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COSYMPLECTIC p-SPHERES

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Abstract: We introduce cosymplectic circles and cosymplectic spheres, which are the analogues in the cosymplectic setting of contact circles and contact spheres. We provide a complete classification of compact 3-manifolds that admit a cosymplectic circle. The properties of tautness and roundness for a cosymplectic p-sphere are studied. To any taut cosymplectic circle on a three-dimensional manifold M we are able to canonically associate a complex structure and a conformal symplectic couple on $M \times \mathbb{R}$. We prove that a cosymplectic circle in dimension three is round if and only if it is taut. On the other hand, we provide examples in higher dimensions of cosymplectic circles which are taut but not round and examples of cosymplectic circles which are round but not taut.

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1. Introduction

The notion of cosymplectic structure was introduced by Libermann in the late 50s as a pair (η, Ω), where η is a closed 1-form and Ω is a closed 2-form on a $(2n+1)$-dimensional manifold M, such that $\eta \wedge \Omega^n$ is a volume form. Cosymplectic manifolds play an important role in the geometric description of time-dependent mechanics (see [8] and references therein). Starting from 1967, when Blair defined an adapted Riemannian structure on a cosymplectic manifold, a study of the metric properties on these manifolds has been also carried forward. Recently, new remarkable results on cosymplectic manifolds and their Riemannian counterpart (sometimes called coKähler manifolds) appeared in [4, 9, 13].

In this paper, we introduce the notion of cosymplectic circle and, more generally, of cosymplectic p-sphere. We were inspired by the concepts of contact circle and contact p-sphere, introduced by Geiges and Gonzalo ([11]) and then generalized by Zessin ([22]). We brought these ideas into the cosymplectic setting.

Let (η_1, Ω_1) and (η_2, Ω_2) be two cosymplectic structures on a manifold M. We say that they generate a cosymplectic circle if the pair $(\lambda_1\eta_1+\lambda_2\eta_2, \lambda_1\Omega_1+\lambda_2\Omega_2)$ is a cosymplectic structure for every $\lambda = (\lambda_1, \lambda_2) \in \mathbb{S}^1$. In [11] Geiges

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and Gonzalo classified closed 3-manifolds that admit a taut contact circle. In the present paper, we provide a complete classification of 3-dimensional compact manifolds that admit a cosymplectic circle. Furthermore, we introduce the notion of tautness and of roundness for a cosymplectic p-sphere. To any taut cosymplectic circle on a three-dimensional manifold M we canonically associate a complex structure and a conformal symplectic couple on $M \times \mathbb{R}$. In dimension three a cosymplectic circle is proven to be round if and only if it is taut. In higher dimensions we provide examples of cosymplectic circles which are taut but not round and examples of cosymplectic circles which are round but not taut.

It is not difficult to show that there exist no cosymplectic circles in any manifold of dimension $4n + 1$ and, a fortiori, no cosymplectic p-spheres. In the last section of the paper, devoted to the relation between Riemannian 3-structures and almost cosymplectic spheres, we show that 3-cosymplectic manifolds ([7]) provide a source of examples of cosymplectic spheres which are both round and taut in any dimension $4n + 3$. We also improve the result of Zessin that any 3-Sasakian manifold admits a contact sphere which is both taut and round (cf. [22, Proposition 4]). Indeed, we show that any 3-Sasakian manifold admits a sphere of Sasakian structures which is both taut and round. We do this by introducing the class of quasi-contact spheres, which include both cosymplectic (rank 1) and contact spheres (rank 2n+1), and we prove that any 3-quasi-Sasakian manifold defines a taut and round quasi-contact sphere.

2. Main definitions and examples

An *almost cosymplectic structure* on a smooth manifold M is given by a pair (η, Ω), where η is a 1-form and Ω a 2-form on M such that $\eta \wedge \Omega^n$ is a volume form. Thus, in particular, $\dim(M) = 2n+1$. The almost cosymplectic structure (η, Ω) is said to be a *contact structure* if $d\eta = \Omega$ and a *cosymplectic structure* if η and Ω are both closed.

On any almost cosymplectic manifold there exists a global vector field ξ uniquely determined by the conditions

$$i_\xi \eta = 1, \quad i_\xi \Omega = 0, \tag{2.1}$$

and called the *Reeb vector field*.

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Definition 2.1. Let $(\eta_1, \Omega_1), \ldots, (\eta_{p+1}, \Omega_{p+1})$ be $p+1$ almost cosymplectic structures on M. Consider the family $\{(\eta_\lambda, \Omega_\lambda)\}_{\lambda \in \mathbb{S}^p}$ where

$$\eta_\lambda := \lambda_1\eta_1 + \ldots + \lambda_{p+1}\eta_{p+1}, \quad \Omega_\lambda := \lambda_1\Omega_1 + \ldots + \lambda_{p+1}\Omega_{p+1}.$$

If the pair $(\eta_\lambda, \Omega_\lambda)$ is an almost cosymplectic structure for every $\lambda = (\lambda_1, \ldots, \lambda_{p+1}) \in \mathbb{S}^p$, then the family $\{(\eta_\lambda, \Omega_\lambda)\}_{\lambda \in \mathbb{S}^p}$ is called an *almost cosymplectic p-sphere generated by* $(\eta_1, \Omega_1), \ldots, (\eta_{p+1}, \Omega_{p+1})$. An almost cosymplectic p-sphere is called an *almost cosymplectic circle* or an *almost cosymplectic sphere* if $p = 2$ or $p = 3$, respectively.

We shall denote an almost cosymplectic p-sphere either by $\{(\eta_\lambda, \Omega_\lambda)\}_{\lambda \in \mathbb{S}^p}$ or by indicating a set of generators, such as $\{(\eta_1, \Omega_1), \ldots, (\eta_{p+1}, \Omega_{p+1})\}$.

There are two interesting types of almost cosymplectic p-spheres, which are introduced by the following definitions.

Definition 2.2. An almost cosymplectic p-sphere $\{(\eta_\lambda, \Omega_\lambda)\}_{\lambda \in \mathbb{S}^p}$ is said to be *taut* if all its elements give the same volume form, i.e.

$$(\lambda_1\eta_1 + \ldots + \lambda_{p+1}\eta_{p+1}) \wedge (\lambda_1\Omega_1 + \ldots + \lambda_{p+1}\Omega_{p+1})^n = (\lambda'_1\eta_1 + \ldots + \lambda'_{p+1}\eta_{p+1}) \wedge (\lambda'_1\Omega_1 + \ldots + \lambda'_{p+1}\Omega_{p+1})^n$$

for any $(\lambda_1, \ldots, \lambda_{p+1}), (\lambda'_1, \ldots, \lambda'_{p+1}) \in \mathbb{S}^p$.

Definition 2.3. An almost cosymplectic p-sphere $\{(\eta_\lambda, \Omega_\lambda)\}_{\lambda \in \mathbb{S}^p}$ is said to be *round* if for any $\lambda = (\lambda_1, \ldots, \lambda_{p+1}) \in \mathbb{S}^p$ the vector field $\lambda_1\xi_1 + \ldots + \lambda_{p+1}\xi_{p+1}$ is the Reeb vector field of the almost cosymplectic structure $(\eta_\lambda, \Omega_\lambda)$.

Let $\{(\eta_\lambda, \Omega_\lambda)\}_{\lambda \in \mathbb{S}^p}$ be an almost cosymplectic p-sphere. Notice that if all the almost cosymplectic structures $(\eta_1, \Omega_1), \ldots, (\eta_{p+1}, \Omega_{p+1})$ are contact structures, then any element $(\eta_\lambda, \Omega_\lambda)$ of the almost cosymplectic p-sphere is a contact structure, since $d\eta_\lambda = \lambda_1 d\eta_1 + \ldots + \lambda_p d\eta_p = \lambda_1\Omega_1 + \ldots + \lambda_p\Omega_p = \Omega_\lambda$. Thus Definition 2.1 generalizes the notion of *contact circles* and *contact spheres*, introduced by Geiges and Gonzalo in [11], and that one of *contact p-spheres*, introduced by Zessin in [22].

On the other hand, if the generating almost cosymplectic structures (η_1, Ω_1), $\ldots$, $(\eta_{p+1}, \Omega_{p+1})$ are cosymplectic, so is any element of the almost cosymplectic p-sphere $\{(\eta_\lambda, \Omega_\lambda)\}_{\lambda \in \mathbb{S}^p}$. In this case we shall speak of *cosymplectic p-sphere*.

Now we describe a class of examples of cosymplectic spheres in dimension 3.

Example 2.4. Let M be any orientable three-dimensional manifold and let η_1, η_2, η_3 be three 1-forms which are linearly independent at every point of M. Notice that such 1-forms always exist, by Stiefel's theorem ([16]). Set

$$\Omega_1 := \eta_2 \wedge \eta_3, \quad \Omega_2 := \eta_3 \wedge \eta_1, \quad \Omega_3 := \eta_1 \wedge \eta_2.$$

Then it is easy to see that (η_1, Ω_1), (η_2, Ω_2), (η_3, Ω_3) are almost cosymplectic structures. Moreover, a direct computation shows that

$$\begin{aligned}(\lambda_1\eta_1 + \lambda_2\eta_2 + \lambda_3\eta_3) \wedge (\lambda_1\Omega_1 + \lambda_2\Omega_2 + \lambda_3\Omega_3) &= \left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\right) \eta_1 \wedge \eta_2 \wedge \eta_3 \\ &= \eta_1 \wedge \eta_2 \wedge \eta_3 \neq 0.\end{aligned}$$

Therefore we can conclude that (η_1, Ω_1), (η_2, Ω_2), (η_3, Ω_3) generate a taut almost cosymplectic sphere on M. Furthermore, such an almost cosymplectic sphere is cosymplectic if the forms η_1, η_2, η_3 are closed. In particular, the above construction applies to the 3-torus $\mathbb{T}^3$ endowed with the 1-forms $\eta_\alpha := d\theta_\alpha$, $\alpha \in \{1, 2, 3\}$, where $\theta_1, \theta_2, \theta_3$ are the coordinates on $\mathbb{T}^3$.

Now we consider another special case of Example 2.4, where the 3-manifold is the Heisenberg group.

Example 2.5. Let H be the Heisenberg group, whose Lie algebra structure is given by

$$[e_1, e_3] = 0, \quad [e_1, e_2] = \gamma e_3, \quad [e_3, e_2] = 0.$$

Let η_1, η_2, η_3 be the 1-forms dual to e_1, e_2, e_3, respectively. We have

$$\mathrm{d}\eta_1 = \mathrm{d}\eta_2 = 0, \qquad \mathrm{d}\eta_3 = -\frac{\gamma}{2}\eta_1 \wedge \eta_2.$$

Notice that the resulting sphere is neither contact not cosymplectic, but (η_1, Ω_1) and (η_2, Ω_2) generate a left-invariant taut cosymplectic circle on H.

The result of Zessin [22] about the non-existence of contact circles in dimension $4n + 1$ can be easily generalized to the almost cosymplectic case.

Theorem 2.6. *Manifolds of dimension $4n + 1$ do not admit any almost cosymplectic circle and thus any almost cosymplectic p-sphere for $p \geq 2$.*

Proof: We will prove the theorem for $n = 1$. Higher dimensions can be dealt with similarly. Assume that $\{(\eta_1, \Omega_1), (\eta_2, \Omega_2)\}$ is an almost cosymplectic circle on a smooth manifold M of dimension 5. Then, for any $\lambda = (\lambda_1, \lambda_2) \in \mathbb{S}^1$ we have

$$\begin{aligned}\eta_\lambda \wedge \Omega_\lambda^2 = {} & \lambda_1^3 \eta_1 \wedge \Omega_1^2 + \lambda_1^2\lambda_2 \left(2\eta_1 \wedge \Omega_1 \wedge \Omega_2 + \eta_2 \wedge \Omega_1^2\right) \\ & + \lambda_1\lambda_2^2 \left(2\eta_2 \wedge \Omega_1 \wedge \Omega_2 + \eta_1 \wedge \Omega_2^2\right) + \lambda_2^3 \eta_2 \wedge \Omega_2^2.\end{aligned} \tag{2.2}$$

Corollary 4.9. *Any 3-cosymplectic manifold admits an almost cosymplectic sphere of cosymplectic structures which is both taut and round.*

Remark 4.10. Notice that Corollary 4.8 improves the result of Zessin that any 3-Sasakian manifold admits a contact sphere which is both taut and round (cf. [22, Proposition 4]).

Arguing in a similar way to the proof of Theorem 4.3 we can find the following classes of examples of cosymplectic spheres.

Example 4.11. Let N be a smooth manifold endowed with a hyperholomorphic symplectic structure in the sense of [1] (note that these geometric structures were also studied in [20], but with a different name). Namely, on N there are defined three symplectic structures ω_1, ω_2, ω_3 related to each other by means of the relations

$$\omega_\alpha^\sharp \circ \omega_\beta^\flat = -\omega_\beta^\sharp \circ \omega_\alpha^\flat$$

for any $\alpha, \beta \in \{1, 2, 3\}$, $\alpha \neq \beta$. As a consequence, one can define three almost complex structures which satisfy the quaternionic identities. On $M := N \times \mathbb{R}^3$ we define for each even permutation (α, β, γ) of $\{1, 2, 3\}$

$$\eta_\alpha := dt_\alpha, \quad \Omega_\alpha := \omega_\alpha + \eta_\beta \wedge \eta_\gamma$$

where (t_1, t_2, t_3) are the global coordinates of $\mathbb{R}^3$. Then, arguing as in the proof of Corollary 4.4, one can prove that $\{(\eta_1, \Omega_1), (\eta_2, \Omega_2), (\eta_3, \Omega_3)\}$ generates a cosymplectic sphere on M.

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manifold $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)_{\alpha\in\{1,2,3\}}$, namely $2p+1 = 4l+3$. We will prove this by showing that $\mathcal{E} = \mathcal{C}$, where the distributions $\mathcal{E}$ and $\mathcal{C}$ are defined by (4.6) and (4.7), respectively. First we prove that $\mathcal{E} \subset \mathcal{C}$. Notice that $\mathcal{E} \subset \mathcal{H} \subset \ker(\eta_\lambda)$. Consequently, since, on $\mathcal{E}$, $d\eta_\alpha = 0$ for any $\alpha \in \{1,2,3\}$, we get that $d\eta_\lambda = 0$ on $\mathcal{E}$. Hence $\mathcal{E} \subset \mathcal{C}$. In order to prove the other inclusion, we observe that the endomorphism of $\mathcal{H}$ defined by $\rho = -\lambda_2 I + \lambda_3\phi_1 + \lambda_1\phi_3$ is in fact an isomorphism. Indeed a straightforward computation shows that $(\phi_2\rho)^2 = -I$, hence the mapping $\phi_2\rho$ is an isomorphism. It follows that ρ is an isomorphism too. Now let us consider an arbitrary $X \in \Gamma(\mathcal{C})$ and let us prove that $X \in \Gamma(\mathcal{E})$. We can decompose X in its components along the horizontal and the Reeb distribution, as follows

$$X = X_{\mathcal{H}} + f_1\xi_1 + f_2\xi_2 + f_3\xi_3$$

where $f_\alpha = \eta_\alpha(X)$ for each $\alpha \in \{1,2,3\}$. The condition $\eta_\lambda(X) = 0$ yields

$$\lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 = 0, \tag{4.21}$$

whereas the conditions $d\eta_\lambda(\xi_1, X) = d\eta_\lambda(\xi_2, X) = d\eta_\lambda(\xi_3, X) = 0$ yield

$$\lambda_2 f_3 - \lambda_3 f_2 = 0, \qquad -\lambda_1 f_3 + \lambda_3 f_1 = 0, \qquad \lambda_1 f_2 - \lambda_2 f_1 = 0. \tag{4.22}$$

It is easy to see that the homogeneous linear system of the four equations (4.21), (4.22) admits non-zero solutions if and only if $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Since this circumstance can not happen, we conclude that $f_1 = f_2 = f_3 = 0$ and hence $X \in \mathcal{H}$. Finally, in order to prove that X belongs to $\mathcal{E}$, it remains to prove that $d\eta_\alpha(X, Y) = 0$ for any $Y \in \Gamma(TM)$ and for any $\alpha \in \{1,2,3\}$. In fact, in view of [5, Lemma 5.4] it is enough to prove that $d\eta_\alpha(X, Y) = 0$ for *some* $\alpha \in \{1,2,3\}$, for instance for $\alpha = 3$. By [5, Corollary 3.8] we have that $d\eta_3(X, \xi_\beta) = 0$ for any $\beta \in \{1,2,3\}$. Thus we may assume that $Y \in \Gamma(\mathcal{H})$. Since ρ is an isomorphism, there exists $Y' \in \Gamma(\mathcal{H})$ such that $Y = \rho Y' = -\lambda_2 Y' + \lambda_3\phi_1 Y' + \lambda_1\phi_3 Y'$. Therefore, by using Lemma 5.2 – Lemma 5.3 of [5], we get

$$\begin{aligned} d\eta_3(X, Y) &= d\eta_3(X, \rho Y') = \lambda_1 d\eta_3(X, \phi_3 Y') - \lambda_2 d\eta_3(X, Y') + \lambda_3 d\eta_3(X, \phi_1 Y') \\ &= \lambda_1 d\eta_1(X, \phi_1 Y') + \lambda_2 d\eta_2(X, \phi_1 Y') + \lambda_3 d\eta_3(X, \phi_1 Y') \\ &= d\eta_\lambda(X, \phi_1 Y') = 0, \end{aligned}$$

where the last equality follows from the fact that $X \in \Gamma(\mathcal{C})$. ∎

Corollary 4.8. *Any 3-Sasakian manifold admits an almost cosymplectic sphere of Sasakian structures which is both taut and round.*

Let p be a point of M and let $\{e_1, e_2, e_3, e_4, e_5\}$ be a basis of T_pM. Then in view of (2.2) we have that the real valued function

$$(\lambda_1, \lambda_2) \in \mathbb{R}^2 \mapsto f(\lambda_1, \lambda_2) := \eta_\lambda \wedge \Omega_\lambda^2 (e_1, e_2, e_3, e_4, e_5)$$

is the polynomial function of a homogeneous polynomial of degree 3 in the indeterminates λ_1, λ_2. Thus, in particular, $f(-\lambda_1, -\lambda_2) = -f(\lambda_1, \lambda_2)$ for any $(\lambda_1, \lambda_2) \in \mathbb{S}^1$. Therefore f should have some zero in $\mathbb{S}^1$, but this contradicts the fact that $(\eta_\lambda, \Omega_\lambda)$ is an almost cosymplectic structure. ∎

Note that in contrast to the case of dimension $4n+1$, considered in Theorem 2.6, we shall see in Section 4 that there are examples of cosymplectic spheres in any dimension $4n+3$.

Example 2.7. Let (η_1, Ω_1) and (η_2, Ω_2) be the cosymplectic structures on $\mathbb{R}^7$ given by

$$\eta_1 := \mathrm{d}x_7, \quad \Omega_1 := \mathrm{d}x_1 \wedge \mathrm{d}x_2 + \mathrm{d}x_3 \wedge \mathrm{d}x_4 + \mathrm{d}x_5 \wedge \mathrm{d}x_6$$
$$\eta_2 := \mathrm{d}x_6, \quad \Omega_2 := (\mathrm{d}x_1 + \mathrm{d}x_2) \wedge \mathrm{d}x_3 + (\mathrm{d}x_4 + \mathrm{d}x_5) \wedge \mathrm{d}x_7 - \mathrm{d}x_2 \wedge \mathrm{d}x_5.$$

A straightforward computation shows that for any $(\lambda_1, \lambda_2) \in \mathbb{S}^1$ one has

$$(\lambda_1\eta_1 + \lambda_2\eta_2) \wedge (\lambda_1\Omega_1 + \lambda_2\Omega_2)^3 = 6\left(\left(\lambda_1^2 - \lambda_2^2\right)^2 + \lambda_1^2\lambda_2^2\right) \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_7.$$

Thus (η_1, Ω_1) and (η_2, Ω_2) define a non-taut cosymplectic circle. Moreover, the cosymplectic circle is also not round. In fact, we have $\xi_1 = \frac{\partial}{\partial x_7}$, $\xi_2 = \frac{\partial}{\partial x_6}$ and for any $\lambda \in \mathbb{S}^1$

$$i_{\lambda_1\xi_1 + \lambda_2\xi_2}(\lambda_1\Omega_1 + \lambda_2\Omega_2) = \lambda_1\lambda_2(i_{\xi_1}\Omega_2 + i_{\xi_2}\Omega_1) = -\lambda_1\lambda_2(dx_4 + 2dx_5)) \neq 0,$$

unless $\lambda_1 = 0$ or $\lambda_2 = 0$.

Proposition 2.8. *Let $\{(\eta_1, \Omega_1), (\eta_2, \Omega_2)\}$ be an almost cosymplectic circle. Then $i_{\xi_2}\Omega_1$ and $i_{\xi_1}\Omega_2$ nowhere vanish.*

Proof: Suppose that

$$(i_{\xi_1}\Omega_2)(p) = 0, \tag{2.3}$$

for some $p \in M$. Now, let $(\eta_\lambda, \Omega_\lambda)$ be the almost cosymplectic structure given by $\eta_\lambda = \lambda_1\eta_1 + \lambda_2\eta_2$ and $\Omega_\lambda = \lambda_1\Omega_1 + \lambda_2\Omega_2$, for some $\lambda = (\lambda_1, \lambda_2) \in \mathbb{S}^1$. Because of (2.3) and $i_{\xi_1}\Omega_1 = 0$ we have $(i_{\xi_1}\Omega_\lambda)(p) = 0$. It follows that $\eta_\lambda(\xi_1) \neq 0$ at p. For, if $\eta_\lambda(\xi_1)(p) = 0$ one would have that $(\eta_\lambda \wedge \Omega_\lambda)(\xi_1, -) = 0$ at p, so $(\eta_\lambda, \Omega_\lambda)$ would not be an almost cosymplectic structure. Thus $\alpha := \eta_\lambda(\xi_1)(p) \neq 0$. We put $\xi_1' := \xi_1/\alpha$. Then $i_{\xi_1'}\Omega_\lambda = 0$ and $i_{\xi_1'}\eta_\lambda = 1$ at

p. Consequently $\xi_1'(p) = \xi_\lambda(p)$ and we conclude that ξ_λ is parallel to ξ_1 at p, for any $\lambda \in \mathbb{S}^1$. For the almost cosymplectic structure $(-\eta_1, -\Omega_1)$, which belongs to the almost cosymplectic circle, the Reeb vector field is $-\xi_1$. Since the function $f\colon \mathbb{S}^1 \longrightarrow \mathbb{R}$, defined by $\xi_\lambda(p) = f(\lambda)\xi_1(p)$, is continuous and takes the value -1 at $(-1,0)$ and the value 1 at $(1,0)$, there exists some $\lambda_0 \in \mathbb{S}^1$ such that $f(\lambda_0) = 0$, that is $\xi_{\lambda_0}(p) = 0$. So we get a contradiction. Thus, the 1-form $i_{\xi_1}\Omega_2$ nowhere vanishes. Similarly one shows that $i_{\xi_2}\Omega_1$ nowhere vanishes. ∎

Corollary 2.9. *Let $\{(\eta_1, \Omega_1), (\eta_2, \Omega_2)\}$ be an almost cosymplectic circle. Then the Reeb vector fields ξ_1 and ξ_2 of (η_1, Ω_1) and (η_2, Ω_2) are everywhere linearly independent.*

Definition 2.10. The *Reeb distribution* of an almost cosymplectic p-sphere $\{(\eta_\lambda, \Omega_\lambda)\}_{\lambda\in\mathbb{S}^p}$ is the distribution $\mathcal{V}$ generated by the Reeb vector fields of the generators $\{(\eta_1, \Omega_1), \dots, (\eta_{p+1}, \Omega_{p+1})\}$, i.e.

$$\mathcal{V} =< \xi_1, \dots, \xi_{p+1} > .$$

A natural question concerns the integrability of the Reeb distribution $\mathcal{V}$. The following proposition shows that in the 3-dimensional case $\mathcal{V}$ is the common kernel of the 1-forms $i_{\xi_1}\Omega_2$ and $i_{\xi_2}\Omega_1$.

Proposition 2.11. *Let $\{(\eta_1, \Omega_1), (\eta_2, \Omega_2)\}$ be a almost cosymplectic circle on a 3-dimensional manifold M^3. Then the Reeb distribution $\mathcal{V} = \langle \xi_1, \xi_2 \rangle$ is given by*

$$\mathcal{V} = \ker(i_{\xi_1}\Omega_2) = \ker(i_{\xi_2}\Omega_1).$$

Proof: By Proposition 2.8 the 1-forms $i_{\xi_1}\Omega_2$ and $i_{\xi_2}\Omega_1$ nowhere vanish, thus their kernels have dimension two. Moreover, one can easily check that $\xi_1, \xi_2 \in \ker(i_{\xi_1}\Omega_2)$ and $\xi_1, \xi_2 \in \ker(i_{\xi_2}\Omega_1)$. Since the two Reeb vector fields ξ_1, ξ_2 are linearly independent by Corollary 2.9, then $\mathcal{V}$ coincides with $\ker(i_{\xi_1}\Omega_2)$ and $\ker(i_{\xi_2}\Omega_1)$ for dimensional reasons. ∎

Recall that the kernel of a 1-form θ is integrable if and only if $\theta \wedge \mathrm{d}\theta = 0$. Therefore we obtain the following corollary.

Corollary 2.12. *Let $\{(\eta_1, \Omega_1), (\eta_2, \Omega_2)\}$ be a almost cosymplectic circle on a 3-dimensional manifoldM^3. Then the Reeb vector fields ξ_1 and ξ_2 generate a two dimensional integrable distribution if and only if*

$$i_{\xi_1}\Omega_2 \wedge \mathrm{d}i_{\xi_1}\Omega_2 = 0,$$

for each $i \in \{1, \dots, 4\}$ and $\alpha, \beta \in \{1, 2, 3\}$, and by defining

$$\begin{gathered}\phi_1 X_1 = X_2, \quad \phi_1 X_2 = -X_1, \quad \phi_1 X_3 = X_4, \quad \phi_1 X_4 = -X_3,\\ \phi_2 X_1 = X_3, \quad \phi_2 X_2 = -X_4, \quad \phi_2 X_3 = -X_1, \quad \phi_2 X_4 = X_2,\\ \phi_3 X_1 = X_4, \quad \phi_3 X_2 = X_3, \quad \phi_3 X_3 = -X_2, \quad \phi_3 X_4 = -X_1,\\ \phi_\alpha \xi_\beta = \epsilon_{\alpha\beta\gamma}\xi_\gamma,\end{gathered}$$

where $\epsilon_{\alpha\beta\gamma}$ is the total antisymmetric symbol. From the definition, it follows that $d\eta_1 = d\eta_2 = 0$. Next, we have $2d\eta_3(X_1, X_4) = X_1(\eta_3(X_4)) - X_4(\eta_3(X_1)) - \eta_3([X_1, X_4]) = -\eta_3([X_1, X_4]) = -\eta_3(\xi_3) = -1$, and $2d\eta_3(\xi_1, \xi_2) = -1$ by a similar computation. One also checks that $d\eta_3$ is zero on any other pair of basis vector fields. Thus $(\eta_3 \wedge (d\eta_3)^2)(\xi_3, \xi_1, \xi_2, X_1, X_4)$ is a non-zero constant at every point and $\eta_3 \wedge (d\eta_3)^3 \equiv 0$. Therefore η_1 and η_2 have rank 1, while η_3 has rank 5.

The above example motivates the following definition.

Definition 4.6. A *quasi-contact* p-sphere of *rank* k is an almost cosymplectic p-sphere $\{(\eta_\lambda, \Omega_\lambda)\}_{\lambda\in\mathbb{S}^p}$ such that $\mathrm{d}\Omega_\lambda = 0$ for any $\lambda \in \mathbb{S}^p$ and there exists a positive integer k such that all 1-forms η_λ have the same constant rank k, i.e. each 1-form η_λ has Cartan class k, for any $\lambda \in \mathbb{S}^p$.

Of course contact p-spheres are examples of almost cosymplectic p-spheres which are quasi-contact, having the maximal possible rank $2n+1 = \dim(M)$. Further, notice that any cosymplectic p-sphere is quasi-contact of minimal rank 1, being η_λ closed and nonsingular. Now we shall present a class of examples of almost cosymplectic p-spheres which are quasi-contact of rank $1 < k < \dim(M)$.

Theorem 4.7. *With the notation of Theorem 4.3, if $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)_{\alpha\in\{1,2,3\}}$ is a 3-quasi-Sasakian manifold of rank $4l+3$ then, for every $\lambda \in \mathbb{S}^2$, $(\phi_\lambda, \xi_\lambda, \eta_\lambda, g)$ defines a quasi-Sasakian structure of the same rank. In particular, the almost cosymplectic sphere generated by $\{(\eta_1, \Omega_1), (\eta_2, \Omega_2), (\eta_3, \Omega_3)\}$ is quasi-contact of rank $4l + 3$.*

Proof: By Theorem 4.3 we already know that $(\phi_\lambda, \xi_\lambda, \eta_\lambda, g)$ is a normal almost contact metric structure on M. We have to prove that the fundamental 2-form Ω_λ is closed. However, notice that $\Omega_\lambda = \lambda_1\Omega_1 + \lambda_2\Omega_2 + \lambda_3\Omega_3$. Since each structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ is quasi-Sasakian, we get that $d\Omega_\lambda = 0$. We pass to prove that the rank of η_λ is the same of that of the 3-quasi-Sasakian

for any $i,j \in \{1,\dots,n\}$. Moreover, one easily finds that

$$\Omega_\lambda(\xi_1,\xi_2) = -\lambda_3, \quad \Omega_\lambda(\xi_1,\xi_3) = \lambda_2, \quad \Omega_\lambda(\xi_2,\xi_3) = -\lambda_1 \tag{4.19}$$

and

$$\Omega_\lambda(\xi_\alpha, X_i) = \Omega_\lambda(\xi_\alpha, Y_i) = \Omega_\lambda(\xi_\alpha, Z_i) = \Omega_\lambda(\xi_\alpha, U_i) = 0 \tag{4.20}$$

for any $\alpha \in \{1,2,3\}$ and $i \in \{1,\dots,n\}$. By using (4.15)–(4.18), (4.19) and (4.20), one has that

$$\begin{aligned}
&\eta_\lambda \wedge \Omega_\lambda^{2n+1}(\xi_1,\xi_2,\xi_3,X_1,\dots,X_n,Y_1,\dots,Y_n,Z_1,\dots,Z_n,U_1,\dots,U_n)\\
&\quad = \eta_\lambda(\xi_1)\Omega_\lambda(\xi_2,\xi_3)\Omega_\lambda^{2n}(X_1,\dots,X_n,Y_1,\dots,Y_n,Z_1,\dots,Z_n,U_1,\dots,U_n)\\
&\qquad - \eta_\lambda(\xi_2)\Omega_\lambda(\xi_1,\xi_3)\Omega_\lambda^{2n}(X_1,\dots,X_n,Y_1,\dots,Y_n,Z_1,\dots,Z_n,U_1,\dots,U_n)\\
&\qquad + \eta_\lambda(\xi_3)\Omega_\lambda(\xi_1,\xi_2)\Omega_\lambda^{2n}(X_1,\dots,X_n,Y_1,\dots,Y_n,Z_1,\dots,Z_n,U_1,\dots,U_n)\\
&\quad = \big(\eta_\lambda(\xi_1)\Omega_\lambda(\xi_2,\xi_3) - \eta_\lambda(\xi_2)\Omega_\lambda(\xi_1,\xi_3) + \eta_\lambda(\xi_3)\Omega_\lambda(\xi_1,\xi_2)\big)\\
&\quad \sum_{i=1}^n \big(\Omega_\lambda(X_i,Y_i)\Omega_\lambda(Z_i,U_i) - \Omega_\lambda(X_i,Z_i)\Omega_\lambda(Y_i,U_i) + \Omega_\lambda(X_i,U_i)\Omega_\lambda(Y_i,Z_i)\big)\\
&\quad = \big(-\lambda_1^2-\lambda_2^2-\lambda_3^2\big)\sum_{i=1}^n\big(\lambda_1^2+\lambda_2^2+\lambda_3^2\big) = -n.
\end{aligned}$$

This shows that the almost cosymplectic sphere is taut. ∎

We now illustrate an example of almost contact 3-structure on a 7-dimensional manifold such that the 1-forms η_1, η_2, η_3 have different constant ranks.

Example 4.5. Let $\mathfrak{g}$ be the 7-dimensional Lie algebra with basis

$$\{X_1, X_2, X_3, X_4, \xi_1, \xi_2, \xi_3\}$$

and non-zero Lie brackets

$$[X_1, X_4] = \xi_3, \quad [\xi_1, \xi_2] = \xi_3.$$

Let G be a Lie group whose Lie algebra is $\mathfrak{g}$. We define an almost contact 3-structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha)$, $\alpha \in \{1,2,3\}$, on G by setting

$$\eta_\alpha(X_i) = 0, \quad \eta_\alpha(\xi_\beta) = \delta_{\alpha\beta}$$

or equivalently if and only if

$$i_{\xi_2}\Omega_1 \wedge \mathrm{d} i_{\xi_2}\Omega_1 = 0.$$

Remark 2.13. An example of a taut cosymplectic circle on a 3-dimensional manifold with non-integrable Reeb distribution is given by Example 2.5 in the case $\gamma \neq 0$ where the condition $i_{\xi_1}\Omega_2 \wedge \mathrm{d} i_{\xi_1}\Omega_2 = 0$ of Corollary 2.12 is not satisfied. In that example we have $\xi_1 = e_1$, $\xi_2 = e_2$ and

$$i_{\xi_1}\Omega_2 = i_{\xi_1}(\eta_3 \wedge \eta_1) = -\eta_3.$$

Hence

$$i_{\xi_1}\Omega_2 \wedge \mathrm{d} i_{\xi_1}\Omega_2 = \eta_3 \wedge \mathrm{d}\eta_3 = -\frac{\gamma}{2}\eta_1 \wedge \eta_2 \wedge \eta_3 \neq 0.$$

The distribution generated by ξ_1 and ξ_2 is indeed not integrable since

$$[\xi_1, \xi_2] = e_3.$$

Concerning the topological properties of a cosymplectic p-sphere on a compact manifold we have the following result.

Proposition 2.14. *Let $\{(\eta_1,\Omega_1),\dots,(\eta_{p+1},\Omega_{p+1})\}$ be a cosymplectic p-sphere on a compact manifold M. Then the classes*

(i) $[\eta_1],\dots,[\eta_{p+1}] \in H^1(M)$ and
(ii) $[\Omega_1],\dots,[\Omega_{p+1}] \in H^2(M)$

are nonzero and linearly independent. Therefore the following conditions on the Betti numbers are fulfilled: $b_1 \geq p+1$, $b_2 \geq p+1$.

Proof: Suppose that

$$\sum_{i=1}^{p+1} \lambda_i [\eta_i] = 0 \text{ in } H^1(M).$$

Upon dividing by $\sum_{i=1}^{p+1}\lambda_i^2 > 0$, we can assume that $(\lambda_1,\dots,\lambda_{p+1}) \in \mathbb{S}^p$. Then, there is a function f such that

$$\mathrm{d}f = \sum_{i=1}^{p+1} \lambda_i \eta_i.$$

On the other hand, we know that $(\sum_{i=1}^{p+1}\lambda_i\eta_i, \sum_{i=1}^{p+1}\lambda_i\Omega_i)$ is a cosymplectic structure on M. Thus,

$$\mathrm{d}\left(f \wedge (\sum_{i=1}^{p+1}\lambda_i\Omega_i)^n\right) = \mathrm{d}f \wedge \left(\sum_{i=1}^{p+1}\lambda_i\Omega_i\right)^n = \left(\sum_{i=1}^{p+1}\lambda_i\eta_i\right) \wedge \left(\sum_{i=1}^{p+1}\lambda_i\Omega_i\right)^n$$

should have a nontrivial class in $H^{2n+1}(M)$. We get a contradiction, hence $[\eta_1], \dots, [\eta_{p+1}] \in H^1(M)$ are nonzero and linearly independent. Similarly one shows that $[\Omega_1], \dots, [\Omega_{p+1}] \in H^2(M)$ are nonzero and linearly independent. ∎

Now we give a full classification of 3-dimensional compact manifolds admitting a cosymplectic circle.

Theorem 2.15. *Let M be a compact 3-dimensional manifold. M admits a cosymplectic circle if and only if M is either a 3-torus or a quotient of the Heisenberg group by a co-compact subgroup.*

Proof: Assume that M carries a cosymplectic circle $(\eta_1, \Omega_1), (\eta_2, \Omega_2)$. Since η_1, η_2 are closed and linearly independent, by Tischler's theorem [18, Corollary 2] M is a locally trivial fibration over the 2-torus

$$F \longrightarrow M \longrightarrow \mathbb{T}^2.$$

The 1-dimensional fiber F is closed, hence compact. Thus it is a finite disjoint union of circles. By eventually passing to a finite cover of the base, we can assume that $F \cong \mathbb{S}^1$. Indeed, one can show that every fibration of a connected space over a connected base can be written as a fibration with a connected fiber followed by a covering (cf. Stein factorization in algebraic geometry). Thus we have

$$M \longrightarrow B \longrightarrow \mathbb{T}^2,$$

where $M \to B$ is an $\mathbb{S}^1$-fiber bundle and B is a finite covering of $\mathbb{T}^2$, and hence $B \cong \mathbb{T}^2$. Therefore M is an $\mathbb{S}^1$-bundle over $\mathbb{T}^2$. Moreover, this $\mathbb{S}^1$-bundle over $\mathbb{T}^2$ is oriented since M is oriented. Now, every oriented $\mathbb{S}^1$-bundle is principal by [14, Proposition 6.15]. Hence M is a principal circle bundle over $\mathbb{T}^2$ and by [15, Theorem 3] we get that M is diffeomorphic to a 2-step nilpotent nilmanifold G/Γ. However, every 3-dimensional 2-step nilpotent Lie group is either the Heisenberg group or the torus $\mathbb{T}^3$.

Conversely, Example 2.5 and Example 2.4 show that both H/Γ and $\mathbb{T}^3$ admit a cosymplectic circle. ∎

3. Taut and round cosymplectic p-spheres

In this section we focus on round and taut cosymplectic p-spheres. We start with a characterization of roundness.

$$\begin{aligned}
&= g(X,Y) - \lambda_1^2\eta_1(X)\eta_1(Y) - \lambda_2^2\eta_2(X)\eta_2(Y) - \lambda_3^2\eta_3(X)\eta_3(Y)\\
&\quad - \lambda_1\lambda_2\eta_2(X)\eta_1(Y) - \lambda_1\lambda_2\eta_1(X)\eta_2(Y) - \lambda_1\lambda_3\eta_3(X)\eta_1(Y)\\
&\quad - \lambda_1\lambda_3\eta_1(X)\eta_3(Y) - \lambda_2\lambda_3\eta_3(X)\eta_2(Y) - \lambda_2\lambda_3\eta_2(X)\eta_3(Y)\\
&= g(X,Y) - \eta_\lambda(X)\eta_\lambda(Y).
\end{aligned}$$

Now we pass to prove the second part of the theorem. By a very long but straightforward computation one can prove the relation

$$N^{(1)}_{\phi_\lambda} = N^{(1)}_{\phi_1} + N^{(1)}_{\phi_2} + N^{(1)}_{\phi_3} + \lambda_1\lambda_2 N^{(1)}_{1,2} + \lambda_1\lambda_3 N^{(1)}_{1,3} + \lambda_2\lambda_3 N^{(1)}_{2,3}.$$

Since $(\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha\in\{1,2,3\}}$ is a hyper-normal almost contact 3-structure, we have that $N^{(1)}_{\phi_1} = N^{(1)}_{\phi_2} = N^{(1)}_{\phi_3} = 0$ and, by Theorem 4.2, $N^{(1)}_{\alpha,\beta} = 0$ for any $\alpha, \beta \in \{1,2,3\}$. Therefore $N^{(1)}_{\phi_\lambda} = 0$. ∎

Corollary 4.4. *Let $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be an almost contact metric 3-structure on a $(4n+3)$-dimensional smooth manifold M. Then $\{(\eta_1, \Omega_1), (\eta_2, \Omega_2), (\eta_3, \Omega_3)\}$, being $\Omega_\alpha := g(\cdot, \phi_\alpha \cdot)$, is an almost cosymplectic sphere which is both round and taut.*

Proof: With the notation of Theorem 4.3, for each $\alpha \in \{1,2,3\}$ let $\Omega_\alpha := g(-, \phi_\alpha -)$ be the fundamental 2-form of the almost contact metric structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha)$. For each $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{S}^2$ let $(\phi_\lambda, \xi_\lambda, \eta_\lambda, g)$ denote the almost contact metric structure defined by (4.14), and let $\Omega_\lambda := (-, \phi_\lambda -)$ be the corresponding fundamental 2-form. Then by the general theory of almost contact metric structures, one has that $\eta_\lambda \wedge \Omega_\lambda^{2n+1} \neq 0$. This proves that the family $\{(\eta_\lambda, \Omega_\lambda)\}_{\lambda\in\mathbb{S}^2}$ defines an almost cosymplectic sphere on M. Directly from Theorem 4.3 it follows that such an almost cosymplectic sphere is round. Thus it only remains to prove that it is taut. To this aim, let us consider a ϕ-basis, i.e. a (local) orthonormal basis

$$(\xi_1, \xi_2, \xi_3, X_1, \dots, X_n, Y_1, \dots, Y_n, Z_1, \dots, Z_n, U_1, \dots, U_n)$$

such that $X_i \in \Gamma(\mathcal{H})$, $Y_i = \phi_1 X_i$, $Z_i = \phi_2 X_i$, $U_i = \phi_3 X_i$ for all $i \in \{1, \dots, n\}$. Using that $\Omega_\lambda = \lambda_1\Omega_1 + \lambda_2\Omega_2 + \lambda_3\Omega_3$ one finds

$$\Omega_\lambda(X_i, Y_j) = \Omega_\lambda(Z_i, U_j)) = -\lambda_1\delta_{ij}, \tag{4.15}$$

$$\Omega_\lambda(X_i, Z_j) = -\Omega_\lambda(Y_i, U_j)) = -\lambda_2\delta_{ij}, \tag{4.16}$$

$$\Omega_\lambda(X_i, U_j) = \Omega_\lambda(Y_i, Z_j)) = -\lambda_3\delta_{ij}, \tag{4.17}$$

$$\Omega_\lambda(X_i, X_j) = \Omega_\lambda(Y_i, Y_j) = \Omega_\lambda(Z_i, Z_j) = \Omega_\lambda(U_i, U_j) = 0, \tag{4.18}$$

Theorem 4.3. *Let $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)_{\alpha\in\{1,2,3\}}$ be an almost contact metric 3-structure on M. Then, for any $\lambda := (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{S}^2$ the tensors*

$$\phi_\lambda := \lambda_1\phi_1+\lambda_2\phi_2+\lambda_3\phi_3, \quad \xi_\lambda := \lambda_1\xi_1+\lambda_2\xi_2+\lambda_3\xi_3, \quad \eta_\lambda := \lambda_1\eta_1+\lambda_2\eta_2+\lambda_3\eta_3, \tag{4.14}$$

define an almost contact structure on M, compatible with the Riemannian metric g. Furthermore, if $(\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha\in\{1,2,3\}}$ is hyper-normal, then $(\phi_\lambda, \xi_\lambda, \eta_\lambda, g)$ is a normal almost contact metric structure on M.

Proof: First, by using the relations $\eta_\alpha(\xi_\beta) = \delta_{\alpha\beta}$ we get

$$\eta_\lambda(\xi_\lambda) = \lambda_1^2\eta_1(\xi_1) + \lambda_2^2\eta_2(\xi_2) + \lambda_3^2\eta_3(\xi_3) = 1.$$

Next, due to (4.4), we have that

$$\begin{aligned}
\phi_\lambda^2 &= \sum_{\alpha=1}^3 \lambda_\alpha^2\phi_\alpha^2 + \sum_{\alpha<\beta} \lambda_\alpha\lambda_\beta(\phi_\alpha\phi_\beta + \phi_\beta\phi_\alpha) \\
&= \sum_{\alpha=1}^3 (-\lambda_\alpha^2 I + \lambda_\alpha^2\eta_\alpha\otimes\xi_\alpha) + \sum_{\alpha<\beta} \lambda_\alpha\lambda_\beta(\eta_\alpha\otimes\xi_\beta + \eta_\beta\otimes\xi_\alpha) \\
&= -I + \sum_{\alpha=1}^3 (\lambda_\alpha\eta_\alpha)\otimes(\lambda_\alpha\xi_\alpha) + \sum_{\alpha\neq\beta} (\lambda_\alpha\eta_\alpha)\otimes(\lambda_\beta\xi_\beta) = -I + \eta_\lambda\otimes\xi_\lambda.
\end{aligned}$$

Finally, we prove that the almost contact structure (ϕ, ξ, η) is compatible with g. Indeed, for any $X, Y \in \Gamma(TM)$ one has

$$\begin{aligned}
g(\phi_\lambda X, \phi_\lambda Y) &= \lambda_1^2 g(\phi_1 X, \phi_1 Y) + \lambda_2^2 g(\phi_2 X, \phi_2 Y) + \lambda_3^2 g(\phi_3 X, \phi_3 Y) \\
&\quad + \lambda_1\lambda_2 g(\phi_1 X, \phi_2 Y) + \lambda_1\lambda_3 g(\phi_1 X, \phi_3 Y) + \lambda_1\lambda_2 g(\phi_2 X, \phi_1 Y) \\
&\quad + \lambda_2\lambda_3 g(\phi_2 X, \phi_3 Y) + \lambda_1\lambda_3 g(\phi_3 X, \phi_1 Y) + \lambda_2\lambda_3 g(\phi_3 X, \phi_2 Y) \\
&= (\lambda_1^2 + \lambda_2^2 + \lambda_3^2) g(X, Y) - \lambda_1^2\eta_1(X)\eta_1(Y) - \lambda_2^2\eta_2(X)\eta_2(Y) \\
&\quad - \lambda_3^2\eta_3(X)\eta_3(Y) - \lambda_1\lambda_2 g(X, \eta_1(Y)\xi_2) - \lambda_1\lambda_2 g(X, \eta_2(Y)\xi_1) \\
&\quad - \lambda_1\lambda_3 g(X, \eta_1(Y)\xi_3) - \lambda_1\lambda_3 g(X, \eta_3(Y)\xi_1) - \lambda_2\lambda_3 g(X, \eta_2(Y)\xi_3) \\
&\quad - \lambda_2\lambda_3 g(X, \eta_3(Y)\xi_2)
\end{aligned}$$

Proposition 3.1. *Let $\{(\eta_1, \Omega_1), \dots, (\eta_{p+1}, \Omega_{p+1})\}$ be a cosymplectic p-sphere on a manifold M and let $\xi_1, \dots, \xi_{p+1}$ denote the Reeb vector fields of the generators. Then $\{(\eta_1, \Omega_1), \dots, (\eta_{p+1}, \Omega_{p+1})\}$ is round if and only if the following conditions are fulfilled*

(i) $\eta_i(\xi_j) + \eta_j(\xi_i) = 0$ for any $i, j \in \{1, \dots, p+1\}$, $i \neq j$
(ii) $i_{\xi_i}\Omega_j + i_{\xi_j}\Omega_i = 0$ for any $i, j \in \{1, \dots, p+1\}$.

Proof: The cosymplectic sphere $\{(\eta_1, \Omega_1), \dots, (\eta_{p+1}, \Omega_{p+1})\}$ is round if and only if the vector field $\xi_\lambda := \lambda_1\xi_1 + \ldots + \lambda_{p+1}\xi_{p+1}$ is the Reeb vector field of $(\eta_\lambda, \Omega_\lambda)$, i.e. $\eta_\lambda(\xi_\lambda) = 1$ and $i_{\xi_\lambda}\Omega_\lambda = 0$. These conditions give

$$\sum_{i=1}^{p+1} \lambda_i^2 + \sum_{i\neq j} \lambda_i\lambda_j\eta_i(\xi_j) = 1, \qquad \sum_{i\neq j} \lambda_i\lambda_j i_{\xi_i}\Omega_j = 0.$$

Substituting $\lambda_k = 0$ for $k \neq i, j$ and $\lambda_i = \lambda_j = 1/\sqrt{2}$, we get

$$\frac{1}{2}(\eta_i(\xi_j) + \eta_j(\xi_i)) = 0, \qquad \frac{1}{2}(i_{\xi_i}\Omega_j + i_{\xi_j}\Omega_i) = 0.$$

■

We now prove that in dimension 3 tautness and roundness are equivalent. We start with the following lemma, which is a characterization of tautness in dimension 3.

Lemma 3.2. *A cosymplectic circle $\{(\eta_1, \Omega_1), (\eta_2, \Omega_2)\}$ on a 3-manifold M is taut if and only if the following conditions are fulfilled*

$$\eta_1 \wedge \Omega_1 = \eta_2 \wedge \Omega_2 \tag{3.1}$$
$$\eta_1 \wedge \Omega_2 = -\eta_2 \wedge \Omega_1. \tag{3.2}$$

Proof: Let us assume that $\{(\eta_1, \Omega_1), (\eta_2, \Omega_2)\}$ is taut, that is

$$(\lambda_1\eta_1 + \lambda_2\eta_2) \wedge (\lambda_1\Omega_1 + \lambda_2\Omega_2) = (\lambda_1'\eta_1 + \lambda_2'\eta_2) \wedge (\lambda_1'\Omega_1 + \lambda_2'\Omega_2)$$

for any $(\lambda_1, \lambda_2), (\lambda_1', \lambda_2') \in \mathbb{S}^1$. Then by taking $(\lambda_1, \lambda_2) = (1, 0)$ and $(\lambda_1', \lambda_2') = (0, 1)$ we get (3.1). Next, by taking $(\lambda_1, \lambda_2) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $(\lambda_1', \lambda_2') = (1, 0)$, and using (3.1), we obtain (3.2). Conversely, if (3.1)–(3.2) hold, then for

any $(\lambda_1, \lambda_2) \in \mathbb{S}^1$ we have

$$\begin{aligned}(\lambda_1\eta_1 + \lambda_2\eta_2) \wedge (\lambda_1\Omega_1 + \lambda_2\Omega_2) &= \lambda_1^2\eta_1 \wedge \Omega_1 + \lambda_1\lambda_2\eta_1 \wedge \Omega_2 \\ &\quad + \lambda_2\lambda_1\eta_2 \wedge \Omega_1 + \lambda_2^2\eta_2 \wedge \Omega_2 \\ &= (\lambda_1^2 + \lambda_2^2)\eta_1 \wedge \Omega_1 + \lambda_1\lambda_2\eta_1 \wedge \Omega_2 + \lambda_2\lambda_1\eta_2 \wedge \Omega_1 \\ &= \eta_1 \wedge \Omega_1.\end{aligned}$$

This proves that $\{(\eta_1, \Omega_1), (\eta_2, \Omega_2)\}$ is taut. ∎

Theorem 3.3. *On a 3-manifold M a cosymplectic p-sphere is taut if and only if it is round.*

Proof: We prove the statement of the theorem only for cosymplectic circles, since the case of cosymplectic spheres can be proved in a very similar way. Let $\{(\eta_1, \Omega_1), (\eta_2, \Omega_2)\}$ be a taut cosymplectic circle, that is (3.1)–(3.2) hold. Let ξ_1 and ξ_2 be the Reeb vector fields of (η_1, Ω_1) and (η_2, Ω_2), respectively. From (3.1) and (2.1) it follows that

$$i_{\xi_2}\Omega_1 = i_{\xi_2}i_{\xi_1}(\eta_1 \wedge \Omega_1) = i_{\xi_2}i_{\xi_1}(\eta_2 \wedge \Omega_2) = -i_{\xi_1}i_{\xi_2}(\eta_2 \wedge \Omega_2) = -i_{\xi_1}\Omega_2. \quad (3.3)$$

Moreover, by applying the interior product by ξ_1 to (3.2) we get

$$\Omega_2 - \eta_1 \wedge i_{\xi_1}\Omega_2 = -i_{\xi_1}\eta_2 \wedge \Omega_1. \quad (3.4)$$

By applying the interior product by ξ_2 to both sides of (3.4) we get

$$-i_{\xi_2}\eta_1 \wedge i_{\xi_1}\Omega_2 = -i_{\xi_1}\eta_2 \wedge i_{\xi_2}\Omega_1. \quad (3.5)$$

Thus, by using (3.3) in the first side of (3.5) we find

$$(i_{\xi_1}\Omega_2)\,(\eta_1(\xi_2) + \eta_2(\xi_1)) = 0,$$

which, by Proposition 2.8, is equivalent to

$$\eta_1(\xi_2) + \eta_2(\xi_1) = 0. \quad (3.6)$$

Notice that (3.6) and (3.3) are, respectively, the conditions (i) and (ii) in Proposition 3.1. Therefore the cosymplectic circle is round.

Conversely, let $\{(\eta_1, \Omega_1), (\eta_2, \Omega_2)\}$ be a round cosymplectic circle. Let ξ_1 and ξ_2 be the Reeb vector fields of the generators (η_1, Ω_1) and (η_2, Ω_2), respectively. By Proposition 3.1 we have

$$\eta_1(\xi_2) = -\eta_2(\xi_1), \quad (3.7)$$

$$i_{\xi_2}\Omega_1 = -i_{\xi_1}\Omega_2 \quad (3.8)$$

Now let $(\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha\in\{1,2,3\}}$, be an almost contact 3-structure on M. For any $\alpha, \beta \in \{1, 2, 3\}$ we define four tensors in the following way

$$N^{(1)}_{\alpha,\beta} := [\phi_\alpha, \phi_\beta] + d\eta_\alpha \otimes \xi_\beta + d\eta_\beta \otimes \xi_\alpha \quad (4.8)$$

$$\begin{aligned}N^{(2)}_{\alpha,\beta}(X, Y) := &\left(\mathcal{L}_{\phi_\alpha X}\eta_\beta\right)(Y) - \left(\mathcal{L}_{\phi_\alpha Y}\eta_\beta\right)(X) \\ &+ \left(\mathcal{L}_{\phi_\beta X}\eta_\alpha\right)(Y) - \left(\mathcal{L}_{\phi_\beta Y}\eta_\alpha\right)(X)\end{aligned} \quad (4.9)$$

$$N^{(3)}_{\alpha,\beta} := \mathcal{L}_{\xi_\alpha}\phi_\beta + \mathcal{L}_{\xi_\beta}\phi_\alpha \quad (4.10)$$

$$N^{(4)}_{\alpha,\beta} := \mathcal{L}_{\xi_\alpha}\eta_\beta + \mathcal{L}_{\xi_\beta}\eta_\alpha. \quad (4.11)$$

These tensors satisfy $N^{(i)}_{\alpha,\beta} = N^{(i)}_{\beta,\alpha}$, $1 \le i \le 4$. Moreover, $N^{(1)}_{\alpha,\alpha} = N^{(1)}_{\phi_\alpha}$ and $2N^{(i)}_{\alpha,\alpha} = N^{(i)}_{\phi_\alpha}$, $2 \le i \le 4$, where $N^{(i)}_{\phi_\alpha}$ are the fundamental tensors of an almost contact manifold (cf. [2, (1.3)]).

The following theorem establishes a general property of hyper-normal almost contact 3-structures.

Theorem 4.2. *Let $(\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha\in\{1,2,3\}}$ be a hyper-normal almost contact 3-structure on M. Then for each $\alpha, \beta \in \{1, 2, 3\}$ the tensors $N^{(1)}_{\alpha,\beta}$, $N^{(2)}_{\alpha,\beta}$, $N^{(3)}_{\alpha,\beta}$, $N^{(4)}_{\alpha,\beta}$ vanish.*

Proof: For each $\alpha \in \{1, 2, 3\}$ we define on the product $M \times \mathbb{R}$ the $(1, 1)$-tensor

$$J_\alpha\left(X, f\frac{d}{dt}\right) = \left(\phi_\alpha X - f\xi_\alpha, \eta_\alpha(X)\frac{d}{dt}\right)$$

where X is a vector field on M, f a smooth function on $M \times \mathbb{R}$ and t denotes the coordinate function on $\mathbb{R}$. It is well-known that (J_1, J_2, J_3) defines a hyper-complex structure on $M \times \mathbb{R}$. Now let us fix $\alpha, \beta \in \{1, 2, 3\}$ and let us evaluate the concomitant $[J_\alpha, J_\beta]$ of the vector fields of type $((X, 0), (Y, 0))$ and $((X, 0), (0, \frac{d}{dt}))$, where X, Y are vector fields on M. After a long computation, by using (4.4) and (4.8)–(4.11), we get

$$[J_\alpha, J_\beta]\,((X, 0), (Y, 0)) = \left(N^{(1)}_{\alpha,\beta}(X, Y), N^{(2)}_{\alpha,\beta}(X, Y)\frac{d}{dt}\right), \quad (4.12)$$

$$[J_\alpha, J_\beta]\left((X, 0), \left(0, \frac{d}{dt}\right)\right) = \left((N^{(3)}_{\alpha,\beta}(X), N^{(4)}_{\alpha,\beta}(X)\frac{d}{dt}\right). \quad (4.13)$$

Because of [21, Theorem 3.1] each Nijenhuis concomitant $[J_\alpha, J_\beta]$ vanishes and so from (4.12)–(4.13) the vanishing of $N^{(1)}_{\alpha,\beta}$, $N^{(2)}_{\alpha,\beta}$, $N^{(3)}_{\alpha,\beta}$, $N^{(4)}_{\alpha,\beta}$ follows. ∎

Theorem 4.1 ([5, 6])**.** *Let $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be a 3-quasi-Sasakian manifold of dimension $4n+3$. Then, for any even permutation (α, β, γ) of $\{1,2,3\}$, the Reeb vector fields satisfy*

$$[\xi_\alpha, \xi_\beta] = c\xi_\gamma, \tag{4.5}$$

for some $c \in \mathbb{R}$ which is zero if and only if the manifold is 3-cosymplectic. Moreover, the 1-forms η_1, η_2, η_3 have the same rank, called the rank *of the 3-quasi-Sasakian manifold M. The rank of M is 1 if and only if M is 3-cosymplectic and it is an integer of the form $4l+3$, for some $l \leq n$, in the other cases.*

Recall that in a quasi-Sasakian manifold of dimension $2n+1$ and of rank $2p+1$, the characteristic distribution

$$\mathcal{C} := \{X \in TM \,|\, i_X\eta = 0, i_X d\eta = 0\} \tag{4.6}$$

is integrable and has dimension $2(n-p)$. Moreover, in a 3-quasi-Sasakian manifold of dimension $4n+3$ and of rank $4l+3$ we can also consider the distribution

$$\mathcal{E} := \{X \in \mathcal{H} \,|\, i_X\eta_\alpha = 0, i_X d\eta_\alpha = 0 \text{ for any } \alpha = 1,2,3\} \tag{4.7}$$

which turns out to be integrable and to have dimension $4(n-l)$.

Recall that given two tensor fields P and Q of type $(1,1)$ on a smooth manifold M, one can define a tensor field of type $(1,2)$, denoted by $[P,Q]$, by setting

$$\begin{aligned}[P,Q](X,Y) = {} & [PX,QY] - P[QX,Y] - Q[X,PY] + [QX,PY] \\ & - Q[PX,Y] - P[X,QY] + (PQ+QP)[X,Y],\end{aligned}$$

where X and Y are arbitrary vector fields on M. The tensor $[P,Q]$ is usually called the *Nijenhuis concomitant* of P and Q.

Moreover, by Corollary 2.9 one has that ξ_1 and ξ_2 are linearly independent. Thus we can complete them to a local basis (ξ_1, ξ_2, Z) defined on an open subset U of M. Then by (3.8) we get

$$\Omega_1(\xi_2, Z) = -\Omega_2(\xi_1, Z). \tag{3.9}$$

Then, by using (3.7) and (3.9) we have, for any $\lambda = (\lambda_1, \lambda_2) \in \mathbb{S}^2$,

$$\begin{aligned}\eta_\lambda \wedge \Omega_\lambda(\xi_1,\xi_2,Z) &= (\lambda_1\eta_1 + \lambda_2\eta_2) \wedge (\lambda_1\Omega_1 + \lambda_2\Omega_2)(\xi_1,\xi_2,Z) \\ &= \lambda_1^2\,\eta_1 \wedge \Omega_1(\xi_1,\xi_2,Z) + \lambda_1\lambda_2\,\eta_1 \wedge \Omega_2(\xi_1,\xi_2,Z) \\ &\quad + \lambda_2\lambda_1\,\eta_2 \wedge \Omega_1(\xi_1,\xi_2,Z) + \lambda_2^2\,\eta_2 \wedge \Omega_2(\xi_1,\xi_2,Z) \\ &= \lambda_1^2\,\eta_1(\xi_1)\Omega_1(\xi_2,Z) + \lambda_1\lambda_2\,\eta_1(\xi_2)\Omega_2(Z,\xi_1) \\ &\quad + \lambda_2\lambda_1\,\eta_2(\xi_1)\Omega_1(\xi_2,Z) + \lambda_2^2\,\eta_2(\xi_2)\Omega_2(Z,\xi_1) \\ &= \left(\lambda_1^2 + \lambda_2^2\right)\Omega_1(\xi_2,Z) = \eta_1 \wedge \Omega_1(\xi_1,\xi_2,Z).\end{aligned}$$

This proves that the cosymplectic circle is taut. ∎

Theorem 3.3 does not hold in higher dimensions, as it is shown in the following example.

Example 3.4. Let us consider the cosymplectic structures (η_1, Ω_1), (η_2, Ω_2) on $\mathbb{T}^7$ given by

$$\begin{aligned}&\eta_1 := dx_7, \quad \eta_2 := dx_2, \qquad \Omega_1 := dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + dx_5 \wedge dx_6, \\ &\Omega_2 := dx_5 \wedge dx_4 - dx_3 \wedge dx_6 + (dx_1 + dx_3) \wedge dx_7.\end{aligned}$$

One can prove that they generate a cosymplectic circle on $\mathbb{T}^7$ which is taut but not round, since the condition (ii) in Proposition 3.1 is not satisfied. On the other hand, the cosymplectic circle generated by

$$\begin{aligned}&\eta_1 := dx_7, \quad \eta_2 := -dx_2, \qquad \Omega_1 := dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + dx_5 \wedge dx_6, \\ &\Omega_2 := dx_3 \wedge (dx_5 + dx_6) + dx_4 \wedge dx_5 + (dx_1 + dx_3) \wedge dx_6 + dx_1 \wedge dx_7,\end{aligned}$$

is round but not taut.

To any taut cosymplectic circle $\{(\eta_1, \Omega_1), (\eta_2, \Omega_2)\}$ on a 3-dimensional smooth manifold M it can be associated a complex structure J on $M \times \mathbb{R}$ in a canonical way. Indeed given an almost cosymplectic structure (η, Ω) on M we can define an almost symplectic structure (i. e. a nondegenerate 2-form) on $M \times \mathbb{R}$ by

$$\omega := dt \wedge \eta + \Omega.$$

It is well known that (M, η, Ω) is cosymplectic if and only if $(M \times \mathbb{R}, \omega)$ is symplectic. Recall that a pair of symplectic structures (ω_1, ω_2) on an oriented 4-manifold is said to be a *symplectic couple* [10] if one has $\omega_1 \wedge \omega_2 \equiv 0$ and ω_1^2, ω_2^2 are volume forms defining the positive orientation. A symplectic couple is called *conformal* if $\omega_1^2 = \omega_2^2$. It is convenient to extend the definition to the case of nondegenerate 2-forms. In that case one simply speaks of a *couple* or a *conformal couple*, respectively, omitting the term 'symplectic'. Two couples are called *equivalent* if at every point they span the same oriented plane of nondegenerate 2-forms. Now, the cosymplectic structures (η_1, Ω_1) and (η_2, Ω_2) give rise to two symplectic structures ω_1 and ω_2 on $M \times \mathbb{R}$. They satisfy the following relations.

$$\omega_1 \wedge \omega_1 = 2dt \wedge \eta_1 \wedge \Omega_1 \tag{3.10}$$

$$\omega_2 \wedge \omega_2 = 2dt \wedge \eta_2 \wedge \Omega_2 \tag{3.11}$$

$$\omega_1 \wedge \omega_2 = 2dt \wedge (\eta_1 \wedge \Omega_2 + \eta_2 \wedge \Omega_1). \tag{3.12}$$

As a consequence, we have $\omega_1 \wedge \omega_1 = \omega_2 \wedge \omega_2$ if and only if $\eta_1 \wedge \Omega_1 = \eta_2 \wedge \Omega_2$.

Hence by Lemma 3.2 we have that (ω_1, ω_2) is a conformal symplectic couple on $M \times \mathbb{R}$ if and only if $\{(\eta_1, \Omega_1), (\eta_2, \Omega_2)\}$ is a taut cosymplectic circle on M. Therefore, by [10, Theorem 2.2] to any taut cosymplectic circle $\{(\eta_1, \Omega_1), (\eta_2, \Omega_2)\}$ on a 3-dimensional smooth manifold M, we can associate a unique complex structure J on $M \times \mathbb{R}$ defined by the property that the forms of type $(2,0)$ with respect to J are precisely those of the type $\tilde{\omega}_1 + i\tilde{\omega}_2$, where $(\tilde{\omega}_1, \tilde{\omega}_2)$ is any conformal couple equivalent to $(\tilde{\omega}_1, \tilde{\omega}_2)$. The obtained complex structure J is a recursion operator in the sense of [1], i.e. it is the unique endomorphism such that $i_X \omega_1 = i_{JX} \omega_2$ for any $X \in TM$.

4. Cosymplectic spheres and 3-structures

In this section we describe a wide class of examples of almost cosymplectic spheres on a manifold of dimension $4n + 3$.

Recall that given an almost cosymplectic structure (η, Ω) there exist a Riemannian metric g and a tensor field ϕ such that

$$\phi^2 = -I + \eta \otimes \xi, \tag{4.1}$$

$$\Omega = g(-, \phi -). \tag{4.2}$$

Then one can prove that $\eta \circ \phi = 0$, $\phi\xi = 0$ and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{4.3}$$

for any $X, Y \in \Gamma(TM)$. An alternative approach consists in taking (4.1) as definition. So one defines an *almost contact structure* (unfortunately, the name is rather misleading, but it is widely used in literature) as the triplet (ϕ, ξ, η) satisfying (4.1) and $\eta(\xi) = 1$. In that case the dimension of M is necessarily odd. Then, a Riemannian metric g is called *compatible* if (4.3) is satisfied. The geometric structure (ϕ, ξ, η, g) is called *almost contact metric structure*. It follows that $g(X, \phi Y) = -g(\phi X, Y)$, so that the bilinear form $\Omega(X, Y) = g(X, \phi Y)$ is a 2-form, usually called the *fundamental 2-form*. Then one can prove that $\eta \wedge \Omega^n \neq 0$, where $\dim(M) = 2n + 1$. For further details we refer the reader to [3] and [8].

Now, when on the same manifold M there are given three distinct almost contact structures (ϕ_1, ξ_1, η_1), (ϕ_2, ξ_2, η_2), (ϕ_3, ξ_3, η_3) satisfying the following relations, for any even permutation (α, β, γ) of $\{1, 2, 3\}$,

$$\begin{aligned} \phi_\gamma &= \phi_\alpha \phi_\beta - \eta_\beta \otimes \xi_\alpha = -\phi_\beta \phi_\alpha + \eta_\alpha \otimes \xi_\beta, \\ \xi_\gamma &= \phi_\alpha \xi_\beta = -\phi_\beta \xi_\alpha, \quad \eta_\gamma = \eta_\alpha \circ \phi_\beta = -\eta_\beta \circ \phi_\alpha, \end{aligned} \tag{4.4}$$

we say that $(\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha \in \{1,2,3\}}$ is an *almost contact 3-structure* on M. One proves that the conditions (4.4) force the dimension of M to be necessarily $4n+3$ for some integer n. This notion was introduced independently by Kuo ([12]) and Udriste ([19]). Kuo proved also that given an almost contact 3-structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha \in \{1,2,3\}}$, there exists a Riemannian metric g compatible with each almost contact structure and hence we can speak of *almost contact metric 3-structure*. It is well known that in any almost 3-contact metric manifold the Reeb vector fields ξ_1, ξ_2, ξ_3 are orthonormal with respect to the compatible metric g. Moreover, by putting $\mathcal{H} = \bigcap_{\alpha=1}^{3} \ker(\eta_\alpha)$ we obtain a codimension 3 distribution on M and the tangent bundle splits as the orthogonal sum $TM = \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{V} = \langle \xi_1, \xi_2, \xi_3 \rangle$. The distributions $\mathcal{H}$ and $\mathcal{V}$ are called, respectively, *horizontal* and *Reeb distribution*.

An almost 3-contact manifold M is said to be *hyper-normal* if each almost contact structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha)$ is normal. An important class of hyper-normal almost contact 3-structures is given by the 3-quasi-Sasakian ones. A *3-quasi-Sasakian structure* is an almost contact metric 3-structure such that each structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ is quasi-Sasakian, i.e. it is normal and the corresponding fundamental 2-form is closed. Remarkable subclasses of 3-quasi-Sasakian manifolds are 3-Sasakian and 3-cosymplectic manifolds.

Many results on 3-quasi-Sasakian manifolds were obtained in [5] and [6]. We collect some of them in the following theorem.