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INDUCED HYPERSYMPLECTIC AND HYPERKÄHLER STRUCTURES ON THE DUAL OF A LIE ALGEBROID

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Abstract: We show that every hypersymplectic structure on a Lie algebroid A determines a hypersymplectic and a hyperkähler structure on the dual A^*. This result is illustrated with an example on the Lie algebroid $T(H^3 \times I)$, where H^3 is the Heisenberg group.

1. Introduction

Hypersymplectic structures on Lie algebroids were defined and studied in our previous paper [1]. Our definition is inspired in Xu's definition of hypersymplectic structure on a manifold [5]. One of the results we got in [1] and we use in the present paper, is the existence of a 1-1 correspondence between hypersymplectic and hyperkähler structures on a Lie algebroid.

A hypersymplectic structure on a Lie algebroid A is a triple $(\omega_1, \omega_2, \omega_3)$ of symplectic forms satisfying a certain condition. The inverse of the symplectic form ω_i, $i = 1, 2, 3$, is a Poisson bivector π_i on A and, using the identification of $(A^*)^*$ with A, we show that π_i can be viewed as a symplectic form with respect to a certain Lie algebroid structure on A^*. Moreover, the triple (π_1, π_2, π_3) is a hypersymplectic structure on A^*. This means that every hypersymplectic structure on a Lie algebroid automatically determines a hypersymplectic structure on its dual and the same happens with hyperkähler structures, as a consequence of the 1-1 correspondence between hypersymplectic and hyperkähler structures.

In [2] the authors present several (para-)hypersymplectic structures in 4-dimensional manifolds arising from the 3-dimensional Bianchi type A groups. In order to obtain, explicitly, the metrics of these structures, the authors solve a system of evolution equations and consider some particular solutions. One of the cases considered in [2] concerns the Heisenberg group H^3. We show that if we treat that case in the Lie algebroid setting, taking $T(H^3 \times I)$ and applying our construction, we get a hypersymplectic structure and

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its corresponding hyperkähler structure recovering, this way, the Heisenberg metric of [2].

The paper is divided into three sections. In Section 2 we recall the definition and the main properties of an $\boldsymbol{\varepsilon}$-hypersymplectic structure on a Lie algebroid, which is the more general structure we work with. In Section 3 we consider the particular case of (para-)hypersymplectic structures and we include an example related to the Heisenberg group. In Section 4, we prove the main result of this paper, showing that each hypersymplectic structure on a Lie algebroid determines a hypersymplectic structure on its dual and that this result also holds for hyperkähler structures. Taking the example of the previous section, we present the hypersymplectic and hyperkähler structures induced on the Lie algebroid $T^*(H^3 \times I)$.

2. $\boldsymbol{\varepsilon}$-hypersymplectic structures on Lie algebroids

In this section we recall the notion of $\boldsymbol{\varepsilon}$-hypersymplectic structure on a Lie algebroid, introduced in [1], and some of its main properties.

Let $(A, \rho, [.,.])$ be a Lie algebroid over M and $N \in \Gamma(A \otimes A^*)$ a $(1,1)$-tensor, seen as a vector bundle morphism $N : A \to A$. The deformation by N of the Lie bracket $[.,.]$ is defined, for all $X, Y \in \Gamma(A)$, by $[X,Y]_N := [NX, Y] + [X, NY] - N[X,Y]$.
We denote by $\mathcal{T}N$ the Nijenhuis torsion of N, which is given, for all $X, Y \in \Gamma(A)$, by $\mathcal{T}N(X,Y) := [NX, NY] - N[X,Y]_N$. If the Nijenhuis torsion of N vanishes, N is a *Nijenhuis* tensor on A.

A vector bundle morphism $N : A \to A$ on a Lie algebroid A that satisfies $N^2 = -\mathrm{Id}_A$ (respectively, $N^2 = \mathrm{Id}_A$) is said to be an *almost complex structure* (respectively, *almost para-complex structure*) on A. If, moreover, $\mathcal{T}N = 0$, then N is a *complex structure* (respectively, *para-complex structure*).

Given a bivector $\pi \in \Gamma(\bigwedge^2 A)$ and a 2-form $\omega \in \Gamma(\bigwedge^2 A^*)$ on a Lie algebroid A, we consider the usual vector bundle maps $\pi^\# : A^* \to A$ and $\omega^\flat : A \to A^*$ and the induced morphisms on sections, denoted by the same symbols, which are defined, for all $\alpha, \beta \in \Gamma(A^*)$ and $X, Y \in \Gamma(A)$, respectively by $\langle \beta, \pi^\#(\alpha) \rangle = \pi(\alpha, \beta)$ and $\langle \omega^\flat(X), Y \rangle = \omega(X, Y)$.

A bivector $\pi \in \Gamma(\bigwedge^2 A)$ on a Lie algebroid $(A, \rho, [.,.])$ is a *Poisson* bivector if $[\pi, \pi] = 0$, where $[.,.]$ denotes the Schouten-Nijenhuis bracket of the Lie algebroid, which is the natural extension, by derivation, of the Lie bracket on $\Gamma(A)$ to a bracket on $\Gamma(\bigwedge^\bullet A)$.

Let $(A, \rho, [.,.])$ be a Lie algebroid and ω_1, ω_2 and ω_3 three symplectic forms on A, with inverse Poisson bivectors π_1, π_2 and π_3, respectively. Then, for all $i \in \{1,2,3\}$, we have $\omega_i^\flat \circ \pi_i^\# = \mathrm{Id}_{A^*}$ and $\pi_i^\# \circ \omega_i^\flat = \mathrm{Id}_A$. The symplectic forms ω_i and Poisson bivectors π_i determine the *transition* $(1,1)$-tensors N_1, N_2 and N_3 on A, defined by

$$N_i := \pi_{i-1}^\# \circ \omega_{i+1}^\flat, \tag{1}$$

where the indices of $\pi^\#$ and $\omega^\flat$ are considered as elements of $\mathbb{Z}_3$.

Remark 2.1. We consider 1, 2 and 3 as the representative elements of the equivalence classes of $\mathbb{Z}_3$, i.e., $\mathbb{Z}_3 := \{[1],[2],[3]\}$. In what follows, although we omit the brackets and write i instead of $[i]$, all the indices (and corresponding computations) must be thought in $\mathbb{Z}_3$.

Definition 2.2. A triple $(\omega_1, \omega_2, \omega_3)$ of symplectic forms on a Lie algebroid $(A, \rho, [.,.])$ is an $\boldsymbol{\varepsilon}$*-hypersymplectic structure* on A if the transition $(1,1)$-tensors N_i, $i = 1,2,3$, given by (1), satisfy $N_i^2 = \varepsilon_i \mathrm{Id}_A$, where the parameters $\varepsilon_i = \pm 1$ form the triple $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$.

Notice that the condition $N_i^2 = \varepsilon_i \mathrm{Id}_A$ in Definition 2.2 can be written as

$$\omega_{i+1}^\flat \circ \pi_{i-1}^\sharp = \varepsilon_i\, \omega_{i-1}^\flat \circ \pi_{i+1}^\sharp. \tag{2}$$

Having an $\boldsymbol{\varepsilon}$-hypersymplectic structure $(\omega_1, \omega_2, \omega_3)$ on a Lie algebroid A over M, we define a map

$$g : \Gamma(A) \times \Gamma(A) \longrightarrow C^\infty(M), \qquad g(X,Y) = \langle g^\flat(X), Y\rangle, \tag{3}$$

where $g^\flat : \Gamma(A) \longrightarrow \Gamma(A^*)$ is given by

$$g^\flat := \varepsilon_3\varepsilon_2\; {\omega_3}^\flat \circ {\pi_1}^\sharp \circ {\omega_2}^\flat. \tag{4}$$

The definition of $g^\flat$ is not affected by a circular permutation of the indices in equation (4), that is,

$$g^\flat = \varepsilon_{i-1}\varepsilon_{i+1}\; {\omega_{i-1}}^\flat \circ {\pi_i}^\sharp \circ {\omega_{i+1}}^\flat, \tag{5}$$

for all $i \in \mathbb{Z}_3$. It is easy to see that (5) is equivalent to

$$g^\flat = \varepsilon_{i-1} {\omega_i}^\flat \circ N_i, \quad i \in \mathbb{Z}_3. \tag{6}$$

Moreover, we have

$$(g^\flat)^* = -\varepsilon_1\varepsilon_2\varepsilon_3\; g^\flat, \tag{7}$$

which means that g is symmetric or skew-symmetric, depending on the sign of $\varepsilon_1\varepsilon_2\varepsilon_3$. Notice that $g^\flat$ is invertible and, using its inverse, we may define a map $g^{-1} : \Gamma(A^*) \times \Gamma(A^*) \longrightarrow C^\infty(M)$, by setting

$$g^{-1}(\alpha, \beta) := \langle \beta, (g^\flat)^{-1}(\alpha) \rangle, \tag{8}$$

for all $\alpha, \beta \in \Gamma(A^*)$.

Several relations between the morphisms $\omega_i^\flat$, $\pi_i^\sharp$, N_i and $g^\flat$ were proved in [1].
One of them, that will be used later, is the following:

$$g(N_i X, N_i Y) = \varepsilon_{i-1}\varepsilon_{i+1}\, g(X, Y), \tag{9}$$

for all sections $X, Y \in \Gamma(A)$ and for all indices in $\mathbb{Z}_3$.

The next theorem gives one of the main properties of an $\boldsymbol{\varepsilon}$-hypersymplectic structure on a Lie algebroid.

Theorem 2.3. *Let $(\omega_1, \omega_2, \omega_3)$ be an $\boldsymbol{\varepsilon}$-hypersymplectic structure on a Lie algebroid A. The transition tensors N_1, N_2 and N_3 are Nijenhuis tensors, i.e., $\mathcal{T} N_i = 0$, for all $i \in \{1, 2, 3\}$.*

3. Hypersymplectic and para-hypersymplectic structures

Among the $\boldsymbol{\varepsilon}$-hypersymplectic structures on a Lie algebroid, the case where $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$ has a relevant importance. In this section we consider that case and recall some results from [1]. Then, we present an example related to the Heisenberg group.

Let $(\omega_1, \omega_2, \omega_3)$ be an $\boldsymbol{\varepsilon}$-hypersymplectic structure on a Lie algebroid A, with $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$. We distinguish two cases:

- If $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = -1$, then $(\omega_1, \omega_2, \omega_3)$ is said to be a *hypersymplectic* structure on A.
- Otherwise, $(\omega_1, \omega_2, \omega_3)$ is said to be a *para-hypersymplectic* structure on A.

It is clear that all para-hypersymplectic structures satisfy, eventually after a cyclic permutation of the indices, $\varepsilon_1 = \varepsilon_2 = 1$ and $\varepsilon_3 = -1$. In the sequel, every para-hypersymplectic structures will be considered in such form.

Now, we recall what a pseudo-metric on a Lie algebroid is. Let A be a Lie algebroid over M. A *pseudo-metric* on $A \to M$ is a symmetric and $C^\infty(M)$-bilinear map $\mathsf{g} : \Gamma(A) \times \Gamma(A) \to C^\infty(M)$, which is non-degenerate at each point of M. Furthermore, if g is positive definite, i.e., if $\mathsf{g}(X, X) > 0$, for any non vanishing section $X \in \Gamma(A)$, then g is a *metric* on A. The (pseudo-)metric g induces an invertible $C^\infty(M)$-linear map,

$$\mathsf{g}^\flat : \Gamma(A) \to \Gamma(A^*), \quad \langle \mathsf{g}^\flat(X), Y\rangle = \mathsf{g}(X, Y).$$

An important point is the following : If $(\omega_1, \omega_2, \omega_3)$ is a para-hypersymplectic or a hypersymplectic structure on a Lie algebroid A then, the morphism g defined by (3) and (5) is a pseudo-metric on A. In fact, g is obviously $C^\infty(M)$-bilinear, non-degenerate and, from (7), g is symmetric.

Remark 3.1. In the sequel, we do not require the metric to be positive definite. However, in order to simplify the terminology we shall omit the "pseudo" prefix, although we deal with pseudo-metrics.

A pair (g, I), formed by a metric g and a complex tensor I on a Lie algebroid A, is said to be a *hermitian* structure on A if $\mathsf{g}(IX, IY) = \mathsf{g}(X, Y)$, for all $X, Y \in \Gamma(A)$. In the case where I is a para-complex tensor, the pair (g, I) is called a *para-hermitian* structure if $\mathsf{g}(IX, IY) = -\mathsf{g}(X, Y)$, for all $X, Y \in \Gamma(A)$.

Now, if $(\omega_1, \omega_2, \omega_3)$ is a hypersymplectic structure on a Lie algebroid A and g is its induced metric, it is clear from (9) that the pairs (g, N_i), $i \in \{1, 2, 3\}$ are hermitian structures on the Lie algebroid A. If $(\omega_1, \omega_2, \omega_3)$ is a para-hypersymplectic structure on A, then (g, N_1) and (g, N_2) are para-hermitian structures on A, while (g, N_3) is a hermitian structure.

Having recalled the notion of hermitian (respectively, para-hermitian) structure on a Lie algebroid, we may now give the definition of a hyperkähler (respectively, para-hyperkähler) structure on the Lie algebroid.

Definition 3.2. A *hyperkähler* (respectively, *para-hyperkähler*) structure on a Lie algebroid $(A, \rho, [.,.])$ is a quadruple $(\mathsf{g}, I_1, I_2, I_3)$, where the pairs $(\mathsf{g}, I_j)_{j=1,2}$ are both hermitian (respectively, para-hermitian) structures on A, the morphisms I_1 and I_2 anti-commute, $I_3 = I_1 I_2$, and ${\omega_i}^\flat := \mathsf{g}^\flat \circ I_i$ are closed 2-forms on A, for $i = 1, 2, 3$.

One of the main results we proved in [1] is the following:

Theorem 3.3. *Let $(A, \rho, [.,.])$ be a Lie algebroid. The triple $(\omega_1, \omega_2, \omega_3)$ is a hypersymplectic (respectively, para-hypersymplectic) structure on A if and only if (g, N_1, N_2, N_3) is a hyperkähler (respectively, para-hyperkähler) structure on A.*

The next example shows that the Heisenberg metric constructed in [2] can be obtained applying our construction, if we treat the problem in the Lie algebroid setting, that is, if we construct the transition tensors and get the metric from formula (6). We should stress that all the cases considered in [2] can be tackled in the same way as the one we will explain next.

Example 3.4 (Heisenberg metric)**.** Consider the Heisenberg Lie group H^3 and $\{e^1, e^2, e^3\}$ a basis of 1-forms on H^3 defined by

$$e^1 := \mathrm{d}x, \qquad e^2 := \mathrm{d}y \qquad \text{and} \qquad e^3 := \mathrm{d}z - \frac{1}{2}x\,\mathrm{d}y + \frac{1}{2}y\,\mathrm{d}x,$$

where x, y, z are global coordinate functions on H^3.

We define a hypersymplectic triple $(\omega_1, \omega_2, \omega_3)$ on the Lie algebroid $T(H^3 \times I)$, where $I \subseteq \mathbb{R}^+$, by setting

$$\omega_1{}^\flat := \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & t \\ 1 & 0 & 0 & 0 \\ 0 & -t & 0 & 0 \end{bmatrix}, \ \omega_2{}^\flat := \begin{bmatrix} 0 & 0 & 0 & -t \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ t & 0 & 0 & 0 \end{bmatrix}, \ \omega_3{}^\flat := \begin{bmatrix} 0 & -t & 0 & 0 \\ t & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

on the basis $\{e_1, e_2, e_3, \partial_t\}$, t being the coordinate on $I \subset \mathbb{R}^+$.

The transition tensors N_i, $i \in \{1, 2, 3\}$, given by (1), correspond to the following matrices:

$$N_1 = \begin{bmatrix} 0 & 0 & -\frac{1}{t} & 0 \\ 0 & 0 & 0 & 1 \\ t & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \ N_2 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -\frac{1}{t} & 0 \\ 0 & t & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \ N_3 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -t \\ 0 & 0 & \frac{1}{t} & 0 \end{bmatrix}.$$

Using (6), the morphism $g^\flat$ associated to the metric g has the following matrix representation:

$$g^\flat = \begin{bmatrix} t & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & \frac{1}{t} & 0 \\ 0 & 0 & 0 & t \end{bmatrix},$$

which corresponds to the Heisenberg metric obtained in [2]. By Theorem 3.3, the quadruple (g, N_1, N_2, N_3) is an hyperkähler structure on the Lie algebroid $T(H^3 \times I)$.

Changing a couple of signs, we may also define a para-hypersymplectic structure on the Lie algebroid $T(H^3 \times I)$. Let us define

$$\omega_1'^\flat := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & t \\ -1 & 0 & 0 & 0 \\ 0 & -t & 0 & 0 \end{bmatrix}, \quad \omega_2'^\flat := \begin{bmatrix} 0 & 0 & 0 & -t \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ t & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \omega_3'^\flat := -\omega_3^\flat.$$

Then, if we denote by $D_{a,b,c,d}$ the 4×4 diagonal matrix with diagonal terms a, b, c and d, the transition tensors are

$$N_1' = N_1 D_{-1,1,1,-1}, \quad N_2' = N_2 D_{-1,1,-1,1} \quad \text{and} \quad N_3' = N_3 D_{1,1,-1,-1}.$$

Note that they satisfy $N_1'^2 = N_2'^2 = \mathrm{Id}_{T(H^3\times I)}$ and $N_3'^2 = -\mathrm{Id}_{T(H^3\times I)}$. Thus, $(\omega_1', \omega_2', \omega_3')$ is a para-hypersymplectic structure on $T(H^3 \times I)$. In this case, the map $(g')^\flat$ associated to the metric g' is given, in matrix form, by $(g')^\flat = g^\flat D_{1,1,-1,-1}$. By Theorem 3.3, the quadruple (g', N_1', N_2', N_3') is a para-hyperkähler structure on $T(H^3 \times I)$.

4. ε-hypersymplectic structures on the dual of a Lie algebroid

In this section we show that an ε-hypersymplectic structure on a Lie algebroid A equipped with a Poisson bivector determines an ε-hypersymplectic structure on the dual A^* of A.

Let $(A, \rho, [.,.])$ be a Lie algebroid and $\pi \in \Gamma(\wedge^2 A)$ a Poisson bivector. Let us consider the bracket $[.,.]_\pi$ on the space of sections $\Gamma(A^*)$, given by

$$[\alpha, \beta]_\pi := \mathcal{L}_{\pi^\#(\alpha)}\beta - \mathcal{L}_{\pi^\#(\beta)}\alpha - \mathrm{d}(\pi(\alpha, \beta)),$$

where d stands for the differential of the Lie algebroid A and $\mathcal{L}$ is the Lie derivative on A. It is well known [4] that the pair $(\rho \circ \pi^\#, [.,.]_\pi)$ is a Lie algebroid structure on A^*. We denote this Lie algebroid by A^*_π. Its differential is the Lichnerowicz differential $\mathrm{d}_\pi = [\pi, .]$.

Proposition 4.1. *Let $(\omega_1, \omega_2, \omega_3)$ be an ε-hypersymplectic structure on a Lie algebroid A and π a Poisson tensor on A, which is compatible with π_i, the inverse of ω_i, for $i \in \{1, 2, 3\}$. Then, (π_1, π_2, π_3) is an ε-hypersymplectic structure on A^*_π.*

Proof: First notice that, since the vector bundles A and $(A^*)^*$ can be identified, each π_i can be viewed as a non-degenerate 2-form on the Lie algebroid

A^*_π. Moreover, because π is compatible with π_i, i.e., $[\pi, \pi_i] = 0$, we have

$$\mathrm{d}_\pi(\pi_i) = [\pi, \pi_i] = 0,$$

and π_i is a symplectic form on A^*_π, for all $i \in \{1, 2, 3\}$. According to (1), the transition tensors are $\mathfrak{N}_i = \omega^\flat_{i-1} \circ \pi^\#_{i+1}$. Using (2), we get

$$\mathfrak{N}_i = \varepsilon_i \omega^\flat_{i+1} \circ \pi^\#_{i-1} = \varepsilon_i \, N_i{}^*, \tag{10}$$

so that $(\mathfrak{N}_i)^2 = (N_i{}^*)^2 = \varepsilon_i \, \mathrm{Id}_{A^*}$. ∎

Now, we want to make the following interesting observation. From [1], we know that if $(\omega_1, \omega_2, \omega_3)$ is an $\boldsymbol{\varepsilon}$-hypersymplectic structure on a Lie algebroid A, then the pairs (π_i, N_j), $i, j \in \{1, 2, 3\}$, $i \neq j$, are Poisson-Nijenhuis structures on A. Therefore, the dual N^*_j of each transition tensor is a Nijenhuis tensor on the Lie algebroid $A^*_{\pi_i}$ [3]. However, under the conditions of Proposition 4.1, we get, from (10), that each N^*_j is also Nijenhuis on the Lie algebroid A^*_π, although the pair (π, N_j) is not, in general, a Poisson-Nijenhuis structure on A.

We recall from [1] that if $(\omega_1, \omega_2, \omega_3)$ is an $\boldsymbol{\varepsilon}$-hypersymplectic structure on a Lie algebroid A, then the Poisson bivectors π_i are pairwise compatible, i.e., $[\pi_i, \pi_j] = 0$, for all $i, j \in \{1, 2, 3\}$. Having this in mind, it is obvious that Proposition 4.1 admits the following corollary:

Corollary 4.2. *If $(\omega_1, \omega_2, \omega_3)$ is an $\boldsymbol{\varepsilon}$-hypersymplectic structure on A, then (π_1, π_2, π_3) is an $\boldsymbol{\varepsilon}$-hypersymplectic structure on $A^*_{\pi_i}$, for all $i \in \{1, 2, 3\}$.*[*]

In other words, every $\boldsymbol{\varepsilon}$-hypersymplectic structure on a Lie algebroid automatically induces an $\boldsymbol{\varepsilon}$-hypersymplectic structure on its dual.

The map $\mathfrak{g}^\flat$ associated to the $\boldsymbol{\varepsilon}$-hypersymplectic structure (π_1, π_2, π_3) on A^*_π obtained from Proposition 4.1, is given by $\mathfrak{g}^\flat = \varepsilon_{i-1}\varepsilon_{i+1} \, \pi_{i-1}{}^\sharp \circ \omega_i{}^\flat \circ \pi_{i+1}{}^\sharp$, with $i \in \mathbb{Z}_3$ (see (5)). We also have

$$\mathfrak{g}^\flat = \varepsilon_1 \varepsilon_2 \varepsilon_3 \, (g^\flat)^{-1},$$

where $g^\flat$ is given by (4).

In the case where $\varepsilon_1 \varepsilon_2 \varepsilon_3 = -1$, Theorem 3.3 asserts that $(\mathfrak{g}, \mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{N}_3)$ is a (para-)hyperkähler structure on A^*_π. Since $\mathfrak{g} = -g^{-1}$, with g^{-1} given by (8), and $\mathfrak{N}_i = \varepsilon_i \, N_i{}^*$, $i = 1, 2, 3$ (see (10)), we realize that every (para-)hyperkähler structure (g, N_1, N_2, N_3) on A induces the (para-)hyperkähler structure $(\mathfrak{g}, \mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{N}_3)$ on A^*_π. From what we have seen so far, we may

[*]Notice that we may replace $A^*_{\pi_i}$ by $A^*_{\pi_i + \pi_j}$, $i, j \in \{1, 2, 3\}$, or by $A^*_{\pi_1 + \pi_2 + \pi_3}$.

conclude that a (para-)hypersymplectic structure on A induces a (para-)hypersymplectic structure and a (para-)hyperkähler structure on A^*_π.

Next, we construct the hypersymplectic structure and the hyperkähler structure on $T^*(H^3 \times I)$ induced by the structures of Example 3.4. The Lie algebroid structure on $T^*(H^3 \times I)$ is defined by π_1, π_2 or π_3, or even by a sum of some of them.

Example 4.3. The hypersymplectic structure $(\omega_1, \omega_2, \omega_3)$ on $T(H^3 \times I)$ considered in Example 3.4 determines the hypersymplectic structure (π_1, π_2, π_3) on $T^*(H^3 \times I)$, with

$$\pi_1{}^\sharp = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{t} \\ -1 & 0 & 0 & 0 \\ 0 & \frac{1}{t} & 0 & 0 \end{bmatrix}, \; \pi_2{}^\sharp = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{t} \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -\frac{1}{t} & 0 & 0 & 0 \end{bmatrix}, \; \pi_3{}^\sharp = \begin{bmatrix} 0 & \frac{1}{t} & 0 & 0 \\ -\frac{1}{t} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

The transition tensors $\mathfrak{N}_i$, given by (10), correspond to the following matrices:

$$\mathfrak{N}_i = -N_i{}^T, \quad i = 1, 2, 3,$$

where $N_i{}^T$ denotes the transpose of the matrix N_i in Example 3.4. The quadruple $(\mathfrak{g}, \mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{N}_3)$ is a hyperkähler structure on $T^*(H^3 \times I)$, where the metric $\mathfrak{g}$ has its associated morphism $\mathfrak{g}^\flat$ given, in matrix form, by

$$\mathfrak{g}^\flat = \begin{bmatrix} -\frac{1}{t} & 0 & 0 & 0 \\ 0 & -\frac{1}{t} & 0 & 0 \\ 0 & 0 & -t & 0 \\ 0 & 0 & 0 & -\frac{1}{t} \end{bmatrix}.$$

When A is a Lie algebroid equipped with a Poisson bivector π, the pair (A, A^*_π) is a triangular Lie bialgebroid [4]. So we have shown how to construct a hypersymplectic and hyperkähler structure on this type of Lie bialgebroids. This can be considered as a first step towards the definition of hypersymplectic and hyperkähler structures on a general Lie bialgebroid. We shall treat this problem in a coming paper.

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