN ORDER-ENRICHED CATEGORIES

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Dedicated to the memory of Daniel M. Kan (1927–2013)

Abstract: Continuous lattices were characterised by Martín Escardó as precisely the objects that are Kan-injective w.r.t. a certain class of morphisms. We study Kan-injectivity in general categories enriched in posets. An example: ω-CPO's are precisely the posets that are Kan-injective w.r.t. the embeddings $\omega \hookrightarrow \omega + 1$ and $0 \hookrightarrow 1$.

For every class $\mathcal{H}$ of morphisms we study the subcategory of all objects Kan-injective w.r.t. $\mathcal{H}$ and all morphisms preserving Kan-extensions. For categories such as Top_0 and Pos we prove that whenever $\mathcal{H}$ is a set of morphisms, the above subcategory is monadic, and the monad it creates is a Kock-Zöberlein monad. However, this does not generalise to proper classes: we present a class of continuous mappings in Top_0 for which Kan-injectivity does not yield a monadic category.

1. Introduction

Dana Scott's result characterising continuous lattices as precisely the injective topological T_0-spaces, see [20], was one of the milestones of domain theory. This was later refined by Alan Day [9] who characterised continuous lattices as the algebras for the open filter monad on the category Top_0 of topological T_0-spaces and by Martín Escardó [10] who used the fact that the category Top_0 of topological T_0-spaces is naturally enriched in the category of posets (shortly: order-enriched).

In every order-enriched category one can define the *left Kan extension* f/h of a morphism $f : A \longrightarrow X$ along a morphism $h : A \longrightarrow A'$

$$\begin{array}{ccc} A & \xrightarrow{\ h\ } & A' \\ & {\scriptstyle f}\searrow \ \leq\ \swarrow{\scriptstyle f/h} & \\ & X & \end{array} \tag{1.1}$$

Received November 13, 2013.

The second author acknowledges the support of the Centre for Mathematics of the University of Coimbra (funded by the program COMPETE and by the Fundação para a Ciência e a Tecnologia, under the project PEst-C/MAT/UI0324/2013).

The third author acknowledges the support of the grant No. P202/11/1632 of the Czech Science Foundation.

as the smallest morphism from A' to X with $f \leq (f/h) \cdot h$. An object X is called *left Kan-injective* w.r.t. h iff for every morphism f the left Kan extension f/h exists and fulfills $f = (f/h) \cdot h$. Martín Escardó proved that in Top_0 the left Kan-injective spaces w.r.t. all subspace inclusions are precisely the continuous lattices endowed with the Scott topology. And w.r.t. all dense subspace inclusions they are precisely the continuous Scott domains (again with the Scott topology), see [10].

Recently, Margarida Carvalho and Lurdes Sousa [8] extended the concept of left Kan-injectivity to morphisms: a morphism is left-Kan injective w.r.t. h if it preserves left Kan extensions along h.

We thus obtain, for every class $\mathcal{H}$ of morphisms in an order-enriched category $\mathscr{X}$, a (not full, in general) subcategory

$$\mathsf{LInj}(\mathcal{H})$$

of all objects and all morphisms that are left Kan-injective w.r.t. every member of $\mathcal{H}$.

Example 1.1. For $\mathcal{H}$ = subspace embeddings in Top_0, $\mathsf{LInj}(\mathcal{H})$ is the category of continuous lattices (endowed with the Scott topology) and meet-preserving continuous maps.

Example 1.2. In the category Pos of posets take $\mathcal{H}$ to consist of the two embeddings $\omega \hookrightarrow \omega + 1$ and $\emptyset \hookrightarrow 1$. Then $\mathsf{LInj}(\mathcal{H})$ is the category of ω-CPOS's, i.e., posets with a least element and joins of ω-chains, and ω-continuous strict functions.

We are going to prove that whenever the subcategory $\mathsf{LInj}(\mathcal{H})$ is reflective, i.e., its embedding into $\mathscr{X}$ has a left adjoint, then the monad $\mathbb{T} = (T, \eta, \mu)$ on $\mathscr{X}$ that this adjunction defines is a Kock-Zöberlein monad, i.e., the inequality $T\eta \leq \eta T$ holds. And $\mathsf{LInj}(\mathcal{H})$ is the Eilenberg-Moore category $\mathscr{X}^{\mathbb{T}}$. Our main result is that in a wide class of order-enriched categories, called *locally ranked categories* (they include Top_0 and Pos), every class $\mathcal{H}$ of morphisms, such that all members of $\mathcal{H}$ but a set are order-epimorphisms, defines a reflective subcategory $\mathsf{LInj}(\mathcal{H})$. However, this does not hold for general classes $\mathcal{H}$: we present a class $\mathcal{H}$ of continuous functions in Top_0 whose subcategory $\mathsf{LInj}(\mathcal{H})$ fails to be reflective.

We also study weak left Kan-injectivity: this means that for every f a left Kan extension f/h exists but in (1.1) equality is not required. We prove

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category, then Kan-injectivity w.r.t. a set of morphisms defines a (not generally full) reflective subcategory. The monad this creates is a Kock-Zöberlein monad whose Eilenberg-Moore category is the given subcategory. And conversely, every Eilenberg-Moore category of a Kock-Zöberlein monad is specified by Kan-injectivity w.r.t. all units of the monad. On the other hand, we have presented a class of continuous maps in Top_0 whose Kan-injectivity class is not reflective.

Our main technical tool was the concept of an inserter-ideal subcategory: we proved that every inserter-ideal reflective subcategory is the Eilenberg-Moore category of a Kock-Zöberlein monad. And given any class of morphisms, Kan-injectivity always defines an inserter-ideal subcategory.

It is easy to see that for every set of morphisms in a locally presentable category the Kan-injectivity subcategory is accessibly embedded, i.e., closed under κ-filtered colimits for some infinite cardinal κ. It is an open problem whether every inserter-ideal, accessibly embedded subcategory closed under weighted limits is the Kan-injectivity subcategory for some set of morphisms. This would generalise the known fact that the orthogonality to sets of morphisms defines precisely the full, accessibly embedded subcategories closed under limits, see [2].

In case of orthogonality, a morphism h is called a consequence of a set $\mathcal{H}$ of morphisms provided that objects orthogonal to $\mathcal{H}$ are also orthogonal w.r.t. h. A simple logic of orthogonality, making it possible to derive all consequences of $\mathcal{H}$, is known [3]. Despite the strong similarity between orthogonality and Kan-injectivity, we have not been so far able to find a (sound and complete) logic for Kan-injectivity.

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that, in a certain sense, this concept can always be substituted by the above (stronger) one.

2. Left Kan-injectivity

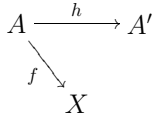
Throughout the paper we work with

(1) *order-enriched categories* $\mathscr{X}$, i.e., all homsets $\mathscr{X}(X, X')$ are partially ordered, and composition is monotone (in both variables)

and

(2) *locally monotone functors* $F : \mathscr{X} \longrightarrow \mathscr{Y}$, i..e, the derived functions from $\mathscr{X}(X, X')$ to $\mathscr{Y}(FX, FX')$ are all monotone.

Notation 2.1. Given morphisms



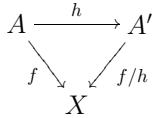
we denote by $f/h : A' \longrightarrow X$ the *left Kan extension* of f along h. That is, we have $f \leq (f/h) \cdot h$ and for all $g : A' \longrightarrow X$

(2.1)

The following definition is due to Escardó [10] for objects and Carvalho and Sousa [8] for morphisms:

Definition 2.2. Let $h : A \longrightarrow A'$ be a morphism of an order-enriched category.

(1) An object X is called *left Kan-injective* w.r.t. h provided that for every morphism $f : A \longrightarrow X$ there is a left Kan extension f/h and it makes the following triangle



(2.2)

commutative.

(2) A morphism $p : X \longrightarrow X'$ is called *left Kan-injective* w.r.t. h if both X and X' are and for every $f : A \longrightarrow X$ the morphism p preserves the left Kan extension f/h. This means that the following diagram

$$\begin{array}{ccc} A & \xrightarrow{h} & A' \\ {\scriptstyle f}\downarrow & \swarrow^{f/h} & \downarrow{\scriptstyle (pf)/h} \\ X & \xrightarrow[p]{} & X' \end{array} \tag{2.3}$$

commutes.

Remark 2.3.

(1) Right Kan-injectivity is briefly mentioned in Section 8 below. (Escardó used "right Kan-injective" for left Kan-injectivity in [10]. We decided to follow the usual terminology, see, e.g., [18].)

(2) A weaker variant of left Kan-injectivity would just require that for every f the left Kan extension f/h exists (i.e., we only have $f \leq f/h \cdot h$, instead of equality). We also turn to this concept in Section 8, but we will show that it can (under mild side conditions) be superseded by the concept of Definition 2.2.

Notation 2.4. Let $\mathcal{H}$ be a class of morphisms of an order-enriched category $\mathscr{X}$. We denote by

$$\mathsf{LInj}(\mathcal{H})$$

the category of all objects and all morphisms that are left Kan-injective w.r.t. all members of $\mathcal{H}$. The category $\mathsf{LInj}(\mathcal{H})$ is order-enriched using the enrichment of $\mathscr{X}$.

Examples 2.5. We give examples of Kan-injectivity in Pos. The order on homsets in Pos is defined pointwise.

(1) *Complete semilattices.* For $\mathcal{H}$ = all order-embeddings (that is, strong monomorphisms) we have

$\mathsf{LInj}(\mathcal{H})$ = complete join-semilattices and join-preserving maps.

Indeed, Bernhard Banaschewski and Günter Bruns proved in [6] that every complete (semi)lattice X is left Kan-injective w.r.t. $\mathcal{H}$ since for every order-embedding $h : A \longrightarrow A'$ and every monotone $f : A \longrightarrow X$ we have f/h given by

$$(f/h)(b) = \bigvee_{h(a) \leq b} f(a) \tag{2.4}$$

we denote by $f\backslash h : A' \longrightarrow X$ the largest morphism with

$$\begin{array}{ccc} A & \xrightarrow{h} & A' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f\backslash h} & \\ & X & \end{array} \quad \geq$$

Definition 8.7.

(1) An object X is *right Kan-injective* w.r.t. $h : A \longrightarrow A'$ provided that for every morphism $f : A \longrightarrow X$ a right Kan extension $f\backslash h$ exists and fulfils

$$f = (f\backslash h) \cdot h.$$

(2) A morphism $p : X \longrightarrow Y$ is right Kan-injective w.r.t. $h : A \longrightarrow A'$ provided that both X and Y are, and for every morphism $f : A \longrightarrow X$ we have

$$p \cdot (f\backslash h) = (pf)\backslash h.$$

Notation 8.8. $\mathsf{RInj}(\mathcal{H})$ is the subcategory of all right Kan-injective objects and morphisms w.r.t. all members of $\mathcal{H}$.

Remark 8.9. If $\mathscr{X}^{co}$ denotes the category obtained from $\mathscr{X}$ by reversing the ordering of homsets (thus leaving objects, morphisms and composition as before), then every class $\mathcal{H}$ of morphisms in $\mathscr{X}$ yields a right Kan-injectivity subcategory $\mathsf{RInj}(\mathcal{H})$ of $\mathscr{X}$ as well as a left Kan-injectivity subcategory $\mathsf{LInj}(\mathcal{H})$ in $\mathscr{X}^{co}$, and we have $\mathsf{RInj}(\mathcal{H}) = (\mathsf{LInj}(\mathcal{H}))^{co}$. Thus, in a sense, right Kan-injectivity is not needed. However, in some examples it is more intuitive to work with this concept.

Example 8.10. We have considered Top_0 above as an ordered category with respect to the specialisation order. Thus Top_0^{co} is the same category with dual of the specialisation order on homsets. This is the prefered enrichment of many authors. The examples of $\mathsf{LInj}(\mathcal{H})$ in Section 2 become, under the last enrichment of Top_0, examples of $\mathsf{RInj}(\mathcal{H})$.

9. Conclusion and open problems

For locally ranked categories (which is a wide class containing all locally presentable categories and Top) it is known that orthogonality w.r.t. a set of morphisms defines a full reflective subcategory. And the latter is the Eilenberg-Moore category of an idempotent monad. In our paper we have proved the order-enriched analogy: given an order-enriched, locally ranked

Its construction is analogous to the construction of pushouts via coequalisers: form a coproduct $A \xrightarrow{i_A} A+B \xleftarrow{i_B} B$ and a coinserter

$$\begin{array}{ccc} D & \xrightarrow{i_B \cdot q} & A+B \\ {\scriptstyle i_A \cdot p}\downarrow & \leq & \downarrow{\scriptstyle c} \\ A+B & \xrightarrow[c]{} & C \end{array}$$

Then put $\overline{p} = c \cdot i_A$ and $\overline{q} = c \cdot i_B$.

(2) The category $\mathsf{LInj}_w(\mathcal{H})$ is reflective. The proof is completely analogous to that of Theorem 5.6, except that Construction 5.2 needs one modification: in diagram (5.2) we do not require equality but inequality:

$$\begin{array}{ccc} A & \xrightarrow{h} & A' \\ {\scriptstyle f}\downarrow & \leq & \downarrow{\scriptstyle f/\!\!/h} \\ X_i & \dashrightarrow & X_{i+1} \end{array}$$

Thus, given h in $\mathcal{H}$ and $f : A \longrightarrow X_i$ we form a cocomma object

$$\begin{array}{ccc} A & \xrightarrow{h} & A' \\ {\scriptstyle f}\downarrow & \leq & \downarrow{\scriptstyle \overline{f}} \\ X_i & \xrightarrow[\overline{h}]{} & C \end{array}$$

Then $X_i \dashrightarrow X_{i+1}$ is the wide pushout of all $\overline{h}$ (with the colimit cocone $c_{f,h} : C \longrightarrow X_{i+1}$) and we put $f/\!\!/h = c_{f,h} \cdot \overline{f}$.

(3) The category $\mathsf{LInj}_w(\mathcal{H})$ is also inserter-ideal: the proof is completely analogous to that of Lemma 4.2. By Theorem 4.11 $\mathsf{LInj}_w(\mathcal{H})$ is a KZ-monadic category.

(4) Let $\overline{\mathcal{H}}$ denote the collection of all reflection maps of objects of $\mathscr{X}$ in $\mathsf{LInj}_w(\mathcal{H})$. Then

$$\mathsf{LInj}_w(\mathcal{H}) = \mathsf{LInj}(\overline{\mathcal{H}})$$

holds by Proposition 4.9. ∎

Remark 8.6. There is another obvious variation of Kan-injectivity, using right Kan extensions instead of left ones. Given $h : A \longrightarrow A'$ and $f : A \longrightarrow X$

And conversely, if X is left Kan-injective, then every set $M \subseteq X$ either has a maximum, which is $\bigvee M$, or we have

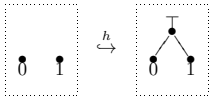
$$M \cap M^+ = \emptyset \text{ for } M^+ = \text{all upper bounds of } M.$$

In the latter case consider $A = M \cup M^+$ as a subposet of X and let A' extend A by a single element a' that is an upper bound of M and a lower bound of M^+. The embedding $f : A \hookrightarrow X$ has a left Kan extension f/h that sends a' to $\bigvee M$.

By using the formula (2.4) it is easy to see that a monotone map $g : X \longrightarrow Y$ between complete join-semilattices is left Kan-injective iff g preserves joins.

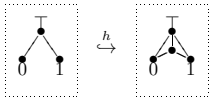
(2) *ωCPOS's.* Posets with joins of ω-chains and $\bot$ and strict functions preserving joins of ω-chains are $\mathsf{LInj}(\mathcal{H})$ for $\mathcal{H}$ consisting of the embeddings $h : \omega \hookrightarrow \omega + 1$ and $h' : \emptyset \hookrightarrow 1$.

(3) *Semilattices.* For the embedding



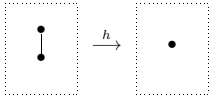
we obtain the category of join-semilattices and their homomorphisms as $\mathsf{LInj}(\{h\})$.

(4) *Conditional semilattices.* For the embedding



we obtain the category of conditional join-semilattices (where every pair with an upper bound has a join) and maps that preserve nonempty finite joins as $\mathsf{LInj}(\{h\})$.

(5) *The category Pos_d of discrete posets.* Form $\mathsf{LInj}(\{h\})$ for the morphism



(6) *The category Pos_1 of posets of cardinality ≤ 1.* Form $\mathsf{LInj}(\{h\})$ for the mapping $h : 1 + 1 \longrightarrow 1$.

Except for the trivial cases Pos_d and Pos_1 all of the examples in 2.5 worked with $\mathcal{H}$ consisting of strong monomorphisms. This is not coincidential:

Lemma 2.6. *Let $\mathcal{H}$ be a class of morphisms of* Pos *such that* $\mathsf{LInj}(\mathcal{H})$ *is neither* Pos_d *nor* Pos_1. *Then all members of $\mathcal{H}$ are strong monomorphisms.*

Proof: Assume the contrary, i.e., suppose there exists $h : A \longrightarrow A'$ in $\mathcal{H}$ such that for some p, q in A we have $h(p) \leq h(q)$ although $p \not\leq q$. Then we prove that every poset X left Kan-injective w.r.t. h is discrete. It then follows easily that $\mathsf{LInj}(\mathcal{H})$ is either Pos_d or Pos_1.

Given elements $x \leq x'$ in X, we prove that $x = x'$. Define $f : A \longrightarrow X$ by

$$f(a) = \begin{cases} x', & \text{if } a \geq p \\ x, & \text{else} \end{cases}$$

which is clearly monotone. Then $p \not\leq q$ implies $f(q) = x$. Consequently, f/h sends $h(p)$ to x' and $h(q)$ to x. Since $h(p) \leq h(q)$, we conclude $x' \leq x$, thus, $x = x'$. ∎

Example 2.7. The category Top_0 of T_0 topological spaces and continuous maps is order-enriched as follows. Recall the *specialisation order* $\sqsubseteq$ that Dana Scott [20] used on every T_0-space:

$x \sqsubseteq y$ iff every neighbourhood of x contains y.

We consider Top_0 to be order-enriched by the opposite of the pointwise specialisation order: for continuous functions $f, g : X \longrightarrow Y$ we put

$f \leq g$ iff $g(x) \sqsubseteq f(x)$ for all x in X.

(1) *Continuous lattices.* For the collection $\mathcal{H}$ of all subspace embeddings in Top_0 we have

$\mathsf{LInj}(\mathcal{H})$ = continuous lattices and meet-preserving continuous maps.

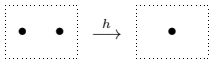
This was proved for objects by Escardo [11] and for morphisms by Carvalho and Sousa [8], we present a proof for the convenience of the reader.

Indeed, Scott proved that a T_0-space X is injective iff its specialisation order is a continuous lattice, i.e., a complete lattice in which every element y satisfies

$$y = \bigsqcup_{U \in nbh(y)} \left(\bigsqcap U \right). \tag{2.5}$$

Moreover, he gave, for every subspace embedding $h : A \longrightarrow A'$ and every continuous map $f : A \longrightarrow X$, a concrete formula for a continuous

Example 8.1. For the morphism



in Pos, the left Kan-injective objects in the above weak sense are precisely the join-semilattices.

Definition 8.2. Let $h : A \longrightarrow A'$ be a morphism.

(1) An object X is called *weakly left Kan-injective* w.r.t. h if for every morphism $f : A \longrightarrow X$ a left Kan extension $f/h : A' \longrightarrow X$ of f along h exists.
(2) A morphism $p : X \longrightarrow Y$ between weakly left Kan-injective objects is called weakly left Kan-injective if $p \cdot (f/h) = (pf)/h$ holds for all $f : A \longrightarrow X$.

Remark 8.3. When comparing Examples 8.1 and 2.5 we see that in some cases (strong) left Kan-injectivity seems more "natural" than the weak one. Theorem 8.5 indicates that the weak notion is, moreover, not really needed.

Notation 8.4. For every class $\mathcal{H}$ of morphisms of an order-enriched category $\mathscr{X}$ we denote by

$$\mathsf{LInj}_w(\mathcal{H})$$

the category of all objects and morphisms of $\mathscr{X}$ that are weakly left Kan-injective w.r.t. all members of $\mathcal{H}$.

Theorem 8.5. *In every locally ranked order-enriched category $\mathscr{X}$, given a set $\mathcal{H}$ of morphisms there exists a class $\overline{\mathcal{H}}$ of morphisms such that*

$$\mathsf{LInj}_w(\mathcal{H}) = \mathsf{LInj}(\overline{\mathcal{H}})$$

Proof:

(1) The category $\mathscr{X}$ has cocomma objects, i.e., given a span $A \xleftarrow{p} D \xrightarrow{q} B$ there exists a couniversal square

$$\begin{array}{ccc} D & \xrightarrow{q} & B \\ {\scriptstyle p}\downarrow & \leq & \downarrow{\scriptstyle \overline{q}} \\ A & \xrightarrow[\overline{p}]{} & C \end{array}$$

implies $f = Eb_{ik} \cdot Ea_{0i} = Ea_{ik}$ which is nonconstant — a contradiction.

(3) Suppose that $r : EA_0 \longrightarrow R$ is a reflection of EA_0 in $\mathsf{LInj}(\mathcal{H})$. We derive a contradiction by proving that there exists a proper class of continuous functions from EC to R.

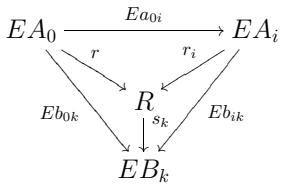
Since r is Kan-injective, for every $i \in \mathrm{Ord}$ we have

$$r_i = r/Ea_{0i} : EA_i \longrightarrow R$$

And the Kan-injectivity of EB_k implies that there exists a Kan-injective morphism

$$s_k : R \longrightarrow EB_k \quad \text{with } Eb_{0k} = s_k \cdot r$$

See the diagram



Then, due to Kan-injectivity of s_k, we have

$$s_k \cdot r_i = s_k \cdot (r/Ea_{0i}) = (Ea_{0i})/(Eb_{0k})$$

and in part (2a) above we have seen that the last morphism is Eb_{ik}. Thus the above diagram commutes. For all $k > i$ we have $b_{kk} \cdot c_k \neq b_{ik} \cdot c_i$, therefore, $Eb_{kk} \cdot Ec_k \neq Eb_{ik} \cdot Ec_i$. Thus

$$s_k \cdot r_k \cdot Ec_k \neq s_k \cdot r_i \cdot Ec_i$$

which implies

$$r_k \cdot Ec_k \neq r_i \cdot Ec_i : EC \longrightarrow R$$

for all $k > i$ in Ord. This is the desired contradiction.

8. Weak Kan-injectivity and right Kan-injectivity

It may seem more natural to define left Kan-injectivity of an object X w.r.t. $h : A \longrightarrow A'$ by requiring only that for every morphism $f : A \longrightarrow X$ a left Kan extension $f/h : A' \longrightarrow X$ exists. Thus, we only have $f \leq (f/h) \cdot h$, but not necessarily an equality.

extension $f' : A' \longrightarrow X$:

$$f'(a') = \bigsqcup_{U \in nbh(a')} \left(\bigsqcap f(h^{-1}(U)) \right) \text{ for all } a' \in A'. \tag{2.6}$$

This is actually the desired left Kan extension $f' = f/h$, as proved by Escardó [10]. His proof uses the filter monad $\mathcal{F}$ on Top_0 whose Eilenberg-Moore algebras are, as proved by Alan Day [9] and Oswald Wyler [22], precisely the continuous lattices: for every continuous lattice X the algebra $\alpha : \mathcal{F}X \longrightarrow X$ is defined by

$$\alpha(F) = \bigsqcup_{U \in F} \left(\bigsqcap U \right) \text{ for all filters } F. \tag{2.7}$$

Every continuous map $p : X \longrightarrow Y$ between continuous lattices preserving meets is Kan-injective. This follows from the formula (2.6) for f/h: given $f : A \longrightarrow X$ we have

$$\begin{aligned} p \cdot (f/h)(a') &= p \left(\bigsqcup_{U \in nbh(a')} \left(\bigsqcap f(h^{-1}(U)) \right) \right) && \text{by (2.6)} \\ &= \bigsqcup_{U \in nbh(a')} p \left(\bigsqcap f(h^{-1}(U)) \right) && \text{since } p \text{ is continuous} \\ &= \bigsqcup_{U \in nbh(a')} \left(\bigsqcap pf(h^{-1}(U)) \right) && \text{since } p \text{ preserves meets} \\ &= (pf)/h(a') && \text{by (2.6)} \end{aligned}$$

Conversely, if a continuous map $p : X \longrightarrow Y$ is Kan-injective, then it preserves meets. Indeed, following Day, p is a homomorphism of the corresponding monad algebras. Given $M \subseteq X$, let F_M be the filter of all subsets containing M, then (2.7) yields $\alpha(F_M) = \bigsqcap M$ — hence, the fact that p is a homomorphism implies that p preserves meets.

(2) *Continuous Scott Domains.* For the collection $\mathcal{H}$ of all dense subspace embeddings we have

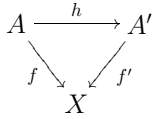
$$\mathsf{LInj}(\mathcal{H}) = \begin{array}{l} \text{continuous Scott domains and continuous} \\ \text{functions preserving nonempty meets.} \end{array}$$

Recall that a continuous Scott domain is a poset with bounded joins (or, equivalently, nonempty meets) satisfying (2.5). Escardó proved that the T_0 spaces Kan-injective w.r.t. dense embeddings are precisely those whose order is a continuous Scott domain. His proof uses the

monad $\mathcal{F}^+$ of proper filters on Top_0. The conclusion that Kan-injective morphisms are precisely those preserving nonempty meets is analogous to (1).

Remark 2.8. The order enrichment of Top_0 above is frequently used in literature. However, some authors prefer the dual enrichment (by the pointwise specialisation order). We mention in Example 8.10 below that this yields the same examples as above but for the right Kan-injectivity.

Example 2.9. Given an ordinary category, we can consider it order-enriched by the trivial order. An object X is then Kan-injective w.r.t. $\mathcal{H}$ iff it is *orthogonal*, i.e., given $h : A \longrightarrow A'$ it fulfills: for every $f : A \longrightarrow X$ there is a unique $f' : A' \longrightarrow X$ such that the triangle



commutes.

And every morphism between orthogonal objects is Kan-injective. Thus, the Kan-injectivity subcategory is precisely

$$\mathcal{H}^\perp = \mathsf{LInj}(\mathcal{H})$$

the full subcategory of all orthogonal objects.

Remark 2.10.

(1) A special case is given by a monad $\mathbb{T} = (T, \eta, \mu)$ on the (ordinary) category which is *idempotent*, i.e., fulfills

$$T\eta = \eta T$$

Consequently, every object X carries at most one structure on an Eilenberg-Moore algebra $x : TX \longrightarrow X$, since $x = \eta_X^{-1}$. Thus, the category $\mathscr{X}^{\mathbb{T}}$ can be considered as a full subcategory of $\mathscr{X}$. For the class $\mathcal{H} = \{\eta_X \mid X \text{ in } \mathscr{X}\}$ of all units of $\mathbb{T}$ we then have

$$\mathscr{X}^{\mathbb{T}} = \mathcal{H}^\perp$$

(2) Conversely, whenever the full subcategory $\mathcal{H}^\perp$ is reflective, i.e., its embedding into $\mathscr{X}$ has a left adjoint, then the corresponding monad $\mathbb{T}$ on $\mathscr{X}$ is idempotent and $\mathscr{X}^{\mathbb{T}} \cong \mathcal{H}^\perp$.

It consists of a transfinite chain $a_{ij} : A_i \longrightarrow A_j$ ($i \leq j$ in Ord) and, for every ordinal k, a cocone $b_{ik} : A_i \longrightarrow B_k$ ($i \in$ Ord) of that chain. Furthermore, there are morphisms $c_i : C \longrightarrow A_i$ (i in Ord) with free composition modulo the equations

$$b_{kk} \cdot c_k = b_{ik} \cdot c_i, \quad \text{for all } i \geq k$$

In particular, we have

$$b_{kk} \cdot c_k \neq b_{ik} \cdot c_i, \quad \text{for all } i < k$$

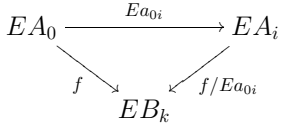
This category is concrete, i.e., it has a faithful functor into Set. For example, take $U : \mathscr{C} \longrightarrow \mathsf{Set}$ with $UB_i = UA_i = \{t \in \text{Ord} \mid t \leq i\}$ and $UC = \{0\}$. The morphisms Ua_{ij} are then the inclusions, $Ub_{ik}(t) = \max(t, k)$ and $Uc_i(0) = i$.

Václav Koubek proved in [16] that every concrete category has an *almost full embedding* $E : \mathscr{C} \longrightarrow \mathsf{Top}_2$ into the category Top_2 of topological Hausdorff spaces. This means that E is faithful and maps morphisms of $\mathscr{C}$ into nonconstant mappings, and every nonconstant continous map $p : EX \longrightarrow EY$ has the form $p = Ef$ for a unique $f : X \longrightarrow Y$ in $\mathscr{C}$.

(2) For the proper class

$$\mathcal{H} = \{Ea_{0i} \mid i \in \text{Ord}\}$$

in Top_0 we prove that the space EA_0 does not have a reflection in $\mathsf{LInj}(\mathcal{H})$. We first verify that all spaces EB_k are Kan-injective:



Given $i \in$ Ord and $f : EA_0 \longrightarrow EB_k$ we find f/Ea_{0i} as follows:

(a) If f is nonconstant, then $f = Eb_{0k}$ and we claim that $f/Ea_{0i} = Eb_{ik}$. For that it is sufficient to recall that EB_k is a Hausdorff space, thus, given $g : EA_i \longrightarrow EB_k$ with $f \leq g \cdot Ea_{0i}$, it follows that $f = g \cdot Ea_{0i}$. Hence, g is also nonconstant. But then $g = Eb_{ik}$.

(b) If f is constant, then we claim that f/Ea_{0i} is the constant function with the same value. For that, take again g with $f \leq g \cdot Ea_{0i}$ and conclude $f = g \cdot Ea_{0i}$. This implies that g is constant (and thus $g = f/Ea_{0i}$) because otherwise $g = Eb_{ik}$, but the latter

we derive, since η_A is an order-epimorphism, that $f^\sharp \leq g^\sharp \cdot Rh$. Since clearly $(g^\sharp Rh)/Rh \leq g^\sharp$, we conclude

$$\begin{aligned} f/h &= (f^\sharp/Rh) \cdot \eta_{A'} \\ &\leq \big((g^\sharp Rh)/Rh\big) \cdot \eta_{A'} \\ &\leq g^\sharp \cdot \eta_{A'} \\ &= g. \end{aligned}$$

(b2) Every morphism $p : X \longrightarrow Y$ of $\mathcal{H}_e^\perp$ Kan-injective w.r.t. $\mathcal{H}_0$ is Kan-injective w.r.t. $\mathcal{H}$. Again, we only need to consider h in $\mathcal{H}_0$. Given $f : A \longrightarrow X$ we have $f/h = (f^\sharp/Rh) \cdot \eta_{A'}$. Put $f_1 = p \cdot f$ and obtain the corresponding $f_1^\sharp : RA \longrightarrow Y$ with $f_1/h = (f_1^\sharp/Rh) \cdot \eta_{A'}$. Then $f_1 = p \cdot f$ implies $f_1^\sharp \cdot \eta_A = p \cdot f^\sharp \cdot \eta_A$, and since η_A is an order-epimorphism, we conclude $f_1^\sharp = p \cdot f^\sharp$. Consequently, from the Kan-injectivity of p w.r.t. Rh we obtain the desired equality:

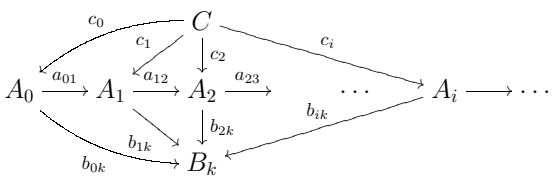
$$\begin{aligned} p \cdot (f/h) &= p \cdot (f^\sharp/Rh) \cdot \eta_{A'} \\ &= \big((pf^\sharp)/Rh\big) \cdot \eta_{A'} \\ &= (f_1^\sharp/Rh) \cdot \eta_{A'} \\ &= f_1/h \\ &= (pf)/h. \end{aligned}$$

■

7. A counterexample

We give an example of a proper class $\mathcal{H}$ of continuous maps in Top_0 for which the Kan-injectivity category $\mathsf{LInj}(\mathcal{H})$ is not reflective. The example is based on ideas of [1].

(1) We denote by $\mathscr{C}$ the following category



(3) The concepts of (i) full reflective subcategory of $\mathscr{X}$, (ii) idempotent monad on $\mathscr{X}$ and (iii) orthogonal subcategory $\mathcal{H}^\perp$ coincide — modulo the *orthogonal subcategory problem*. This is the problem whether given a class $\mathcal{H}$ of morphisms the subcategory $\mathcal{H}^\perp$ is reflective. Some positive solutions can be found in [12] and [3], for a negative solution in $\mathscr{X} = \mathsf{Top}$ see [1].

The situation with order-enriched categories is completely analogous, as we prove below. The following can be found in [10] and [8].

Example 2.11. Let $\mathbb{T} = (T, \eta, \mu)$ be a *Kock-Zöberlein monad* on an order-enriched category $\mathscr{X}$, i.e., one satisfying

$$T\eta \leq \eta T.$$

Kock-Zöberlein monads over order-enriched categories are a particular case of the monads on 2-categories, independently introduced by Anders Kock [15] and Volker Zöberlein [23].

Every object X carries at most one structure of an Eilenberg-Moore algebra $\alpha : TX \longrightarrow X$, since α is left adjoint to η_X. Thus, $\mathscr{X}^\mathbb{T}$ can be considered as a (not necessarily full) subcategory of $\mathscr{X}$. Then the category of $\mathbb{T}$-algebras consists precisely of all objects and morphisms Kan-injective to all units:

$$\mathscr{X}^\mathbb{T} = \mathsf{LInj}(\mathcal{H}) \text{ for } \mathcal{H} = \{\eta_X \mid X \text{ in } \mathscr{X}\}$$

see Proposition 4.9 below. Conversely, whenever the subcategory $\mathsf{LInj}(\mathcal{H})$ is reflective, i.e., its (possibly non-full) embedding into $\mathscr{X}$ has a left adjoint, then it is monadic and the corresponding monad $\mathbb{T}$ satisfies the Kock-Zöberlein property, see Corollary 4.12 below.

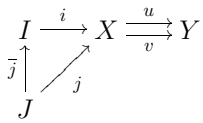
3. Inserters and coinserters

Since inserters and coinserters play a central role in our paper, we recall the facts about them we need (in our special case of order-enriched categories) in this section. Throughout this section we work in an order-enriched category.

Definition 3.1.

(1) We call a morphism $i : I \longrightarrow X$ an *order-monomorphism* provided that for all $f, g : I' \longrightarrow I$ we have: $i \cdot f \leq i \cdot g$ implies $f \leq g$.

(2) An *inserter* of a parallel pair $u, v : X \longrightarrow Y$ in an order-enriched category is a morphism $i : I \longrightarrow X$ universal w.r.t. $u \cdot i \leq v \cdot i$.



Universality means the following two conditions:
(a) Given j with $u \cdot j \leq v \cdot j$, there exists a unique $\overline{j}$ with $j = i \cdot \overline{j}$.
(b) i is an order-monomorphism.

Example 3.2. In Top_0 the inserter of $u, v : X \longrightarrow Y$ is the embedding $I \hookrightarrow X$ of the subspace of X on all elements $x \in X$ with $u(x) \leq v(x)$. In general, every subspace embedding is an order-monomorphism.

In Pos, analogously, the inserter of $u, v : X \longrightarrow Y$ is the embedding $I \hookrightarrow X$ of the subposet of X on all elements $x \in X$ with $u(x) \leq v(x)$. In general, every subposet embedding is an order-monomorphism — and vice versa (up to isomorphism).

Lemma 3.3. *For a morphism i in Pos the following conditions are equivalent:*

(1) *i is an order-monomorphism.*
(2) *i is a strong monomorphism.*
(3) *i is a subposet embedding (up to isomorphism).*
(4) *i is an inserter of some pair.*

Proof: It is easy to see that (2) and (3) are both equivalent to the validity of the implication "$i(x) \leq i(y)$ implies $x \leq y$". Therefore (1) implies (3). To prove (3) implies (4), given a subposet embedding $i : X \hookrightarrow Y$, let Z be the poset obtained from Y by splitting every element outside of $i[X]$ to two incomparable elements. The two obvious embeddings of Y into Z have i as their inserter. Finally, (4) implies (1) by the definition. ∎

Definition 3.4.

(1) An *order-epimorphism* is a morphism $e : X \longrightarrow Y$ such that for all $f, g : Y \longrightarrow Z$ we have: $f \cdot e \leq g \cdot e$ implies $f \leq g$.
(2) A *coinserter* of a parallel pair $u, v : X \longrightarrow Y$ is a morphism $c : Y \longrightarrow C$ couniversal w.r.t. $c \cdot u \leq c \cdot v$. That is, the following two conditions hold:
   (a) Given $d : Y \longrightarrow Z$ with $d \cdot u \leq d \cdot v$ there exists a unique $\overline{d} : C \longrightarrow Z$ with $d = \overline{d} \cdot c$.

$\widehat{f} = f/Rh$ above, and analogously for $f_1 = p \cdot f : RA \longrightarrow Y$ we have $\widehat{f_1}$, defined by $\widehat{f_1} \cdot \eta_{A'} = (f_1 \eta_A)/h$, satisfying $\widehat{f_1} = f_1/Rh$. Since p is Kan-injective w.r.t. $\mathcal{H}$, we have

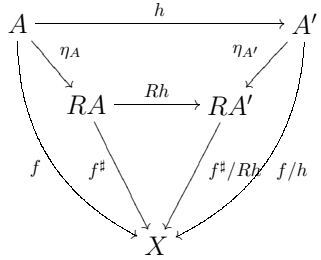
$$p \cdot \widehat{f} \cdot \eta_{A'} = p \cdot (f\eta_A)/h = (pf\eta_A)/h = (f_1\eta_A)/h = \widehat{f_1} \cdot \eta_{A'}$$

and this implies $p \cdot \widehat{f} = \widehat{f_1}$ since $\eta_{A'}$ is order-epic. Thus

$$p \cdot (f/Rh) = p \cdot \widehat{f} = \widehat{f_1} = (pf)/Rh$$

as required.

(b1) Every object X of $\mathcal{H}_e^{\perp}$ Kan-injective w.r.t. $\widehat{\mathcal{H}_0}$ is Kan-injective w.r.t. $\mathcal{H}$. We only need to consider $h : A \longrightarrow A'$ in $\mathcal{H}_0$.



Given $f : A \longrightarrow X$, since X is in $\mathcal{H}_e^{\perp}$, we have a unique $f^\sharp : RA \longrightarrow X$ with $f = f^\sharp \eta_A$. And we define

$$f/h = (f^\sharp/Rh) \cdot \eta_{A'}.$$

This morphism has both of the required properties: firstly

$$\begin{aligned} (f/h) \cdot h &= (f^\sharp/Rh) \cdot \eta_{A'} \cdot h \\ &= (f^\sharp/Rh) \cdot Rh \cdot \eta_A \\ &= f^\sharp \cdot \eta_A \\ &= f. \end{aligned}$$

Secondly, given $g : A' \longrightarrow X$ with $f \leq g \cdot h$, there exists a unique $g^\sharp : RA' \longrightarrow X$ with $g = g^\sharp \cdot \eta_{A'}$. From

$$f^\sharp \cdot \eta_A = f \leq g \cdot h = g^\sharp \cdot \eta_{A'} \cdot h = g^\sharp \cdot Rh \cdot \eta_A$$

an order-epimorphism. Thus, if $R : \mathscr{X} \longrightarrow \mathcal{H}_e^\perp$ denotes the reflector, the units $\eta_X : X \longrightarrow RX$ are all order-epimorphisms.

(2) The set

$$\widehat{\mathcal{H}_0} = \{Rh \mid h \text{ in } \mathcal{H}_0\}$$

of morphisms of the locally ranked category $\mathcal{H}_e^\perp$ fulfills, by Theorem 6.10, that

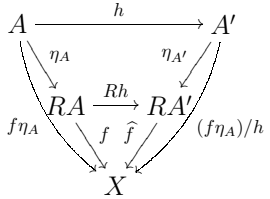
$$\mathsf{LInj}_{\mathcal{H}_e^\perp}(\widehat{\mathcal{H}_0}) \text{ is reflective in } \mathcal{H}_e^\perp.$$

(The lower index is used to stress in which category the injectivity is considered.) Consequently, $\mathsf{LInj}_{\mathcal{H}_e^\perp}(\widehat{\mathcal{H}_0})$ is a reflective subcategory of $\mathscr{X}$. The theorem will be proved by verifying that

$$\mathsf{LInj}_{\mathscr{X}}(\mathcal{H}) = \mathsf{LInj}_{\mathcal{H}_e^\perp}(\widehat{\mathcal{H}_0}).$$

We prove that (a) $\mathsf{LInj}_{\mathscr{X}}(\mathcal{H})$ is a subcategory of $\mathsf{LInj}_{\mathcal{H}_e^\perp}(\widehat{\mathcal{H}_0})$ and (b) the other way round.

(a1) Every object X of $\mathscr{X}$ Kan-injective w.r.t. $\mathcal{H}$ is clearly an object of $\mathcal{H}_e^\perp$; we prove that it is Kan-injective w.r.t. Rh in $\widehat{\mathcal{H}_0}$.



Given $f : RA \longrightarrow X$, the morphism $(f\eta_A)/h$ factorises, since X is in $\mathcal{H}_e^\perp$, through $\eta_{A'}$: we have a unique $\widehat{f}$ such that the diagram above commutes. Then

$$\widehat{f} = f/Rh.$$

Indeed, $\widehat{f} \cdot Rh = f$. And given $g : RA' \longrightarrow X$ with $f \leq g \cdot Rh$, then $f \cdot \eta_A \leq g \cdot Rh \cdot \eta_A = g \cdot \eta_{A'} \cdot h$ which implies $(f\eta_A)/h \leq g \cdot \eta_{A'}$. Recall that R is a reflector of $\mathcal{H}_e^\perp$ and $\eta_{A'}$ is an order-epimorphism. Thus $\widehat{f} \leq g$, as desired.

(a2) Every morphism $p : X \longrightarrow Y$ of $\mathscr{X}$ Kan-injective w.r.t. $\mathcal{H}$ lies in the (full) subcategory $\mathcal{H}_e^\perp$, and we must prove that p is Kan-injective w.r.t. Rh. Given $f : RA \longrightarrow X$ we have seen that

(b) c is an order-epimorphism.

Examples 3.5.

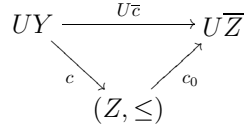
(1) In Pos every surjection (= epimorphism) is an order-epimorphism, see Lemma 3.6 below.

(2) In Top_0 also every epimorphism is an order-epimorphism. We can describe coinserters by using those in Pos and applying the forgetful functor

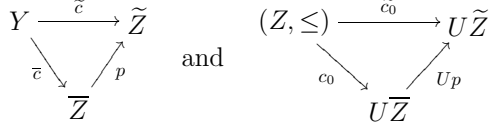
$$U : \mathsf{Top}_0 \longrightarrow \mathsf{Pos}$$

of Example 2.7.

This functor has the following universal property: given a monotone function $c : UY \longrightarrow (Z, \leq)$ where Y is a T_0 space, there exists a *semifinal solution* in the sense of 25.7 [4], which means a pair consisting of $\overline{c} : Y \longrightarrow \overline{Z}$ in Top_0 and $c_0 : (Z, \leq) \longrightarrow U\overline{Z}$ in Pos universal w.r.t.



Thus given another pair $\widetilde{c} : Y \longrightarrow \widetilde{Z}$ and $\widetilde{c_0} : (Z, \leq) \longrightarrow U\widetilde{Z}$ with $U\widetilde{c} = \widetilde{c_0} \cdot c$ there exists a unique $p : \overline{Z} \longrightarrow \widetilde{Z}$ in Top_0 making the diagrams



commutative.

Indeed, to construct $\overline{c}$, let τ be the topology on Z of all lowersets whose inverse image under c is open in Y. Let $r : (Z, \tau) \longrightarrow \overline{Z}$ be a T_0-reflection, then put $\overline{c} = r \cdot c$. Consequently, we see that each such $\overline{c}$ is an order-epimorphism in Pos.

The coinserter of $u, v : X \longrightarrow Y$ in Top_0 is obtained by first forming a coinserter $c : UY \longrightarrow (Z, \leq)$ of Uu, Uv in Pos and then taking the semifinal solution $\overline{c} : Y \longrightarrow \overline{Z}$.

Lemma 3.6. *For a morphism e in Pos the following conditions are equivalent:*

(1) *e is an order-epimorphism.*
(2) *e is an epimorphism.*
(3) *e is surjective.*
(4) *e is a coinserter of some pair.*

Proof: The equivalence of (2) and (3) is well-known, see, e.g., Example 7.40(2) in [4].

It is clear that (1) implies (2) and (4) implies (1). To prove that (3) implies (4), choose a surjective map $e : A \longrightarrow B$ and define the poset A_0 as follows: its elements are pairs (x, x') such that $e(x) \leq e(x')$, the pairs are ordered pointwise. Denote by $d_0, d_1 : A_0 \longrightarrow A$ the obvious monotone projections. Then it follows easily that e is a coinserter of the pair (d_0, d_1), using the fact that e is surjective. ∎

Definition 3.7. An order-enriched category is said to have *conical products* if it has products $\prod_{i\in I} X_i$ and the projections π_i are collectively order-monic. That is, given a parallel pair $f, g : Y \longrightarrow \prod_{i\in I} X_i$ we have that

$$\pi_i \cdot f \leq \pi_i \cdot g \text{ for all } i \in I \text{ implies } f \leq g. \tag{3.1}$$

Example 3.8. In Top_0 and Pos products are clearly conical.

Remark 3.9. Throughout Section 4 we work with order-enriched categories having inserters and conical products. This can be expressed more compactly by saying that weighted limits exist. We recall this fact (that can be essentially found in Max Kelly's book [14]) for convenience of the reader. However, we are not going to apply any weighted limits except inserters and conical limits in our paper.

Given order-enriched categories $\mathscr{X}$ and $\mathscr{D}$, where $\mathscr{D}$ is small, we denote by

$$\mathscr{X}^{\mathscr{D}}$$

the order-enriched category of all locally monotone functors from $\mathscr{D}$ to $\mathscr{X}$ and all natural transformations between them (the order on natural transformations is objectwise: given $\alpha, \beta : F \longrightarrow G$ then $\alpha \leq \beta$ means $\alpha_d \leq \beta_d$ for every d in $\mathscr{D}$).

Definition 3.10. Let $\mathscr{X}$ and $\mathscr{D}$ be order-enriched categories, $\mathscr{D}$ small. Given a locally monotone functor $D : \mathscr{D} \longrightarrow \mathscr{X}$, its *limit weighted by* $W : \mathscr{D} \longrightarrow \mathsf{Pos}$, also locally monotone, is an object

$$\{W, D\}$$

for all $h \in \mathcal{H}$ and all f. For that, we first prove the equalities

$$(x_{ik} \cdot f)/h = x_{i+1,k} \cdot f /\!\!/ h, \qquad i < k. \tag{6.2}$$

From Lemma 6.8 we have that $x_{ik} \cdot f = x_{\hat{i}k} \cdot (\beta_i \cdot \gamma_i \cdot f)$, that is, $x_{ik} f = m_i(\gamma_i f)$. Then, by (6.1), we know that

$$(x_{ik} \cdot f)/h = x_{\hat{i}+1,k} \cdot (\beta_i \cdot \gamma_i \cdot f) /\!\!/ h = x_{\hat{i}+1,k} \cdot (x_{i\hat{i}} \cdot f) /\!\!/ h. \tag{6.3}$$

By Remark 5.5, the morphism $x_{\hat{i}+1,\hat{i}+2}$ merges $(x_{i,\hat{i}} \cdot f) /\!\!/ h$ and $x_{i+1,\hat{i}+1} \cdot f /\!\!/ h$. Thus, $x_{\hat{i}+1,k} \cdot (x_{i,\hat{i}} \cdot f) /\!\!/ h = x_{i+1,k} \cdot f /\!\!/ h$. That is, by (6.3), $(x_{ik} \cdot f)/h = x_{i+1,k} \cdot f /\!\!/ h$.

Now, due to the equality $p_k \cdot x_{ik} = q \cdot x_{ik}$, we have $(p_k \cdot x_{ik})/h = (q \cdot x_{ik})/h$, hence $p_k \cdot (x_{ik})/h = q \cdot (x_{ik})/h$, because both p_k and q belong to $\mathsf{LInj}(\mathcal{H})$. Using (6.2), we obtain then that $p_k \cdot x_{i+1,k} \cdot f /\!\!/ h = q \cdot x_{i+1,k} \cdot f /\!\!/ h$.

(3b) From $i + 1$ to $i + 2$. This is clear, since $x_{i+1,i+2}$ is an order-epimorphism.

(4) From (2) and (3) we know that $\mathsf{LInj}(\mathcal{H})$ is reflective, therefore KZ-monadic by Corollary 4.12.

∎

Theorem 6.11. *In every locally ranked, order-enriched category $\mathscr{X}$ the subcategory $\mathsf{LInj}(\mathcal{H})$ is KZ-monadic for every class*

$$\mathcal{H} = \mathcal{H}_0 \cup \mathcal{H}_e$$

of morphisms with $\mathcal{H}_0$ small and $\mathcal{H}_e$ consisting of order-epimorphisms.

Proof:

(1) Since the members of $\mathcal{H}_e$ are order-epimorphisms, the category $\mathsf{LInj}(\mathcal{H}_e)$ is simply the orthogonal (full) subcategory $\mathcal{H}_e^{\perp}$, see Example 2.9. It was proved in 2.4(c) of [3] that $\mathcal{H}_e^{\perp}$ is again a locally ranked category w.r.t. $\mathcal{E}$ = all epis and $\mathcal{M}$ = all monics lying in $\mathcal{H}_e^{\perp}$. (The proof concerned ordinary categories, but it adapts immediately to the order-enriched setting.)

Moreover, $\mathcal{H}_e^{\perp}$ is a reflective subcategory of $\mathscr{X}$ whose units are order-epimorphisms. Indeed, the reflection of an object X of $\mathscr{X}$ is the wide pushout of all morphisms $\overline{h}$ in all pushouts (5.3).

Since h is an order-epimorphism and $\mathscr{X}$ has weighted colimits (thus, $\overline{h}$ and $\overline{f}$ are collectively order-epic), it is clear that $\overline{h}$ is also an order-epimorphism. Analogously, a wide pushout of order-epimorphisms is

And we may choose this j to be even and fulfill $j \geq i$. Then the inequality $f \leq gh$ yields $m_j \cdot y_{ij} \cdot f' \leq m_j \cdot g' \cdot h$, and, since m_j is order-monic, $y_{ij} \cdot f' \leq g' \cdot h$. Consequently, composing with $x_{\hat{j},\hat{j+1}} \cdot \beta_j$, and using the naturality of β, we obtain

$$x_{\hat{i},\hat{j+1}} \cdot \beta_i \cdot f' = x_{\hat{j},\hat{j+1}} \cdot \beta_j \cdot y_{ij} \cdot f' \leq x_{\hat{j},\hat{j+1}} \cdot \beta_j \cdot g' \cdot h.$$

This is an instance of the inequality (5.4) with $\beta_i \cdot f'$ in place of f and $x_{\hat{j},\hat{j+1}} \cdot \beta_j \cdot g'$ in place of g. Hence, taking into account the universal property of the morphism $X_{\hat{j+1}} \dashrightarrow X_{\hat{j+2}}$, we conclude that

$$x_{\hat{j+1},\hat{j+2}} \cdot x_{\hat{i},\hat{j+1}} \cdot (\beta_i \cdot f') /\!\!/ h \leq x_{\hat{j+1},\hat{j+2}} \cdot x_{\hat{j},\hat{j+1}} \cdot \beta_j \cdot g'$$

from which it follows that $f/h \leq m_j \cdot g' = g$.

(2) Let $p : X_0 \longrightarrow P$ be a morphism with $P \in \mathsf{LInj}(\mathcal{H})$. Then we know that p gives rise to a cocone $p_i : X_i \longrightarrow P$ of the chain $X : \mathrm{Ord} \longrightarrow \mathscr{X}$ as in Lemma 5.4. We show that the morphism $p_k : X_k \longrightarrow P$ belongs to $\mathsf{LInj}(\mathcal{H})$, i.e., the bottom triangle in the following diagram

$$\begin{array}{ccc} A & \xrightarrow{h} & A' \\ {\scriptstyle f}\downarrow & \swarrow^{f/h} & \downarrow{\scriptstyle (p_k f)/h} \\ X_k & \xrightarrow[p_k]{} & P \end{array}$$

is commutative.

Indeed, given $f = m_i \cdot f'$, as in (1) above, then, recalling from (1) that $f/h = x_{\hat{i+1},k} \cdot (\beta_i f') /\!\!/ h$, and applying Lemma 5.4, we have that:

$$p_k \cdot f/h = p_{\hat{i+1}} \cdot (\beta_i f') /\!\!/ h = [p_{\hat{i}} \cdot (\beta_i f')]/h = (p_k \cdot x_{\hat{i},k} \cdot \beta_i \cdot f')/h = (p_k \cdot f)/h.$$

(3) In order to conclude that p_k is unique, let $q : X_k \longrightarrow P$ be another morphism of $\mathsf{LInj}(\mathcal{H})$ with $q \cdot x_{0k} = p$. We prove that $q = p_k$ by showing, by transfinite induction, that $q \cdot x_{ik} = p_k \cdot x_{ik}$ for all $i \leq k$.

For $i = 0$, this is the assumption. For limit ordinals the inductive step is trivial, by the universal property of the colimit. So we prove the property for $i + 1$ and $i + 2$ with i even.

(3a) From i to $i + 1$. Since $x_{i,i+1}$ and all $f /\!\!/ h$ are collectively epic, we only need proving

$$p_k \cdot x_{i+1,k} \cdot f /\!\!/ h = q \cdot x_{i+1,k} \cdot f /\!\!/ h$$

together with an isomorphism

$$\mathscr{X}(X, \{W, D\}) \cong \mathsf{Pos}^{\mathscr{D}}(W, \mathscr{X}(X, D-))$$

natural in X in $\mathscr{X}$.

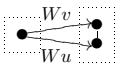
Examples 3.11.

(1) *Conical limits* (which means limits whose limit cones fulfill (3.1)) are precisely the weighted limits with weight constantly 1 (the terminal poset).

(2) Inserters are weighted limits with the scheme

$$\mathscr{D} : \quad d \underset{u}{\overset{v}{\rightrightarrows}} d'$$

and the weight W given by



Remark 3.12. A category with conical products and inserters has conical equalisers, hence all conical limits. Indeed, an equaliser of a pair $f, g : X \longrightarrow Y$ is obtained as an inserter of the pair

$$X \underset{\langle g,f\rangle}{\overset{\langle f,g\rangle}{\rightrightarrows}} X \times Y$$

Just observe that a morphism $i : I \longrightarrow X$ fulfills $\langle f, g\rangle \cdot i \leq \langle g, f\rangle \cdot i$ iff it fulfills $f \cdot i = g \cdot i$. Moreover, we see that equalisers are order-monomorphisms (since inserters are).

Lemma 3.13. *An order-enriched category has weighted limits iff it has conical products and inserters.*

Proof: The necessity follows from Examples 3.11. For the sufficiency, we use Theorem 3.73 of [14]. In fact, it suffices to prove that a particular type of weighted limits, called *cotensors*, exists in $\mathscr{X}$. Given a poset P and an object X, then the P-th cotensor of X is an object $P \pitchfork X$, together with an isomorphism

$$\mathscr{X}(X', P \pitchfork X) \cong \mathsf{Pos}(P, \mathscr{X}(X', X))$$

natural in X'.

Observe that, for a discrete poset P, the cotensor $P \pitchfork X$ is just the P-fold conical product of X. Hence the category $\mathscr{X}$ has cotensors with discrete posets, since it has products.

A general poset P can be described as a coinserter in Pos of a parallel pair

$$P_1 \underset{d_0}{\overset{d_1}{\rightrightarrows}} P_0$$

where P_0 is the discrete poset on elements of P, P_1 is the discrete poset on all pairs (x, x') such that $x \leq x'$ holds, and d_0 and d_1 are the obvious projections. Then one can define $P \pitchfork X$ as an inserter of

$$P_0 \pitchfork X \underset{d_0 \pitchfork X}{\overset{d_1 \pitchfork X}{\rightrightarrows}} P_1 \pitchfork X$$

in $\mathscr{X}$. ∎

Whereas inserters and conical products are required in Section 4, we work with the dual concepts in Section 5.

Definition 3.14. An order-enriched category is said to have *conical coproducts* if it has coproducts $\coprod_{i\in I} X_i$ and the injections γ_i are collectively order-epic. That is, given a parlallel pair $f, g : \coprod_{i\in I} X_i \longrightarrow Y$, we have that $f \cdot \gamma_i \leq g \cdot \gamma_i$ for all $i \in I$ implies $f \leq g$.

Example 3.15. The categories Pos and Top_0 clearly have conical coproducts. Therefore, they have conical colimits. This is dual to Remark 3.12.

Again, the dual notions can be subsumed by the concept of a weighted colimit.

Definition 3.16. Let $\mathscr{X}$ and $\mathscr{D}$ be order-enriched categories, $\mathscr{D}$ small. Given a locally monotone functor $D : \mathscr{D} \longrightarrow \mathscr{X}$, its *colimit weighted by* $W : \mathscr{D}^{op} \longrightarrow \mathsf{Pos}$, also locally monotone, is an object

$$W \star D$$

together with an isomorphism

$$\mathscr{X}(W \star D, X) \cong \mathsf{Pos}^{\mathscr{D}^{op}}(W, \mathscr{X}(D-, X))$$

natural in X in $\mathscr{X}$.

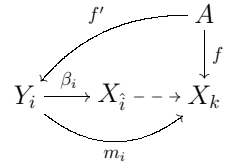
Lemma 3.17. *An order-enriched category has weighted colimits iff it has conical coproducts and coinserters.*

Proof: Since $\mathcal{H}$ is a set, there exists a cardinal λ such that for every $h : A \longrightarrow A'$ in $\mathcal{H}$ both A and A' have rank λ. Put

$$k = \varphi(\lambda).$$

We show that the connecting map $X_0 \dashrightarrow X_k$ of the Kan-injective reflection chain, see Construction 5.2, is a reflection of $X = X_0$ in $\mathsf{LInj}(\mathcal{H})$.

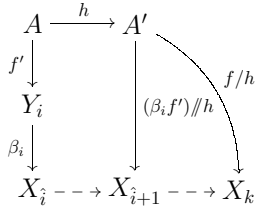
(1) X_k belongs to $\mathsf{LInj}(\mathcal{H})$. Indeed, given $h : A \longrightarrow A'$ in $\mathcal{H}$ and $f : A \longrightarrow X_k$, since A has rank λ, there is some $i < \lambda$ making the diagram



commutative. And we may choose this i to be even. Put

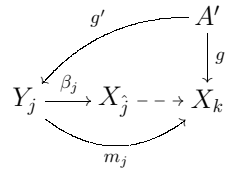
$$f/h = x_{\hat{i}+1,k} \cdot (\beta_i f') /\!\!/ h \tag{6.1}$$

We show that it is the desired f/h.



(1a) $(f/h)\cdot h = x_{\hat{i}+1,k}\cdot(\beta_i f')/\!\!/ h \cdot h = x_{\hat{i}+1,k}\cdot x_{\hat{i},\hat{i}+1}\cdot\beta_i\cdot f' = x_{\hat{i},k}\cdot\beta_i\cdot f' = f$.

(1b) Let $g : A' \longrightarrow X_k$ fulfil the inequality $f \leq gh$. We show that $f/h \leq g$.
Again, the rank λ of A' ensures a factorization of g for some ordinal $j < \gamma$:



is obvious. From Examples 3.5, 3.15 and Lemma 3.17 we know that Top_0 has weighted colimits.

Remark 6.7. In Theorem 6.10 below we use the following trick of Jan Reiterman, see [19] or [17]. Given a transfinite chain $X : \mathrm{Ord} \longrightarrow \mathscr{X}$ and an ordinal i, factorize all connecting maps

$$\begin{array}{ccc} X_i & \xrightarrow{x_{ij}} & X_j \\ {\scriptstyle e_{ij}}\downarrow & \nearrow_{m_{ij}} & \\ E_{ij} & & \end{array}$$

in the $(\mathcal{E}, \mathcal{M})$ factorization system. Since $\mathscr{X}$ is $\mathcal{E}$-cowellpowered there exists an ordinal i^* such that all e_{ij} with $j \geq i^*$ represent the same quotient of X_i. Define $\varphi : \mathrm{Ord} \longrightarrow \mathrm{Ord}$ by $\varphi(0) = 0$, $\varphi(i+1) = \varphi(i)^*$ and $\varphi(i) = \bigvee_{j<i} \varphi(j)$ for limit ordinals i. This gives a new transfinite chain

$$Y_i = E_{i,\varphi(i)}$$

and natural transformations $\beta_i = m_{i,\varphi(i+1)}$ and $\gamma_i = e_{i,\varphi(i+1)}$ with the following properties that were explicitly formulated by Max Kelly [13], Proposition 4.1.

Lemma 6.8. *For every transfinite chain $X : \mathrm{Ord} \longrightarrow \mathscr{X}$ there exists a monotone function $\varphi : \mathrm{Ord} \longrightarrow \mathrm{Ord}$ preserving joins, a transfinite chain $Y : \mathrm{Ord} \longrightarrow \mathscr{X}$ of $\mathcal{M}$-monomorphisms and natural transformations $\gamma_i : X_i \longrightarrow Y_i$ and $\beta_i : Y_i \longrightarrow X_{\hat{i}}$, where $\hat{i} = \varphi(i+1)$, such that*

(1) *$\beta_i \cdot \gamma_i = x_{i\hat{i}}$ for all $i \in \mathrm{Ord}$.*

(2) *For all $j \geq \hat{i}$ we have a morphism of $\mathcal{M}$*

$$Y_i \xrightarrow{\beta_i} X_{\hat{i}} \xrightarrow{x_{ij}} X_j$$

and

(3) *For every limit ordinal j the union of the chain Y_i $(i \leq j)$ is given by*

$$Y_i \xrightarrow{\beta_i} X_{\hat{i}} \xrightarrow{x_{i\varphi(j)}} X_{\varphi(j)}$$

Remark 6.9. Without loss of generality we choose φ so that $\hat{i}$ is an even ordinal for every ordinal i.

Theorem 6.10. *For every set $\mathcal{H}$ of morphisms of a locally ranked category, $\mathsf{LInj}(\mathcal{H})$ is a KZ-monadic subcategory.*

Proof: This is dual to Lemma 3.13. ∎

4. KZ-monadic subcategories and inserter-ideals

In this section we prove that whenever the Kan-injectivity subcategory $\mathsf{LInj}(\mathcal{H})$ is reflective, then the monad $\mathbb{T}$ this generates is a Kock-Zöberlein monad and the Eilenberg-Moore category $\mathscr{X}^{\mathbb{T}}$ is precisely $\mathsf{LInj}(\mathcal{H})$. In the subsequent sections we prove that for small collections $\mathcal{H}$ in "reasonable" categories $\mathsf{LInj}(\mathcal{H})$ is always reflective. A basic concept we need is that of an inserter-ideal subcategory.

Definition 4.1. A subcategory of an order-enriched category $\mathscr{X}$ is *inserter-ideal* provided that it contains with every morphism u also inserters of the pairs (u, v), where v is any morphism in $\mathscr{X}$ parallel to u.

Lemma 4.2. *Every Kan-injectivity subcategory $\mathsf{LInj}(\mathcal{H})$ is inserter-ideal.*

Proof: Suppose that we have an inserter i of (u, v) in $\mathscr{X}$. It is our task to prove that if u is left Kan-injective w.r.t. $h : A \longrightarrow A'$ in $\mathcal{H}$, then so is i. We first verify that I is left Kan-injective. Consider an arbitrary $f : A \longrightarrow I$. In the following diagram

$$\begin{array}{ccccc} A & & \xrightarrow{h} & & A' \\ {\scriptstyle f}\downarrow & \overset{k}{\swarrow}_{f^*} & & & \downarrow{\scriptstyle (if)/h} \\ I & & \xrightarrow[i]{} & X & \underset{u}{\overset{v}{\rightrightarrows}} YY \end{array}$$

the morphism $(if)/h : A' \longrightarrow X$ exists since X is left Kan-injective. Also, u is left Kan-injective and therefore we have

$$u \cdot (if)/h = (uif)/h \leq (vif)/h \leq v \cdot (if)/h$$

proving that $(if)/h$ factorises through i as indicated above.

That the morphism $f^* : A' \longrightarrow I$ is f/h follows immediately from the two aspects of the universal property of an inserter. This proves that the object I is left Kan-injective w.r.t. h.

Moreover, we also have the equality $(if)/h = i \cdot f^* = i \cdot f/h$, proving that the morpism $i : I \longrightarrow X$ is left Kan-injective w.r.t. h, as desired. ∎

Corollary 4.3. *$\mathsf{LInj}(\mathcal{H})$ is closed under weighted limits.*

Proof: Indeed, it is closed under inserters by Lemma 4.2 and under conical limits by [8], Proposition 2.10. The rest is analogous to the proof of Lemma 3.13 above. ∎

Definition 4.4. A subcategory of an order-enriched category $\mathscr{X}$ is called *KZ-monadic* if it is the Eilenberg-Moore category $\mathscr{X}^{\mathbb{T}}$ of a Kock-Zöberlein monad $\mathbb{T}$ on $\mathscr{X}$.

Example 4.5.

(1) Continuous lattices, see Example 2.7(1), are KZ-monadic for the filter monad on Top_0, as proved by Escardó [10].
(2) Complete semilattices, see Example 2.5(1), are KZ-monadic w.r.t. the lowerset monad $\mathbb{T} = (T, \eta, \mu)$ on Pos. More in detail: TX is the poset of all lowersets on a poset X, $\eta_X : X \longrightarrow TX$ assigns the principal lowerset $\downarrow x$ to every $x \in X$, $\mu_X : TTX \longrightarrow TX$ is the union.

Remark 4.6. Recall the concept of a projection-embedding pair of Mike Smyth and Gordon Plotkin [21]. We use the dual concept and call a morphism $r : C \longrightarrow X$ a *coprojection* if there exists $s : X \longrightarrow C$ with

$$r \cdot s = id_C \text{ and } id_X \leq s \cdot r.$$

In the terminology of [8] the morphism r would be called reflective left adjoint.

Definition 4.7. A subcategory $\mathscr{C}$ of an order-enriched category $\mathscr{X}$ is said to be *closed under coprojections* if (a) for every coprojection $r : C \longrightarrow X$ whenever C is in $\mathscr{C}$, then so is X, and (b) for any commutative square in $\mathscr{X}$

$$\begin{array}{ccc} C_1 & \xrightarrow{f} & C_2 \\ {\scriptstyle r_1}\downarrow & & \downarrow{\scriptstyle r_2} \\ X_1 & \xrightarrow[g]{} & X_2 \end{array}$$

whenever f is in $\mathscr{C}$ and r_1, r_2 are coprojections, then also g is in $\mathscr{C}$.

Proposition 4.8 (Proposition 2.13 of [8])**.** *Every Kan-injectivity subcategory* $\mathsf{LInj}(\mathcal{H})$ *is closed under coprojections.*

Proposition 4.9 (See [7] and [8])**.** *Every KZ-monadic category is the Kan-injectivity subcategory w.r.t. all units, i.e.,*

$$\mathscr{X}^{\mathbb{T}} = \mathsf{LInj}(\mathcal{H}) \text{ for } \mathcal{H} = \{\eta_X : X \longrightarrow TX \mid X \text{ in } \mathscr{X}\}.$$

This follows from Proposition 1.5 and Corollary 1.6 in [7], as well as from Theorem 3.9 and Remark 3.10 in [8].

inequality $mu \leq mv$ with m a strong monomorphism, and let c be the coinserter of u and v.

$$\begin{array}{ccccc} X & \underset{u}{\overset{v}{\rightrightarrows}} & A & \xrightarrow{m} & B \\ & & {\scriptstyle c}\downarrow & \nearrow{\scriptstyle m'} & \\ & & C & & \end{array}$$

Then m factorizes through c. But c is an epimorphism and m a strong monomorphism, thus c is invertible. Equivalently, $u \leq v$.

Definition 6.4. Let $\mathscr{X}$ be an order-enriched category with a proper factorization system $(\mathcal{E}, \mathcal{M})$. We call $\mathscr{X}$ *locally ranked* if it has weighted colimits, is $\mathcal{E}$-cowellpowered, and every object has a rank.

Remark 6.5. Explicitly, an object A has rank λ iff given a union $X = \bigcup_{i<\lambda} m_i$ of a λ-chain $m_i : M_i \longrightarrow X$ of subobjects in $\mathcal{M}$, then every morphism $p : A \longrightarrow X$ factorizes through some m_i.

This concept is "automatically enriched": given p, $q : A \longrightarrow X$ with $p \leq q$, it follows that there exists i such that they both factorize through m_i:

$$\begin{array}{ccc} & & M_i \\ & {\scriptstyle q}\nearrow \; {\scriptstyle p}\nearrow & \downarrow{\scriptstyle m_i} \\ A & \underset{p'}{\overset{q'}{\rightrightarrows}} & X \end{array}$$

and we get $p' \leq q'$ from m_i being an order-monomorphism.

In other words: if the hom-functor into Set preserves λ-unions of $\mathcal{M}$-subobjects, it follows that the hom-functor into Pos also does.

Example 6.6.

(1) Pos is a locally ranked category w.r.t. (*Epi*, *Strong Mono*). Indeed, in the non-enriched sense all locally presentable categories are locally ranked, see [5], and, by Example 6.3, (*Epi*, *Strong Mono*) is proper. From Examples 3.5, 3.15 and Lemma 3.17 we know that Pos has weighted colimits.
(2) Top_0 is locally ranked w.r.t. (*Surjection*, *Subspace Embedding*). Indeed, every space A of cardinality less than λ has rank λ — this follows from unions of subspace embeddings in Top_0 being carried by their unions in Set. Cowellpoweredness w.r.t. surjective morphisms

Remark 5.7. The construction above can also be performed, assuming the base category $\mathscr{X}$ is cowellpowered, with every *class* $\mathcal{H}$ of morphisms, provided that it has the form $\mathcal{H} = \mathcal{H}_0 \cup \mathcal{H}_e$ where $\mathcal{H}_0$ is small and $\mathcal{H}_e$ is a class of epimorphisms.

Indeed, in the isolated step $i \mapsto i+1$ with i even the conical colimit exists because $x_{i,i+1}$ is the wide pushout of all the morphisms $\overline{h}$. If h lies in $\mathcal{H}_e$ then $\overline{h}$ is an epimorphism. Thus cowellpoweredness guarantees that X_{i+1} is obtained as a small wide pushout. The isolated step $i+1 \mapsto i+2$ with i even also makes no problem because $x_{i+1,i+2}$ is an epimorphism, and we obtain it as the cointersection of the corresponding epimorphisms over all subsets of $\mathcal{H}$.

6. Locally ranked categories

Our main result, proved in Theorem 6.11 below, states that for every class $\mathcal{H}$ of morphisms in an order-enriched category $\mathscr{X}$ such that all but a set of members of $\mathcal{H}$ are order-epic, the subcategory $\mathsf{LInj}(\mathcal{H})$ is KZ-reflective. For that we need to assume that $\mathscr{X}$ is locally ranked, a concept introduced in [5]. It is based on a factorization system $(\mathcal{E}, \mathcal{M})$ in a (non-enriched) category $\mathscr{X}$ which is *proper*, i.e., all morphisms in $\mathcal{E}$ are epimorphisms and all morphisms in $\mathcal{M}$ are monomorphisms. An object X of $\mathscr{X}$ has *rank* λ, where λ is an infinite regular cardinal, provided that its hom-functor preserves unions of λ-chains of subobjects in $\mathcal{M}$.

Definition 6.1 (See [5])**.** An ordinary category $\mathscr{X}$ with a proper factorization system $(\mathcal{E}, \mathcal{M})$ is called *locally ranked* if it is cocomplete and $\mathcal{E}$-cowellpowered, and every object has a rank.

Remark 6.2. In order-enriched categories *proper* is defined for a factorization system $(\mathcal{E}, \mathcal{M})$ to mean that all morphisms in $\mathcal{E}$ are epimorphisms, and all morphisms in $\mathcal{M}$ are order-monomorphisms.

Example 6.3. Recall from [4] that every cocomplete, cowellpowered category has the factorization system (*Epi*, *Strong Mono*). In every order-enriched category this factorization system is proper. Indeed, consider the

Remark 4.10. For the larger collection $\mathcal{H}'$ of all morphisms i with Ti having a right adjoint $Ti \dashv j$ such that $j \cdot Ti = id$ it also holds that $\mathscr{X}^{\mathbb{T}} = \mathsf{LInj}(\mathcal{H}')$, see [11] and [8].

Theorem 4.11. *A subcategory of an order-enriched category is KZ-monadic iff it is*

(1) *reflective,*
(2) *inserter-ideal, and*
(3) *closed under coprojections.*

Proof: We first recall from [8], Theorems 3.13 and 3.4 that a subcategory $\mathscr{C}$ is KZ-monadic iff it is

(a) reflective, with reflections $\eta_X : X \longrightarrow FX$ (X in $\mathscr{X}$)
(b) closed under coprojections,
(c) a subcategory of $\mathsf{LInj}(\mathcal{H})$ for $\mathcal{H} = \{\eta_X \mid X \text{ in } \mathscr{X}\}$,

and such that

(d) every morphism $f : FX \longrightarrow A$ in $\mathscr{C}$ fulfils $(f\eta_X)/\eta_X = f$.

Indeed, Theorem 3.4 states that (a), (c) and (d) are equivalent to $\mathscr{C}$ being KZ-reflective, thus Theorem 3.13 applies.

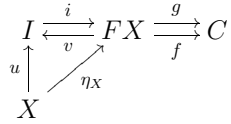
Every KZ-monadic category is inserter ideal by Lemma 4.2 and Proposition 4.9, thus it has all the properties of our Theorem: see Conditions (a) and (b) above.

For the converse implication, we only need to verify Conditions (c) and (d) above. For (c) see Proposition 4.9. Condition (d) easily follows from the implication

$$f\eta_X \leq g\eta_X \text{ implies } f \leq g$$

for all pairs $f, g : FX \longrightarrow A$ with f in $\mathscr{C}$.

In order to prove the implication, form the inserter i of the pair (f, g):



Thus, we have a morphism u in $\mathscr{X}$ with $\eta_X = i \cdot u$. Since f lies in the inserter-ideal subcategory $\mathscr{C}$, so does I. Therefore u factorises through the reflection η_X:

$$u = v \cdot \eta_X$$

and both v and i are morphisms of $\mathscr{C}$. Thus so is $i \cdot v$ and from $(i \cdot v) \cdot \eta_X = \eta_X$ we therefore conclude $i \cdot v = id$. Now i is monic as well as split epic, therefore it is invertible. This gives the desired inequality $f \leq g$. ∎

From Lemma 4.2, Theorem 4.11, and Proposition 4.8, we obtain the following:

Corollary 4.12. *Whenever* $\mathsf{LInj}(\mathcal{H})$ *is a reflective subcategory, then it is KZ-monadic.*

5. Kan-injective reflection chain

Here we show how a reflection of an object X in the Kan-injectivity subcategory $\mathsf{LInj}(\mathcal{H})$ is constructed: we define a transfinite chain X_i ($i \in \mathrm{Ord}$) with $X_0 = X$ such that with increasing i the objects X_i are "nearer" to being Kan-injective. This chain is said to *converge* if for some ordinal k the connecting map $X_k \dashrightarrow X_{k+2}$ is invertible. When this happens, X_k is Kan-injective, and a reflection of X is given by the connecting map $X_0 \dashrightarrow X_k$. In Section 6 sufficient conditions for the convergence of the reflection chain are discussed.

Assumption 5.1. Throughout this section $\mathscr{X}$ denotes an order-enriched category with weighted colimits.

Construction 5.2 (Kan-injective reflection chain). Let $\mathscr{X}$ be an order-enriched category with weighted colimits, and $\mathcal{H}$ a set of morphisms in $\mathscr{X}$. Given an object X, we construct a chain of objects X_i ($i \in \mathrm{Ord}$). We denote the connecting maps by $x_{ij} : X_i \longrightarrow X_j$ or just by $X_i \dashrightarrow X_j$, for all $i \leq j$.

The first step is the given object $X_0 = X$. Limit steps X_i, i a limit ordinal, are defined by (conical) colimits of i-chains:

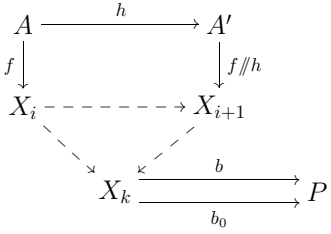
$$X_i = \operatorname*{colim}_{j<i} X_j.$$

Isolated steps: given X_i we define both X_{i+1} and X_{i+2}, thus, we can restrict ourselves to even ordinals i (having distance $2n$, $n < \omega$, from 0 or a limit ordinal).

(1) To define X_{i+1} and the connecting map $X_i \dashrightarrow X_{i+1}$, consider all spans

$$\begin{array}{ccc} A & \xrightarrow{h} & A' \\ {\scriptstyle f}\downarrow & & \\ X_i & & \end{array} \tag{5.1}$$

(a) From i to $i+1$.



Since $x_{i,i+1}$ and all $f \pitchfork h$ are collectively order-epic, we only need proving

$$b_0 \cdot x_{i+1,k} \cdot f \pitchfork h \leq b \cdot x_{i+1,k} \cdot f \pitchfork h$$

The formula (5.7) for $x_{ik}f$ in place of f yields

$$(x_{ik}f)/h = x^{-1}_{k,k+2} \cdot x_{k+1,k+2} \cdot (x_{ik}f) \pitchfork h.$$

And, since $x_{k+1,k+2}$ merges $(x_{ik}f) \pitchfork h$ and $x_{i+1,k+1} \cdot f \pitchfork h$, see Remark 5.5, we get

$$\begin{aligned} (x_{ik}f)/h &= x^{-1}_{k,k+2} \cdot x_{k+1,k+2} \cdot x_{i+1,k+1} \cdot f \pitchfork h \\ &= x^{-1}_{k,k+2} \cdot x_{k,k+2} \cdot x_{i+1,k} \cdot f \pitchfork h \\ &= x_{i+1,k} \cdot f \pitchfork h. \end{aligned}$$

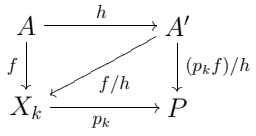
Since b_0 lies in $\mathsf{LInj}(\mathcal{H})$, we know that $b_0[(x_{ik}f)/h] = (b_0 x_{ik}f)/h$. And, since by induction hypothesis $b_0 x_{ik} \leq b x_{ik}$, we then obtain that $(b_0 x_{ik})/h \leq (b x_{ik})/h$. Consequently:

$$\begin{aligned} b_0 \cdot x_{i+1,k} \cdot f \pitchfork h &= b_0 \cdot [(x_{ik}f)/h] \\ &= (b_0 x_{ik}f)/h \\ &\leq (b x_{ik}f)/h \\ &\leq b \cdot (x_{ik}f)/h \\ &= b \cdot x_{i+1,k} \cdot f \pitchfork h \end{aligned}$$

(b) From $i+1$ to $i+2$. This is trivial because $x_{i+1,i+2}$ is order-epic.

∎

(2) Given $p : X_0 \longrightarrow P$ where P lies in $\mathsf{LInj}(\mathcal{H})$, we prove that the morphism p_k of Lemma 5.4 belongs to $\mathsf{LInj}(\mathcal{H})$. For every span (5.1) we want to prove that the bottom triangle in the following diagram



is commutative. Indeed,

$$\begin{aligned} p_k \cdot (f/h) &= p_k \cdot x_{k,k+2}^{-1} \cdot x_{k+1,k+2} \cdot (f /\!\!/ h), && \text{by (5.7)} \\ &= (p_{k+2} \cdot x_{k,k+2}) \cdot x_{k,k+2}^{-1} \cdot x_{k+1,k+2} \cdot (f /\!\!/ h) && \text{by Lemma 5.4} \\ &= p_{k+2} \cdot x_{k+1,k+2} \cdot (f /\!\!/ h) \\ &= p_{k+1} \cdot (f /\!\!/ h), && \text{by Lemma 5.4} \\ &= (p_k \cdot f)/h && \text{again by Lemma 5.4} \end{aligned}$$

(3) We have, for every p as in (2), the morphism p_k of $\mathsf{LInj}(\mathcal{H})$ with $p = p_k \cdot x_{0,k}$. Now we prove the unicity of p_k. It suffices to show that, given morphisms $b, b_0 : X_k \longrightarrow P$ with b_0 in $\mathsf{LInj}(\mathcal{H})$, then

$$b_0 \cdot x_{0k} \leq b \cdot x_{0k} \qquad \text{implies} \qquad b_0 \leq b.$$

Indeed, in the case where b is also a morphism of $\mathsf{LInj}(\mathcal{H})$ then the equality $b_0 \cdot x_{0k} = b \cdot x_{0k}$ will imply $b_0 = b$. We are going to verify the above implication by proving that

$$b_0 \cdot x_{0k} \leq b \cdot x_{0k} \qquad \text{implies} \qquad b_0 \cdot x_{ik} \leq b \cdot x_{ik}$$

for all $i \leq k$. We use transfinite induction. The first step $i = 0$ is clear. Also limit steps are clear since the colimit cocones are collectivelly order-epic.

It remains to check the isolated steps $i + 1$ and $i + 2$ for i an even ordinal.

where h is in $\mathcal{H}$ and f is arbitrary. We form the colimit of this diagram and call the colimit morphisms $X_i \dashrightarrow X_{i+1}$ and $f /\!\!/ h$ (because they "approximate" f/h), respectively:

$$\begin{array}{ccc} A & \xrightarrow{h} & A' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f /\!\!/ h} \\ X_i & \dashrightarrow & X_{i+1} \end{array} \tag{5.2}$$

More detailed: given h in $\mathcal{H}$ and $f : A \longrightarrow X_i$ we form a pushout

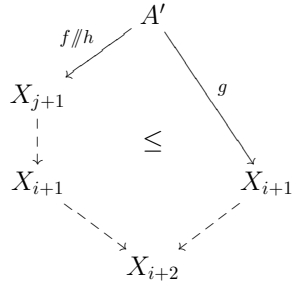
$$\begin{array}{ccc} A & \xrightarrow{h} & A' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \overline{f}} \\ X_i & \xrightarrow[\overline{h}]{} & C \end{array} \tag{5.3}$$

Then $X_i \dashrightarrow X_{i+1}$ is the wide pushout of all $\overline{h}$ (with the colimit cocone $c_{f,h} : C \longrightarrow X_{i+1}$) and we put $f /\!\!/ h = c_{f,h} \cdot \overline{f}$.

(2) To define X_{i+2} and the connecting map $X_{i+1} \dashrightarrow X_{i+2}$, consider all inequalities

$$\begin{array}{ccc} A & \xrightarrow{h} & A' \\ {\scriptstyle f}\downarrow & \leq & \downarrow{\scriptstyle g} \\ X_j & \dashrightarrow & X_{i+1} \end{array} \tag{5.4}$$

where $h \in \mathcal{H}$, $j \leq i$ is an even ordinal, and f, g are arbitrary. We let $X_{i+1} \dashrightarrow X_{i+2}$ be the universal map such that (5.4) implies the inequality



(5.5)

In other words, $X_{i+1} \dashrightarrow X_{i+2}$ is the wide pushout of all the coinserters

$$\mathrm{coins}(x_{j+1,i+1} \cdot (f /\!\!/ h), g).$$

Example 5.3. In case of join semilattices (where h is the embedding of Example 2.5(3)) the even step from X_i to X_{i+1} adds to every pair x, y of elements of X_i an upper bound compatible only with all elements under x or y. And the odd step from X_{i+1} to X_{i+2} is a quotient that turns this upper bound into a join of x and y. After ω steps we get the join-semilattice reflection of X.

Lemma 5.4. *Given a morphism $p_0 : X_0 \longrightarrow P$ where P is Kan-injective, there exists a unique cocone $p_i : X_i \longrightarrow P$ ($i \in$ Ord) such that for all spans (5.1) the following triangle*

$$\begin{array}{ccc} A' & & \\ {\scriptstyle f/\!\!/h}\downarrow & \searrow^{(p_i f)/h} & \\ X_{i+1} & \xrightarrow[p_{i+1}]{} & P \end{array} \tag{5.6}$$

commutes.

Proof: We only need to prove the isolated step: given p_i for i even, we have unique p_{i+1} and p_{i+2}. For p_{i+1} we observe that the morphisms $p_i : X_i \longrightarrow P$ and $(p_i f)/h : A' \longrightarrow P$ form a cocone of the diagram defining $X_i \overset{-}{\to} X_{i+1}$. Indeed, the square

$$\begin{array}{ccc} A & \xrightarrow{h} & A' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle (p_i f)/h} \\ X_i & \xrightarrow[p_i]{} & P \end{array}$$

clearly commutes. It follows that there is a unique p_{i+1} for which the above triangle commutes and which prolongs the given cocone.

Next we prove the existence of p_{i+2} (uniqueness is clear since $X_{i+1} \overset{-}{\to} X_{i+2}$ is epic) by verifying that p_{i+1} has the universal property of $X_{i+1} \overset{-}{\to} X_{i+2}$: for every square (5.4) we have

$$\begin{array}{ccc} A' & \xrightarrow{g} & X_{i+1} \\ {\scriptstyle f/\!\!/h}\downarrow & \leq & \downarrow{\scriptstyle p_{i+1}} \\ X_{j+1} & \xrightarrow[p_{j+1}]{} & P \end{array}$$

Indeed, by (5.6), the lower passage is $(p_j \cdot f)/h$, hence, it is sufficient to verify $p_j \cdot f \leq p_{i+1} \cdot g \cdot h$. To that end, compose the given inequality (5.4) with p_{i+1}. ∎

Remark 5.5. In the Kan-injective reflection chain, for every pair i, j of even ordinals with $j \leq i$ and every span as in (5.1) with j in place of i, the connecting map $x_{i+1,i+2}$ merges the morphisms $(x_{ji}f)/\!\!/h$ and $x_{j+1,i+1} \cdot (f/\!\!/h)$.

Indeed, the equality (5.2) for f implies clearly the equality

$$((x_{ji}f)/\!\!/h) \cdot h = x_{j+1,i+1} \cdot (f/\!\!/h) \cdot h$$

decomposes into two inequalities which by the universal property of the morphism $x_{i+1,i+2}$ gives rise to

$x_{i+1,i+2} \cdot x_{j+1,i+1} \cdot f/\!\!/h \leq x_{i+1,i+2} \cdot (x_{ji}f)/\!\!/h$ (putting $g = (x_{ji}f)/\!\!/h$ in (5.4)),

and

$x_{i+1,i+2} \cdot (x_{ji}f)/\!\!/h \leq x_{i+1,i+2} \cdot x_{j+1,i+1} \cdot f/\!\!/h$ (putting $g = x_{j+1,i+1} \cdot f/\!\!/h$ in (5.4)).

Theorem 5.6. *If the Kan-injective reflection chain converges at an even ordinal k (i.e., $x_{k,k+2}$ is invertible), then X_k lies in $\mathsf{LInj}(\mathcal{H})$ and $x_{0k} : X_0 \longrightarrow X_k$ is a reflection of X_0 in $\mathsf{LInj}(\mathcal{H})$.*

Proof:

(1) We prove the Kan-injectivity of X_k. Given $h : A \longrightarrow A'$ in $\mathscr{X}$ and $f : A \longrightarrow X_k$, the square (5.2) allows us to define a morphism

$$f/h = x_{k,k+2}^{-1} \cdot x_{k+1,k+2} \cdot (f/\!\!/h) : A' \longrightarrow X_k \tag{5.7}$$

and we verify the two properties needed. The first one is clear by applying (5.2) to $i = k$:

$$\begin{aligned} (f/h) \cdot h &= x_{k,k+2}^{-1} \cdot x_{k+1,k+2} \cdot (f/\!\!/h) \cdot h \\ &= x_{k,k+2}^{-1} \cdot x_{k+1,k+2} \cdot x_{k,k+1} \cdot f \\ &= x_{k,k+2}^{-1} \cdot x_{k,k+2} \cdot f \\ &= f. \end{aligned}$$

For the second one let $g : A' \longrightarrow X_k$ fulfil $gh \geq f$. Then we prove $g \geq f/h$. The morphism $\overline{g} = x_{k,k+1} \cdot g$ fulfils $\overline{g}h \geq x_{k,k+1} \cdot f$, thus, the universal property of $x_{k+1,k+2}$ implies

$$x_{k+1,k+2} \cdot \bar{g} \geq x_{k+1,k+2} \cdot (f/\!\!/h).$$

That is,

$$x_{k,k+2} \cdot g \geq x_{k+1,k+2} \cdot (f/\!\!/h).$$

By composing with $x_{k,k+2}^{-1}$ we get $g \geq x_{k,k+2}^{-1} \cdot x_{k+1,k+2} \cdot (f/\!\!/h)$, as desired.