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ON THE REPRESENTABILITY OF ACTIONS FOR TOPOLOGICAL ALGEBRAS

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Dedicated to Manuela Sobral

ABSTRACT: The actions of a group B on a group X correspond bijectively to the group homomorphisms $B \to \mathrm{Aut}(X)$, proving that the functor "actions on X" is representable by the group of automorphisms of X. Making the detour through pseudotopological spaces, we generalize this result to the topological case, for quasi-locally compact groups and some other algebraic structures. We investigate next the case of arbitrary topological algebras for a semi-abelian theory and prove that the representability of topological actions reduces to the preservation of coproducts by the functor $\mathrm{Act}(-, X)$.

KEYWORDS: actors, split extensions, topological algebras.
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1. Introduction

An action of a group $(B, \cdot)$ on a group $(X, +)$ is a mapping

$$B \times X \to X, \quad 1x = x, \quad b(x + x') = bx + bx', \quad (bb')x = b(b'x).$$

This is equivalent to giving a group homomorphism $B \to \mathrm{Aut}(X)$ to the group of automorphisms of X; this is further equivalent to giving a *split extension with kernel* X, that is, a short exact sequence with kernel X, provided with a splitting of the quotient map:

$$0 \longrightarrow X \xrightarrow{k} A \underset{q}{\overset{s}{\leftrightarrows}} B \longrightarrow 0; \quad qs = \mathrm{id}_B$$

(see [3]). In the case of groups, the functor $\mathbf{Grp} \to \mathbf{Set}$ mapping a group B on the set $\mathrm{Act}(B, X)$ of B-actions on X is thus represented by the group $\mathrm{Aut}(X)$. The purpose of this paper is to investigate analogous results, in

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the case of topological groups, topological Lie algebras and more generally, topological algebras for some semi-abelian algebraic theory $\mathbb{T}$. In those topological settings, the notion of split extension makes at once sense and it is equivalent to the one of *internal action* [7]. Here we identify split extensions with topological actions.

It is mentioned in [7] that group actions are representable in every Cartesian closed category $\mathbf{C}$; we first develop an explicit proof of that result, based on a private communication of G. Janelidze. The basic idea is that $\mathrm{Aut}(X)$ can be defined as a subobject of X^X in $\mathbf{C}$. We apply next this result in the Cartesian closed category **PsTop** of pseudotopological spaces, which contains the category **Top** of topological spaces as a full subcategory. When the topological space X is quasi-locally compact, it is exponentiable in **Top** and the result established in **PsTop** implies at once that actions on X are representable in the category of topological groups. We transpose the same kind of arguments to prove analogous results in the case of topological Lie algebras and in the case of topological groups with operations.

We switch next to a completely different approach, in order to investigate the case of topological $\mathbb{T}$-algebras, for an arbitrary semi-abelian algebraic theory $\mathbb{T}$. The representability of the functor $\mathrm{Act}(-,X)$, via the special adjoint functor theorem, reduces at once to the preservation of colimits. We prove that $\mathrm{Act}(-,X)$ always preserves sufficiently many coequalizers, to reduce the problem to the preservation of arbitrary coproducts. We give some necessary and sufficient conditions for that preservation of coproducts, conditions based on particular amalgamation properties in the category $\mathbf{Top}^{\mathbb{T}}$ of topological $\mathbb{T}$-algebras. We also make some observations towards a splitting of the problem between the case of binary coproducts and that of filtered colimits, which are computed in $\mathbf{Top}^{\mathbb{T}}$ as in **Set**.

2. Action representative categories

We start by recalling from [4] the following notions.

Definition 2.1. *Let $\mathbf{C}$ be a pointed protomodular category. Given an object $X \in \mathbf{C}$, actions on X are said to be* representable *if there exists an object $\mathrm{Act}(X) \in \mathbf{C}$, called the* actor *of X, and a split extension*

$$0 \longrightarrow X \xrightarrow{\kappa} \mathrm{Hol}(X) \underset{\pi}{\overset{\sigma}{\leftrightarrows}} \mathrm{Act}(X) \longrightarrow 0,$$

called the split extension classifier *of X, such that, for any split extension with kernel X:*

$$0 \longrightarrow X \xrightarrow{k} A \underset{p}{\overset{s}{\leftrightarrows}} B \longrightarrow 0$$

there exists a unique morphism $\varphi\colon B \to \mathrm{Act}(X)$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[d, equal] & A \arrow[r, shift right, "p"'] \arrow[d, "\varphi_1"'] & B \arrow[l, shift right, "s"'] \arrow[d, "\varphi"] \\ X \arrow[r] & \mathrm{Hol}(X) \arrow[r, shift right] & \mathrm{Act}(X), \arrow[l, shift right] \end{tikzcd}$$

where the morphism φ_1 is uniquely determined by φ and the identity on X (since k and s are jointly epimorphic).

When an actor exists for any $X \in \mathbf{C}$, the category $\mathbf{C}$ is said to be action representative.

Let us recall that a split extension with kernel X means a short exact sequence with kernel X, whose quotient part is provided with a splitting. Morphisms of split extensions commute with the specified splittings.

The name *representable* comes from the fact that, when an object X has an actor, then the functor

$$\mathrm{SplExt}(-, X)\colon \mathbf{C} \to \mathbf{Set},$$

associating with every $C \in \mathbf{C}$ the set of isomorphic classes of split extensions with codomain C and kernel X, is representable, as it was observed in [6], where this representability was studied in the context of semi-abelian categories [15]. When the functor $\mathrm{SplExt}(-, X)$ is representable, the representing object is the actor $\mathrm{Act}(X)$.

It is well known that the category **Grp** of groups is action representative. The actor of a group X is the group $\mathrm{Aut}(X)$ of automorphisms of X. The object $\mathrm{Hol}(X)$ is the classical holomorph of the group X, i.e. the semidirect product of X and $\mathrm{Aut}(X)$ with respect to the evaluation action. This fact justifies the notation $\mathrm{Hol}(X)$ that we are using for the split extension classifier. Moreover, we have the following result, that was already observed in [7] without proof.

Theorem 2.1. *If $\mathbf{E}$ is a finitely complete Cartesian closed category, then the category $\mathbf{Grp}(\mathbf{E})$ of internal groups in $\mathbf{E}$ is action representative.*

Proof: Given $X, Y \in \mathbf{Grp}(\mathbf{E})$, we first build the internal object $\mathrm{Hom}(X,Y)$. Consider the morphism $u = Y^{m_X} \colon Y^X \to Y^{X\times X}$ induced by the multiplication m_X of X. In other terms, u corresponds, via the universal property of the exponential, to the morphism

$$Y^X \times X \times X \xrightarrow{1\times m_X} Y^X \times X \xrightarrow{\mathrm{ev}} Y,$$

where ev is the evaluation morphism. In set-theoretical terms, $u(f)(x_1,x_2) = f(x_1x_2)$. Consider then the morphism $v \colon Y^X \to Y^{X\times X}$ which corresponds to the morphism

$$\begin{tikzcd} Y^X \times X \times X \arrow[r, "\Delta\times 1\times 1"] & Y^X \times Y^X \times X \times X \arrow[d, "1\times\mathrm{tw}\times 1"] & & \\ & Y^X \times X \times Y^X \times X \arrow[r, "\mathrm{ev}\times\mathrm{ev}"] & Y \times Y \arrow[r, "m_Y"] & Y, \end{tikzcd}$$

where Δ is the diagonal and tw is the twisting isomorphism. In set-theoretical terms, $v(f)(x_1,x_2) = f(x_1)f(x_2)$. We define then $\mathrm{Hom}(X,Y)$ as the equalizer of u and v:

$$\begin{tikzcd} Hom(X,Y) \arrow[r, "l"] & Y^X \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & Y^{X\times X}. \end{tikzcd}$$

We observe then that, if $X \in \mathbf{Grp}(\mathbf{E})$, the object X^X is an internal monoid in $\mathbf{E}$. The multiplication $\mu_X \colon X^X \times X^X \to X^X$ is given, via the universal property of the exponential, by:

$$X^X \times X^X \times X \xrightarrow{1\times\mathrm{ev}} X^X \times X \xrightarrow{\mathrm{ev}} X.$$

We prove now that the object $E(X) = \mathrm{Hom}(X,X)$ is a submonoid of X^X. Since l is the equalizer of u and v, in order to prove that the composition μ_X in X^X restricts to a composition in $E(X)$ it suffices to show that the composite

$$E(X) \times E(X) \rightarrowtail^{l\times l} X^X \times X^X \xrightarrow{\mu_X} X^X$$

equalizes u and v. The fact that l equalizes u and v can be expressed, using the universal property of the exponential, by the commutativity of the

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conditions in Proposition 8.3 are valid in this specific case. But we were unable to provide direct proofs of these conditions. Finding such proofs could possibly throw some light on the way to prove representability results for more general semi-abelian theories.

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following diagram:

$$
\begin{tikzcd}
E(X)\times X\times X \arrow[r, "1\times m_X"] \arrow[d, "\Delta\times 1\times 1"'] & E(X)\times X \arrow[r, "\bar{\mathrm{ev}}"] & X \arrow[dd, equal] \\
E(X)\times E(X)\times X\times X \arrow[d, "1\times \mathrm{tw}\times 1"'] & & \\
(E(X)\times X)\times(E(X)\times X) \arrow[r, "\bar{\mathrm{ev}}\times\bar{\mathrm{ev}}"'] & X\times X \arrow[r, "m_X"'] & X,
\end{tikzcd}
\tag{2.1}
$$

where $\bar{\mathrm{ev}}$ is the composite

$$E(X)\times X \xrightarrow{l\times 1} X^X\times X \xrightarrow{\mathrm{ev}} X.$$

We need to prove that the whole diagram below commutes:

$$
\begin{tikzcd}
E(X)\times E(X)\times X\times X \arrow[r, "1\times 1\times m_X"] \arrow[d, "1\times\Delta\times 1\times 1"'] & E(X)\times E(X)\times X \arrow[r, "1\times\bar{\mathrm{ev}}"] & E(X)\times X \arrow[r, "\bar{\mathrm{ev}}"] \arrow[dd, equal] & X \arrow[dd, equal] \\
E(X)\times E(X)\times E(X)\times X \arrow[d, "1\times 1\times\mathrm{tw}\times 1"'] \arrow[r, phantom, "(1)"] & {} & {} \arrow[r, phantom, "(2)"] & {} \\
E(X)\times(E(X)\times X)\times(E(X)\times X) \arrow[r, "1\times\bar{\mathrm{ev}}\times\bar{\mathrm{ev}}"'] \arrow[d, "\Delta\times 1\times 1"'] \arrow[dr, phantom, "(3)"] & E(X)\times X\times X \arrow[r, "1\times m_X"'] \arrow[d, "\Delta\times 1\times 1"] & E(X)\times X \arrow[r, "\bar{\mathrm{ev}}"'] & G \arrow[dd, equal] \\
E(X)\times E(X)\times(E(X)\times X)\times(E(X)\times X) \arrow[d, "1\times\mathrm{tw}\times 1"'] & E(X)\times E(X)\times X\times X \arrow[d, "1\times\mathrm{tw}\times 1"] \arrow[r, phantom, "(4)"] & {} & \\
(E(X)\times E(X)\times X)\times(E(X)\times E(X)\times X) \arrow[r, "(1\times\bar{\mathrm{ev}})\times(1\times\bar{\mathrm{ev}})"'] & (E(X)\times X)\times(E(X)\times X) \arrow[r, "\bar{\mathrm{ev}}\times\bar{\mathrm{ev}}"'] & X\times X \arrow[r, "m_X"'] & X.
\end{tikzcd}
\tag{2.2}
$$

Diagrams (2) and (3) obviously commute, while the commutativity of (1) and (4) follows immediately from the commutativity of Diagram 2.1. Hence the whole diagram commutes and $E(X)$ is a submonoid of X^X.

The object $\mathrm{Aut}(X)$ is then the internal group of invertible elements of $E(X)$; it is given by the following pullback:

$$
\begin{tikzcd}
\mathrm{Aut}(X) \arrow[r] \arrow[d] & 1 \arrow[d, "\langle \mathrm{id}_X, \mathrm{id}_X\rangle"] \\
E(X)\times E(X) \arrow[r, "\langle \mu_X, \mu_X^{op}\rangle"'] & E(X)\times E(X),
\end{tikzcd}
$$

with $\mu_X^{\mathrm{op}} = \mu_X\,\mathrm{tw}$, where tw is the twisting isomorphism of the product $E(X) \times E(X)$, while id_X is the morphism which determines the unit of the monoid $E(X)$.

It remains to show that $\mathrm{Aut}(X)$ is an actor of X. In the category **Grp** of groups, split extensions are equivalent to actions, as already mentioned in the Introduction. Indeed, the equivalence between actions and split extensions is obtained via the classical semidirect product construction. As it was observed in [7] and made explicit in [18], both the definition of an action and the semidirect product construction only involve finite limits, so they are Yoneda invariant. This means that the same equivalence holds in **Grp**(**E**) for any finitely complete category **E**. When **E** is Cartesian closed, it is not difficult to see that an action of B on X in **Grp**(**E**) is nothing but a morphism $B \to \mathrm{Aut}(X)$, and this concludes the proof. ■

3. Examples

This section is devoted to the description of other examples of action representative categories, and of objects which admit an actor even when the whole category is not action representative.

(1) The category R-**Lie** of Lie algebras, over a commutative ring R with unit, is action representative. The actor of a Lie algebra X is the Lie algebra $\mathrm{Der}(X)$ of derivations of X. We recall that a derivation of a Lie algebra X is a linear map (i.e. a homomorphism of R-modules) $\delta\colon X \to X$ such that, for any $x, y \in X$, $\delta([x,y]) = [\delta(x), y] + [x, \delta(y)]$.

As for the case of groups, we have the following result, already mentioned in [6]:

Theorem 3.1. *If* **E** *is a finitely complete Cartesian closed category, then the category R-**Lie**(**E**) of internal Lie algebras in* **E** *over an internal commutative ring R with unit is action representative.*

Proof: We only give a sketch of the construction of the internal object of derivations of an internal Lie algebra X. We start by building, exactly as in the proof of Theorem 2.1, the object $E(X)$ of endomorphisms of the additive group of X. We build then the object $L(X)$ as

If we did not insist much on this result, it is because the colimits involved are not very easy to cope with in $\mathbf{Top}^{\mathbb{T}}$, especially as far as their topological structure is concerned. This is why the second part of our paper has been organized to focus instead on the following result:

Proposition 8.2. *Let $\mathbb{T}$ be a theory of interest. The actions on $X \in \mathbf{Top}^{\mathbb{T}}$ are representable if and only if*

(1) *the normal amalgamation property holds for pairs of protosplit monomorphisms with domain X;*
(2) *in a filtered colimit of split extensions, the colimit of the kernels remains an embedding.*

Proof: By Propositions 6.4 and 7.1, $\mathrm{Act}(-, X)$ preserves finite coproducts and filtered colimits, thus preserves all coproducts. One concludes by Proposition 5.5. ■

One can even particularize Proposition 8.2 to reduce the problem to a purely topological one: the fact that some monomorphisms in $\mathbf{Top}^{\mathbb{T}}$, constructed from some given embedding (in fact, from some protosplit monomorphisms), remain embeddings.

Proposition 8.3. *Let $\mathbb{T}$ be a theory of interest and $X \in \mathbf{Top}^{\mathbb{T}}$. Suppose that in $\mathbf{Set}^{\mathbb{T}}$ the actions on X are representable. The actions on X are representable in $\mathbf{Top}^{\mathbb{T}}$ if and only if*

(1) *in the pushout of two protosplit monomorphisms with domain X, the inclusion of X in the pushout is an embedding;*
(2) *in a filtered colimit of split extensions, the colimit of the kernels remains an embedding.*

Proof: Since actions on X are representable in $\mathbf{Set}^{\mathbb{T}}$, the amalgamation property for pairs of morphisms with domain X is valid in $\mathbf{Set}^{\mathbb{T}}$ (see [6]) and thus, as observed in the proof of Proposition 6.4, the normal amalgamation property holds for these pairs. So the inclusion of X in the pushout is a normal monomorphism in $\mathbf{Set}^{\mathbb{T}}$ and, since it is an embedding by assumption, it is a normal monomorphism in $\mathbf{Top}^{\mathbb{T}}$ as well. One concludes by Proposition 8.2. ■

We want to conclude this paper by pointing out two open problems which puzzled us quite a lot. The first part of the paper proves that actions on a quasi-locally compact group (for example) are representable: thus both

is an isomorphism. Thus the filtered colimit (k', q', s') of the split extensions (k_i, q_i, s_i) is a split extension isomorphic to (k, q, s).

Condition 2 implies at once Condition 3. Assume now Condition 3 and, in the first diagram of the proof, define the bottom line to be the filtered colimit of the upper lines. This forces at once $qs = \mathrm{id}_Q$ and by commutativity of colimits, $q = \operatorname{coker} k$. Again by commutativity of kernels with filtered colimits in $\mathbf{Set}^{\mathbb{T}}$, $k = \ker q$ in $\mathbf{Set}^{\mathbb{T}}$ and, since by assumption k is an embedding, we have further $k = \ker q$ in $\mathbf{Top}^{\mathbb{T}}$. So the bottom line is a split extension. Moreover, since q and the various q_i have the same kernel X, the squares $(*)$ are pullbacks (see [3]). It remains to prove that (k, q, s) is the only split extension with that property. But just as above, if

$$0 \longrightarrow X \xrightarrow{k'} A' \underset{q'}{\overset{s'}{\leftrightarrows}} Q \longrightarrow 0$$

is another such sequence, we get a factorization $\gamma\colon A \to A'$ through the colimit A and, by the split short five lemma, this factorization is an isomorphism. ■

8. Some representability criteria . . . and more open problems

This paper contains two strikingly different approaches to the problem of representing topological actions. The first approach uses essentially the precise form of the theory: groups, Lie algebras, groups with operations, concluding that topological actions on X are representable as soon as X is quasi-locally compact. In our second approach, we handle the case of an arbitrary semi-abelian theory $\mathbb{T}$; in the case of a theory of interest, we end up with various "if and only if" criteria for the representability of actions.

Proposition 8.1. *Let $\mathbb{T}$ be a theory of interest. The actions on $X \in \mathbf{Top}^{\mathbb{T}}$ are representable if and only if the normal amalgamation property holds for all families of protosplit monomorphisms with domain X.*

Proof: As mentioned after Definition 6.2, all results and proofs of Section 6 transfer at once to the case of arbitrary families of protosplit monomorphisms with domain X, yielding in the analogue of Proposition 6.4, the preservation of all coproducts. By 5.5, this is precisely the condition needed for the representability of $\mathrm{Act}(-, X)$. ■

the equalizer of the morphisms u' and v' as below:

$$L(X) \overset{l'}{\rightarrowtail} X^X \underset{v'}{\overset{u'}{\rightrightarrows}} X^{R\times X},$$

where u' corresponds to the morphism

$$X^X \times R \times X \xrightarrow{1\times \mathrm{sm}} X^X \times X \xrightarrow{\mathrm{ev}} X$$

and v' corresponds to

$$X^X \times R \times X \xrightarrow{\mathrm{tw}\times 1} R \times X^X \times X \xrightarrow{1\times\mathrm{ev}} R \times X \xrightarrow{\mathrm{sm}} X,$$

where ev is the evaluation, sm is the scalar multiplication and tw is the twisting isomorphism. In set-theoretical terms, $u'(f)(\lambda, x) = f(\lambda x)$ and $v'(f)(\lambda, x) = \lambda f(x)$. Finally, we build the object $D(X)$ as the equalizer of the morphisms u'' and v'' as below:

$$D(X) \overset{l''}{\rightarrowtail} X^X \underset{v''}{\overset{u''}{\rightrightarrows}} X^{X\times X},$$

where u'' corresponds to the morphism

$$X^X \times X \times X \xrightarrow{\mathrm{br}} X^X \times X \xrightarrow{\mathrm{ev}} X$$

and v'' corresponds to

$$\begin{tikzcd}
X^X \times X \times X \arrow[r, "\Delta\times\Delta\times\Delta"] & X^X \times X^X \times X \times X \times X \times X \arrow[d, "\mathrm{tw}"] & & \\
& X^X \times X \times X \times X \times X^X \times X \arrow[d, "\mathrm{ev}\times 1\times 1\times \mathrm{ev}"] & & \\
& X \times X \times X \times X \arrow[r, "\mathrm{br}\times\mathrm{br}"] & X \times X \arrow[r, "+"] & X,
\end{tikzcd}$$

where br is the Lie bracket, $+$ is the additive group operation and tw is the suitable twisting. In set-theoretical terms, $u''(f)(x, y) = f([x, y])$, while $v''(f)(x, y) = [f(x), y] + [x, f(y)]$. The object $\mathrm{Der}(X)$ of derivations of X is then the intersection of $E(X)$, $L(X)$ and $D(X)$. ■

(2) Let R-**CAss** be the category of commutative associative algebras over a commutative ring R with unit. Given $X \in R$-**CAss**, a *multiplier* [13] of X is a linear map $\delta\colon X \to X$ such that

$$\delta(xy) = \delta(x)y = x\delta(y) \quad \text{for any } x, y \in X$$

The set $\mathrm{Mul}(X)$ of multipliers of X is an associative algebra, under the usual sum, scalar multiplication and composition. However, it is not commutative, in general. According to Theorem 2.6 in [6], X has an actor in R-**CAss** if and only if $\mathrm{Mul}(X)$ is commutative. When it is the case, $\mathrm{Mul}(X)$ is the actor of X. A sufficient condition for X to have an actor is the following (Proposition 2.7 in [6]): if $XX = X$, then X has an actor. Here XX is the algebra generated by elements of the form xy, for $x, y \in X$. In particular, this condition is satisfied when X has a unit element, because then $x = 1x$ for all $x \in X$. (we observe that in [6] the multipliers are called endomorphisms, although they are not homomorphisms of associative algebras)

(3) A commutative associative algebra X is a *von Neumann regular algebra* if, for all $x \in X$, there exists $y \in X$ such that $x = xyx$. If X is a von Neumann regular algebra, then obviously $XX = X$, and hence $\mathrm{Mul}(X)$ is an actor of X in R-**CAss**. Moreover (see Lemma 4.3 in [6]), $\mathrm{Mul}(X)$, in this case, is a von Neumann regular algebra, and hence it is an actor of X also in the category of commutative von Neumann regular algebras. This category is then action representative.

(4) A commutative associative algebra X is a *Boolean algebra* if, for all $x \in X$, $x = xx$. If X is a Boolean algebra, then obviously $XX = X$, and hence $\mathrm{Mul}(X)$ is an actor of X in R-**CAss**. Moreover (see Proposition 3.1 in [6]), $\mathrm{Mul}(X)$, in this case, is a Boolean algebra, and hence it is an actor of X also in the category R-**Bool** of Boolean algebras. R-**Bool** is then action representative.

(5) Let R-**Ass** be the category of associative algebras over a commutative ring R with unit. Given $X \in R$-**Ass**, a *bimultiplier* [13] of X is a pair (δ, d) of linear maps $\delta, d \colon X \to X$ such that the following conditions are satisfied:
(a) $\delta(xy) = \delta(x)y$;
(b) $d(xy) = xd(y)$;
(c) $x\delta(y) = d(x)y$.
The set $\mathrm{Bimul}(X)$ of bimultipliers of X is an associative algebra, under the usual componentwise sum and scalar multiplication, while the

both lines coincide. And since k' is also an embedding, $k' = \ker q'$ in $\mathbf{Top}^{\mathbb{T}}$. The split short five lemma allows to conclude that γ is an isomorphism in $\mathbf{Top}^{\mathbb{T}}$. ■

7. The preservation of filtered colimits

One way to switch from finite coproducts to arbitrary coproducts is just to look at an arbitrary coproduct as the filtered colimit of its finite subcoproducts. Moreover filtered colimits in $\mathbf{Top}^{\mathbb{T}}$, like all colimits, are calculated as in $\mathbf{Set}^{\mathbb{T}}$, thus filtered colimits in $\mathbf{Top}^{\mathbb{T}}$ are computed as in **Set**.

Proposition 7.1. *For an object $X \in \mathbf{Top}^{\mathbb{T}}$, the following conditions are equivalent:*
(1) *the functor* $\mathrm{Act}(-, X)$ *preserves (contravariantly) filtered colimits;*
(2) *split extensions with kernel X are stable under filtered colimits;*
(3) *in a filtered colimit of split extensions, the colimit of the kernels remains an embedding.*

Proof: Suppose that $\mathrm{Act}(-, X)$ preserves filtered colimits. Given a filtered diagram (k_i, q_i, s_i) of split extensions, there exists thus a unique split extension (k, q, s) allowing to recapture all the sequences (k_i, q_i, s_i) by pullbacks along the morphisms σ_i.

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "k_i"] \arrow[d, equal] & A_i \arrow[r, shift right, "q_i"'] \arrow[d, "\alpha_i"'] \arrow[dr, phantom, "(*)"] & Q_i \arrow[l, shift right, "s_i"'] \arrow[r] \arrow[d, "\sigma_i"] & 0 \\
0 \arrow[r] & X \arrow[r, "k"] & A \arrow[r, shift right, "q"'] & Q \arrow[l, shift right, "s"'] \arrow[r] & 0
\end{tikzcd}$$

Consider now the filtered colimit (k', q', s') of the upper lines. By the colimit property, we obtain a factorization γ to the bottom line.

$$\begin{tikzcd}
0 \arrow[r, dotted] & X \arrow[r, "k'"] \arrow[d, equal] & A' \arrow[r, shift right, "q'"'] \arrow[d, "\gamma"'] & Q \arrow[l, shift right, "s'"'] \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r, "k"] & A \arrow[r, shift right, "q"'] & Q \arrow[l, shift right, "s"'] \arrow[r] & 0
\end{tikzcd}$$

We have at once $q's' = \mathrm{id}_Q$ and by commutativity of colimits, $q' = \operatorname{coker} k'$. But in $\mathbf{Set}^{\mathbb{T}}$, kernels commute with filtered colimits, thus $k' = \ker q'$ in $\mathbf{Set}^{\mathbb{T}}$. But since $\gamma k' = k$ with k an embedding in $\mathbf{Top}^{\mathbb{T}}$, k' is an embedding as well. This implies that $k' = \ker q'$ in $\mathbf{Top}^{\mathbb{T}}$ and, by the split short five lemma, γ

the notation above and that of Definition 6.2, we have

$$\forall i \in \{1,2\} \;\; \forall x \in X \;\; \forall a \in A_i \;\; \forall t \in \mathcal{N} \;\; t(x,a) \in X.$$

But the set theoretical union $S = \bigcup_{i\in\{1,2\}} A_i$ generates the pushout C and the formula above tells us in particular that

$$\forall x \in X \;\; \forall a \in S \;\; \forall t \in \mathcal{N} \;\; t(x,a) \in X.$$

Thus X is normal in C.

Let us come back to the statement of the Proposition and assume that $\mathrm{Act}(-,X)$ preserves (contravariantly) finite coproducts. The assumption means – with the notation of the proof of Proposition 6.2, given two split extensions (k_i, q_i, s_i) – that there exists a unique split extension (k,q,s) whose pullbacks along the morphisms σ_i recapture the original sequences (k_i, q_i, s_i). Forgetting the topologies, we obtain exactly an analogous situation in $\mathbf{Set}^{\mathbb{T}}$.

By Proposition 6.2, we know that the amalgamation property holds in $\mathbf{Top}^{\mathbb{T}}$ for the pair (k_1, k_2), thus it holds also in $\mathbf{Set}^{\mathbb{T}}$ since the forgetful functor is topological (see [5]). But as we have just seen, in $\mathbf{Set}^{\mathbb{T}}$, this forces the pair (k_1, k_2) to satisfy the normal amalgamation property. Repeating then in $\mathbf{Set}^{\mathbb{T}}$ the argument proving Proposition 6.3, we conclude that in $\mathbf{Set}^{\mathbb{T}}$ the sequence (k,q,s) is that obtained by the pushout process.

Since (k,q,s) is a split extension in $\mathbf{Top}^{\mathbb{T}}$, X is normal in A and it remains to check that A, which is the pushout of (k_1, k_2) in $\mathbf{Set}^{\mathbb{T}}$, is also provided with the pushout topology. To see that, consider, as in the first diagram in the proof of Proposition 6.3, the sequence (k', q', s') obtained via the pushout process in $\mathbf{Top}^{\mathbb{T}}$. We know that A' and A are the same $\mathbb{T}$-algebra, possibly with two different topologies. We know also that $q's' = \mathrm{id}_Q$ and by commutativity of colimits, $q' = \mathrm{coker}\, k'$. We would like to prove further that $k' = \ker q'$. But the pushout property in $\mathbf{Top}^{\mathbb{T}}$ forces the existence of a factorization γ

$$\begin{tikzcd}
0 \arrow[r, dotted] & X \arrow[r, "k'"] \arrow[d, equal] & A' \arrow[r, shift right, "q'"'] \arrow[d, "\gamma"'] & Q \arrow[l, shift right, "s'"'] \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r, "k"] & A \arrow[r, shift right, "q"'] & Q \arrow[l, shift right, "s"'] \arrow[r] & 0
\end{tikzcd}$$

and this factorization γ is the identity mapping, since in $\mathbf{Set}^{\mathbb{T}}$ both lines coincide. But k is a normal monomorphism in $\mathbf{Top}^{\mathbb{T}}$, thus an embedding; this forces k' to be an embedding as well. But in $\mathbf{Set}^{\mathbb{T}}$, $k' = \ker q'$ since

multiplication is given by:

$$(\delta, d)(\gamma, c) = (\delta\gamma, cd).$$

According to Proposition 2.4 in [6], if $XX = X$, then $\mathrm{Bimul}(X)$ is an actor of X. Again, this sufficient condition is satisfied when X has a unit element. In [8] it is proved that another sufficient condition for $\mathrm{Bimul}(X)$ to be an actor is that the annihilator $\mathrm{Ann}(X)$ of X is the trivial algebra 0.

(6) A (right) Leibniz algebra [16] over a commutative ring R with unit is a R-module X equipped with a bilinear binary operation $[\;,\;]\colon X \times X \to X$ satisfying the following axiom:

$$[x,[y,z]] = [[x,y],z] - [[x,z],y] \quad \text{for all } x,y,z \in X.$$

A *biderivation* of a Leibniz algebra X is a pair (δ, d) of linear maps $\delta, d\colon X \to X$ such that the following conditions are satisfied:
(a) $\delta([x,y]) = [x,\delta(y)] + [\delta(x),y]$;
(b) $d([x,y]) = [d(x),y] - [d(y),x]$;
(c) $[x,\delta(y)] = [x,d(y)]$.
The set $\mathrm{Bider}(X)$ of biderivations of X is a Leibniz algebra, under the usual componentwise sum and scalar multiplication, while the bracket is given by:

$$[(\delta,d),(\gamma,c)] = (\delta\gamma - \gamma\delta, d\gamma - \gamma d).$$

It is proved in [8] that, if X is such that $[X,X] = X$ or the annihilator $\mathrm{Ann}(X)$ is the trivial algebra, then $\mathrm{Bider}(X)$ is an actor of X in the category R-$\mathbf{Leib}$ of Leibniz algebras.

We conclude this section by observing that all the categories considered in the examples above are categories of groups with operations in the sense of Porter [21]. It was observed in [21] that, in any category of groups with operations, every split epimorphism with codomain B and kernel X corresponds to a set of actions, i.e. to a set of functions $B \times X \to X$, indexed by the set Ω_2 of binary operations of the underlying algebraic theory, satisfying suitable conditions. This implies that, in any category of groups with operations, if an object X has representable actions, then the actor $\mathrm{Act}(X)$ is necessarily a subset of the Cartesian product of copies of X^X. This fact will be useful in the following section.

4. Actors for topological algebras ... and some open problems

In order to study the representability of actions for topological algebras, we first analyse the problem for algebras equipped with a pseudotopology. We recall that a pseudotopological space (X, R_X) is a set X equipped with a convergence relation R_X between ultrafilters on X and points of X, so that, for every $x \in X$, the principal ultrafilter $\dot{\{x\}}$ defined by $\{x\}$ converges to x (we will use $\mathfrak{x} \to x$ to denote that the ultrafilter $\mathfrak{x}$ converges to x). A map $f\colon (X, R_X) \to (Y, R_Y)$, between pseudotopological spaces, is continuous if $f(\mathfrak{x}) \to x$ whenever $\mathfrak{x} \to x$ (here $f(\mathfrak{x})$ denotes the ultrafilter generated by $\{f(A), A \in \mathfrak{x}\}$ on Y).

We recall that a topology on a set X can be defined also via a convergence relation R_X between ultrafilters and points on X, satisfying:

$$\dot{\{x\}} \to x,$$

$$\mathfrak{X} \to \mathfrak{x} \text{ and } \mathfrak{x} \to x \Rightarrow \mu(\mathfrak{X}) \to x$$

for every ultrafilter $\mathfrak{X}$ on the ultrafilters of X, every ultrafilter $\mathfrak{x}$ and every point $x \in X$, where μ is the Kowalsky sum of $\mathfrak{X}$ (that is, μ is the multiplication of the ultrafilter monad on **Set**): see [1, 9] for details. The category **Top** of topological spaces and continuous maps is a full subcategory of the category **PsTop** of pseudotopological spaces and continuous maps.

The category **PsTop** is Cartesian closed, that is, for every object X the functor

$$(\ ) \times X\colon \mathbf{PsTop} \to \mathbf{PsTop}$$

has a right adjoint $(\ )^X\colon \mathbf{PsTop} \to \mathbf{PsTop}$, and therefore Theorems 2.1 and 3.1 apply, giving that actions of internal groups and internal Lie algebras in **PsTop** are representable. The category **Top** is not Cartesian closed. Indeed, as shown essentially in [11] (see also [14, 20, 9]), the functor $(\ ) \times X : \mathbf{Top} \to \mathbf{Top}$ has a right adjoint if, and only if, X is *quasi-locally compact*, that is, for each $x \in X$ and each neighborhood U of x there exists a neighborhood V of x that is relatively compact in U. By *relatively compact in* U it is meant that, for every open cover $(U_i)_{i \in I}$ of U, there is a finite subset F of I such that $\bigcup_{i \in F} U_i \supseteq V$. If X is sober, in particular if X is Hausdorff,

we get at once a factorization $\gamma\colon C \to C'$ such that $\gamma\kappa_i = \kappa'_i$, just because C has been defined as the pushout of the k_i. This implies further

$$\gamma k = \gamma\kappa_i k_i = \kappa'_i k_i = k'.$$

On the other hand

$$q'\gamma\kappa_i = q'\kappa'_i = \sigma_i q_i = q\kappa_i, \quad \gamma s\sigma_i = \gamma\kappa_i s_i = \kappa'_i s_i = s'\sigma_i$$

from which $q'\gamma = q$ and $\gamma s = s'$. We get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "k"] \arrow[d, equal] & C \arrow[r, shift right, "q"'] \arrow[d, "\gamma"'] & Q \arrow[l, shift right, "s"'] \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r, "k'"] & C' \arrow[r, shift right, "q'"'] & Q \arrow[l, shift right, "s'"'] \arrow[r] & 0
\end{tikzcd}$$

and by the split short five lemma, γ is an isomorphism. ■

Proposition 6.3 can be turned in an "if and only if" condition in a special case "of interest": the precisely so-called *categories of interest* (see [19]). These categories of interest cover major examples of semi-abelian categories: groups, rings, Lie algebras, as well as all the concrete examples of Section 3. Throughout by an algebraic theory of interest we mean an algebraic theory whose category of models in **Set** is a category of interest. We point out that an algebraic theory of interest is automatically semi-abelian.

Proposition 6.4. *Let $\mathbb{T}$ be an algebraic theory of interest and $X \in \mathbf{Top}^{\mathbb{T}}$. The functor* $\mathrm{Act}(-, X)$ *transforms finite coproducts into finite products if and only if the normal amalgamation property holds for the pairs of protosplit monomorphisms with domain X.*

Proof: Very roughly speaking, the algebraic categories of interest are characterized by a "good" theory of ideals. Such a theory admits in particular a set $\mathcal{N}$ of binary terms with the property that a subalgebra $X \subseteq A$ is normal if and only if there exists a subset $S \subseteq A$ that generates A and is such that

$$\forall x \in X \quad \forall a \in S \quad \forall t \in \mathcal{N} \quad t(x, a) \in X.$$

Of course in the case of the theory of groups, it suffices to choose the single term $t(x, a) = a + x - a$.

Observe first that, under these conditions, in $\mathbf{Set}^{\mathbb{T}}$ the normal amalgamation property follows at once from the amalgamation property. Indeed with

the morphisms α_i are thus monomorphisms as well. With the notation of Definition 6.2, since we have $\alpha_i k_i = k$ for both indices i, we get a factorization $\alpha\colon C \to A$ through the pushout such that $\alpha\kappa_i = \alpha_i$. Since each α_i is a monomorphism, so is each κ_i. ∎

As usual, we shall use the term *embedding* to indicate in $\mathbf{Top}^{\mathbb{T}}$ the inclusion of a topological subalgebra provided with the induced topology.

Proposition 6.3. *Let X be an object in $\mathbf{Top}^{\mathbb{T}}$. The normal amalgamation property in the case of pairs of protosplit monomorphisms with domain X is a sufficient condition for the transformation of finite coproducts into finite products by the functor* $\mathrm{Act}(-,X)$.

Proof: With the notation of Definition 6.2, let us consider the diagram, for both indices i

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, shift right] \arrow[d, "k_i"'] & 0 \arrow[l, shift right] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X \arrow[r, "k_i"] \arrow[d, equal] & A_i \arrow[r, shift right, "q_i"'] \arrow[d, "\kappa_i"'] \arrow[dr, phantom, "(*)"] & Q_i \arrow[l, shift right, "s_i"'] \arrow[r] \arrow[d, "\sigma_i"] & 0 \\
0 \arrow[r] & X \arrow[r, "k"] & C \arrow[r, shift right, "q"'] & Q \arrow[l, shift right, "s"'] \arrow[r] & 0
\end{tikzcd}$$

The bottom line is objectwise the pushout the upper part of the diagram, when i runs through $\{1,2\}$; we define k, q, s to be the factorizations through the pushouts. In particular $qs = \mathrm{id}_Q$ and $q = \operatorname{coker} k$, by commutativity of colimits. By the normal amalgamation property, k is a normal monomorphism. Since the category $\mathbf{Top}^{\mathbb{T}}$ is homological, the normal monomorphism k is the kernel of its cokernel (see [3]), that is, $k = \ker q$. Thus the bottom line is a split extension. And since q_i and q have the same kernel, the squares $(*)$ are pullbacks (see [3] again).

It remains to prove that (k,q,s) is – up to an isomorphism – the unique split extension restricting to each (k_i,q_i,s_i) by pullbacks along the monomorphisms σ_i. If (k',q',s') is another such sequence

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "k_i"] \arrow[d, equal] & A_i \arrow[r, shift right, "q_i"'] \arrow[d, "\kappa'_i"'] \arrow[dr, phantom, "(*)"] & Q_i \arrow[l, shift right, "s_i"'] \arrow[r] \arrow[d, "\sigma_i"] & 0 \\
0 \arrow[r] & X \arrow[r, "k'"] & C' \arrow[r, shift right, "q'"'] & Q \arrow[l, shift right, "s'"'] \arrow[r] & 0
\end{tikzcd}$$

X is quasi-locally compact if and only if it is *locally compact*, that is if every point of X has a neighborhood base consisting of compact subsets.

We recall from [22] (see also [12]) that, since **Top** is finally dense in **PsTop**, for every topological space X,

$$X^X = \{f\colon X \to X \mid f \text{ is continuous}\}$$

has a topological structure making the evaluation map

$$\mathrm{ev}\colon X^X \times X \to X$$

continuous and universal if and only if its pseudotopological structure, given by the right adjoint to the functor $(\ )\times X\colon \mathbf{PsTop} \to \mathbf{PsTop}$, is a topology. Moreover, **Top** is closed, in **PsTop**, under embeddings and products. Hence, for a quasi-locally compact space X we can make the construction of $E(X)$ in **PsTop** as in the proof of Theorem 2.1, and, since the pseudotopology in $E(X)$ is a topology, conclude:

Theorem 4.1. *If X is a quasi-locally compact topological group, then the functor* $\mathrm{SplExt}(-,X)\colon \mathbf{Grp}(\mathbf{Top}) \to \mathbf{Set}$ *is representable.*

In the same way, adapting the construction of the object of internal derivations given in Theorem 3.1, we get:

Theorem 4.2. *If X is a quasi-locally compact topological Lie algebra over a topological commutative ring R with unit, then the functor* $\mathrm{SplExt}(-,X)\colon R\text{-}\mathbf{Lie}(\mathbf{Top}) \to \mathbf{Set}$ *is representable.*

Proof: We repeat the same construction as in Theorem 3.1 for $\mathbf{E} = \mathbf{PsTop}$, and the quasi-local compactness of X guarantees that $L(X)$, as a subspace of X^X, is a topological space, without hypotheses on the ring R. ∎

Moreover, using the remarks at the end of Section 3, we can make analogous constructions of the internal actors of objects X with representable actions in a category of groups with operations, because these actors are always subobjects of a product of copies of X^X. We obtain then the following:

Theorem 4.3. *Let $\mathbf{V}$ be a variety of groups with operations, and $\mathbf{V}(\mathbf{Top})$ the corresponding category of topological models. Let $X \in \mathbf{V}(\mathbf{Top})$ be a quasi-locally compact topological algebra such that the functor*

$$\mathrm{SplExt}(-,X)\colon \mathbf{V} \to \mathbf{Set}$$

is representable. Then also the functor

$$\mathrm{SplExt}(-,X)\colon \mathbf{V}(\mathbf{Top}) \to \mathbf{Set}$$

is representable.

We do not know whether the condition of X being quasi-locally compact is necessary. In the pseudotopology of X^X an ultrafilter $\mathfrak{f}$ converges to f if, for every $\mathfrak{x} \to x$ on X and every ultrafilter $\mathfrak{w}$ on $X^X \times X$ such that $\pi_1(\mathfrak{w}) = \mathfrak{f}$ and $\pi_2(\mathfrak{w}) = \mathfrak{x}$, $\mathrm{ev}(\mathfrak{w}) = f(x)$ on X (where $\pi_1\colon X^X \times X \to X^X$ and $\pi_2\colon X^X \times X \to X$ are the product projections, see, e.g., [9] for details). Hence, it is an open problem, for topological groups, to know whether the quasi-local compactness of X is necessary for this pseudotopology, when restricted to

$$E(X) = \{\, f\colon X \to X \mid f \text{ is an auto-homeomorphism}\,\},$$

being a topology. An analogous problem arises for the actor of an object in a category of topological groups with operations.

5. A formal criterion for representing actions

We intend now to exhibit a formal necessary and sufficient condition for the representability of the functor $\mathrm{Act}(-,X)$, when defined on the category $\mathbf{Top}^{\mathbb{T}}$ of topological $\mathbb{T}$-algebras, for an arbitrary semi-abelian theory $\mathbb{T}$.

The well-known abstract theorems for the representability of a $\mathbf{Set}$-valued functor yield at once:

Proposition 5.1. *Let X be a fixed object in $\mathbf{Top}^{\mathbb{T}}$. The following conditions are equivalent:*

(1) *the functor $\mathrm{Act}(-,X)$ is representable;*
(2) *the functor*

$$\mathrm{Act}(-,X)\colon (\mathbf{Top}^{\mathbb{T}})^{\mathrm{op}} \to \mathbf{Set}$$

preserves products and equalizers of pairs of morphisms with a common retraction.

Proof: The category $\mathbf{Top}^{\mathbb{T}}$ is cocomplete. The free algebra on one generator, provided with the discrete topology, is a generator. Moreover the category $\mathbf{Set}^{\mathbb{T}}$ is co-well-powered and the forgetful functor $\mathbf{Top}^{\mathbb{T}} \to \mathbf{Set}^{\mathbb{T}}$ is topological (see [5]), thus preserves epimorphisms. But each $\mathbb{T}$-algebra admits only a set of topologies turning it into a topological $\mathbb{T}$-algebra. Thus the category $\mathbf{Top}^{\mathbb{T}}$

Definition 6.2. *In a homological category, consider a pair of monomorphisms $k_i\colon X \rightarrowtail A_i$, $(i = 1,2)$ and their pushout*

$$\begin{tikzcd} X \arrow[r, tail, "k_1"] \arrow[d, tail, "k_2"'] & A_1 \arrow[d, "\kappa_1"] \\ A_2 \arrow[r, "\kappa_2"'] & C \end{tikzcd}$$

(1) *The* amalgamation property *holds for the pair (k_1,k_2) when the morphisms κ_i are monomorphisms as well.*
(2) *The property of* normal amalgamation *holds for the pair (k_1,k_2) when moreover, if k_1 and k_2 are normal monomorphisms, the composite $\kappa_1 k_1 = \kappa_2 k_2$ is a normal monomorphism as well (here by normal monomorphism we mean a kernel).*

Notice that these definitions could have been stated as well for an arbitrary family of monomorphisms $(k_i\colon X \rightarrowtail A_i)_{i\in I}$ and the observant reader will notice that all proofs in the present section have been written in such a way that they transfer trivially as such to that more general context.

Proposition 6.2. *Let X be an object in $\mathbf{Top}^{\mathbb{T}}$. The amalgamation property in the case of pairs of protosplit monomorphisms with domain X is a necessary condition for the transformation of finite coproducts into finite products by the functor $\mathrm{Act}(-,X)$.*

Proof: Consider two protosplit monomorphisms $k_i\colon X \rightarrowtail A_i$, kernel parts of two split extensions, which we choose as the upper lines in the diagram below. Suppose that $\mathrm{Act}(-,X)$ is representable; it transforms thus the coproduct Q of Q_1 and Q_2 into a product, which means that

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, tail, "k_i"] \arrow[d, equal] & A_i \arrow[r, shift right, "q_i"'] \arrow[d, tail, "\alpha_i"'] \arrow[dr, phantom, "(*)"] & Q_i \arrow[l, shift right, "s_i"'] \arrow[r] \arrow[d, tail, "\sigma_i"] & 0 \\ 0 \arrow[r] & X \arrow[r, tail, "k"] & A \arrow[r, shift right, "q"'] & Q \arrow[l, shift right, "s"'] \arrow[r] & 0 \end{tikzcd}$$

there exists a unique bottom split extension allowing to recapture the upper sequences via a pullback process along the canonical morphisms $\sigma_i\colon Q_i \rightarrowtail Q$ of the coproduct. Notice that these canonical morphisms are monomorphisms, since σ_i admits as a retraction the morphism which restricts to the identity on Q_i and to 0 on the other term of the coproduct. By pullback,

as observed above via Proposition 5.2. ■

Using the results of this section, we can thus conclude with a formal criterion for the representability of the actions on X:

Proposition 5.5. *For an object $X \in \mathbf{Top}^{\mathbb{T}}$, the following conditions are equivalent:*

(1) *the functor* $\mathrm{Act}(-,X)$ *is representable;*
(2) *the functor* $\mathrm{Act}(-,X)$ *transforms coproducts into products.*

6. The preservation of finite coproducts

Proposition 5.5 reduces the question of the representability of actions to the (contravariant) preservation of coproducts. This section investigates further the preservation of finite coproducts by the functors $\mathrm{Act}(-,X)$. The next section will take care of the case of arbitrary coproducts.

Of course the case of finite coproducts can be split in three cases

- the empty coproduct (the initial object);
- the "one term" coproduct, for which there is nothing to prove;
- binary coproducts, which allow to reconstruct inductively the coproduct of n terms ($n \geq 2$).

The case of the initial object is trivial:

Proposition 6.1. *Each functor* $\mathrm{Act}(-,X)$ *on* $\mathbf{Top}^{\mathbb{T}}$ *maps the initial object to the singleton.*

Proof: The only split extension with kernel X and quotient 0 is

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, equal] & X \arrow[r, shift right] & 0 \arrow[l, shift right] \arrow[r] & 0. \end{tikzcd}$$

■

To facilitate the language, let us borrow the following terminology from [6].

Definition 6.1. *Let $\mathbf{C}$ be a homological category. A monomorphism k is* protosplit *when it is the kernel part of a split extension*

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "k"] & A \arrow[r, shift right, "q"'] & Q \arrow[l, shift right, "s"'] \arrow[r] & 0. \end{tikzcd}$$

Let us further recall the classical amalgamation properties.

is co-well-powered as well. So $(\mathbf{Top}^{\mathbb{T}})^{\mathrm{op}}$ is complete, admits a cogenerator and is well-powered.

A representable functor preserves limits. Conversely, the functor $\mathrm{Act}(-,X)$ is representable when the singleton admits a universal reflection along $\mathrm{Act}(-,X)$, which is of course the case when $\mathrm{Act}(-,X)$ admits a left adjoint. By the Freyd's special adjoint functor theorem, it remains to prove that $\mathrm{Act}(-,X)$ preserves limits. This is the case by assumption since in a complete category, every limit can be reconstructed as the equalizer of a pair of morphisms, defined between two products, and having a common retraction (see [17]). ■

To avoid any ambiguity, we shall always work with the contravariant functor $\mathrm{Act}(-,X)$ defined on $\mathbf{Top}^{\mathbb{T}}$, which has thus to transform coproducts into products and coequalizers of pairs of morphisms with a common section into equalizers.

Proposition 5.2. *Each functor* $\mathrm{Act}(-,X)$ *defined on* $\mathbf{Top}^{\mathbb{T}}$ *transforms the coequalizer of a kernel pair into an equalizer.*

Proof: We consider the following diagram

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, tail, "k''"] \arrow[d, equal] & A'' \arrow[r, shift right, "q''"'] \arrow[d, shift right, "u'"'] \arrow[d, shift left, "v'"] & B'' \arrow[l, shift right, "s''"'] \arrow[r] \arrow[d, shift right, "u"'] \arrow[d, shift left, "v"] & 0 \\ 0 \arrow[r] & X \arrow[r, tail, "k'"] \arrow[d, equal] & A' \arrow[r, shift right, "q'"'] \arrow[d, "p'"] & B' \arrow[l, shift right, "s'"'] \arrow[r] \arrow[d, "p"] & 0 \\ 0 \arrow[r] & X \arrow[r, tail, "k"] & A \arrow[r, shift right, "q"'] & B \arrow[l, shift right, "s"'] \arrow[r] & 0 \end{tikzcd}$$

where (u,v) is a kernel pair with coequalizer p; in particular, (u,v) is the kernel pair of p. We must prove that

$$\begin{tikzcd} \mathrm{Act}(B,X) \arrow[r, "\mathrm{Act}(p,X)"] & \mathrm{Act}(B',X) \arrow[r, shift left, "\mathrm{Act}(u,X)"] \arrow[r, shift right, "\mathrm{Act}(v,X)"'] & \mathrm{Act}(B'',X) \end{tikzcd}$$

is an equalizer. Clearly

$$\mathrm{Act}(u,X) \circ \mathrm{Act}(p,X) = \mathrm{Act}(v,X) \circ \mathrm{Act}(p,X)$$

since $pu = pv$.

To prove that $\mathrm{Act}(p, X)$ is injective, it suffices to prove that (k, q, s) is entirely determined by its pullback (k', q', s') along p. Indeed, by regularity of $\mathbf{Top}^{\mathbb{T}}$, $p' = \mathrm{coeq}(u', v')$. Computing the pullback of the middle line along u and v, we obtain the same upper line (k'', q'', s''), because $pu = pv$. The bottom line is then determined by the other two lines via a coequalizer process.

Consider now (k', q', s') such that the pullbacks along u and v are equal. We must prove that (k', q', s') is the pullback along p of a split exact sequence (k, q, s). Again we obtain (k, q, s) as factorization through the coequalizers. A classical result on regular categories (see 6.10 in [2]) implies that the square $pq' = qp'$ is a pullback because both upper squares are pullbacks. ■

Proposition 5.3. *Each functor* $\mathrm{Act}(-, X)$ *on* $\mathbf{Top}^{\mathbb{T}}$ *transforms surjective morphisms into injections.*

Proof: Every regular epimorphism is the coequalizer of its kernel pair, thus, by Proposition 5.2, is transformed into an injection.

If f is an arbitrary surjective morphism, let us consider its image factorization in $\mathbf{Top}^{\mathbb{T}}$

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "p"'] & & B. \\ & B' \arrow[ur, "i"'] & \end{tikzcd}$$

Since p is a regular epimorphism, $\mathrm{Act}(p, X)$ is injective. But the monomorphism i is also surjective, since so is f; it is thus a bijection. To conclude, it remains to prove that $\mathrm{Act}(i, X)$ is injective as well. And since i is bijective, there is no restriction in assuming that i is the identity on B in $\mathbf{Set}^{\mathbb{T}}$.

We must consider the situation, for $j = 1, 2$,

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "k'"] \arrow[d, equal] & A' \arrow[r, shift right, "q'"'] \arrow[d, "\mathrm{id}"'] & B' \arrow[l, shift right, "s'"'] \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \\ 0 \arrow[r] & X \arrow[r, "k_j"] & A_j \arrow[r, shift right, "q_j"'] & B \arrow[l, shift right, "s_j"'] \arrow[r] & 0 \end{tikzcd}$$

where both lower split exact sequences produce the same upper line via a pullback process. We must prove that the two lower lines are equal. In $\mathbf{Set}^{\mathbb{T}}$ they are equal since $A_1 = A' = A_2$, $q_1 = q' = q_2$, $s_1 = s' = s_2$. And via pullbacks, the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$, of A_1 and A_2 respectively, yield the same topology on A'. As shown in [10], in both cases $A_1 = A_2$ may be described as a subset of $X^n \times B$ for a suitable natural number n, equipped with the product topology. Hence $\mathcal{T}_1$ and $\mathcal{T}_2$ must coincide. ■

Proposition 5.4. *Each functor* $\mathrm{Act}(-, X)$ *on* $\mathbf{Top}^{\mathbb{T}}$ *transforms the coequalizer of those pairs* (u, v) *of parallel morphisms with a common section in an equalizer.*

Proof: Consider the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "u"] \arrow[r, shift right=2, "v"'] \arrow[d, "p"'] \arrow[dr, "{(u,v)}"'] & B \arrow[l, "s"'] \arrow[r, "q"] & Q \\ R \arrow[r, "r"'] & B \times B \arrow[u, shift left, "p_1"] \arrow[u, shift right, "p_2"'] & \end{tikzcd}$$

where

$$us = \mathrm{id}_B = vs, \quad q = \mathrm{coeq}(u, v).$$

Consider further the factorization of (u, v) through the product and the image R if this factorization in $\mathbf{Set}^{\mathbb{T}}$. Choose on R the topology induced by that of $B \times B$. The factorization p is thus a continuous surjection.

Since the morphisms u, v have a common section, the relation R is reflexive. But $\mathbf{Top}^{\mathbb{T}}$ is homological (see [5]), thus in particular a Mal'tsev category (see [3]): so R is an equivalence relation. In particular, in the exact category $\mathbf{Set}^{\mathbb{T}}$, R is the kernel pair of the corresponding quotient. But since R is provided with the topology induced by that of $B \times B$, R is also the kernel pair of that quotient in $\mathbf{Top}^{\mathbb{T}}$. By Proposition 5.2, the coequalizer $\mathrm{coeq}(p_1 r, p_2 r)$ is transformed by $\mathrm{Act}(-, X)$ into an equalizer.

But p is a surjection in $\mathbf{Top}^{\mathbb{T}}$. Thus

$$q = \mathrm{coeq}(u, v) = \mathrm{coeq}(p_1 r p, p_2 r p) = \mathrm{coeq}(p_1 r, p_2 r)$$

and we have just seen that this last coequalizer is transformed by $\mathrm{Act}(-, X)$ into an equalizer. Proposition 5.3 implies further that $\mathrm{Act}(p, X)$ is injective. Therefore

$$\begin{aligned} &\mathrm{eq}\big(\mathrm{Act}(u, X), \mathrm{Act}(v, X)\big) \\ &= \mathrm{eq}\big(\mathrm{Act}(p, X) \circ \mathrm{Act}(r, X) \circ \mathrm{Act}(p_1, X), \mathrm{Act}(p, X) \circ \mathrm{Act}(r, X) \circ \mathrm{Act}(p_2, X)\big) \\ &= \mathrm{eq}\big(\mathrm{Act}(r, X) \circ \mathrm{Act}(p_1, X), \mathrm{Act}(r, X) \circ \mathrm{Act}(p_2, X)\big) \\ &= \mathrm{eq}\big(\mathrm{Act}(p_1 r, X), \mathrm{Act}(p_2 r, X)\big) \\ &= \mathrm{Act}(q, X) \end{aligned}$$