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MONOIDS AND POINTED S-PROTOMODULAR CATEGORIES

DOMINIQUE BOURN, NELSON MARTINS-FERREIRA, ANDREA MONTOLI AND MANUELA SOBRAL

Abstract: We investigate the notion of pointed S-protomodular category, with respect to a suitable class S of split epimorphisms, and we prove that these categories satisfy, relatively to the class S, many partial aspects of the properties of Mal'tsev and protomodular categories, like the split short five lemma for the S-split exact sequences, or the fact that a reflexive S-relation is transitive. The main examples of S-protomodular categories are the category of monoids and, more generally, any category of monoids with operations, where the class S is the class of Schreier split epimorphisms.

Keywords: fibration of points, Mal'tsev and protomodular categories, monoids with operations, Schreier split epimorphisms, pointed S-protomodular categories.
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1. Introduction

The notion of semi-abelian category [10] allowed to describe intrinsically many classical properties and results in group theory (see, for example, [1]), and to point out the similarities with other algebraic structures, like rings, associative algebras, Lie algebras and many others.

From a categorical point of view, much less is known for other algebraic structures, like monoids. However, as Mac Lane observed in the preface of [11], the notion of monoid is fundamental in category theory. Until now, the most important categorical property of monoids that has been pointed out is unitality [2]; this property allows to describe the algebraic notion of commutativity of subobjects and, more generally, of morphisms.

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In the recent paper [12], the three last authors introduced the algebraic context of *monoids with operations*, inspired by the analogous notion, introduced by Porter [14], of groups with operations. This new context includes, among other examples, monoids, commutative monoids, semirings, join-semilattices with a bottom element and distributive lattices with a bottom element (or a top one). The study of the semidirect products in this setting allowed to identify a class of split epimorphisms, called *Schreier split epimorphisms* (the name is inspired by the notion of Schreier internal category introduced by Patchkoria [13] in the category of monoids). Schreier split epimorphisms correspond to actions via the semidirect product construction, as it is proved in [12].

In the monograph [7], and in the paper [6], the present authors observed moreover that, in the case of monoids, Schreier split epimorphisms satisfy some important properties that are classically known to be satisfied by all split epimorphisms of groups (but not by all split epimorphisms of monoids), like the split short five lemma. Defining a *Schreier reflexive relation* as a reflexive relation such that the split epimorphism, given by the first projection and the reflexivity morphism, is a Schreier one, it was proved that any Schreier reflexive relation is transitive. Moreover, many others interesting properties of Schreier split epimorphisms of monoids were studied, and they were extended to the case of semirings. In particular, it was shown that special Schreier extensions with a fixed abelian kernel form an abelian group, as it happens for all extensions with abelian kernel in the category of groups.

All these results gave evidence of the need of a conceptual notion which captures this algebraic context; it was introduced, in the pointed case, in [7], under the name of *S-protomodularity*, where S is a suitable class of split epimorphisms. The aim of the present paper is to investigate the properties of this intrinsic setting and to show how it conceptually allows to recover many partial aspects of the properties of Mal'tsev [8] and protomodular [1] categories. By [7], the main examples of such a situation are the category Mon of monoids and $SRng$ of semirings with the class S of Schreier split epimorphisms. We show here that the same is true for the general context of monoids with operations.

Nelson Martins-Ferreira
Departamento de Matemática, Escola Superior de Tecnologia e Gestão, Centro para o Desenvolvimento Rápido e Sustentado do Produto, Instituto Politécnico de Leiria, Leiria, Portugal
E-mail address: martins.ferreira@ipleiria.pt

Andrea Montoli
CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal
E-mail address: montoli@mat.uc.pt

Manuela Sobral
CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal
E-mail address: sobral@mat.uc.pt

To prove the implication $3 \Rightarrow 1$, suppose that we have $[R[d_0], R[d_1]] = 0$. As we observed in Section 5, in order to equip our S-reflexive graph with a structure of internal category, we only need to give the composition map $d_1 \colon X_2 \to X_1$ satisfying the equalities $d_1 s_0 = d_1 s_1 = 1_{X_1}$. The map d_1 can be defined as in the case of Mal'tsev categories (Theorem 3.6 in [9]). In set-theoretical terms, d_1 is given by:

$$d_1(\alpha, \beta) = p(\beta R[d_0] 1_{d_1(\alpha)} R[d_1]\alpha).$$

It is easy to verify that it satisfies the desired equalities. ∎

References

[1] F. Borceux, D. Bourn, *Mal'cev, protomodular, homological and semi-abelian categories*, Mathematics and its applications, vol. 566 (2004), Kluwer.

[2] D. Bourn, *Mal'cev categories and fibration of pointed objects*, Appl. Categorical Structures 4 (1996), 307-327.

[3] D. Bourn, *Intrinsic centrality and associated classifying properties*, J. Algebra 256 (2002), 126-145.

[4] D. Bourn, *On the monad of internal groupoids*, Theory Appl. Categories 28 (2013), 150-165.

[5] D. Bourn, M. Gran *Centrality and connectors in Maltsev categories*, Algebra Universalis 48 (2002), 309-331.

[6] D. Bourn, N. Martins-Ferreira, A. Montoli, M. Sobral, *Schreier split epimorphisms between monoids*, Semigroup Forum, in press, preprint DMUC 13-41.

[7] D. Bourn, N. Martins-Ferreira, A. Montoli, M. Sobral, *Schreier split epimorphisms in monoids and in semirings*, Textos de Matemática (Série B), Departamento de Matemática da Universidade de Coimbra, vol. 45 (2014).

[8] A. Carboni, J. Lambek. M.C. Pedicchio, *Diagram chasing in Mal'cev categories*, J. Pure Appl. Algebra 69 (1990), 271-284.

[9] A. Carboni, M.C. Pedicchio, N. Pirovano, *Internal graphs and internal groupoids in Mal'cev categories*, CMS Conference Proceedings 13, Category Theory 1991, (1992), 97-109.

[10] G. Janelidze, L. Márki, W. Tholen, *Semi-abelian categories*, J. Pure Appl. Algebra 168 (2002), no. 2-3, 367-386.

[11] S. Mac Lane, *Categories for the working mathematician*, second edition, Graduate Texts in Mathematics, vol. 5 (1998), Springer.

[12] N. Martins-Ferreira, A. Montoli, M. Sobral, *Semidirect products and crossed modules in monoids with operations*, J. Pure Appl. Algebra 217 (2013), 334-347.

[13] A. Patchkoria, *Crossed semimodules and Schreier internal categories in the category of monoids*, Georgian Math. Journal 5 n.6 (1998), 575-581.

[14] T. Porter, *Extensions, crossed modules and internal categories in categories of groups with operations*, Proceedings of the Edinburgh Math. Society 30 (1987), 373-381.

Dominique Bourn
Laboratoire de Mathématiques pures et appliquées, Université du Littoral Côte d'Opale, Calais, France

E-mail address: bourn@lmpa.univ-littoral.fr

Among other things, the definition of S-special morphism (see Section 6) allows to associate with any S-protomodular category $\mathbb{C}$ a protomodular subcategory $S^\sharp\mathbb{C}$, called the *protomodular core* of $\mathbb{C}$ relatively to S. If $\mathbb{C}$ is the category of monoids, equipped with the class of Schreier split epimorphisms, its protomodular core is the category of groups. This gives then a characterization of groups among monoids. In the same way, we prove that the protomodular core of the category of semirings is the category of rings, and we generalize this result to any category of monoids with operations. The notion of S-special morphism permits also a characterization of reflexive graphs (relatively to the class S) that are internal groupoids. This characterization is completely analogous to the one known for Mal'tsev categories (see [9]).

The paper is organized as follows. In Section 2 we recall from [2] the notion of unital category, and from [7] a generalization of it, namely the notion of $\mathbb{C}'$-unital category, which will be used later to describe some Mal'tsev-type properties of S-protomodular categories. In Section 3 we define S-protomodular categories and we study their first properties. In Section 4 we recall the notion of monoids with operations and of the class S of Schreier split epimorphisms, and we show that they are examples of S-protomodular categories. In Section 5 we prove that a S-reflexive graph has at most one structure of internal category, and that any S-reflexive relation is transitive, relating S-protomodular categories with Mal'tsev ones. In Section 6 we define S-special morphisms and we use them to characterize internal groupoids among internal S-categories and equivalence relations among S-reflexive relations. Moreover, we define the protomodular core of a S-protomodular category, and describe it in the examples of monoids with operations. In Section 7 we describe other Mal'tsev aspects of S-protomodular categories, mainly related with the centrality for reflexive relations. Moreover, we show that a S-reflexive graph such that the domain morphism is S-special is an internal groupoid if and only if the kernel pairs of the domain and the codomain centralize each other.

2. Unital and $\mathbb{C}'$-unital categories

We start by recalling from [2] the following definition.

Definition 2.1. *Let $\mathbb{C}$ be a pointed category with finite products. Given two objects A and B in $\mathbb{C}$, consider the following diagram*

$$A \underset{\langle 1_A, 0\rangle}{\overset{\pi_A}{\rightleftarrows}} A \times B \underset{\langle 0, 1_B\rangle}{\overset{\pi_B}{\rightleftarrows}} B. \tag{1}$$

The category $\mathbb{C}$ is said to be unital if, for every pair of objects $A, B \in \mathbb{C}$, the morphisms $\langle 1_A, 0\rangle$ and $\langle 0, 1_B\rangle$ are jointly strongly epimorphic.

If moreover $\mathbb{C}$ is finitely complete, then any pair $(\langle 1_A, 0\rangle, \langle 0, 1_B\rangle)$ is jointly epimorphic. Hence finitely complete unital categories are a setting where it is possible to express a categorical notion of commutativity.

Definition 2.2 ([3])**.** *Let $\mathbb{C}$ be a finitely complete unital category. Two morphisms with the same codomain $f\colon X \to Z$ and $g\colon Y \to Z$ are said to cooperate (or to commute) if there exists a morphism $\varphi\colon X \times Y \to Z$ such that both triangles in the following diagram commute:*

$$\begin{array}{ccccc} X & \xrightarrow{\langle 1_X, 0\rangle} & X \times Y & \xleftarrow{\langle 0, 1_Y\rangle} & Y \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \varphi} & \swarrow{\scriptstyle g} & \\ & & Z. & & \end{array}$$

The morphism φ is necessarily unique, because $\langle 1_X, 0\rangle$ and $\langle 0, 1_Y\rangle$ are jointly epimorphic, and it is called the cooperator *of f and g.*

The uniqueness of the cooperator makes commutativity a property and not an additional structure in the category $\mathbb{C}$.

Definition 2.3 ([3])**.** *An object A of a finitely complete unital category $\mathbb{C}$ is said to be* commutative *if the identity 1_A cooperates with itself.*

A generalization of the notion of unital category, that we shall need later on, is given by the following definition, that we recall from [7].

Definition 2.4. *Let $\mathbb{C}'$ be a full subcategory of any pointed category $\mathbb{C}$ with finite products. The category $\mathbb{C}$ is said to be* $\mathbb{C}'$-unital *when, for any object $A \in \mathbb{C}'$ and any object $B \in \mathbb{C}$, the morphisms $\langle 1_A, 0\rangle$ and $\langle 0, 1_B\rangle$ in the following diagram are jointly strongly epimorphic:*

$$A \underset{\langle 1_A, 0\rangle}{\overset{\pi_A}{\rightleftarrows}} A \times B \underset{\langle 0, 1_B\rangle}{\overset{\pi_B}{\rightleftarrows}} B. \tag{2}$$

$[R, W] = 0$. Indeed, given a double centralizing relation

$$\begin{array}{ccc} C & \underset{p_0}{\overset{p_1}{\rightleftarrows}} & W \\ {\scriptstyle p_0}\downarrow\uparrow\downarrow\uparrow{\scriptstyle p_1} & & {\scriptstyle d_0^W}\downarrow\uparrow\downarrow\uparrow{\scriptstyle d_1^W} \\ R & \underset{d_0^R}{\overset{d_1^R}{\rightleftarrows}} & X \end{array}$$

i.e. a reflexive relation C both on R and W such that the square $d_1^R p_0 = d_0^W p_1$ is a pullback, the morphism $d_1^R p_0\colon C \to X$ is the (necessarily unique) connector.

We can now prove the following

Proposition 7.5. *Consider a reflexive graph such that d_0 is S-special:*

$$X_1 \underset{d_1}{\overset{d_0}{\underset{\leftarrow s_0}{\rightrightarrows}}} X_0$$

The following conditions are equivalent:

1. *the graph is underlying a S-category;*
2. *the graph is underlying a S-groupoid;*
3. *the kernel equivalence relations $R[d_0]$ and $R[d_1]$ centralize each other.*

Proof: Since d_0 is S-special, the graph is a S-reflexive graph. Moreover $R[d_0]$ is a S-equivalence relation, and we can talk about centralization of it with any reflexive relation on X_1.

The equivalence between conditions 1 and 2 was already proved (see Proposition 6.6).

To prove the implication $2 \Rightarrow 3$ consider the following diagram:

$$\begin{array}{ccccc} R[d_2] & \underset{p_0}{\overset{p_1}{\rightrightarrows}} & R[d_0] & \xrightarrow{d_2} & X_1 \\ {\scriptstyle R(d_0)}\downarrow\uparrow & & {\scriptstyle d_0}\downarrow\uparrow & & {\scriptstyle d_0}\downarrow\uparrow{\scriptstyle s_0} \\ R[d_1] & \underset{p_0}{\overset{p_1}{\rightrightarrows}} & X_1 & \xrightarrow[d_1]{} & X_0 \end{array}$$

As we observed in Section 6, the right hand side square is a pullback, and hence the left hand side part of the diagram gives a double centralizing relation, which says that $[R[d_0], R[d_1]] = 0$.

(σ_0^R, σ_0^W) is jointly strongly epimorphic, the morphism j is an isomorphism, and for every $xRyWz \in R \times_X W$ we have $xWp(xRyWz)$.

We have a similar result concerning the subobject $V \rightarrowtail R \times_X W$ defined by the following pullback:

$$\begin{array}{ccc} V & \xrightarrow{\ j\ } & R \times_X W \\ \downarrow & & \downarrow{\scriptstyle (p,\, d_1^W p_1^R)} \\ R & \xrightarrow[(d_0^R,\, d_1^R)]{} & X \times X \end{array}$$

This give us $p(xRyWz)Rz$ for any $xRyWz \in R \times_X W$. ∎

In set-theoretical terms, the previous proposition says that, with any triple $xRyWz$, we can associate a square of related elements:

$$\begin{array}{ccc} x & \overset{W}{\cdots\!\!\rightarrow} & p(x,y,z) \\ {\scriptstyle R}\downarrow & & \vdots{\scriptstyle R} \\ y & \xrightarrow[W]{} & z. \end{array}$$

This says that any connected pair of reflexive relations (R, W) on the object X, where W is a S-reflexive relation, produces the following diagram of reflexive relations in $\mathbb{C}$:

$$\begin{array}{ccc} R \times_X W & \overset{p_1}{\underset{}{\rightleftarrows}} & W \\ {\scriptstyle p_0}\downarrow\uparrow & {\scriptstyle (d_0^R p_0,\, p)\ (p,\, d_1^W p_1)\ d_0^W\ d_1^R} & \downarrow\uparrow{\scriptstyle d_1^W} \\ R & \underset{d_0^R}{\rightleftarrows} & X \end{array}$$

It is called the *centralizing double relation* associated with the connector. When R and W are equivalence relations, all the reflexive relations in this diagram are equivalence relations, and, moreover, any commutative square is a pullback (thanks to Proposition 7.1).

As in the case of Mal'tsev categories (see [5]), in the context of S-protomodular categories the existence of a double centralizing relation between a reflexive relation R and a S-reflexive relation W characterizes the fact that

In a finitely complete $\mathbb{C}'$-unital category we can still speak of cooperating pairs (f, g) of morphisms, provided that the domain X of f belongs to $\mathbb{C}'$. More generally, $X \times Y$ being isomorphic to $Y \times X$, we can speak of cooperating pair as soon as the domain of one of the two maps is in $\mathbb{C}'$. Accordingly, we can still speak of commutative objects in $\mathbb{C}'$.

Proposition 2.5. *Suppose that $\mathbb{C}$ is $\mathbb{C}'$-unital and that $\mathbb{C}'$ is closed under finite products (in particular, it contains the zero object 0). Then $\mathbb{C}'$ is unital.*

Proof: Straightforward. ∎

3. S-protomodular categories

From now on, we will denote by $\mathbb{C}$ a pointed finitely complete category. Let S be a class of split epimorphisms in $\mathbb{C}$ which is stable under pullbacks in the following sense: given a downward pullback

$$\begin{array}{ccc} X' & \xrightarrow{\ g'\ } & X \\ {\scriptstyle f'}\downarrow\uparrow{\scriptstyle s'} & & {\scriptstyle f}\downarrow\uparrow{\scriptstyle s} \\ Y' & \xrightarrow[\ g\ ]{} & Y, \end{array}$$

where the two vertical morphisms are split epimorphisms and the upward square commutes (or, in other terms, the pair (g, g') is a morphism of points), if (f, s) belongs to S, then (f', s') belongs to S, too. Accordingly this class determines a subfibration $\P_{\mathbb{C}}^S$ of the fibration of points $\P_{\mathbb{C}} \colon Pt\mathbb{C} \to \mathbb{C}$. Let us denote by $SPt\mathbb{C}$ the full subcategory of $Pt\mathbb{C}$ whose objects are those which are in S:

$$\begin{array}{ccc} SPt\mathbb{C} & \xrightarrow{\ j\ } & Pt\mathbb{C} \\ & {\scriptstyle \P_{\mathbb{C}}^S}\searrow \quad \swarrow{\scriptstyle \P_{\mathbb{C}}} & \\ & \mathbb{C} & \end{array}$$

Given a split epimorphism $A \underset{f}{\overset{s}{\rightleftarrows}} B$ in $\mathbb{C}$, we say that it is a *strongly split epimorphism* (see [4], and [12], where strongly split epimorphisms were introduced under the name of regular points) if the pair (k, s), where k is a kernel of f, is jointly strongly epimorphic.

Definition 3.1. *The pointed category $\mathbb{C}$ will be said to be* S-protomodular *when:*

(1) *any object in $SPt\mathbb{C}$ is a strongly split epimorphism;*

(2) *$SPt\mathbb{C}$ is closed under finite limits in $Pt\mathbb{C}$ (in particular, it contains the terminal object $0 \rightleftarrows 0$ of $Pt\mathbb{C}$).*

So, S is a class of strongly split epimorphisms. The first part of condition 2 implies that any fiber $SPt_Y\mathbb{C}$ is closed under finite limits in the fiber $Pt_Y\mathbb{C}$ and that any change-of-base functor with respect to $\P^S_{\mathbb{C}}$ is left exact. The fact that $SPt\mathbb{C}$ contains the terminal object implies that the class S contains the isomorphisms (because S is stable under pullbacks, and any isomorphism can be seen as a pullback of the terminal object $0 \rightleftarrows 0$ of $Pt\mathbb{C}$). Hence, any fiber $SPt_Y\mathbb{C}$ is pointed.

Theorem 3.2. *Let $\mathbb{C}$ be a pointed finitely complete category and S a class of split epimorphisms stable under pullbacks. Then:*

(1) *when S satisfies the condition 1 of Definition 3.1, any fiber $Pt_Y\mathbb{C}$ is $SPt_Y\mathbb{C}$-unital;*
(2) *when $\mathbb{C}$ is S-protomodular, any fiber $SPt_Y\mathbb{C}$ is unital;*
(3) *when $\mathbb{C}$ is S-protomodular, any change of base functor with respect to the fibration $\P^S_{\mathbb{C}}$ is conservative.*

Proof: (1) Consider the following left hand side downward pullback of split epimorphisms, where the split epimorphism (f, s) is in the fiber $SPt_Y\mathbb{C}$:

$$\begin{array}{ccc} X' & \underset{g'}{\overset{t'}{\rightleftarrows}} & X \\ {\scriptstyle f'}\downarrow\uparrow{\scriptstyle s'} & & {\scriptstyle f}\downarrow\uparrow{\scriptstyle s} \\ Y' & \underset{g}{\overset{t}{\rightleftarrows}} & Y \end{array} \qquad\qquad K[f] \overset{k}{\rightarrowtail} X \underset{f}{\overset{t'}{\rightleftarrows}} X' \qquad (3)$$

Then (f', s') belongs to $SPt_{Y'}\mathbb{C}$, since $\P^S_{\mathbb{C}}$ is a subfibration of $\P_{\mathbb{C}}$. So the split epimorphism (f', s') is a strongly split epimorphism. On the other hand, the right hand side square is still a pullback, so the map $t'k$ (where k is a kernel of f) is a kernel of f'. Accordingly the pair $(t'k, s')$ is jointly strongly epimorphic. So this is equally the case for the pair (t', s'). Accordingly the fibre $Pt_Y\mathbb{C}$ is $SPt_Y\mathbb{C}$-unital.

(2) This comes immediately from 1, because, as we already observed, if $\mathbb{C}$ is S-protomodular, then $SPt_Y\mathbb{C}$ is closed under finite products in $Pt_Y\mathbb{C}$.

(3) Since any change-of-base functor with respect to $\P^S_{\mathbb{C}}$ is left exact, it is enough to prove that it is conservative on monomorphisms (see

$p\colon R \times_X W \to X$, where $R \times_X W$ is defined by the following pullback:

$$\begin{array}{ccc} R \times_X W & \underset{\sigma_0^R}{\overset{p_1^R}{\rightleftarrows}} & W \\ {\scriptstyle p_0^W}\downarrow\uparrow{\scriptstyle \sigma_0^W} & & {\scriptstyle d_0^W}\downarrow\uparrow{\scriptstyle s_0^W} \\ R & \underset{s_0^R}{\overset{d_1^R}{\rightleftarrows}} & X \end{array}$$

such that $p\sigma_0^R = d_1^W$ and $p\sigma_0^W = d_0^R$. In set-theoretical terms, this means that we have both $p(xRxWy) = y$ and $p(xRyWy) = x$. The morphisms σ_0^R and σ_0^W, defined by the universal property of the pullback, are explicitly given by $\sigma_0^R(yWz) = yRyWz$ and $\sigma_0^W(xRy) = xRyWy$. We denote this situation by $[R, W] = 0$.

Since W is a S-reflexive relation, the split epimorphism (d_0^W, s_0^W) is in S, and consequently the pair (σ_0^R, σ_0^W) is jointly strongly epimorphic. This implies that the connector p is unique.

Example 7.3. *Given the order $\mathcal{O}_{\mathbb{N}}$ on $\mathbb{N}$ in Mon, with the class S of Schreier split epimorphisms, we have that $[\mathcal{O}_{\mathbb{N}}, \mathcal{O}_{\mathbb{N}}] = 0$; in this case, the connector is the morphism p defined by $p(x \leq y \leq z) = z - y + x$.*

When we have $[R, W] = 0$, we recover a well-known result in Mal'tsev categories:

Proposition 7.4. *Let $\mathbb{C}$ be a S-protomodular category. Suppose the reflexive relation R and the S-reflexive relation W on X centralize each other in $\mathbb{C}$. We have necessarily $xWp(xRyWz)$ and $p(xRyWz)Rz$.*

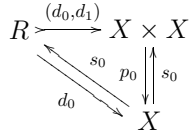
Proof: Let us consider the following pullback:

$$\begin{array}{ccc} U & \rightarrowtail & W \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle (d_0^W, d_1^W)} \\ R \times_X W & \underset{(d_0^R p_0^W, p)}{\rightarrowtail} & X \times X \end{array}$$

It defines U as the subobject of those $xRyWz \in R \times_X W$ such that we have $xWp(xRyWz)$. For any $yWz \in W$, the element $yRyWz \in R \times_X W$ belongs to U, since we have $y = p(yRyWz)$ (as we observed in Definition 7.2). This means that σ_0^R factors through U. In the same way, for any $xRy \in R$, the element $xRyWy \in R \times_X W$ belongs to U, since we have $x = p(xRyWy)$. This means that σ_0^W factors through U. Since the pair

along s_0 is the isomorphism 1_{X_0}. According to Theorem 3.2, the map θ is itself an isomorphism. ∎

7.3. Centrality with respect to S-reflexive relations. More importantly, the Mal'tsev context fits very well to the notion of centrality between equivalence relations. The equivalence relations R on an object X, coinciding with the reflexive relations on X, are just the subobjects of the object $(p_0, s_0)\colon X \times X \rightleftarrows X$ in the fibre $Pt_X\mathbb{C}$:

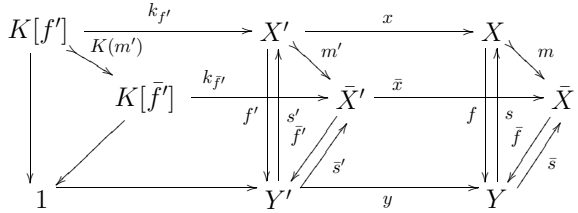


Two equivalence relations R and W on X centralize each other in a Mal'tsev category $\mathbb{C}$ when the subobjects $(d_1^R, d_0^R)\colon R \rightarrowtail X \times X$ and $(d_0^W, d_1^W)\colon W \rightarrowtail X \times X$ commute in the unital fiber $Pt_X\mathbb{C}$. In set-theoretical terms, the cooperator $R \times_X W \to X \times X$ in the fiber is necessarily of the form $\phi(xRyWz) = (x, p(xRyWz))$, with the two equations $p(xRxWy) = y$ and $p(xRyWy) = x$. The morphism $p\colon R \times_X W \to X$ satisfying these two equations, which is characteristic of the fact that R and W centralize each other (see [5]), is called the *connector* between the relations R and W. It is well known that, in the category Gp of groups, two equivalence relations R and W on a group G centralize each other if and only if the normal subgroups $\overline{1}_R$ and $\overline{1}_W$ given by the equivalence classes of the unit element commute inside the group G.

In a S-protomodular category $\mathbb{C}$, since any fiber $Pt_Y\mathbb{C}$ is $SPt_Y\mathbb{C}$-unital, we can keep the same definition provided, now, that, one of the domains, let us choose W, is a S-reflexive relation:

Definition 7.2. *Given a reflexive relation R and a S-reflexive relation W on the same object X in a S-protomodular category $\mathbb{C}$, we say that R and W* centralize each other *when there is a (necessarily unique) morphism*

Lemma 3.3 below). Let us consider the following diagram, where all the quadrangles are pullbacks and all the split epimorphisms are in $SPt\mathbb{C}$:



Suppose moreover that m', and consequently $K(m')$, are isomorphisms. Since $K(m) \simeq K(m')$ is an isomorphism, and the pairs (k_f, s) and $(k_{\bar{f}}, \bar{s})$ are jointly strongly epimorphic, then m is a strong epimorphism, and hence it is an isomorphism.

∎

Lemma 3.3. *Suppose that $U\colon \mathbb{C} \to \mathbb{D}$ is a left exact functor such that, for any monomorphism m in $\mathbb{C}$, if Um is an isomorphism in $\mathbb{D}$ then m is an isomorphism. Then U is conservative.*

Proof: Given any morphism f in $\mathbb{C}$, consider the kernel pair of f:

$$R[f] \underset{p_1}{\overset{p_0}{\underset{\longleftarrow}{\overset{\longrightarrow}{\longleftarrow}}}} X \xrightarrow{f} Y.$$

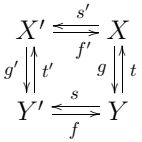
Since U is left exact, we have that $UR[f]$ is the kernel pair of Uf:

$$UR[f] = R[Uf] \underset{Up_1}{\overset{Up_0}{\underset{\longleftarrow}{\overset{\longrightarrow}{\longleftarrow}}}} UX \xrightarrow{Uf} UY.$$

Suppose that Uf is an isomorphism. Then Us_0 is an isomorphism. Since s_0 is a monomorphism, our hypothesis implies that s_0 is an isomorphism. But then f is a monomorphism, hence an isomorphism by our hypothesis. ∎

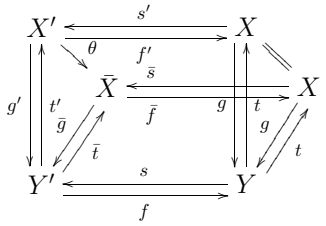
Proposition 3.4. *Let $\mathbb{C}$ be a pointed finitely complete category, and S a class of strongly split epimorphisms which is stable under pullbacks. Given*

any commutative square of split epimorphisms:

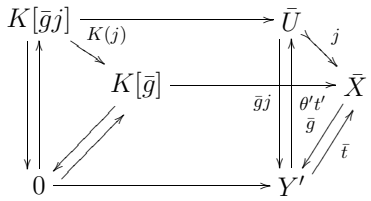


where the split epimorphism (g,t) is in S, the induced factorization to the pullback of (g,t) along f is an extremal epimorphism.

Proof: Consider the following diagram:



where the square $f\bar{g} = g\bar{f}$ is a pullback. Since S is stable under pullbacks, $(\bar{g},\bar{t})$ belongs to S. Moreover, since the category $Pt_Y\mathbb{C}$ is $SPt_Y\mathbb{C}$-unital (by Theorem 3.2) and the pullback considered above is actually the product of the two split epimorphisms $Y' \underset{f}{\overset{s}{\rightleftarrows}} Y$ and $X \underset{g}{\overset{t}{\rightleftarrows}} Y$ in the category $Pt_Y\mathbb{C}$, the pair $(\bar{t},\bar{s})$ is jointly strongly epimorphic. Now let θ be the factorization in question. Suppose $j\colon \bar{U} \rightarrowtail \bar{X}$ is a monomorphism such that θ factors through it by a map θ': $j\theta' = \theta$. Consider the following diagram:



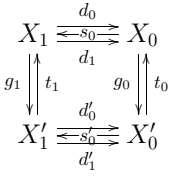
We have that $(\bar{g}j, \theta' t')$ is a split epimorphism, because

$$\bar{g}j\theta' t' = \bar{g}\theta t' = \bar{g}\bar{t} = 1.$$

relation R is an equivalence relation if and only if d_0 is S-special (Corollary 6.7 above).

In a Mal'tsev category, on a reflexive graph, there is at most one structure of internal category, which is necessarily an internal groupoid. In Section 5 we showed that, on a S-reflexive graph, there is again at most one structure of internal category, but there are S-categories which are not groupoids. An internal S-category is a groupoid if and only if, again, d_0 is S-special (Proposition 6.6).

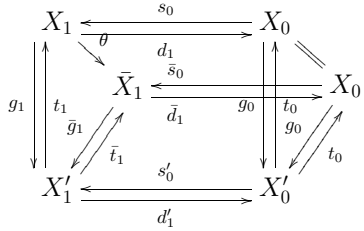
In a Mal'tsev category we have also the following useful result (see [2]): given any split epimorphism of reflexive graphs,



the commutative square $g_0 d_1 = d_1' g_1$ is a pullback as soon as so is the square $g_0 d_0 = d_0' g_1$. Here we have:

Proposition 7.1. *Let $\mathbb{C}$ be a S-protomodular category. Given a split epimorphism of reflexive graphs in $\mathbb{C}$ as in the diagram above, where the split epimorphism (g_0, t_0) is in S, the commutative square $g_0 d_1 = d_1' g_1$ is a pullback as soon as so is the square $g_0 d_0 = d_0' g_1$.*

Proof: If the square $g_0 d_0 = d_0' g_1$ is a pullback and the split epimorphism (g_0, t_0) is in S, so are the split epimorphism (g_1, t_1) and the pullback $(\bar{g}_1, \bar{t}_1)$ of (g_0, t_0) along d_1' in the following diagram:



Let θ be the induced factorization. The two leftward commutative squares are pullbacks; this means that the image of θ by the change-of-base functor

$$= (x, q_f(x') + sf(x)) = (x, q_f(x') + sf(x')) = (x, x'),$$

and, thanks to Proposition 4.5,

$$q_{p_0}((0,k)+s_0(x)) = q_{p_0}((0,k)+(x,x)) = q_{p_0}(x, k+x) = (0, q_f(k+x)-q_f(x)) =$$

$$= (0, q_f(k) + q_f(sf(k) + q_f(x)) - q_f(x)) = (0, q_f(k) + q_f(x) - q_f(x)) = (0,k).$$

The thesis follows then from Proposition 4.4. ∎

Example 6.9. The previous proposition implies that the morphism $abs\colon \mathbb{Z}^* \to \mathbb{N}^*$ of example 4.10 is a special S-morphism.

Corollary 6.10. *Let $\mathbb{C}$ be a category of monoids with operations, and let S be the class of Schreier split epimorphisms. A S-category*

$$X_1 \underset{d_1}{\overset{d_0}{\underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}}}} X_0$$

is a S-groupoid if and only if $K[d_0]$ is a S-special object. A S-reflexive relation is a S-equivalence relation if and only if the kernel $K[d_0]$ of the first projection is a S-special object, which is equivalent to say that $(K[d_0], +)$ is a group.

Proof: Thanks to the previous proposition, this is an immediate consequence of Proposition 6.6 and Corollary 6.7. ∎

7. Mal'tsev aspects of S-protomodular categories

7.1. Mal'tsev categories. We recall that a category $\mathbb{C}$ is a Mal'tsev category [8, 9] when any internal reflexive relation is an equivalence relation; this is equivalent to the property that any fiber $Pt_Y\mathbb{C}$ of the fibration $\P_{\mathbb{C}}$ is unital (see [2]). The category Gp of groups is a Mal'tsev one. The natural order $\mathcal{O}_{\mathbb{N}}$ of natural numbers (Example 5.4) shows that the category Mon of monoids is not a Mal'tsev one.

On the other hand, in the context of S-protomodular categories, any fiber $Pt_Y\mathbb{C}$ is $SPt_Y\mathbb{C}$-unital and, consequently, any fiber $SPt_Y\mathbb{C}$ is unital. In this section, we shall be interested in what is remaining of the properties of Mal'tsev categories in this new structural context.

7.2. Elementary observations. We already observed that, in a S-protomodular category, any S-reflexive relation $(d_0, d_1)\colon R \rightrightarrows X$ is only necessarily transitive. The natural order on $\mathbb{N}$ gives an example of S-reflexive relation (in the category of monoids) which is not an equivalence relation. A S-reflexive

$K(j)$ is an isomorphism, because, being j a monomorphism, $K[\bar{g}j] \simeq K[\bar{g}]$. Using the same argument as in the proof point 3 of Theorem 3.2, we can conclude that j is a strong epimorphism, and hence an isomorphism. ∎

4. Schreier split epimorphisms in monoids with operations

The aim of this section is to introduce an important class of examples of the situation described in the previous one. We start by recalling from [12] the following definition, which was inspired by the analogous one of *groups with operations* introduced by Porter in [14].

Definition 4.1. *Let Ω be a set of finitary operations such that the following conditions hold: if Ω_i is the set of i-ary operations in Ω, then:*

(1) $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$;
(2) *There is a binary operation $+ \in \Omega_2$ (not necessarily commutative) and a constant $0 \in \Omega_0$ satisfying the usual axioms for monoids;*
(3) $\Omega_0 = \{0\}$;
(4) *Let $\Omega_2' = \Omega_2 \backslash \{+\}$; if $* \in \Omega_2'$, then $*^\circ$ defined by $x *^\circ y = y * x$ is also in Ω_2';*
(5) *Any $* \in \Omega_2'$ is left distributive w.r.t. $+$, i.e.:*

$$a * (b + c) = a * b + a * c;$$

(6) *For any $* \in \Omega_2'$ we have $b * 0 = 0$;*
(7) *Any $\omega \in \Omega_1$ satisfies the following conditions:*
- $\omega(x + y) = \omega(x) + \omega(y)$;
- *for any $* \in \Omega_2'$, $\omega(a * b) = \omega(a) * b$.*

Let moreover E be a set of axioms including the ones above. We will denote by $\mathbb{C}$ the category of (Ω, E)-algebras. We call the objects of $\mathbb{C}$ monoids with operations.

Remark. The definition above does not include the case of groups, or more generally, the one of groups with operations. Indeed, the unary operation given by the group inverses, denoted by $-$, does not satisfy Condition 7. However, in order to recover all these structures, it suffices to add another condition: if the base monoid structure (given by the operations $+$ and 0) is a group, then the operation $-$ should be distinguished from the other unary operations. In other terms, Condition 7 should be satisfied only by operations

in $\Omega_1' = \Omega_1 \backslash \{-\}$. In this way, our definition becomes a generalization of the concept of groups with operations.

Example 4.2. Apart from the known structures covered by Porter's definition, such as groups, rings, associative algebras, Lie algebras and many others, our definition includes the cases of monoids, commutative monoids, semirings (i.e. rings where the additive structure is not necessarily a group, but just a commutative monoid), join-semilattices with a bottom element, distributive lattices with a bottom element (or a top one).

Let us observe that, if $\mathbb{C}$ is a category of monoids with operations, then it is pointed, complete and unital.

We now introduce the split epimorphisms that will form the desired class S. For the rest of the section, $\mathbb{C}$ will denote a category of monoids with operations.

Definition 4.3 ([12])**.** *A split epimorphism* $A \underset{f}{\overset{s}{\leftrightarrows}} B$ *in* $\mathbb{C}$ *is said to be a* Schreier split epimorphism *when, for any* $a \in A$*, there exists a unique* α *in the kernel* $K[f]$ *of* f *such that* $a = \alpha + sf(a)$*.*

As it is shown in [12], in the category Mon of monoids Schreier split epimorphisms are equivalent to monoid actions, where an action of a monoid B on a monoid X is a monoid homomorphism $B \to End(X)$, being $End(X)$ the monoid of endomorphisms of X.

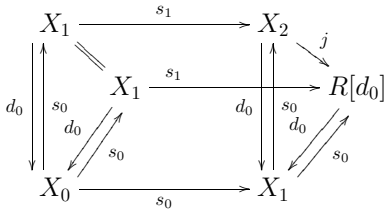
Proposition 4.4. *A split epimorphism* $A \underset{f}{\overset{s}{\leftrightarrows}} B$ *is a Schreier split epimorphism if and only if there exists a set-theoretical map* $q\colon A \dashrightarrow K[f]$ *such that:*

$$\begin{aligned} q(a) + sf(a) &= a \\ q(\alpha + s(b)) &= \alpha \end{aligned}$$

for every $a \in A$*,* $\alpha \in K[f]$ *and* $b \in B$*.*

Proof: Suppose that for every $a \in A$, there exists a unique $\alpha \in K[f]$ such that $a = \alpha + sf(a)$. This property defines a map $q\colon A \to K[f]$, by $q(a) = \alpha$ such that $a = q(a) + sf(a)$, for every $a \in A$. In order to prove that $q(\alpha + s(b)) = \alpha$ for any $\alpha \in K[f]$, it suffices to observe that $sf(\alpha + s(b)) = s(b)$.

diagram:



The two right hand side split epimorphisms are in S. The two commutative squares are pullbacks along $s_0\colon X_0 \rightarrowtail X_1$. The diagram means that the image of the map j by the change-of-base functor along s_0 is the isomorphism 1_{X_1}. According to Theorem 3.2, the map j is an isomorphism, and $\underline{X}_1$ is a groupoid. ∎

The previous proposition, according to Proposition 5.3, gives the following

Corollary 6.7. *Let* $\mathbb{C}$ *be a* S*-protomodular category. A* S*-reflexive relation*

$$X_1 \underset{d_1}{\overset{d_0}{\underset{\leftarrow s_0 \rightarrow}{\rightrightarrows}}} X_0$$

is a S*-equivalence relation if and only if the morphism* $d_0\colon X_1 \to X_0$ *is* S*-special.*

When $\mathbb{C}$ is a category of monoids with operations, and S is the class of Schreier split epimorphisms, the converse of Proposition 6.5 holds:

Proposition 6.8. *Let* $\mathbb{C}$ *be a category of monoids with operations, and let* S *be the class of Schreier split epimorphisms. Given a Schreier split epimorphism*

$$K[f] \underset{k_f}{\overset{q_f}{\leftrightarrows}} X \underset{f}{\overset{s}{\leftrightarrows}} Y,$$

if $K[f]$ *is a* S*-special object (or, in other terms, if* $(K[f], +)$ *is a group), then* f *is a* S*-special morphism.*

Proof: We have to show that the split epimorphism $R[f] \underset{p_0}{\overset{s_0}{\leftrightarrows}} X$ is a Schreier split epimorphism. Let us define $q_{p_0}(x, x') = (0, q_f(x') - q_f(x))$. We can check that

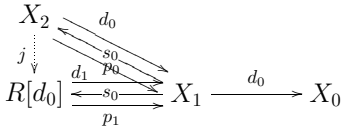
$q_{p_0}(x, x') + s_0 p_0(x, x') = (0, q_f(x') - q_f(x)) + (x, x) = (x, q_f(x') - q_f(x) + x) =$

the terminal morphism $K[f] \to 1$ is S-special as so is f, and then $K[f]$ is a S-special object. ∎

An internal category $\underline{X}_1$ in a finitely complete category $\mathbb{C}$ is a groupoid when, moreover, the following square determined by the composition map d_1 is a pullback:

$$\begin{array}{ccc} X_2 & \xrightarrow{d_1} & X_1 \\ {\scriptstyle d_0}\downarrow & & \downarrow{\scriptstyle d_0} \\ X_1 & \xrightarrow[d_0]{} & X_0, \end{array} \tag{7}$$

or, in other words, when the following vertical comparison morphism j is an isomorphism:



In this case we have a discrete fibration between groupoids:

$$\begin{array}{ccc} R[d_0] & \xrightarrow{d_2} & X_1 \\ {\scriptstyle d_0}\downarrow\uparrow{\scriptstyle d_1} & & {\scriptstyle d_0}\downarrow\uparrow{\scriptstyle d_1} \\ X_1 & \xrightarrow[d_1]{} & X_0 \end{array}$$

Proposition 6.6. *Let $\mathbb{C}$ be a S-protomodular category. A S-category*

$$X_1 \underset{d_1}{\overset{d_0}{\underset{\leftarrow}{\overset{\rightarrow}{\leftarrow}}}} X_0$$

is a S-groupoid if and only if the morphism $d_0 \colon X_1 \to X_0$ is S-special.

Proof: Let $\underline{X}_1$ be a S-groupoid. Then the split epimorphism $(d_0, s_0) \colon X_1 \rightleftarrows X_0$ is in S. By the pullbacks of the previous fibration diagram, the split epimorphism $(d_0, s_0) \colon R[d_0] \rightleftarrows X_1$ is in S and consequently the morphism $d_0 \colon X_1 \to X_0$ is S-special. Conversely, suppose that the map $d_0 \colon X_1 \to X_0$ of the S-category $\underline{X}_1$ is S-special. Consider the following

Conversely, given a set-theoretical map $q \colon A \to B$ satisfying the asserted identities, we can choose $\alpha = q(a)$ for every $a \in A$ by the first identity; suppose now that $a = \alpha' + sf(a)$, then we get:

$$q(a) = q(\alpha' + sf(a)) = \alpha'$$

by the second identity. ∎

We shall call the following diagram:

$$K[f] \underset{k}{\overset{q}{\underset{\rightarrow}{\overset{\leftarrow}{\rightarrow}}}} A \underset{f}{\overset{s}{\underset{\rightarrow}{\overset{\leftarrow}{\rightarrow}}}} B,$$

the canonical *Schreier split sequence* associated with the Schreier split epimorphism and q the associated *Schreier retraction*. The following properties of the retraction q will be useful later. For the sake of simplicity, we consider k just as an inclusion.

Proposition 4.5. *Given a Schreier split epimorphism (A, B, f, s), we have:*

(a) $qk = 1_{K[f]}$;
(b) $qs = 0$;
(c) $q(0) = 0$;
(d) *if $b \in B$ and $\alpha \in K[f]$, then* $q(s(b) + \alpha) + s(b) = s(b) + \alpha$;
(e) *for every $a, a' \in A$* $q(a + a') = q(a) + q(sf(a) + q(a'))$.

Proof:
(a) is a straightforward consequence of the second identity in Proposition 4.4.
(b) for $b \in B$ we have that $s(b) = 0 + sf(s(b))$, and the uniqueness of q gives that $qs(b) = 0$ for every $b \in B$.
(c) obviously we have $0 = 0 + sf(0)$.
(d) for any $b \in B$ and any $\alpha \in K[f]$ we have:

$$s(b) + \alpha = q(s(b) + \alpha) + sf(s(b) + \alpha) =$$
$$= q(s(b) + \alpha) + sfs(b) + sf(\alpha) = q(s(b) + \alpha) + s(b).$$

(e) $q(a + a')$ is the unique element of $K[f]$ such that

$$a + a' = q(a + a') + sf(a + a') = q(a + a') + sf(a) + sf(a'),$$

so it suffices to prove that

$$q(a) + q(sf(a) + q(a')) + sf(a) + sf(a') = a + a'.$$

By point (d), we have that

$$q(sf(a) + q(a')) + sf(a) = sf(a) + q(a')$$

and hence

$$q(a)+q(sf(a)+q(a'))+sf(a)+sf(a') = q(a)+sf(a)+q(a')+sf(a') = a+a'.$$

■

We are now going to show that any category $\mathbb{C}$ of monoids with operations is S-protomodular, where S is the class of Schreier split epimorphisms. The results below were already proved in [7] for the particular cases of monoids and semirings.

Proposition 4.6. *Schreier split epimorphisms are stable under pullbacks along any morphism.*

Proof: Consider the following diagram, where the lower row is a Schreier split sequence and the right hand side square is a pullback:

$$\begin{array}{ccccc} K[\pi_Y] & \underset{\langle k,0\rangle}{\overset{q'}{\rightleftarrows}} & A\times_B Y & \underset{\pi_Y}{\overset{\langle sh,1_Y\rangle}{\rightleftarrows}} & Y \\ \simeq\downarrow & & \pi_A\downarrow & & \downarrow h \\ K[f] & \underset{k}{\overset{q}{\rightleftarrows}} & A & \underset{f}{\overset{s}{\rightleftarrows}} & B. \end{array}$$

The map q' defined by $q'(a,y) = (q(a),0)$ satisfies the conditions of Proposition 4.4 since:

$$(a,y) = (q(a),0) + (sf(a),y) = (q(a),0) + (sh(y),y)$$

for any $(a,y) \in A \times_B Y$. Moreover, the elements of $K[\pi_Y]$ are of the form $(\alpha,0)$, with $\alpha \in K[f]$, and then:

$$q'((\alpha,0) + (sh(y),y)) = q'(\alpha + sh(y),y) = (q(\alpha + sh(y)),0) = (\alpha,0)$$

Accordingly the upper row is a Schreier split epimorphism. ■

Lemma 4.7. *A Schreier split epimorphism is a strongly split epimorphism.*

Proof: Given a Schreier split epimorphism $A \underset{f}{\overset{s}{\rightleftarrows}} B$, the formula $a = q(a) + sf(a)$ proves that the pair (k,s) is jointly strongly epimorphic. ■

Proposition 4.8. *Given any direct product diagram*

$$X \underset{\langle 1_X,0\rangle}{\overset{\pi_X}{\rightleftarrows}} X\times B \underset{\langle 0,1_B\rangle}{\overset{\pi_B}{\rightleftarrows}} B,$$

where q is the Schreier retraction associated with the Schreier split epimorphism (p_0,s_0). Let $x \in X$; according to the Schreier condition, the pair $(x,0) \in X \times X$ can be written as

$$(x,0) = q(x,0) + s_0p_0(x,0) = q(x,0) + (x,x).$$

Since $q(x,0)$ is an element of $\mathrm{Ker}(p_0)$, it is an element of the form $(0,y)$, for some $y \in X$. Hence we have

$$(x,0) = (0,y) + (x,x) = (x,y+x),$$

and from this equality we get $y + x = 0$. So y is a left inverse for x. Doing the same thing for all $x \in X$ we prove that $(X,+)$ is a group.

Conversely, suppose that $(X,+)$ is a group. The needed Schreier retraction is simply given by

$$q(x_1,x_2) = (0, x_2 - x_1).$$

■

As a consequence, we have that, if $\mathbb{C}$ is the category Mon of monoids, its protomodular core is the category Gp of groups. If $\mathbb{C}$ is the category $SRng$ of semirings, the protomodular core is the category Rng of (not necessarily unitary) rings. More generally, given any category $\mathbb{C}$ of monoids with operation, the protomodular core is the corresponding category of groups with operations, obtained from $\mathbb{C}$ by adding the condition that $+$ is a group operation.

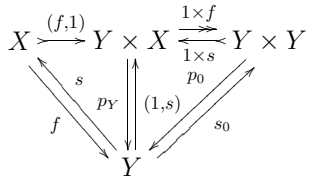
Proposition 6.5. *Let $\mathbb{C}$ be a S-protomodular category. Any split epimorphism $f\colon X \to Y$ which is a S-special morphism belongs to S and its kernel is a S-special object.*

Proof: Let s be the splitting of f. Consider the following diagram:

$$\begin{array}{ccccc} K[f] & \xrightarrow{k_f} & X & \xrightarrow{s_1} & R[f] \\ \downarrow & & f\downarrow\uparrow s & & p_0\downarrow\uparrow s_0 \\ 1 & \xrightarrow[\alpha_Y]{} & Y & \xrightarrow[s]{} & X, \end{array}$$

where s_1 is the morphism $(1,sf)\colon X \to R[f]$. The right hand side square is a pullback. If the morphism f is S-special then, by definition, the split epimorphism (p_0,s_0) is in S. By stability under pullbacks, the split epimorphism (f,s) is in S, too. The left hand side square is a pullback as well, so

Proof: Let us recall that any split epimorphism $(f, s\colon X \rightleftarrows Y)$ produces a kernel diagram in the fibre $Pt_Y\mathbb{C}$:



When Y is in $S^\sharp\mathbb{C}$, the right hand side split epimorphism is in S. The following pullback:

$$\begin{array}{ccc} Y \times X & \xrightarrow{s\times 1} & X \times X \\ {\scriptstyle p_Y}\uparrow\downarrow{\scriptstyle (1,s)} & & {\scriptstyle p_0}\uparrow\downarrow{\scriptstyle s_0} \\ Y & \xrightarrow[s]{} & X \end{array}$$

shows that, when X is in $S^\sharp\mathbb{C}$, the middle split epimorphism is in S. Since the fibre $SPt_Y\mathbb{C}$ is closed under finite limits, the kernel (f, s) is in S. So, according to Theorem 3.2, the change-of-base functor with respect to the fibration $Pt(S^\sharp\mathbb{C})$ is conservative, and consequently $S^\sharp\mathbb{C}$ is protomodular. On the other hand, since $Pt(S^\sharp\mathbb{C})$ is stable under finite limits, the kernel equivalence relation of f lies in $Pt(S^\sharp\mathbb{C})$, and the split epimorphism $(p_0, s_0)\colon R[f] \rightleftarrows X$ is in S. Accordingly $R[f]$ is a S-equivalence relation, and f is a S-special morphism. ∎

Definition 6.3. *Given a S-protomodular category $\mathbb{C}$, we will call the subcategory $S^\sharp\mathbb{C}$ the* protomodular core *of $\mathbb{C}$ relatively to S.*

We are now going to describe the protomodular core when $\mathbb{C}$ is a category of monoids with operations and S is the class of Schreier split epimorphisms.

Proposition 6.4. *Let $\mathbb{C}$ be a category of monoids with operations and S the class of Schreier split epimorphisms. Given an object $X \in \mathbb{C}$, it is S-special if and only if $(X, +)$ is a group.*

Proof: Suppose that X is S-special. Consider the following diagram:

$$X \underset{\langle 0,1\rangle}{\overset{q}{\leftrightarrows}} X \times X \underset{p_1}{\overset{p_0}{\underset{\leftarrow s_0 \rightarrow}{\rightrightarrows}}} X,$$

the canonical split epimorphism $(X \times B, B, \pi_B, \langle 0, 1_B\rangle)$ is a Schreier split epimorphism.

Proof: It suffices to observe that $(x, 0) + (0, b) = (x, b)$ for any $x \in X$ and any $b \in B$. Here the Schreier retraction π_X is a monoid homomorphism. ∎

Corollary 4.9. *For any $X \in \mathbb{C}$, the identity split epimorphism $X \underset{1_X}{\overset{1_X}{\leftrightarrows}} X$, and more generally any isomorphism, is a Schreier split epimorphism.*

Example 4.10. We denote by $\mathbb{Z}^*$ the monoid of non-zero integers with the usual multiplication, and by $\mathbb{N}^*$ its submonoid whose elements are the numbers greater than 0. Then the split epimorphism

$$\mathbb{Z}^* \underset{abs}{\overset{i}{\leftrightarrows}} \mathbb{N}^*,$$

where i is the inclusion and *abs* associates with any integer its absolute value, is a Schreier split epimorphism. In fact $K[abs] = \{\pm 1\}$, and it is immediate to see that any non-zero integer z can be written in a unique way as $z = \pm 1 \cdot |z| = |z| \cdot \pm 1$.

Proposition 4.11. *Schreier split epimorphisms are closed under products, i.e. the product of two Schreier split epimorphisms is a Schreier one.*

Proof: Consider the two Schreier split exact sequences

$$K[f] \underset{k}{\overset{q}{\leftrightarrows}} A \underset{f}{\overset{s}{\leftrightarrows}} B$$

and

$$K[f'] \underset{k'}{\overset{q'}{\leftrightarrows}} A' \underset{f'}{\overset{s'}{\leftrightarrows}} B',$$

Their term by term product

$$K[f] \times K[f'] \underset{k\times k'}{\overset{q\times q'}{\leftrightarrows}} A \times A' \underset{f\times f'}{\overset{s\times s'}{\leftrightarrows}} B \times B',$$

clearly satisfies the conditions of Proposition 4.4. ∎

Lemma 4.12. *Consider a (vertical) map (h, l) in $Pt\mathbb{C}$:*

$$\begin{array}{ccc} X & \underset{f}{\overset{s}{\rightleftarrows}} & Y \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle h} \\ X' & \underset{f'}{\overset{s'}{\rightleftarrows}} & Y'. \end{array}$$

Suppose that the two rows are Schreier split epimorphisms, then the Schreier retractions are compatible, i.e. the following leftward left hand side diagram commutes (in the category Set of sets):

$$\begin{array}{ccccc} K[f] & \underset{k}{\overset{q}{\leftrightarrows}} & X & \underset{f}{\overset{s}{\rightleftarrows}} & Y \\ {\scriptstyle K(l)}\downarrow & & {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle h} \\ K[f'] & \underset{k'}{\overset{q'}{\leftrightarrows}} & X' & \underset{f'}{\overset{s'}{\rightleftarrows}} & Y'. \end{array}$$

Proof: We have to show that $q'l(x) = lq(x)$ for any x in X. It is true since we have:

$$lq(x) + s'f'l(x) = lq(x) + lsf(x) = l(q(x) + sf(x)) = l(x) = q'l(x) + s'f'l(x).$$

■

Proposition 4.13. *Schreier split epimorphisms are closed also under equalizers, i.e. the equalizer of two parallel morphisms between Schreier split epimorphisms is a Schreier one. Hence Schreier split epimorphisms are closed under finite limits.*

Proof: Given two parallel morphisms of Schreier split epimorphisms:

$$\begin{array}{ccc} A & \underset{g}{\overset{h}{\rightrightarrows}} & A' \\ {\scriptstyle f}\uparrow\downarrow{\scriptstyle s} & & {\scriptstyle f'}\uparrow\downarrow{\scriptstyle s'} \\ B & \underset{g'}{\overset{h'}{\rightrightarrows}} & B', \end{array}$$

epimorphism (d_0, s_0) is in S, because $X_1 \underset{d_1}{\overset{d_0}{\underset{\leftarrow s_0 \rightarrow}{\rightrightarrows}}} X_0$ is a S-reflexive relation and the rightward horizontal square is a pullback. According to Proposition 3.4, the factorization $(p_0, p_2)\colon K[d_0, d_1] \rightarrowtail X_2$ is an extremal epimorphism, and hence an isomorphism; accordingly the morphism $X_2 \xrightarrow{(p_0,p_2)^{-1}} K[d_0, d_1] \xrightarrow{p_2} X_1$ produces the desired transitivity map. ■

A S-reflexive relation doesn't need to be an equivalence relation, because symmetry can fail. The following is a concrete counterexample in the category Mon of monoids, equipped with the class of Schreier split epimorphisms described in section 4.

Example 5.4 ([6], Example 5.3)**.** *The internal order in Mon given by the usual order between natural numbers:*

$$\mathcal{O}_{\mathbb{N}} \underset{p_1}{\overset{p_0}{\underset{\leftarrow s_0 \rightarrow}{\rightrightarrows}}} \mathbb{N},$$

where

$$\mathcal{O}_{\mathbb{N}} = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid x \leq y\},$$

is a Schreier order relation.

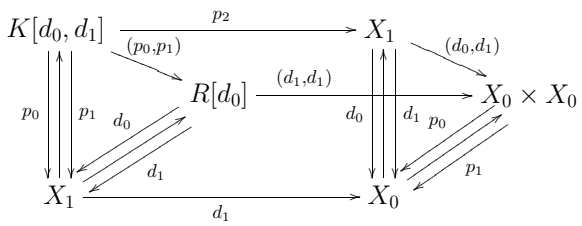
6. S-special morphims and internal S-groupoids

Definition 6.1. *Let $\mathbb{C}$ be a S-protomodular category. A morphism $f\colon X \to Y$ in $\mathbb{C}$ will be called* S-special *when the kernel equivalence relation $R[f]$ is a S-equivalence relation. An object X will be called* S-special *when the terminal morphism $\tau_X\colon X \to 1$ is S-special.*

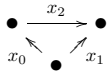
In a S-protomodular category, the S-special morphisms are stable under pullbacks (because the class S is stable under pullbacks). Moreover, the full subcategory $S^\sharp\mathbb{C} \subseteq \mathbb{C}$ of S-special objects is closed under finite limits in $\mathbb{C}$ (this comes from Condition 2 of Definition 3.1).

Proposition 6.2. *Let $\mathbb{C}$ be a S-protomodular category. Any split epimorphism between S-special objects is in S and, consequently, is a S-special morphism. The subcategory $S^\sharp\mathbb{C}$ of S-special objects is protomodular.*

In a finitely complete category $\mathbb{C}$, it is obtained by the following pullback of reflexive graphs:



In set-theoretical terms, $K[d_0, d_1]$ is the set of triples $(x_0, x_1, x_2) \in X_1$ whose incidence conditions are given by the following drawing:

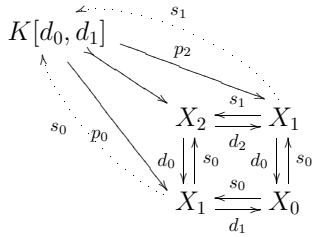


Proposition 5.3. *Let $\mathbb{C}$ be a S-protomodular category. Any S-reflexive relation is transitive.*

Proof: Let us consider the following S-reflexive relation:

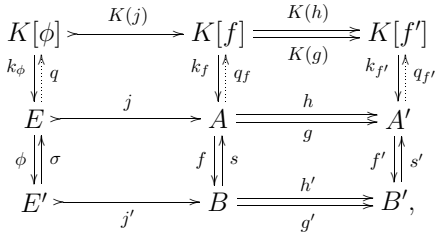
$$X_1 \underset{d_1}{\overset{d_0}{\underset{\longleftarrow}{\overset{\longrightarrow}{\longleftarrow}}}} X_0$$

The square $d_1p_0 = d_0p_2$ in the diagram above determines a factorization $(p_0, p_2)\colon K[d_0, d_1] \to X_2$ to the following vertical pullback:



Since $(d_0, d_1)\colon X_1 \rightarrowtail X_0 \times X_0$ is a relation, and hence d_0 and d_1 are jointly monomorphic, the factorization (p_0, p_2) is a monomorphism. In order to prove this fact, it suffices to observe that it is true in set-teoretical terms, and that it is invariant under the Yoneda embedding. The left hand side split

consider the following diagram:



where j is an equalizer of h and g and j' is an equalizer of h' and g' in $\mathbb{C}$. Then the lower part of the diagram is an equalizer diagram in $Pt\mathbb{C}$. Since the kernel functor preserves equalizers, $K(j)$ is an equalizer of $K(h)$ and $K(g)$ in $\mathbb{C}$, and hence in the category Set of sets. By the previous lemma, the set-theoretical Schreier retractions q_f and $q_{f'}$ make the upward right hand side square commute; hence we get a factorization q which satisfies the conditions of a Schreier retraction for the split epimorphism (ϕ, σ) and makes it a Schreier split epimorphism. ∎

5. Internal S-reflexive relations and S-categories

We recall that an internal reflexive graph in a category $\mathbb{C}$ is a diagram of the form

$$X_1 \underset{d_1}{\overset{d_0}{\underset{\longleftarrow}{\overset{\longrightarrow}{\longleftarrow}}}} X_0$$

such that $d_0s_0 = 1_{X_0} = d_1s_0$. A reflexive relation is a reflexive graph such that the pair (d_0, d_1) is jointly monomorphic.

Definition 5.1. *An internal reflexive graph (resp. category, groupoid) in a S-protomodular category $\mathbb{C}$*

$$X_1 \underset{d_1}{\overset{d_0}{\underset{\longleftarrow}{\overset{\longrightarrow}{\longleftarrow}}}} X_0$$

*is said to be a S-*reflexive graph *(resp. S-category, S-groupoid) if the split epimorphism (d_0, s_0) is in S.*

As a consequence of condition 2 of the definition of an S-protomodular category, S-reflexive graphs are closed under finite limits inside the category of

internal reflexive graphs. The same is true for S-categories and S-groupoids. Let us recall that an internal category $\underline{X}_1$ in $\mathbb{C}$ is a reflexive graph:

$$X_1 \underset{d_1}{\overset{d_0}{\underset{\longleftarrow}{\overset{\longrightarrow}{\longleftarrow}}}} X_0$$

such that the following pullback of split epimorphisms, which defines X_2 as the internal object of the composable pairs

$$\begin{array}{ccc} X_2 & \underset{d_2}{\overset{s_1}{\rightleftarrows}} & X_1 \\ {\scriptstyle d_0}\downarrow\uparrow{\scriptstyle s_0} & & {\scriptstyle d_0}\downarrow\uparrow{\scriptstyle s_0} \\ X_1 & \underset{d_1}{\overset{s_0}{\rightleftarrows}} & X_0, \end{array} \tag{4}$$

is endowed with a composition map $d_1 \colon X_2 \to X_1$ satisfying the remaining simplicial identities:
(1) $d_0d_1 = d_0d_0$, $d_1d_1 = d_1d_2$ (incidence axioms)
(2) $d_1s_0 = 1_{X_1}$, $d_1s_1 = 1_{X_1}$ (composition with identities)
This composition must satisfy the associativity axiom; for that consider the following pullback of split epimorphisms (where X_3 is the object of composable triples):

$$\begin{array}{ccc} X_3 & \underset{d_3}{\overset{s_2}{\rightleftarrows}} & X_2 \\ {\scriptstyle d_0}\downarrow\uparrow{\scriptstyle s_0} & & {\scriptstyle d_0}\downarrow\uparrow{\scriptstyle s_0} \\ X_2 & \underset{d_2}{\overset{s_1}{\rightleftarrows}} & X_1 \end{array} \tag{5}$$

The composition map d_1 induces a couple of maps $(d_1, d_2) \colon X_3 \rightrightarrows X_2$ such that $d_0d_1 = d_0d_0$, $d_2d_1 = d_1d_3$ and $d_0d_2 = d_1d_0$, $d_2d_2 = d_2d_3$. The associativity is given by the remaining simplicial axiom:
(3) $d_1d_1 = d_1d_2$.

Proposition 5.2. *Let $\mathbb{C}$ be a S-protomodular category. On a S-reflexive graph there is at most one structure of internal category. It is sufficient to have the composition map $d_1 \colon X_2 \to X_1$ with axiom* (2)*, the axioms* (1) *and* (3) *come for free.*

Proof: Let us consider the following S-reflexive graph:

$$X_1 \underset{d_1}{\overset{d_0}{\underset{\longleftarrow}{\overset{\longrightarrow}{\longleftarrow}}}} X_0$$

Consider now the diagram (4):

$$\begin{array}{ccc} X_2 & \underset{d_2}{\overset{s_1}{\rightleftarrows}} & X_1 \\ {\scriptstyle d_0}\downarrow\uparrow{\scriptstyle s_0} & & {\scriptstyle d_0}\downarrow\uparrow{\scriptstyle s_0} \\ X_1 & \underset{d_1}{\overset{s_0}{\rightleftarrows}} & X_0 \end{array}$$

Since the rightward horizontal square is a pullback, and the right hand side split epimorphism is in S, the left hand side one is in S, too. Moreover, since the category $Pt_{X_0}\mathbb{C}$ is $SPt_{X_0}\mathbb{C}$-unital (by Theorem 3.2) and the pullback above is actually the product of the two split epimorphisms $X_1 \underset{d_0}{\overset{s_0}{\rightleftarrows}} X_0$ and $X_1 \underset{d_1}{\overset{s_0}{\rightleftarrows}} X_0$ in the category $Pt_{X_0}\mathbb{C}$, the pair (s_0, s_1) is jointly strongly epimorphic. Hence there is at most one map d_1 satisfying axiom (2). Axiom (1) can be also verified by composition with the pair (s_0, s_1). Axiom (3) comes by composition with the pair (s_0, s_2) of diagram (5), which is jointly strongly epimorphic as well. ∎

Let a reflexive graph be given:

$$X_1 \underset{d_1}{\overset{d_0}{\underset{\longleftarrow}{\overset{\longrightarrow}{\longleftarrow}}}} X_0$$

Let us recall that its simplicial kernel is the upper part of the universal 2-simplicial object associated with it:

$$K[d_0, d_1] \underset{p_2}{\overset{p_0}{\underset{\longleftarrow}{\overset{\longrightarrow}{\longleftarrow}}}} X_1 \underset{d_1}{\overset{d_0}{\underset{\longleftarrow}{\overset{\longrightarrow}{\longleftarrow}}}} X_0 \tag{6}$$