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DOMINANT SHI REGIONS WITH A FIXED SEPARATING WALL: BIJECTIVE ENUMERATION

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Abstract: We present a purely combinatorial proof by the design of an explicit bijection, of the exact number of dominant regions having as a separating wall the hyperplane associated to the longest root in the m–extended Shi hyperplane arrangement of type A and dimension $n-1$.

AMS Subject Classification (2000): 05A17, 05A19, 32S22, 52C35.

1. Overview

Presently, a very active research area is the investigation of the combinatorics of hyperplane arrangements, see e.g. [18, 25] for an introduction on the topic. In particular, it is well know, see [1, 2, 6, 20, 21, 22, 23, 24], that there is a useful and efficient way to encode dominant regions and dominant alcoves of the m–extended Shi arrangement in the affine space of type A with dimension $n-1$, based on some Young tableaux of a staircase partition whose filling must obey to several arithmetic conditions, which are called Shi tableaux. Incidentally, we remark that for the m–extended Shi arrangements of type B, viz. in the affine space spanned by the generators of a type B group, very little is known, see [11, 13, 17].

Shi tableaux together with another combinatorial object called *abacus*, see e.g. [12, 26], proved to be very a powerful tool to investigate hyperplane arrangements, but since they are quite not very easy to handle, because of the arithmetic constraints on the filling, recently some bijections were designed between some Shi tableaux, for instance the one associated to some specific regions, and other families of tableaux which are much easier to work on, for instance cores, see [4, 9, 10, 14, 15, 19], and m–staggered staircases, see [7, 8].

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Nevertheless, unfortunately at the moment it is often impossible to prove enumerative results on hyperplane arrangements in a purely bijective way. Usually in order to count selected families of regions of the m–extended Shi arrangement, one uses the bijection between these regions and an associated family of Young tableaux, but then in order to enumerate the corresponding class of Young tableaux some algebraic non–constructive method is frequently used, namely abstract results connected with representation theory.

In particular, in [8, Corollary 4.5] it is proved that there are exactly m^{n-2} dominant regions having as a separating wall the hyperplane associated to the longest root, but the enumeration of the Young tableaux associated to such regions is made in a very abstract non–constructive way combining the use of generating functions with some deep results from Algebraic Geometry.

In this brief note we present a purely combinatorial proof of the aforementioned result designing an explicit bijection between the involved Young tableaux and a class of words over a finite alphabet having a fixed length.

We remark that the only bijective proofs known so far, viz. [3, 5, 16], deal with the whole set of dominant regions without further constraints, therefore as far as we are aware this is the first bijective result involving dominant regions with additional interesting properties.

2. Notation

We follow closely the notation used in [7, 8]. For more comprehensive references we refer to [1, 2, 25].

Let $\{\varepsilon_1, \ldots, \varepsilon_n\}$ be the standard basis of $\mathbb{R}^n$ and $\langle , \rangle$ be the bilinear form for which this is an orthonormal basis. Let $\alpha_j = \varepsilon_j - \varepsilon_{j+1}$. Then $\Pi = \{\alpha_1, \ldots, \alpha_{n-1}\}$ is a basis of

$$V = \{(a_1, \ldots, a_n) \in \mathbb{R}^n \mid \sum_{j=1}^{n} a_j = 0\}.$$

For $i \leq j$, we define $\alpha_{i,j} = \alpha_i + \ldots + \alpha_j$, so in this notation we have that $\alpha_i = \alpha_{i,i} = \varepsilon_i - \varepsilon_{i+1}$.

The elements of $\Delta = \{\varepsilon_i - \varepsilon_j \mid i \neq j\}$ are called *roots*, and a root α is defined to be *positive*, written $\alpha > 0$, if $\alpha \in \Delta^+ = \{\varepsilon_i - \varepsilon_j \mid i < j\}$. We let $\Delta^- = -\Delta^+$ and write $\alpha < 0$ if $\alpha \in \Delta^-$. Then Π is the set of simple roots, and the highest root is $\theta = \alpha_{1,n-1} = \sum_{j=1}^{n-1} \alpha_j$.

For each $\alpha \in \Delta$, $k \in \mathbb{Z}$ we define an affine reflecting hyperplane

$$H_{\alpha,k} = \{v \in V \mid \langle v, \alpha \rangle = k\}.$$

Note that $H_{-\alpha,-k} = H_{\alpha,k}$, so without loss of generality we might consider $k \in \mathbb{N}$.

The *m–extended Shi hyperplane arrangement* is defined as the collection of hyperplanes

$$\mathcal{H}_m = \{H_{\alpha,k} \mid \alpha \in \Delta^+, -m < k \leq m\}.$$

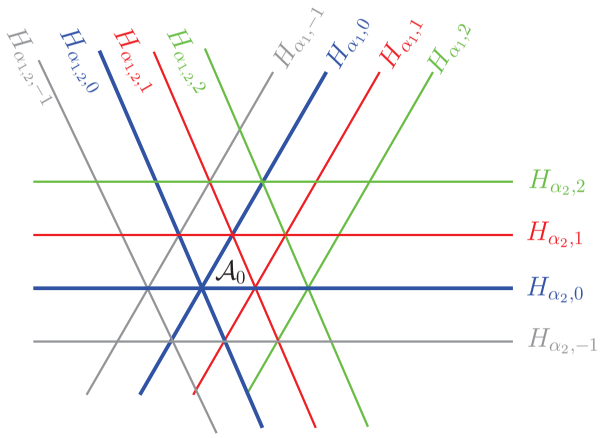
Note that this arrangement can actually be defined for all crystallographic root systems of finite type, but here we are concerned only with type A.

The connected components of the hyperplane arrangement complement $V \setminus \bigcup_{H \in \mathcal{H}m} H$ are called *regions* of the m–extended Shi hyperplane arrangement.

Denote the closed half–spaces $\{v \in V \mid \langle v, \alpha \rangle \geq k\}$ and $\{v \in V \mid \langle v, \alpha \rangle \leq k\}$ by $H^+_{\alpha,k}$ and $H^-_{\alpha,k}$, respectively; then the *dominant* (or *fundamental*) chamber of V is $\bigcap_{i=1}^{n-1} H^+_{\alpha_i,0}$. A *dominant region* of the m–extended Shi hyperplane arrangement is a region contained in the dominant chamber.

Each connected component of $V \setminus \bigcup_{\substack{\alpha \in \Delta^+ \\ k \in \mathbb{Z}}} H_{\alpha,k}$ is called an *alcove*, and the *fundamental alcove* is $\mathcal{A}_0$, the interior of $H^-_{\theta,1} \cap \bigcap_{i=1}^{n-1} H^+_{\alpha_i,0}$. A *dominant alcove* is an alcove contained in the dominant chamber.

The following example shows $\mathcal{H}_2$, with $n = 3$.



Given n, m positive integers, there is a known way, see e.g. [2, 6, 8], to represent a dominant region of m–Shi hyperplane arrangement in $\mathbb{R}^n$ as a Shi tableau, i.e. the Youngs diagram (using the English notation) of a staircase partition $\{(n-j)_{j=1}^{n-1}\}$, with a special filling, i.e. the value $k_{i,j}$ associated to the cell in the row i and column $n-j$ must satisfy some special requirements. Namely, the entries $k_{i,j}$ are nonnegative integers not greater than m, they are non-increasing along rows and columns, and for any $i \leq t < j$ the following conditions must hold: if $k_{i,j} = m$ then $k_{i,t} + k_{t+1,j} \geq m-1$, whereas if $k_{i,j} \leq m-1$ then either

$$k_{i,j} = k_{i,t} + k_{t+1,j} \tag{1}$$

or

$$k_{i,j} = k_{i,t} + k_{t+1,j} + 1. \tag{2}$$

We present as an example the Shi tableaux for $n = 5$, where the filling must satisfy the conditions described above.

$k_{1,4}$	$k_{1,3}$	$k_{1,2}$	$k_{1,1}$
$k_{2,4}$	$k_{2,3}$	$k_{2,2}$	
$k_{3,4}$	$k_{3,3}$		
$k_{4,4}$			

For instance, this means that given m if $k_{1,4} = m$ then $\min\{k_{1,1}+k_{2,4}, k_{1,2}+k_{3,4}, k_{1,3}+k_{4,4}\} \geq m-1$ whereas if $k_{1,4} \leq m-1$ then

$$k_{1,4} = k_{1,1} + k_{2,4} + b_1 = k_{1,2} + k_{3,4} + b_2 = k_{1,3} + k_{4,4} + b_3,$$

where $b_1, b_2, b_3 \in \{0, 1\}$.

So an explicit example of a m–Shi tableau for $n = 5$ and any $m \geq 4$ is

4	3	2	2
2	1	0	
1	1		
0			

.

Very sketchily, given a dominant region R in the m–Shi arrangement, $k_{i,j}$ is the minimum number of integer translates of $H_{\alpha_{i,j},0}$ that separates R from

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For the sake of clarity, we present an example of such bijection: given the word $w = 61513 \in \{0, 1, 2, 3, 4, 5, 6\}^*$, its corresponding 7–Shi tableau is

7	6	5	3	1	1
5	5	4	1	0	
5	5	3	1		
3	3	2			
1	1				
0					

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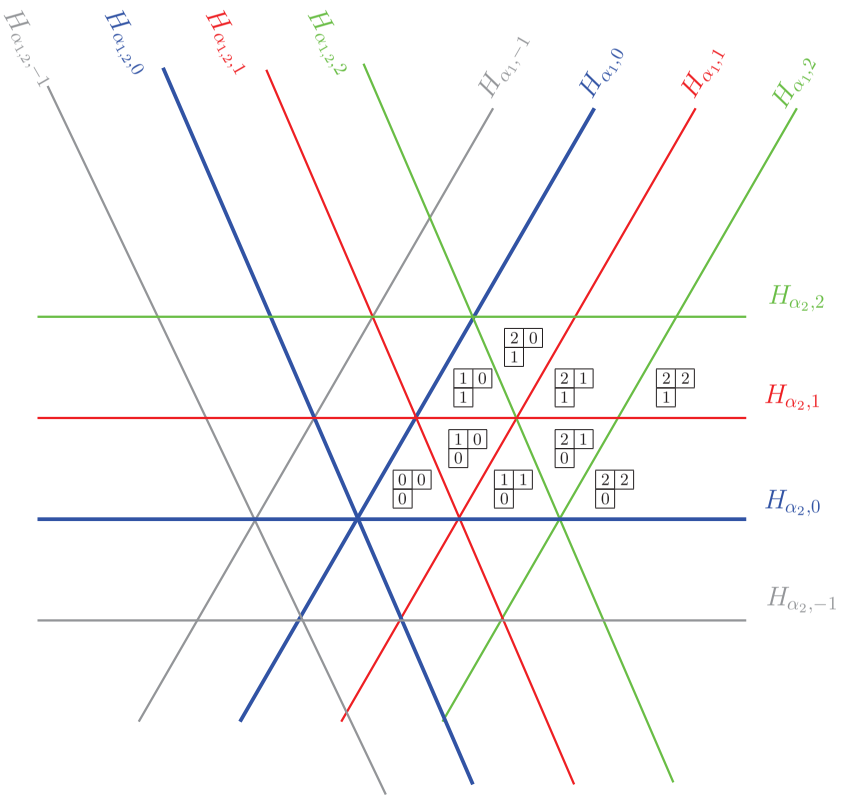
the origin (or equivalently from $\mathcal{A}_0$), whereas given a m–Shi tableau, the corresponding region R consists of those points x such that

$$\begin{cases} k_{i,j} < \langle \alpha_{i,j}, x \rangle < k_{i,j} + 1 & \text{if } 0 \leq k_{i,j} < m, \\ k_{i,j} < \langle \alpha_{i,j}, x \rangle & \text{if } k_{i,j} = m, \end{cases}$$

for all $1 \leq i \leq j \leq n$.

We show some examples in $\mathcal{H}_2$ with $n = 3$.



Two open regions are *separated* by a hyperplane H if they lie in different closed half-spaces relative to H, and a *wall* of a region is a hyperplane in $\mathcal{H}_m$

which supports a facet of that region. A *separating wall* for a region R is a wall of R which separates R from the fundamental alcove $\mathcal{A}_0$.

Hence, for instance, there are two dominant regions having $H_{\alpha_{1,2},2}$ as a separating wall, namely the regions associated to the Young tableaux

$$\begin{array}{|c|c|}\hline 2&0\\\hline 1\\\cline{1-1}\end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline 2&1\\\hline 0\\\cline{1-1}\end{array},$$

and three dominant regions having $H_{\alpha_2,1}$ as a separating wall, namely the regions associated to the Young tableaux

$$\begin{array}{|c|c|}\hline 1&0\\\hline 1\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 2&1\\\hline 1\\\cline{1-1}\end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline 2&2\\\hline 1\\\cline{1-1}\end{array}.$$

3. Main result

We give a purely combinatorial proof of the following result.

Theorem 3.1. *There are exactly m^{n-2} dominant regions having $H_{\theta,m}$ as a separating wall.*

Proof: It is known, see e.g. [7, 8], that a dominant region admits $H_{\theta,m}$ as a separating wall if and only if its corresponding Shi tableau has the following property:

$$\begin{cases} k_{1,n-1} = m, \\ k_{1,t} + k_{t+1,n-1} = m-1 \qquad \text{for all } 1 \le t \le n-2. \end{cases} \tag{3}$$

We design a bijection between these Shi tableaux and the set $W_{n,m}$ of all words on an alphabet with m symbols having length $n-2$, i.e. $n-2$ letters. Without loss of generality we consider the set $\{0, 1, \dots, m-1\}$ to be the alphabet, so there is a natural total order on it.

We describe the map which associate to each such word a suitable Shi tableau, i.e. we give a procedure to univocally and bijectively determine the filling of the tableau from the given word.

Given a word $w = w_1 \cdots w_{n-2}$ let $\widetilde{w} = \widetilde{w}_1 \cdots \widetilde{w}_{n-2}$ be its reordering, i.e. a permutation of the letters such that $\widetilde{w}_j < \widetilde{w}_{j+1}$ for all $j = 1, \dots, n-3$. Then for all $j = 1, \dots, n-2$ we set

$$k_{1,j} = \widetilde{w}_j$$

and

$$k_{j+1,n-1} = m - 1 - k_{1,j} = m - 1 - \widetilde{w}_j.$$

In other words, we are filling the tableau in the following way: according to the property (3) we set $k_{1,n-1} = m$, and the remaining cells of the first row are filled with the decreasing reordering of w. Then we fill the first column of the tableau according to the property (3).
Therefore, since now property (3) is completely fulfilled, in order to prove the Theorem it is enough to fill the remaining cells of the tableau in such a way that the entries are nonnegative integers strictly less than m and non-increasing along rows and columns, each of them satisfies either condition (1) or condition (2), and such that the whole operation is invertible.

For all $t = 2, \dots, n-2$ and all $j = t, \dots, n-2$ we set

$$k_{t,j} = \begin{cases} k_{1,j} - k_{1,t-1} & \text{if } w_{t-1} \le w_j \\ k_{1,j} - k_{1,t-1} - 1 & \text{if } w_{t-1} > w_j; \end{cases}$$

in other words, we define

$$k_{t,j} = k_{1,j} - k_{1,t-1} - b_{t,j},$$

where

$$b_{t,j} = \begin{cases} 0 & \text{if } w_{t-1} \le w_j \\ 1 & \text{if } w_{t-1} > w_j, \end{cases}$$

and therefore the numbers $b_{t,j}$ encode the inversion table of the word $w = w_1 \cdots w_{n-2}$.
We remark that each entry $k_{t,j}$ of this filling is a nonnegative integer strictly less than m and satisfies either condition (1) or condition (2) with $i = 1$ (and with a rescaled t, i.e. shifted by 1).

It is not hard to check that this filling produce a Shi tableau: all entries are nonnegative integers which are non-increasing along rows and columns, and each entry, save $k_{1,n-1} = m$, is strictly less than m and satisfies either condition (1) or condition (2) for all $i = 1, \dots, n-1$. Moreover this operation is invertible, viz. that there is a unique and bijective way to recover the word given the Shi tableau, since the Shi tableau encodes both a sorted word and its inversion table. ∎