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GROWTH DIAGRAMS AND NON-SYMMETRIC CAUCHY IDENTITIES ON NW OR SE NEAR STAIRCASES

O. AZENHAS AND A. EMAMI

Abstract: Mason has introduced an analogue of the Robinson-Schensted-Knuth (RSK) correspondence to produce a bijection between biwords and pairs of semi-skyline augmented fillings whose shapes, compositions, are rearrangements of each other. That pair of shapes encode the right keys for the pair of semi-standard Young tableaux produced by the usual Robinson-Schensted-Knuth (RSK) correspondence. We have shown that this analogue of RSK restricted to multisets of cells in staircases or truncated staircases allows expansions of non-symmetric Cauchy kernels in the basis of Demazure characters of type A, and the basis of Demazure atoms. One considers now a near staircase, in French convention, where the top leftmost and the bottom rightmost boxes of a staircase are deleted and also possibly some boxes in the diagonal layer. The conditions imposed on the pairs of shapes for the semi-skyline augmented fillings are described by inequalities in the Bruhat order, w.r.t. the symmetric group. The bijection is then used to provide a combinatorial expansion of an expansion formula, due to A. Lascoux, of a non-symmetric Cauchy kernel, over near staircases, in the basis of Demazure characters of type A, and the basis of Demazure atoms, under the action of appropriate Demazure operators. The analysis is made in the framework of Fomin's growth diagrams for Robinson-Schensted-Knuth correspondences. On one hand, one gives a formulation of the analogue of RSK, via reverse RSK, to obtain pairs of semi-skyline augmented fillings, and, on the other hand, an interpretation of the action of crystal operators on biwords whose biletters are cells on a Ferrers shape. This sheds light on the aforesaid expansion provided by A. Lascoux.

Keywords: Non-symmetric Cauchy kernels, Demazure character, key polynomial, Demazure operator, semi-skyline augmented filling, RSK analogue, crystal.
AMS Subject Classification (2010): 05E05,05E10.

1. Introduction

The Robinson-Schenested-Knuth (RSK) correspondence [12] is a bijection between biwords (an array of two words), on two finite totally ordered alphabets, and pairs of semi-standard Young tableaux (SSYT), of the same

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shape, with entries in the same alphabets [12]. Let $\mathbb{N}$ denote the set of non-negative integers, and, as usual, if n is a positive integer, let $[n]$ be the set $\{1,\dots,n\}$. Given a positive integer n, let m and k be fixed positive integers where $1\le m\le n$, $1\le k\le n$, $m+k\ge n+1$, and let $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)$ be two sequences of indeterminates. The well-known Cauchy identity [24] expresses the Cauchy kernel $\prod_{i=1}^k\prod_{j=1}^m(1-x_iy_j)^{-1}$ as a sum of products of Schur polynomials s_λ in $(x_1,x_2,\dots,x_k)$ and $(y_1,y_2,\dots,y_m)$, respectively,

$$\prod_{i=1}^k\prod_{j=1}^m(1-x_iy_j)^{-1}=\sum_\lambda s_\lambda(x_1,\dots,x_k)s_\lambda(y_1,\dots,y_m), \tag{1}$$

over all partitions λ of length $\le \min\{k,m\}$. Schur polynomials in a finite number of indeterminates $(x_1,x_2,\dots,x_k)$ are indexed by partitions λ of length $\le k$. They are combinatorially described by semistandard Young tableaux, SSYTs, of shape λ, over the alphabet $[k]$, [7, 30, 31],

$$s_\lambda(x_1,\dots,x_k)=\sum_{\substack{T\ SSYT\\ sh(T)=\lambda}}x^T,$$

where $sh(T)$ denotes the shape of the SSYT, T, and $x^T=x_1^{c_1}\cdots x_k^{c_k}$, with c_i the multiplicity of i in T. Thereby, the right hand side of (1) can be written as $\sum\limits_{(P,Q)}x^Py^Q$, where the sum runs over all pairs (P,Q) of SSYTs of the same shape with length $\le\min\{k,m\}$. On the other hand, expanding the product of formal power series, on the left hand side, of (1), and identifying each monomial x_iy_j, $i\in[k]$, $j\in[m]$, with the biletter $\binom{j}{i}$, the RSK correspondence, over the finite alphabets $[k]$ and $[m]$, provides a bijective proof for identity (1). Key polynomials, or Demazure characters, κ_α, with $\alpha\in\mathbb{N}^n$, [4, 16, 28], and Demazure atoms $\widehat{\kappa}_\alpha$, with $\alpha\in\mathbb{N}^n$, [16, 26], both of each are indexed by weak compositions and form a $\mathbb{Z}$-linear basis for the ring of integer polynomials $\mathbb{Z}[x_1,\dots,x_n]$. When the vectors $\alpha\in\mathbb{N}^n$ are anti-dominant, key polynomials lift the basis of Schur polynomials for the subring of symmetric polynomials $\mathbb{Z}[x_1,\dots,x_n]^{\mathfrak{S}_n}$, where $\mathfrak{S}_n$ denotes the symmetric group of degree n. The Cauchy kernel $\prod_{i=1}^k\prod_{j=1}^m(1-x_iy_j)^{-1}$ in (1) can be written as $\prod_{(i,j)\in(m^k)}(1-x_iy_j)^{-1}$, an expansion over the rectangle shape (m^k) of height k and width m. Cauchy kernels over arbitrary Ferrers shapes are no more symmetric in the indeterminates and their expansions are not on the basis of Schur polynomials but rather over the basis of key polynomials and the basis of Demazure atoms. A. Lascoux has studied Cauchy kernels expansions over staircases which he then generalized for arbitrary Ferrers shapes [19]. For staircases the expansion is explicit in the SSYTs for which he provided both an algebraic proof, with Fu, [6], and a combinatorial proof based on the fact that, in type A, – not known for other Weyl groups [21]–, RSK can be translated in the language of bicrystals [19, 11, 14, 22]. For other shapes the expansion is not entirely explicit in the SSYTs and only an algebraic explanation was provided in [19].

Mason [25, 26] has defined an analogue of RSK where the output is a pair of semi-skyline augmented fillings (SSAFs) whose shapes, vectors of $\mathbb{N}^n$, are a rearrangement of each other. The SSAFs, combinatorial objects introduced in [9], are in bijection with SSYTs in a way that the shape assigns to each SSYT its right key [16]. Key polynomials and Demazure atoms (or standard bases) were first described combinatorially by Lascoux and Schützenberger in [15, 16], for which they have introduced the key notion of the right key of a SSYT. Thus, they can also be combinatorially described by semi-skylines (SSAF) [26]

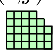
$$\widehat{\kappa}_\nu(x) = \sum_{\substack{F\ SSAF \\ sh(F)=\nu}} x^F, \quad \kappa_\nu(x) = \sum_{\substack{F\ SSAF \\ sh(F)\leq\nu}} x^F, \tag{2}$$

where the inequality regarding the shape of F, $sh(F)$, is in the Bruhat order. In [1, 2], we have proved that the analogue of RSK correspondence, restricted to multisets of cells in a staircase of size n, gives pairs (F, G) of semi-skylines, with entries $\leq n$, whose shapes satisfy $sh(G) \leq \omega sh(F)$ in Bruhat order, with ω the longest permutation of $\mathfrak{S}_n$. As a consequence, using (2), we can write

$$\prod_{\substack{i+j\leq n+1 \\ 1\leq i,j\leq n}} (1 - x_i y_j)^{-1} = \sum_{\substack{(F,G) \\ sh(G)\leq\omega sh(F)}} x^F y^G = \sum_{\nu\in\mathbb{N}^n} \widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y), \tag{3}$$

where the sum runs over pairs of semi-skylines (F, G) with entries $\leq n$. More generally, the restriction of the analogue of RSK [1, 2] can be considered to the cells (i, j) with $i \in [k]$, $j \in [m]$, that is, to the truncated staircases with height k and width m. This allows a bijective proof for the following non-symmetric Cauchy kernel expansion, over a truncated staircase of size n, with height k and width $m \geq k$, a special shape of the more general formula for Ferrers shapes, due to A. Lascoux [19],

$$\prod_{(i,j)\in} (1-x_iy_j)^{-1} = \sum_{\mu\in\mathbb{N}^k} \widehat{\kappa}_\mu(x)\pi_{\sigma(\lambda,SE)}\kappa_{\omega\mu}(y), \tag{4}$$



where $\pi_{\sigma(\lambda,SE)}$ is the Demazure operator indexed by the reduced expression $\sigma(\lambda,SE)=\prod_{i=1}^{k-(n-m)-1}(s_{i+n-k-1}\dots s_i)\prod_{i=0}^{n-m}(s_{m-1}\dots s_{k-(n-m)+i})$ of $\mathfrak{S}_n$, and as usual s_i denotes the simple transposition $(i\ i+1)$. Recall that Demazure operators π_i act on key polynomials κ_μ via elementary bubble sorting operators on the entries of the vector μ [28], that is,

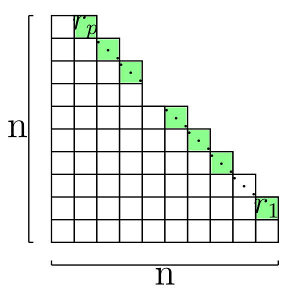
$$\pi_i\kappa_\mu = \begin{cases} \kappa_{s_i\mu} & \text{if } \mu_i>\mu_{i+1} \\ \kappa_\mu & \text{if } \mu_i\le\mu_{i+1} \end{cases}. \tag{5}$$

It is possible then to determine explicitly $\pi_{\sigma(\lambda,SE)}\kappa_{\omega\mu}$, and obtain the explicit expansion

$$(4) = \sum_{\mu\in\mathbb{N}^k} \widehat{\kappa}_\mu(x)\kappa_{(0^{m-k},\alpha)}(y), \tag{6}$$

where α depends on μ in a certain way as explained in [2]. In particular, when $k=m=n$, one recovers the identity for staircases (3).

In this work, we give a combinatorial expansion for the non-symmetric Cauchy kernel $\prod_{(i,j)\in\lambda}(1-x_iy_j)^{-1}$, being the product over all cells (i,j) of the near staircase λ, in French convention, as shown below,



(7)

where one layer of p green boxes, $1\le p<n$, is sited on the stairs of the staircase of size n, at most one box in each stair, avoiding the top and the basement. The label r_i indicates that the row index is r_i+1. The analogue of RSK analogue restricted to the multiset of cells in the near staircase produces

pairs (F,G) of SSAFs with entries $\leq n$ where the shapes satisfy inequalities in the Bruhat order regarding the p boxes sited on the stairs of staircase (7)

$$\begin{aligned} sh(G) &\nleq \omega s_{r_p}\cdots\widehat{s}_{r_i}\cdots s_{r_2}s_{r_1}sh(F),\ i=1,2,\ldots,p,\\ sh(G) &\leq \omega s_{r_p}\cdots s_{r_2}s_{r_1}sh(F), \end{aligned}$$

where $\widehat{\ }$ means omission. Recalling the action of Demazure operators on Demazure atoms $\widehat{\kappa}_\mu$

$$\pi_i\widehat{\kappa}_\mu = \begin{cases} \widehat{\kappa}_{s_i\mu}+\widehat{\kappa}_\mu & \text{if } \mu_i>\mu_{i+1}\\ \widehat{\kappa}_\mu & \text{if } \mu_i=\mu_{i+1}\\ 0 & \text{if } \mu_i<\mu_{i+1} \end{cases}, \tag{8}$$

this bijection produces the following combinatorial formula expansion in terms of pairs of SSAFs

$$\begin{aligned} \prod_{(i,j)\in\lambda}(1-x_iy_j)^{-1} &= \sum_{(F,G)\in\mathcal{A}} x^Fy^G + \sum_{1\leq z\leq p}\sum_{H_z}\sum_{(F,G)\in\mathcal{A}_z^{Hz}} x^Fy^G\\ &= \sum_{\nu\in\mathbb{N}^n}(\pi_{r_1}\ldots\pi_{r_p}\widehat{\kappa}_\nu(x))\kappa_{\omega\nu}(y), \end{aligned} \tag{9}$$

where $H_z=\{i_1<i_2<\cdots<i_z\}\in\binom{[p]}{z}$, and, for $0\leq z\leq p$,

$$\mathcal{A}_z^{H_z} := \left\{ (F,G)\in SSAF_n^2\colon \begin{matrix} sh(G)\nleq\omega s_{r_{i_z}}\cdots\widehat{s}_{r_{i_m}}\cdots s_{r_{i_1}}sh(F),\ m=1,2,\ldots,z\\ sh(G)\leq\omega s_{r_{i_z}}\cdots s_{r_{i_1}}sh(F)\end{matrix} \right\},$$

where $SSAF_n$ denotes the set of all SSAFs with entries $\leq n$. In particular, $\mathcal{A}:=\mathcal{A}_0^{\emptyset}:=\{\,(F,G)\in SSAF_n^2\colon\ sh(G)\leq\omega sh(F)\,\}$.

2. SSYT, reverse SSYT and SSAF

Semi-skyline augmented fillings (SSAFs) have been introduced in [9, 10], to describe combinatorially (non-symmetric) Macdonald polynomials. Mason has defined in [26] a weight preserving bijection, ϱ, between RSSYTs and SSAFs. We shall use this map to define SSAFs which allows later to translate the analogue of RSK [25] for growth diagrams via the usual reverse RSK.

2.1. SSYT and reverse SSYT. A weak composition $\gamma=(\gamma_1\ldots,\gamma_n)$ is a vector in $\mathbb{N}^n$. A weak composition γ whose entries are in weakly decreasing order, that is, $\gamma_1\geq\cdots\geq\gamma_n$, is said to be a partition. Every weak composition γ determines a unique partition obtained by arranging the entries in weakly decreasing order. More precisely, it is the unique partition in the orbit of γ regarding the usual action of symmetric group $\mathfrak{S}_n$ on $\mathbb{N}^n$. A partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ is identified with its Young diagram (or Ferrers shape) $dg(\lambda)$ in French convention, an array of left-justified cells (boxes) with λ_i cells in row i from the bottom, for $1 \leq i \leq n$. The cells are located in the diagram $dg(\lambda)$ by their row and column indices (i, j), where $1 \leq i \leq n$ and $1 \leq j \leq \lambda_i$.

A filling of shape λ (or a filling of $dg(\lambda)$), in the alphabet $[n]$, is a map $T : dg(\lambda) \to [n]$. A semi-standard Young tableau (SSYT) T of shape $sh(T) = \lambda$, in the alphabet $[n]$, is a filling of $dg(\lambda)$ which is weakly increasing in each row from left to right and strictly increasing up in each column. The column word of SSYT T is the word, which consists of the entries of each column, read top to bottom and left to right. The content or weight of T is the content or weight of its column word, which is the weak composition $c(T) = (\alpha_1, \ldots, \alpha_n)$ such that α_i is the multiplicity of i in the column word of T. A key tableau is a SSYT such that the set of entries in the $(j+1)^{th}$ column is a subset of the set of entries in the j^{th} column, for all j. There is a bijection [28] between weak compositions in $\mathbb{N}^n$ and keys in the alphabet $[n]$ given by $\gamma \to key(\gamma)$, where $key(\gamma)$ is the SSYT such that for all j, the first γ_j columns contain the letter j. Any key tableau is of the form $key(\gamma)$ where γ is the content and the shape is the unique partition in its $\mathfrak{S}_n$-orbit.

A reverse semi-standard Young tableau (RSSYT), $\widetilde{T}$, of shape $sh(\widetilde{T}) = \lambda$, in the alphabet $[n]$, is a filling of $dg(\lambda)$ such that the entries in each row are weakly decreasing from left to right, and strictly decreasing from bottom to top.

Example 1. *The Ferrers diagram of* $\lambda = (4, 2, 1)$*, a SSYT and a RSSYT of shape* $\lambda = (4, 2, 1)$*, with respectively* $c(T) = (2, 2, 1, 1, 1)$ *and* $c(\widetilde{T}) = (1, 2, 2, 1, 1)$*,*

$$\begin{array}{|c|c|c|c|}\hline \ & & & \\ \hline \ & \ & & \\ \hline \ & \ & \ & \ \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|}\hline 4 & & & \\ \hline 2 & 5 & & \\ \hline 1 & 1 & 2 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & & & \\ \hline 4 & 2 & & \\ \hline 5 & 3 & 3 & 2 \\ \hline \end{array}$$

$dg(\lambda)$ *SSYT* T *RSSYT* $\widetilde{T}$

2.2. SSAFs are in bijection with RSSYTs. A weak composition $\gamma = (\gamma_1, \ldots, \gamma_n)$ is visualised as a diagram consisting of n columns, with γ_j boxes in column j, for $1 \leq j \leq n$. Formally, the *column diagram* of γ is the set $dg'(\gamma) = \{(i, j) \in \mathbb{N}^2 : 1 \leq j \leq n, 1 \leq i \leq \gamma_j\}$ where the coordinates are in French convention, i indicates the vertical coordinate, indexing the rows, and j the horizontal coordinate, indexing the columns. (The prime reminds that the components of γ are the columns.) The number of cells in a column is called the height of that column, and a cell a in a column diagram is written $a = (i, j)$, where i is the row index and j the column index. The augmented diagram of γ, $\widehat{dg}(\gamma) = dg'(\gamma) \cup \{(0, j) : 1 \leq j \leq n\}$, is the column diagram with n extra cells adjoined in row 0. This adjoined row is called the basement and it always contains the numbers 1 through n in strictly increasing order. The shape of $\widehat{dg}(\gamma)$ is defined to be γ. The empty augmented diagram consists of the basement elements from 1 through n.

We introduce now the semi-skyline augmented filling (SSAF) object as the output of the injective map ϱ, in [26], acting on RSSYTs. Let $\widetilde{P}$ be a RSSYT. Define the empty semi-skyline augmented filing (SSAF) as the empty augmented diagram with basement elements from 1 through n, where n is the maximum element of RSSYT, $\widetilde{P}$. Pick the first column of $\widetilde{P}$, say, P_1. Put the largest element of the first column in the top of the leftmost basement to avoid increasing columns from bottom to top, then put the next largest element in the top of the leftmost basement to have a weakly decreasing columns property, continue this manner to put all the elements of the first column of $\widetilde{P}$ to the top of the same basement elements. The new diagram is called the semi-skyline augmented filling corresponding to the first column of $\widetilde{P}$ and is denoted by SSAF. Assume that the first i columns of $\widetilde{P}$, denoted $\{P_1, P_2, \ldots, P_i\}$ have been mapped to a SSAF. Consider the largest element, α_1, in the $(i+1)$-th column P_{i+1}. There exists an element greater than or equal to α_1 in the i-th row of the SSAF. Place α_1 on top of the leftmost such element. Assume that the largest $k-1$ entries in P_{i+1} have been placed into the SSAF. The k-th largest element, α_k, of P_{i+1} is then placed into the SSAF. Place α_k on top of the leftmost entry β in row $k-1$ such that $\beta \geq \alpha_k$ and the cell immediately above β is empty. Continue this procedure until all entries in P_{i+1} have been mapped into the $(i+1)$-th row and then repeat for the remaining columns of $\widetilde{P}$ to obtain the semi-skyline augmented filling F.

It is clear that rotating 90 degrees F, sliding down the boxes in each column, and reordering them, in decreasing order from bottom to top, we obtain $\widetilde{P}$.

We can associate to each SSAF, F, a weak composition that records the length of the columns of F, and defines the shape of F, $sh(F)$. The content of the SSAF F is the vector $c(F) = c(\widetilde{P}) \in \mathbb{N}^n$ whose i-th entry is the multiplicity of the letter i in the SSAF F.

Example 2. *The SSAF corresponding to the RSSYT $\widetilde{P}$ defined by ϱ*

$$\begin{array}{cc} \begin{array}{|c|c|c|c|} \hline 1 & \multicolumn{3}{c}{} \\ \cline{1-1} 4 & 2 & \multicolumn{2}{c}{} \\ \hline 5 & 3 & 3 & 2 \\ \hline \end{array} & \quad \begin{array}{ccccc} & & & \boxed{2} & \\ & & & \boxed{3} & \\ & & & \boxed{3} & \boxed{2} \\ \boxed{1} & & & \boxed{4} & \boxed{5} \\ 1 & 2 & 3 & 4 & 5 \end{array} \\ \widetilde{P} & F = \varrho(\widetilde{P}) \end{array}$$

$$sh(\widetilde{P}) = (4,2,1) \qquad sh(F) = (1,0,0,4,2)$$

$$c(F) = (1,2,2,1,1) = c(\widetilde{P})$$

3. Reverse RSK, analogue of RSK and growth diagrams

The formulation of RSK in terms of growth diagrams is due to Fomin [5], subsequently developed by Roby [29] and van Leewven [23], and applied to enumeration by Krattenthaler [13]. The bijection ϱ between SSAFs and RSSYTs allows a growth diagrammatic formulation of the analogue of RSK for SSAFs [25] via reverse Schensted insertion.

3.1. The reverse RSK. The reverse Schensted insertion applied to the word $b_1 \dots b_m$, over the alphabet $[n]$, consists of reversing the roles of $\leq$ and $\geq$ in defining the Schensted insertion of $b_1 \dots b_m$, to get the reverse SSYT, $\widetilde{P}$. Equivalently, apply Schensted insertion to $-b_m, \dots, -b_1$ to get the SSYT, $P(-b_m, \dots, -b_1)$, and then change the sign back to positive in all entries of $P(-b_m, \dots, -b_1)$, to obtain the reverse SSYT $\widetilde{P}$ [31].

The two line array $w = \begin{pmatrix} j_1 & j_2 & \cdots & j_l \\ i_1 & i_2 & \cdots & i_l \end{pmatrix}$, $j_r < j_{r+1}$, or $j_r = i_{r+1}$ & $i_r \leq i_{r+1}$, $1 \leq i,j \leq l-1$, with i_r, $j_r \in [n]$, is called a biword in lexicographic order over the alphabet $[n]$. The reverse RSK algorithm is the obvious variant of the RSK algorithm [31], where we apply the RSK to the biword $\widetilde{w} = \begin{pmatrix} -j_n & \cdots & -j_1 \\ -i_n & \cdots & -i_1 \end{pmatrix}$, instead of $w = \begin{pmatrix} j_1 & \cdots & j_n \\ i_1 & \cdots & i_n \end{pmatrix}$, and then change the sign back to positive of all entries of the pair of SSYTs. We will obtain a pair $(\widetilde{P}, \widetilde{Q})$ of reverse SSYTs.

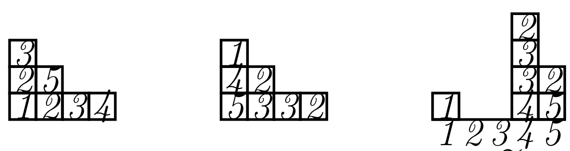
3.2. Analogue of Schensted insertion and reverse Schensted insertion. SSAFs in bijection with SSYTs by assigning the right key.

The fundamental operation of the Robinson-Schensted-Knuth [12] (RSK) algorithm is Schensted insertion which is a procedure for inserting a positive integer k into a semi-standard Young tableau T. In [25] it is defined a similar procedure for inserting a positive integer k into a SSAF F, which is used to describe an analogue of the RSK algorithm. Based on this Schensted insertion analogue, it is given a weight preserving and a shape rearranging bijection Ψ between SSYT and SSAF over the alphabet $[n]$. The bijection Ψ is defined to be the insertion, from right to left, of the column word which consists of the entries of each column, read top to bottom from columns left to right, of a SSYT into the empty SSAF with basement $1, \ldots, n$. The shape of $\Psi(T)$ provides the right key, $K_+(T)$, of T, a notion due to Lascoux and Schützenberger [16].

Theorem 1. [26] *Given an arbitrary SSYT T, let γ be the shape of $\Psi(T)$. Then $K_+(T) = key(\gamma)$.*

On the other hand, applying the reverse Schensted insertion to the column word of the SSYT, T, gives the RSSYT $\widetilde{T}$. Then $\rho(\widetilde{T})$ is a SSAF and $\rho(\widetilde{T}) = \Psi(T)$ [25]. We have then two equivalent weight preserving and shape rearranging bijections between SSYTs and SSAFs.

Example 3. T *SSYT* $\to \widetilde{T}$ *RRSYT* $\to \rho(\widetilde{T})$ *SSAF*



T $\qquad\widetilde{T}\qquad$ $\rho(\widetilde{T})$

$$K_+(T) = key(1, 0, 0, 4, 2)$$

3.3. RSK, reverse RSK and analogue of RSK for SSAFs. Given the alphabet $[n]$, the RSK algorithm is a bijection between biwords in lexicographic order and pairs of SSYT of the same shape over $[n]$. The analogue of Schensted insertion is applied in [25] to find an analogue Φ of the RSK for SSAF. The map Φ defines a bijection between the set of all biwords w in lexicographic order in the alphabet $[n]$, and pairs of SSAFs whose shapes are rearrangements of a same partition in $\mathbb{N}^n$, and the contents are respectively those of the second and first rows of w. The bijection Φ applied to a biword

w is the same as applying the reverse RSK to w and then apply ρ to each reverse SSYT of the output pair $(\widetilde{P}, \widetilde{Q})$, that is, $\Phi(w) = (\varrho(\widetilde{P}), \varrho(\widetilde{Q}))$.

Corollary 1. [25, 26] *The RSK algorithm commutes with the above analogue* Φ. *That is, if* (P, Q) *is the pair of SSYTs produced by RSK algorithm applied to biword* w, *then* $(\Psi(P), \Psi(Q)) = \Phi(w)$, *and* $K_+(P) = key(sh(\Psi(P)))$, $K_+(Q) = key(sh(\Psi(Q)))$.

The relation between RSK, the reverse RSK and the RSK analogue Φ, is summarised in the following scheme from which, in particular, it is clear the RSK analogue Φ also shares the symmetry of RSK,

$$\begin{tikzcd} & w \arrow[dl, dash, "\text{reverse RSK}"'] \arrow[d, dash, "\Phi"] \arrow[dr, dash, "\text{RSK}"] & \\ (\widetilde{P}, \widetilde{Q}) \arrow[r, dash, "\rho"'] & (F, G) \arrow[r, dash, "\Psi"'] & (P, Q) \end{tikzcd}$$

$$\begin{gathered} sh(F)^+ = sh(G)^+ = sh(P) = sh(Q) = sh(\widetilde{P}) = sh(\widetilde{Q}) \\ key(sh(F)) = K_+(P),\ key(sh(G)) = K_+(Q) \\ c(F) = c(P) = c(\widetilde{P}),\ c(G) = c(Q) = c(\widetilde{Q}) \end{gathered}$$

3.4. Reverse RSK and analogue of RSK in terms of Fomin's growth diagrams. In this subsection we follow very close [13, 31]. Let w be the biword in the lexicographic order over alphabet $[n]$. We can represent a biword w in the $n \times n$ square grid by putting the number r in the cell (i, j) of the square, when the biletter $\begin{pmatrix} j \\ i \end{pmatrix}$ appears r times in the biword w.

Example 4. *The diagram corresponding to* $w = \begin{pmatrix} 1 & 1 & 2 & 3 & 4 & 4 & 5 & 7 & 7 \\ 2 & 7 & 2 & 4 & 1 & 3 & 3 & 1 & 1 \end{pmatrix}$ *is*

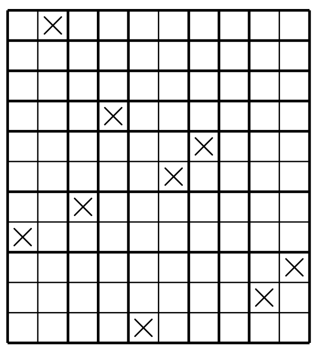
1						
		1				
			1	*1*		
1	*1*					
			1			*2*

where the rows are counted from bottom to top and the columns from left to right.

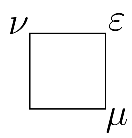
We would like to have a 01-filling of the diagram, that is, at most one 1 in each row and each column. To remedy this, the entries in the diagram are separated in the following way.

Construct a rectangle diagram with more rows and columns so that entries which are originally in the same column or in the same row are put in different columns and rows in the larger diagram, and that an entry m is replaced by m 1's in the new diagram all of them placed in different rows and columns. Separate the entries in a row from bottom/left to top/right, as before the 1's are represented by $\times$'s and 0's are suppressed. If there should be several entries in a column as well, separate entries in a column from bottom/left to top/right. In the cell with entry m we replace m by a chain of m $\times$'s arranged from bottom/left to top/right. In the figure, the original columns and rows are indicated by thick lines, whereas the newly created columns and rows are indicated by thin lines. This process of transforming a filling into a $0-1$ filling is called standardization. The biword defined by the standard filling is said to be the standardisation of the biword w.

Example 5. *The standardisation of w is* $\widetilde{w} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 9 & 10 \\ 4 & 11 & 5 & 8 & 1 & 6 & 7 & 2 & 3 \end{pmatrix}$, *corresponding to the standardised diagram*



To give an interpretation of reverse RSK in terms of growth diagrams, we start by assigning the empty partition $\emptyset$ to each corner on the right and top edges of the 01-filling F. Then assign the partitions to the other corners inductively by applying the following backward local rules. Consider the cell below, labeled by the partitions ε, μ, ν, where $\varepsilon \subseteq \mu$ and $\varepsilon \subseteq \nu$, μ and ε differ by one box, and ν and ε differ by one box. Then λ is determined as follows:



- If $\varepsilon = \mu = \nu$, and if there is no cross in the cell, then $\lambda = \varepsilon$.
- If $\varepsilon = \mu \neq \nu$, then $\lambda = \nu$.
- If $\varepsilon = \nu \neq \mu$, then $\lambda = \mu$.
- If ε, μ, ν are pairwise different, then $\lambda = \mu \cup \nu$.
- If $\varepsilon \neq \mu = \nu$, then λ is formed by adding a square to the $(k+1)$-st row of $\mu = \nu$, given that $\mu = \nu$ and ε differ in the k-th row.
- If $\varepsilon = \mu = \nu$, and if there is a cross in the cell, then λ is formed by adding a square to the first row of $\varepsilon = \mu = \nu$.

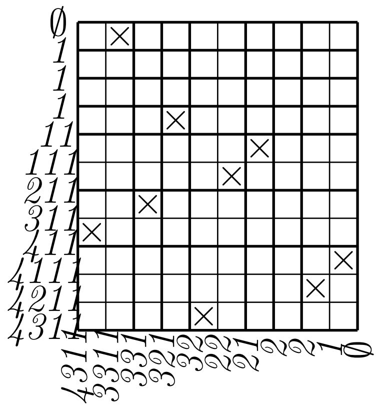
Applying backward the local rules leads to a pair of sequences of partitions on the left and in the bottom of growth diagram. The partitions of each sequence are related by containment. Let λ_i be the partition associated to the i-th thick column line on the bottom of the growth diagram when we scan the thick column lines from right to left. Then the bottom side of the growth diagram is a sequence of partitions $\emptyset = \lambda_0 \subseteq \lambda_1 \subseteq \cdots \subseteq \lambda_l$, where l is the maximum element in the first row of the biword w and λ_i/λ_{i-1} is a horizontal strip. Let $\underline{\lambda}_i$ be the partition associated to the i-th row thick line on the left of the growth diagram when we scan the thick row lines from top to bottom. Then the left hand side of the growth diagram is a sequence of partitions $\emptyset = \underline{\lambda}_0 \subseteq \underline{\lambda}_1 \subseteq \cdots \subseteq \underline{\lambda}_t$, where t is the maximum element in the second row of the biword w and $\underline{\lambda}_i/\underline{\lambda}_{i-1}$ is a horizontal strip. Fill with $n+1-i$ all the squares in λ_i/λ_{i-1} and $\underline{\lambda}_i/\underline{\lambda}_{i-1}$, for $i \geq 1$. This pair of nested sequences of partitions defines a pair $(\widetilde{P}, \widetilde{Q})$ of RSSYTs of the same shape with contents, respectively, of the second and the first rows of w which is the same as applying the reverse RSK to the biword w.

In addition there is a global description of the backward local rules as a consequence of a variant of Greene's theorem [8] and Theorem 2 in [13]. A SW-chain of a 01-filling is a sequence of 1's such that any 1 is below and to the left of the preceding 1 in the sequence. The length of a SW-chain is defined to be the number of 1's in the chain. Another way to find the nested sequences of partitions on the bottom and on the left of the diagram is just looking for the k SW-chains by using the following natural version of the Theorem 2 in [13].

Theorem 2. *Given a diagram with empty partitions labelling all the corners along the right side and the top side of a rectangle shape, which has been completed according to the reverse RSK local rules, the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ labelling corner c satisfies the following property:*

For any k, the maximal cardinality of the union of k SW-chains situated in the rectangular region to the right and above of c is equal to $\lambda_1 + \lambda_2 + \cdots + \lambda_k$. In particular, λ_1 is the length of the longest SW-chain in the rectangular region to the right and above c.

Example 6. *The growth diagram corresponding to the reverse RSK for w and the pair of RSSYTs are,*



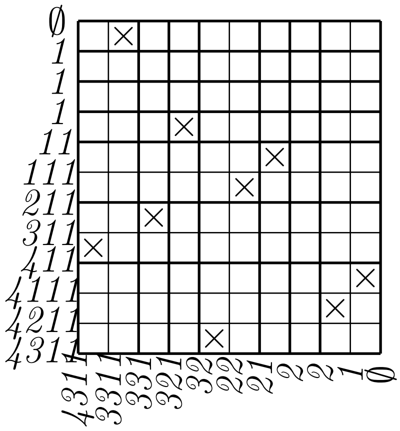
$$\widetilde{P} = \begin{array}{llll} 1 & & & \\ 3 & & & \\ 4 & 1 & 1 & \\ 7 & 3 & 2 & 2 \end{array} \qquad \widetilde{Q} = \begin{array}{llll} 1 & & & \\ 3 & & & \\ 5 & 4 & 2 & \\ 7 & 7 & 4 & 1 \end{array}$$

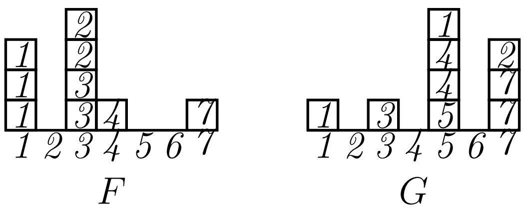
The map ρ allows us to find the pair of SSAFs from the growth diagram corresponding to the reverse RSK. Recall that the shape of a SSAF is the weak composition γ that records the length of the columns. The partition that we get in each step of the growth diagram is the shape of the RSSYT, a rearrangement of the shape of SSAF. Let λ_i be the partition associated to the i-th thick column line on the bottom of the growth diagram when we scan the thick column lines from right to left, and consider the sequence of partitions $\{\lambda_{i_j}\}$, where $\lambda_i = \lambda_{i_1} \subseteq \cdots \subseteq \lambda_{i_{l_i}}$, associated respectively with the thick column line i, and with the $l_i - 1$ thin column lines strictly in between the two thick column lines i and $i+1$, counted from right to left. Start with the rightmost partition of the bottom sequence in the growth diagram and also with an empty SSAF. When we arrive to the partition λ_{i_j} we put a cell, filled with $n+1-i$, in the leftmost possible place of the SSAF with basement 1 through n, such that the shape of the new SSAF becomes a rearrangement of the partition λ_{i_j} and the decreasing property on the columns of SSAF, from bottom to top, is preserved.

Similarly let $\underline{\lambda}_i$ be the partition associated to the i-th thick row line on the left hand of the growth diagram when we scan the thick row lines from top to bottom, and consider the sequence of partitions $\{\underline{\lambda}_{i_j}\}$, where $\underline{\lambda}_i = \underline{\lambda}_{i_1} \subseteq \cdots \subseteq \underline{\lambda}_{i_{\underline{l}_i}}$, associated respectively with the thick row line i, and with the thin

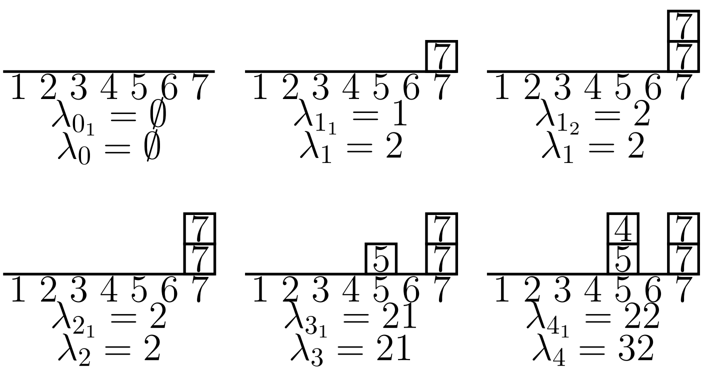
$\underline{l}_i - 1$ row lines strictly in between the two thick row lines i and $i+1$, counted from top to bottom. Start with the topmost partition of the left sequence in the growth diagram and also with an empty SSAF. When we arrive to the partition $\underline{\lambda}_{i_j}$ we put a cell filled with $n+1-i$ in the leftmost possible place of the SSAF with basement 1 through n, such that the shape of the new SSAF becomes a rearrangement of the partition $\underline{\lambda}_{i_j}$ and the decreasing property on the columns of SSAF, from bottom to top, is preserved.

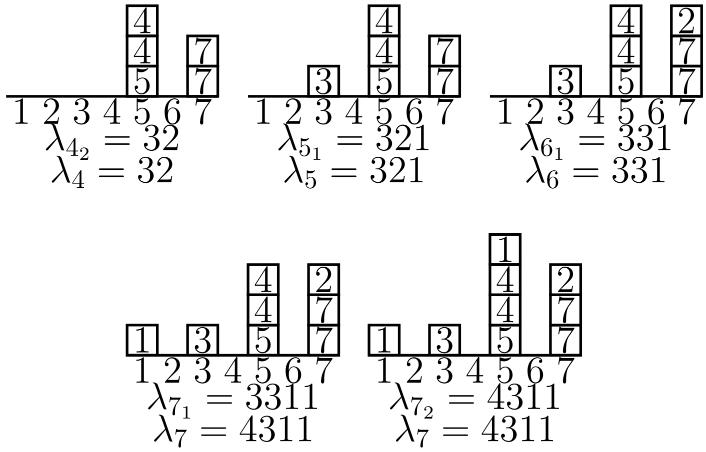
Example 7. *The growth diagram corresponding to the analogue of RSK for w and the corresponding pair of SSAF are,*





Below we can see how each step, on the left hand side of the growth diagram, produces a SSAF.





4. Bruhat order in $\mathfrak{S}_n$ and in a $\mathfrak{S}_n$-orbit

Let $\theta = \theta_1 \dots \theta_n \in \mathfrak{S}_n$, written in one line notation. A pair (i, j), with $i < j$, such that $\theta_i > \theta_j$, is said to be an inversion of θ, and $\ell(\theta)$ denotes the number of inversions of θ. The Bruhat order in $\mathfrak{S}_n$ is the partial order in $\mathfrak{S}_n$ defined by the transitive closure of the relations

$$\theta < t\theta, \text{ if } \ell(\theta) < \ell(t\theta), (t \text{ transposition}, \theta \in \mathfrak{S}_n).$$

We may write $\alpha < \beta$ in the Bruhat ordering of $\mathfrak{S}_n$ if $\ell(\alpha) < \ell(\beta)$ and $\beta = \tau\alpha$ for some permutation τ in $\mathfrak{S}_n$ that can be written as a product of transpositions each increasing the number of inversions when passing from α to β.

Let $\theta = s_{i_N} \cdots s_{i_1}$ be a decomposition of θ into simple transpositions $s_i = (i\, i+1)$, $1 \le i < n$. When $N = \ell(\theta)$, the number N in a such decomposition is minimised, and we say that we have a reduced decomposition of θ.

Let λ be a partition in $\mathbb{N}^n$. The Bruhat ordering of the orbit of λ, $\mathfrak{S}_n\lambda$, is defined by taking the transitive closure of the relations

$$\alpha < t\alpha, \text{ if } \alpha_i > \alpha_j,\ i < j, \text{ and } t \text{ the transposition } (i\, j),\ (\alpha \in \mathfrak{S}_n\lambda).$$

Given $\alpha \in \mathbb{N}^n$, a pair (i, j), with $i < j$, such that $\alpha_i < \alpha_j$, is called an inversion of α, and $\iota(\alpha)$ denotes the number of inversions of α. We may write $\alpha < \beta$ if $\iota(\alpha) < \iota(\beta)$ and $\beta = \tau\alpha$ for some permutation τ in $\mathfrak{S}_n$ that can be written as a product of transpositions each increasing the number of inversions when passing from α to β.

5. Demazure operators, Demazure characters and Demazure atoms

Isobaric divided difference operators [20], or Demazure operators [4], π_i and $\widehat{\pi}_i$, $1 \le i < n$, act on $\mathbb{Z}[x_1, \dots, x_n]$ by

$$\pi_i f = \frac{x_i f - s_i(x_i f)}{x_i - x_{i+1}}, \tag{10}$$

$$\widehat{\pi}_i f = (\pi_i - 1) f = \pi_i f - f, \tag{11}$$

where the simple transposition s_i of $\mathfrak{S}_n$ acts on f swapping x_i with x_{i+1}, and 1 is the identity operator on $\mathbb{Z}[x_1, \dots, x_n]$. It follows from the definition that $\pi_i(f) = f$ and $\widehat{\pi}_i(f) = 0$ if and only if $s_i f = f$. They both satisfy the commutation and the braid relations of $\mathfrak{S}_n$, $\pi_i \pi_j = \pi_j \pi_i$, $\hat{\pi}_i \hat{\pi}_j = \hat{\pi}_j \hat{\pi}_i$ for $|i - j| > 1$, and $\pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1}$, $\hat{\pi}_i \hat{\pi}_{i+1} \hat{\pi}_i = \hat{\pi}_{i+1} \hat{\pi}_i \hat{\pi}_{i+1}$, and this guarantees that, for any permutation $\sigma \in \mathfrak{S}_n$, there exists a well defined isobaric divided difference $\pi_\sigma := \pi_{i_N} \cdots \pi_{i_2} \pi_{i_1}$ and $\hat{\pi}_\sigma := \hat{\pi}_{i_N} \cdots \hat{\pi}_{i_2} \hat{\pi}_{i_1}$, where $s_{i_N} \dots s_{i_2} s_{i_1}$ is any reduced expression of $\mathfrak{S}_n$. In addition, they satisfy the quadratic relations $\pi_i^2 = \pi_i$ and $\hat{\pi}_i^2 = -\hat{\pi}_i$.

The 0-Hecke algebra $H_n(0)$ of $\mathfrak{S}_n$, a deformation of the group algebra of $\mathfrak{S}_n$, is an associative $\mathbb{C}$-algebra generated by $T_1, \dots, T_{n-1}$ satisfying the commutation and the braid relations of the symmetric group $\mathfrak{S}_n$, and the quadratic relation $T_i^2 = T_i$ for $1 \le i < n$. Setting $\widehat{T_i} := T_i - 1$, for $1 \le i < n$, one obtains another set of generators of the 0-Hecke algebra $H_n(0)$. The sets $\{T_\sigma : \sigma \in \mathfrak{S}_n\}$ and $\{\hat{T}_\sigma : \sigma \in \mathfrak{S}_n\}$ are both linear basis for $H_n(0)$, where $T_\sigma = T_{i_N} \cdots T_{i_2} T_{i_1}$ and $\hat{T}_\sigma := \hat{T}_{i_N} \cdots \hat{T}_{i_2} \hat{T}_{i_1}$, for any reduced expression $s_{i_N} \cdots s_{i_2} s_{i_1}$ in $\mathfrak{S}_n$. Since Demazure operators (10) or bubble sort operators satisfy the same relations as T_i, and similarly for isobaric divided difference operators (11) and $\widehat{T_i}$, the 0-Hecke algebra $H_n(0)$ of $\mathfrak{S}_n$ may be viewed as an algebra of operators realised either by any of the two isobaric divided differences, or by bubble sort operators, swapping entries i and $i + 1$ in a weak composition α, if $\alpha_i > \alpha_{i+1}$, and doing nothing, otherwise. Therefore, the two families $\{\pi_\sigma : \sigma \in \mathfrak{S}_n\}$ and $\{\hat{\pi}_\sigma : \sigma \in \mathfrak{S}_n\}$ are both linear basis for $H_n(0)$, and from the relation $\widehat{\pi}_i = \pi_i - 1$, the change of basis from the first to the second is given by a sum over the Bruhat order in $\mathfrak{S}_n$, $\pi_\sigma = \sum_{\theta \le \sigma} \hat{\pi}_\theta$ [17, 27]. Key polynomials and Demazure atoms can be defined through Demazure operators, $\kappa_\alpha = \pi_\sigma x^\lambda$ where $\alpha = \sigma\lambda$ and λ a partition, and similarly $\widehat{\kappa}_\alpha = \widehat{\pi}_\sigma x^\lambda$ (assume σ a minimal coset representative modulo stabiliser of

λ). Thereby, key polynomials or Demazure characters are decomposed into Demazure atoms [16, 20],

$$\kappa_\alpha = \sum_{\beta \leq \alpha} \widehat{\kappa}_\beta. \tag{12}$$

6. Crystal operators and growth diagrams

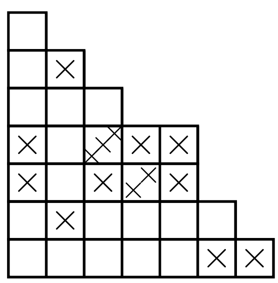
Crystal operators or coplactic operations e_r, f_r, $1 \leq r < n$, can be defined on any word over the alphabet $[n]$. These operations can be also extended to biwords. For details see [14, 18]. Consider the following biword in lexicographic order over the alphabet [7],

$$w = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 5 & 5 & 6 & 7 \\ 3 & 4 & 2 & 6 & 3 & 4 & 4 & 4 & 3 & 3 & 4 & 3 & 4 & 1 & 1 \end{pmatrix}.$$

The crystal operator e_3 acts on the second row of w as follows: ignoring all the entries different from 3 and 4, from the second row of the biword w, one gets 34344433434; match in the usual way all 43 (in blue in the example below) and it remains the subword 344; change to 3 the leftmost 4. For example, applying twice e_3 to w, it means to apply twice e_3 to the second row, and the the subword 344 change to 333, and one obtains

$$34344433434 \xrightarrow{e_3} 34334433434 \xrightarrow{e_3} 34334433433$$

Recalling the presentation of a biword in a rectangle defined in Section 3.4, we represent the biword w in the Ferrers shape $\lambda = (7, 6, 5, 5, 3, 2, 1)$ by putting a cross $\times$ in the cell (i, j) of λ if $\begin{pmatrix} \mathrm{j} \\ \mathrm{i} \end{pmatrix}$ is a biletter of w. The biword

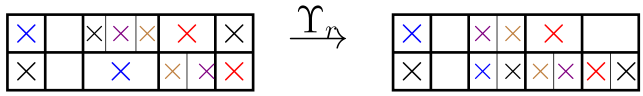


$\lambda = (7, 6, 5, 5, 3, 2, 1)$

FIGURE 1. Reperesentation of the biword w in a Ferrers shape.

w can be recovered, from this representation, by scanning the columns of the Ferrers shape λ, left to right, and bottom to top.

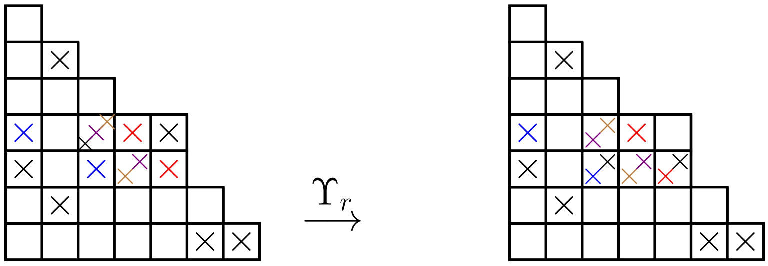
Let w be a biword in lexicographic order represented in the Ferrers shape λ. We introduce an operation Υ_r in the rows r and $r+1$ of λ which consists of matching crosses in rows r and $r+1$, and then sliding down the unmatched crosses from row $r+1$ to row r. This slide of crosses translates to the action of the operator e_r, as long as it is possible, on the second row of the biword w. The operation Υ_r is the analogue of applying m times the crystal operator e_r, to the second row of w, where m is the number of unmatched $r+1$ in the second row of w. Therefore, we also write $\Upsilon_r w$ to mean the biword obtained by applying m times the crystal operator e_r, to the second row of w, where m is the number of unmatched $r+1$ in the second row of w. We scan from left to right and match crosses in rows $r+1$ and r that are in the following way: $\begin{smallmatrix}\times & \\ & \times\end{smallmatrix}$, it means that we match a cross of row $r+1$ with the cross to its SE in row r such that there is no unmatched cross in the columns between them, within the rows under consideration. If there are more than one cross in the same cell then order them from left to right and consider them in different sub columns. Next move all the unmatched crosses of the row $r+1$ to the row r.



$$\begin{pmatrix}1&1&3&3&3&3&4&4&4&5&5\\3&4&3&4&4&4&3&3&4&3&4\end{pmatrix}\begin{array}{c}\xleftarrow{f_r^2}\\ \xrightarrow[e_r^2]{}\end{array}\begin{pmatrix}1&1&3&3&3&3&4&4&4&5&5\\3&4&3&3&4&4&3&3&4&3&3\end{pmatrix}$$

The action of the crystal operator f_r on the biword w is defined by the action of f_r on the second row.

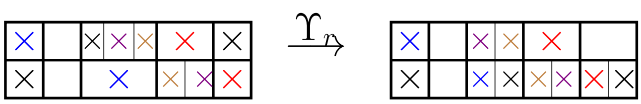
The new set of cells of λ, defined by the crosses, yields a new biword $\Upsilon_r w$, scanning λ along columns from left to right and bottom to top. The biword $\Upsilon_r w$ is obtained from the biword w by applying the crystal operator e_r as long as it is possible to the second row of the biword w.



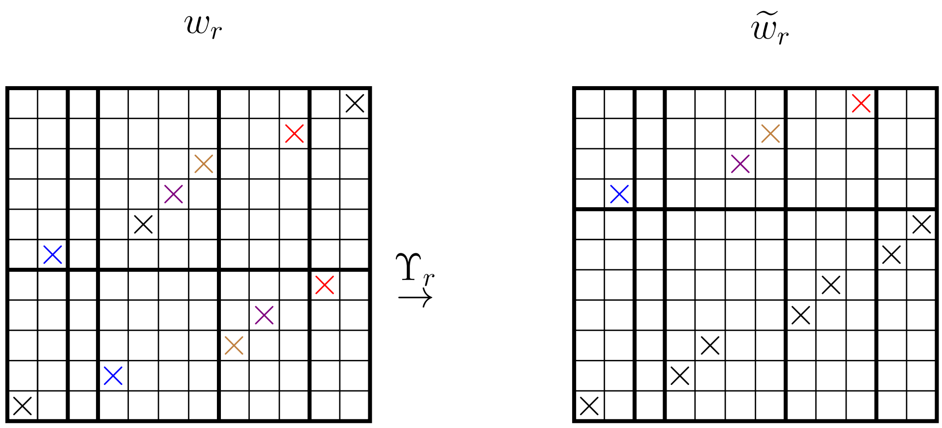
$$\begin{pmatrix} 1\,1\,2\,2\,3\,3\,3\,3\,4\,4\,4\,5\,5\,6\,7 \\ 3\,4\,2\,6\,3\,4\,4\,4\,3\,3\,4\,3\,4\,1\,1 \end{pmatrix} \xrightarrow{\Upsilon_r} \begin{pmatrix} 1\,1\,2\,2\,3\,3\,3\,3\,4\,4\,4\,5\,5\,6\,7 \\ 3\,4\,2\,6\,3\,3\,4\,4\,3\,3\,4\,3\,3\,1\,1 \end{pmatrix}$$

Consider now the two 01-fillings of the biwords w and $\Upsilon_r w$ represented in the Ferrers shapes λ, and apply the backward local rules to them, as defined in Sections 3.4. Notice that in the 01-filling of $\Upsilon_r w$, we match a cross of row $r+1$ with a cross to the SE, in row r, such that in these two rows there is no unmatched cross in a column between them.

These two growth diagrams have the same bottom sequences of partitions and the left sequences are different only in the partitions associated to the rows r and $r+1$. It is proved in [18] that the bottom sequence is preserved by the operations e_r and f_r, when the entries of the first row of the biword w are distinct. In the 01-filling we have standardized the first row of the biword w thus the bottom sequence is preserved, and therefore the same happens when the first row of the biword has repeated letters. Let w_r and $\widetilde{w}_r$ be the biwords that are obtained from w and $\Upsilon_r w$, after deleting all the biletters whose second rows are different from r and $r+1$. The translation of the movement of the cells in the Ferrers shape to the 01-filling is as follows: in the 01-filling of w_r, move up, without changing of columns, the matched crosses of row $r+1$, say s crosses, to the top most s rows such that they form SW chain. Then slide down the remaining unmatched crosses, from row $r+1$ to row r, without changing of columns, such that these crosses and all the crosses of row r form a SW chain. The result is the 01-filling corresponding to $\widetilde{w}_r$.



$$\begin{pmatrix} 1&1&3&3&3&3&4&4&4&5&5\\ 3&4&3&4&4&4&3&3&4&3&4\end{pmatrix} \underset{\xrightarrow[e_r^2]{}}{\xleftarrow{f_r^2}} \begin{pmatrix} 1&1&3&3&3&3&4&4&4&5&5\\ 3&4&3&3&4&4&3&3&4&3&3\end{pmatrix}$$



It is clear that the longest SW chain in the first k columns, from right to left, of the 01-filling of w_r and of $\widetilde{w}_r$, has length equal to the total number of crosses in row r and row $r+1$, of those columns, minus the number of matched crosses in row $r+1$, of those columns. It means that the length of the longest SW chain in the first k columns, from right to left is preserved. Theorem 2 implies that the bottom sequences in growth diagrams corresponding to w_r and $\widetilde{w}_r$ are the same. Let Θ_r be the analogue operator of f_r for SSAF defined in [26]. Since the operator Θ_r has the same behaviour as the operator f_r, the next scheme shows the relation between the action of crystal operator f_r, its analogue Θ_r, the RSK and the analogue of RSK

$$\begin{tikzcd} w = \begin{pmatrix} a \\ b \end{pmatrix} \arrow[r, leftrightarrow, "RSK"] & (P,Q) \arrow[r, leftrightarrow, "\Psi"] & (F,G) \\ \widetilde{w} = \begin{pmatrix} a \\ e_r(b) \end{pmatrix} \arrow[r, leftrightarrow, "RSK"] & (e_rP,Q) \arrow[u, "f_r"] \arrow[r, leftrightarrow, "\Psi"] & (\widetilde{F},G) \arrow[u, "\Theta_r"] \end{tikzcd} \tag{13}$$

If F is SSAF, put $\Upsilon_r F$ such that $\Theta_r^m(\Upsilon_r F) = F$ where m is the number of unmatched $r+1$ in the row reading (left to right and top to bottom) of the SSAF F. Equivalently, if $F = \Psi(P)$ with P a SSYT, then $\Upsilon_r F = \Psi(e_r^m P)$ where m is the number of unmatched $r+1$ in P.

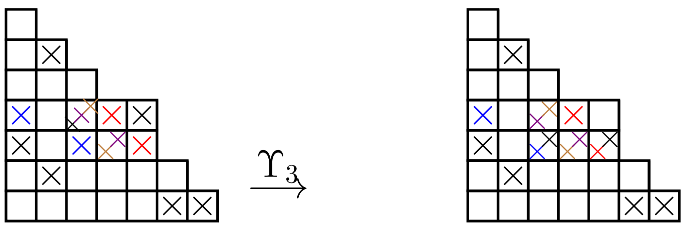
Next theorem is therefore a consequence of our discussion.

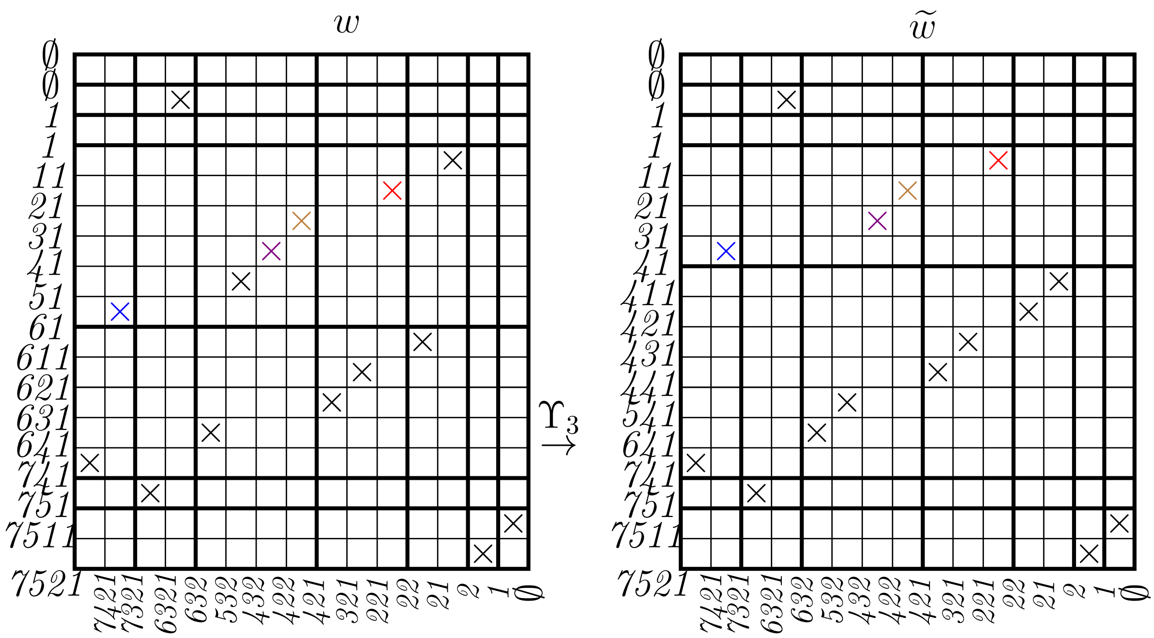
Theorem 3. *Let w be a biword in lexicographic order. If $\Phi(w) = (F, G)$ then $\Phi(\Upsilon_r w) = (\Upsilon_r F, G)$.*

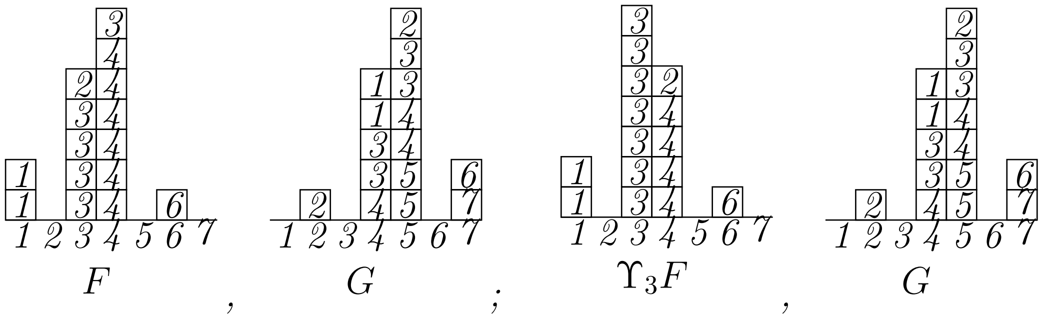
Example 8. *The procedure of passing from a biword under the action of the operator Υ_r to a pair of SSAFs, where $n = 7$, $r = 3$.*

$$w = \begin{pmatrix} 1\,1\,2\,2\,3\,3\,3\,3\,4\,4\,4\,5\,5\,6\,7 \\ 3\,4\,2\,6\,3\,4\,4\,4\,3\,3\,4\,3\,4\,1\,1 \end{pmatrix} \xrightarrow{\Upsilon_3} \begin{pmatrix} 1\,1\,2\,2\,3\,3\,3\,3\,4\,4\,4\,5\,5\,6\,7 \\ 3\,4\,2\,6\,3\,3\,4\,4\,3\,3\,4\,3\,3\,1\,1 \end{pmatrix}$$

The biletters (i, j) satisfy $i + j \leq 8 + 1$ in w, and $i + j \leq 7 + 1$ in $\Upsilon_3 w$. In particular, the biletter $(r+1, 5) = (4, 5)$ in w is transformed to $(r, 5) = (3, 5)$ in $\Upsilon_3 w$.







$$sh(F) = (2,0,5,7,0,1,0) \quad ; \quad sh(\Upsilon_3 F) = (2,0,7,5,0,1,0) = s_3 sh(F)$$

Theorem 4. [3] *Let λ be a Ferrers shape where $\lambda_r = \lambda_{r+1} > \lambda_{r+2} \geq 0$, for some $r \geq 1$. Let w be a biword consisting of a multiset of cells of λ containing the cell $(r+1, \lambda_{r+1})$. Let $\Phi(w) = (F, G)$. If $sh(F) = \nu$ then $\nu_r < \nu_{r+1}$ and $sh(\Upsilon_r F) = s_r \nu$. Moreover, $\Upsilon_r w$ does not contain the biletter $\binom{\lambda_{r+1}}{r+1}$ and therefore fits the Ferrers shape λ with the cell $(r+1, \lambda_{r+1})$ deleted.*

Example 8 illustrates Theorem 4.

Transposing the Ferrers shape λ means to swap the first row and the second row of the biword w, and to transpose, through the secondary diagonal, the growth diagram of the 01-filling of w. Therefore, the move of crosses on rows can be translated to a move of crosses on columns. As a consequence of the symmetry of the growth diagram we have the following versions of Theorem 3 and Theorem 4. Swap the rows of w and then rearrange it in lexicographic order. This new biword is denoted by w^*. Let $\Upsilon_r^* w := \Upsilon_r w^*$.

Corollary 2. [3] *If $\Phi(w) = (F, G)$ then $\Phi(\Upsilon_r^* w) = (F, \Upsilon_r G)$.*

Corollary 3. [3] *Let λ be a Ferrers shape and let $\overline{\lambda} = (\lambda'_1, \lambda'_2, \ldots, \lambda'_{\lambda_1})$ be the conjugate of λ where $\lambda'_r = \lambda'_{r+1} > \lambda'_{r+2}$. Let w be a biword consisting of a multiset of cells of λ containing the cell $(\lambda'_{r+1}, r+1)$. Let $\Phi(w) = (F, G)$. If $sh(G) = \nu$ then $\nu_r < \nu_{r+1}$ and $sh(\Upsilon_r G) = s_r \nu$. Moreover, $\Upsilon_r^* w$ does not contain the biletter $\binom{r+1}{\lambda'_{r+1}}$ and therefore fits the Ferrers shape λ with the cell $(\lambda'_{r+1}, r+1)$ deleted.*

Proposition 1. [3] *Let F be a SSAF with shape ν, and $\nu_r < \nu_{r+1}$, for some $r \geq 1$. Then $sh(\Upsilon_r F) = s_r \nu$.*

6.1. The bijection. Next theorem characterizes the biwords whose biletters constitute a multiset of cells in a staircase possibly plus a layer of boxes sited on the stairs of the staircase, in French convention, leaving free the top of the first column and the end of the first row.

Let $SSAF_n$ be the set of all SSAFs with basement $1,\dots,n$.

Theorem 5. (1 NW inner layer) *Let w be a biword in lexicographic order on the alphabet $[n]$, and let $\Phi(w)=(F,G)\in SSAF_n^2$, with $sh(F)=\nu$ and $sh(G)=\beta$. Let $0\le p<n$, $1\le r_1<\cdots<r_p<n$. For each biletter $\binom{i}{j}$ in w one has $i+j\le n+1$ except for the biletters $\binom{n-r_1+1}{r_1+1}$, ..., $\binom{n-r_p+1}{r_p+1}$, if and only if*

(a) $\beta\le\omega s_{r_p}\cdots s_{r_2}s_{r_1}\nu$,

(b) $\beta\not\le\omega s_{r_p}\cdots\widehat{s}_{r_i}\cdots s_{r_2}s_{r_1}\nu$, for $i=1,2,\dots,p$,

where $\widehat{\ }$ means omission.

Proof: By induction on p. For $p=0$, it is the main Theorem in [2]. For $p>0$, use Theorem 4 and Proposition 1. ■

7. A non-symmetric Cauchy kernel over near staircases

7.1. Some notation and a lemma. Given a finite set S and an integer $m\ge 0$, let $\binom{S}{m}$ denote the set of all m-element subsets of S.

Let $0\le p<n$ and $1\le r_1<r_2<\cdots<r_p<n$. For each $0\le z\le p$, and each $H_z=\{i_1<i_2<\cdots<i_z\}\in\binom{[p]}{z}$, define

$$\mathcal{A}_z^{H_z}:=\left\{(F,G)\in SSAF_n^2 : \begin{matrix} sh(G)\not\le\omega s_{r_{i_z}}\cdots\widehat{s}_{r_{i_m}}\cdots s_{r_{i_1}}sh(F),\ m=1,2,\dots,z \\ sh(G)\le\omega s_{r_{i_z}}\cdots s_{r_{i_1}}sh(F)\end{matrix}\right\}.$$

Put $\mathcal{A}:=\mathcal{A}_0^{\emptyset}=\left\{(F,G)\in SSAF_n^2 : sh(G)\le\omega sh(F)\right\}$.

For each $z=0,\dots,p-1$, and $H_z=\{2\le i_1<\cdots<i_z\}\in\binom{[2,p]}{z}$, where $[2,p]=[p]\setminus\{1\}$, let

$$\mathcal{B}_z^{H_z}:=\left\{(F,G)\in SSAF_n^2 : \begin{matrix} sh(F)_{r_1}<sh(F)_{r_1+1} \\ sh(G)\not\le\omega s_{r_{i_z}}\cdots\widehat{s}_{r_{i_m}}\cdots s_{r_{i_1}}s_{r_1}sh(F), m=1,2,\dots,z \\ sh(G)\le\omega s_{r_{i_z}}\cdots s_{r_{i_1}}s_{r_1}sh(F)\end{matrix}\right\}.$$

Lemma 1. *Given $1\le p<n$, for each $z=0,\dots,p-1$, and $H_z=\{2\le i_1<\cdots<i_z\}\in\binom{[2,p]}{z}$, let $H_{z+1}^1:=\{1\}\cup H_z$. Then*

$$\mathcal{B}_z^{H_z}=\{(F,G)\in\mathcal{A}_z^{H_z} : sh(F)_{r_1}<sh(F)_{r_1+1}\}\cup\mathcal{A}_{z+1}^{H_{z+1}^1}.$$

Let $x=(x_1,\ldots,x_n)$ and $y=(y_1,\ldots,y_n)$ be two sequences of indeterminates. From (8), with π_{r_1} the isobaric divided difference with respect to x, one has

$$\sum_{\nu\in\mathbb{N}^n}\pi_{r_1}\widehat{\kappa}_\nu(x)=\sum_{\nu\in\mathbb{N}^n}\pi_{r_1}\sum_{\substack{F\in SSAF_n\\ sh(F)=\nu}}x^F=\sum_{\substack{\nu\in\mathbb{N}^n\\ \nu_{r_1}\geq\nu_{r_1+1}}}\pi_{r_1}\sum_{\substack{F\in SSAF_n\\ sh(F)=\nu}}x^F$$

$$=\sum_{\substack{\nu\in\mathbb{N}^n\\ \nu_{r_1}\geq\nu_{r_1+1}}}\sum_{\substack{F\in SSAF_n\\ sh(F)=\nu}}x^F+\sum_{\substack{\nu\in\mathbb{N}^n\\ \nu_{r_1}>\nu_{r_1+1}}}\sum_{\substack{F\in SSAF_n\\ sh(F)=s_{r_1}\nu}}x^F.$$

Thereby

$$\sum_{\nu\in\mathbb{N}^n}\left(\pi_{r_1}\sum_{\substack{(F,G)\in\mathcal{A}_z^{H_z}\\ sh(F)=\nu}}x^Fy^G\right)=\sum_{\substack{(F,G)\in\mathcal{A}_z^{H_z}\\ sh(F)_{r_1}\geq sh(F)_{r_1+1}}}x^Fy^G+\sum_{(F,G)\in\mathcal{B}_z^{H_z}}x^Fy^G. \quad (14)$$

7.1.1. *The combinatorial formula.* In [19], Lascoux gives a Cauchy kernel expansion formula for any Ferrers shapes which produces, in particular, the following Cauchy kernel expansion over near staircases (7), $\prod_{(i,j)\in\lambda}(1-x_iy_j)^{-1}=\sum_{\nu\in\mathbb{N}^n}(\pi_{r_1}\ldots\pi_{r_p}\widehat{\kappa}_\nu(x))\kappa_{\omega\nu}(y)$. Next theorem gives a bijective explanation.

Theorem 6. *Let* $0\leq p<n$ *and* $1\leq r_1<r_2<\cdots<r_p<n$. *Let* λ *be the near staircases* (7). *Then*

(1)

$$\sum_{\nu\in\mathbb{N}^n}(\pi_{r_1}\ldots\pi_{r_p}\widehat{\kappa}_\nu(x))\kappa_{\omega\nu}(y)=\sum_{(F,G)\in\mathcal{A}}x^Fy^G+\sum_{1\leq z\leq p}\sum_{H_z}\sum_{(F,G)\in\mathcal{A}_z^{H_z}}x^Fy^G, \quad (15)$$

where $H_z\in\binom{[p]}{z}$.

(2)

$$\prod_{(i,j)\in\lambda}(1-x_iy_j)^{-1}=\sum_{\nu\in\mathbb{N}^n}\left(\pi_{r_1}\ldots\pi_{r_p}\widehat{\kappa}_\nu(x)\right)\kappa_{\omega\nu}(y).$$

Proof: 1. The proof is by induction on p. If $p=0$, we get,

$$\sum_{\nu\in\mathbb{N}^n}\widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y)=\sum_{\nu\in\mathbb{N}^n}\sum_{\substack{(F,G)\in SSAF_n^2\\ sh(G)\leq\omega sh(F)\\ sh(F)=\nu}}x^Fy^G=\sum_{\nu\in\mathbb{N}^n}\sum_{\substack{(F,G)\in\mathcal{A}\\ sh(F)=\nu}}x^Fy^G=\sum_{(F,G)\in\mathcal{A}}x^Fy^G.$$

Let $p\geq 1$ and suppose that identity (15) is true for $p-1$ operators π_i. Then, since π_{r_1} is linear,

$$\sum_{\nu\in\mathbb{N}^n}(\pi_{r_1}\pi_{r_2}\dots\pi_{r_p}\widehat{\kappa}_\nu(x))\kappa_{\omega\nu}(y)=\pi_{r_1}\left(\sum_{\nu\in\mathbb{N}^n}(\pi_{r_2}\dots\pi_{r_p}\widehat{\kappa}_\nu(x))\kappa_{\omega\nu}(y)\right)$$

$$=\pi_{r_1}\left(\sum_{z=0}^{p-1}\sum_{H_z\in\binom{[2,p]}{z}}\sum_{(F,G)\in\mathcal{A}_z^{H_z}}x^Fy^G\right)=\sum_{z=0}^{p-1}\sum_{H_z\in\binom{[2,p]}{z}}\left(\sum_{\nu\in\mathbb{N}^n}\pi_{r_1}\sum_{\substack{(F,G)\in\mathcal{A}_z^{H_z}\\ sh(F)=\nu}}x^Fy^G\right)$$

$$=\sum_{z=0}^{p-1}\sum_{H_z\in\binom{[2,p]}{z}}\left(\sum_{\substack{(F,G)\in\mathcal{A}_z^{H_z}\\ sh(F)_{r_1}\geq sh(F)_{r_1+1}}}x^Fy^G+\sum_{(F,G)\in\mathcal{B}_z^{H_z}}x^Fy^G\right).\tag{16}$$

Using Lemma 1,

$$(16)\;=\;\sum_{z=0}^{p-1}\sum_{H_z\in\binom{[2,p]}{z}}\left(\sum_{\substack{(F,G)\in\mathcal{A}_z^{H_z}\\ sh(F)_{r_1}\geq sh(F)_{r_1+1}}}x^Fy^G+\sum_{\substack{(F,G)\in\mathcal{A}_z^{H_z}\\ sh(F)_{r_1}<sh(F)_{r_1+1}}}x^Fy^G+\sum_{(F,G)\in\mathcal{A}_{z+1}^{H^1_{z+1}}}x^Fy^G\right)$$

$$=\sum_{z=0}^{p-1}\sum_{H_z\in\binom{[2,p]}{z}}\left(\sum_{(F,G)\in\mathcal{A}_z^{H_z}}x^Fy^G+\sum_{(F,G)\in\mathcal{A}_{z+1}^{H^1_{z+1}}}x^Fy^G\right)=\sum_{z=0}^{p}\sum_{H_z\in\binom{[p]}{z}}\sum_{(F,G)\in\mathcal{A}_z^{H_z}}x^Fy^G.$$

2. Let λ_0 the biggest staircase inside of λ. Then, identifying x_iy_j with the biletter $\binom{j}{i}$, and using the bijection in Theorem 5, it follows

$$\begin{aligned}\prod_{(i,j)\in\lambda}(1-x_iy_j)^{-1}\;&=\;\prod_{(i,j)\in\lambda_0}(1-x_iy_j)^{-1}\prod_{i=1}^{p}(1-x_{r_i+1}y_{n-r_i+1})^{-1}\\ &=\;\sum_{(F,G)\in\mathcal{A}}x^Fy^G+\sum_{z=1}^{p}\sum_{H_z\in\binom{[p]}{z}}\sum_{(F,G)\in\mathcal{A}_z^{H_z}}x^Fy^G.\quad\blacksquare\end{aligned}$$

The combinatorial expansion formula for the SE part can be obtained by using the change of basis (12).

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References

[1] Azenhas, O., Emami, A.: Semi-skyline augmented fillings and non-symmetric Cauchy kernels for stair-type shapes. FPSAC'13, DMTCS proc. AS, 981–992 (2013)
[2] Azenhas, O., Emami, A.: An analogue of the Robinson-Schensted-Knuth correspondence and non-symmetric Cauchy kernels for truncated staircases, arXiv:1310.0341
[3] Emami, A.: An analogue of the Robinson-Schensted-Knuth correspondence, growth-diagrams and non-symmetric Cauchy kernels, Ph.D. Thesis, Universidade de Coimbra, Coimbra (2014)
[4] Demazure, M.: Désingularisation des variétés de Schubert généralisées. Ann. Sci. École Norm. Sup., 4: 7: 53–88 (1974)
[5] Fomin, S.: The generalised Robinson-Schensted-Knuth correspondence. J. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), Vol. **155**, N. 195. 156–175 (1986)
[6] Fu, A. M., Lascoux, A.: Non-symmetric Cauchy kernels for the classical groups. J. Comb. Theory, Ser. A 116(4): 903-917 (2009)
[7] Fulton, W.: Young Tableaux with Applications to Representation Theory and Geometry. London Math. Soc. Student Texts, Vol. **35**, Cambridge Univ. Press (1997)
[8] Greene, C.: An extension of Schensted's theorem, Adv. in Math. 14, 254–265 (1974)
[9] Haglund, J., Haiman, M., Loehr, N.: A combinatorial formula for Macdonald polynomials, J. Amer. Math. Soc. vol. **18**, 735–761 (2005)
[10] Haglund, J., Haiman, M., Loehr, N.: A combinatorial formula for non-symmetric Macdonald polynomials, Amer. J. Math. vol. **130**, 359–383 (2008)
[11] Hong, J., Kang, S-J.: Introduction to Quantum Groups and Crystal Bases, Graduate Studies in Mathematics, 42 AMS (2002)
[12] Knuth, D. E.: Permutations, matrices and generalized Young tableaux. Pacific J. Math; **34**, 709–727 (1970)
[13] Krattenthaler, C.: Growth diagrams, and increasing and decreasing chains in fillings of ferrers shapes. Advances in Applied Mathematics, vol **37**, 404–431 (2006)
[14] Kwon, J. H.: Crystal graphs and the combinatorics of Young tableaux, Handbook of Algebra, vol. **6**, Editor M. Hazewinkel, North-Hollands (2009)
[15] Lascoux, A., Schützenberger, M.-P. :Tableaux and noncommutative Schubert polynomials, Funct. Anal. Appl. **23**, 63–64 (1989).
[16] Lascoux, A., Schützenberger, M.-P.: Keys and Standard Bases, Invariant Theory and Tableaux, IMA Vol. in Math and its Appl. **19**, 125–144 (1990)
[17] Lascoux, A.: Anneau de Grothendieck de la variété de drapeaux. In Pierre Cartier, Luc Illusie, Nicholas M. Katz, Gérard Laumon, Yuri I. Manin, and Kenneth A. Ribet, Editors, The Grothendieck Festschrift, Birkhäuser Boston, Vol. III:1-34 (1990)
[18] Lascoux, A., Leclerc, B., Thibon, J.-Y.: The plactic monoid, (Chapter 6) in M. Lothaire, Algebraic Combinatorics on Words, Cambridge University Press, Cambridge (2002)
[19] Lascoux, A.: Double crystal graphs. Studies in Memory of Issai Schur, in: Progr. Math. Birkhäuser, vol. **210**, 95–114 (2003)
[20] Lascoux, A.: Polynômes, (2012) `http://phalanstere.univ-mlv.fr/~al`

[21] Lecouvey, C.: Combinatorics of crystal graphs for the root systems of types A_n, B_n, C_n, D_n and G_2, Kuniba, Atsuo (ed.) et al., Combinatorical aspects of integrable systems. Based on the workshop, RIMS, Kyoto, Japan, July 26–30 (2004). Tokyo: Mathematical Society of Japan. MSJ Memoirs, Vol. **17**:11-41 (2007)

[22] Lecouvey, C.: Crystal bases and combinatorics of infinite rank quantum groups. *Trans. Amer. Math. Soc.* **361** no. 1, 297–329 (2009)

[23] Leeuwen, M. V.: Spin-preserving Knuth correspondences for ribbon tableaux, Electron. J. Combin. 12 (1) Article R10, 65 pp (2005)

[24] Littlewood, D. E.: The theory of group characters, Oxford (1940)

[25] Mason, S.: A decomposition of Schur functions and an analogue of the Robinson-Schensted-Knuth algorithm. Sém. Lothar. Combin. 57: B57e, 24 (2006/08)

[26] Mason, S.: An explicit construction of type A Demazure atoms. J. Algebra Comb. 29(3): 295–313 (2009)

[27] Pons, V.: Interval structure of the Pieri formula for Grothendieck polynomials. IJAC, Vol. **2**3 N. 01, 123-146 (2013) arXiv:1206.6204

[28] Reiner, V., Shimozono, M.: Key polynomilas and flagged Littlewood-Richardson rule, J. Combin. Theory Ser. A, 70:1, 107–143 (1995)

[29] Roby, T. W.: Applications and extensions of Fomin's generalization of the Robinson-Schensted correspondence to differential posets, Ph.D. thesis, M.I.T. Cambridge, Massachusetts, (1991)

[30] Sagan, B.: The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions, Graduate Texts in Mathematics, Vol. **203**, Springer (2001)

[31] Stanley, R. P.: Enumerative combinatorics. Vol. **2**, Cambridge Studies in Advanced Mathematics, vol. **62**, Cambridge University Press, Cambridge (1998)

O. Azenhas
CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal
E-mail address: oazenhas@mat.uc.pt

A. Emami
CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal
E-mail address: aramee@mat.uc.pt