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ON KAN-INJECTIVITY OF LOCALES AND SPACES

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Abstract: In the category Top_0 of T_0-spaces and continuous maps, embeddings are just those morphisms with respect to which the Sierpiński space is Kan-injective, and the Kan-injective hull of the Sierpiński space is the category of continuous lattices and maps preserving directed suprema and arbitrary infima. In the category Loc of locales and localic maps, we give an analogous characterization of flat embeddings; more generally, we characterize n-flat embeddings, for each cardinal n, as those morphisms with respect to which a certain finite subcategory is Kan-injective. Moreover, we show that there is a subcategory of spatial locales whose Kan-injective hull is the entire category Loc. As a consequence, we obtain similar characterizations of the n-flat embeddings in the category Top_0, and we show that several well-known subcategories of Loc and Top_0 are Kan-injective hulls of finite subcategories.

1. Introduction

Two well-known important facts on injectivity are the characterization of the continuous lattices as the spaces injective with respect to the embeddings in the category Top_0 of T_0-spaces ([27]), and the characterization of the compact and stably locally compact locales as the locales injective with respect to the flat embeddings in the category Loc of locales ([16, 17], see also [10]). In [7, 8] Escardó pointed out that in many examples, including the above two, more than injectivity, we have Kan-injectivity (see also [10, 9]). In an order-enriched category, we say that an object A is left Kan-injective with respect to a morphism $h : X \to Y$ if every morphism $f : X \to A$ has a left Kan extension along h, denoted by f/h, and moreover, $(f/h)h = f$. In [5] we extended the concept of Kan-injectivity to morphisms: a morphism is left Kan-injective with respect to h if it preserves left Kan extensions along h. This way (left) Kan-injectivity plays for

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KZ-monads (i.e., Kock-Zöberlein monads [21, 34], also called lax idempotent monads) the same role as orthogonality plays for idempotent monads, that is, Kan-injectivity may be regarded as a "lax" generalization of orthogonality. Indeed, in [5] and [3] several well-known results on orthogonality were recaptured in this more general environment in the scope of order-enriched categories. In particular, in [3] we gave an answer to the Kan-injective Subcategory Problem resembling the one of Freyd and Kelly [11] to the Orthogonal Subcategory Problem.

In this paper we concentrate on Kan-injectivity in the categories Loc of locales and localic maps, and Top_0 of T_0 topological spaces and continuous maps. We show that embeddings, dense embeddings and flat embeddings may be characterized by means of a finite category via Kan-injectivity, both in Loc and in Top_0. More generally, this is shown for n-flat embeddings, for every cardinal n (with "2-flat" meaning just "flat"). As a consequence, we obtain the category Loc, and also the category Sob of sober spaces, expressed as the union of an increasing chain of Kan-injective hulls of finite subcategories, which moreover, are KZ-monadic subcategories. In Loc, the objects of all these finite subcategories are spatial, and thus, Loc is the Kan-injective hull of a subcategory of spatial locales.

2. Preliminaries on Kan-injectivity

In this section we recall the general notions and results on Kan-injectivity in order-enriched categories, needed in the subsequent study on locales and spaces (cf. [5, 3]).

Let Pos be the category of partially ordered sets and monotone maps. A category $\mathcal{X}$ is said to be *enriched in* Pos, or just *order-enriched*, if every hom-set $\mathcal{X}(X, Y)$ of all morphisms from X to Y is endowed with a partial order such that, for every pair of morphisms $f, g : X \to Y$, and morphisms $h : Z \to X$ and $k : Y \to W$, the inequality $f \leq g$ implies that $fh \leq gh$ and $kf \leq kg$. A functor $F : \mathcal{X} \to \mathcal{Y}$ between order-enriched categories is said to be *locally monotone*, or *order-enriched*, if the inequality $f \leq g$ implies that $Ff \leq Fg$ for all morphisms f and g with common domain and codomain.

Definition 2.1. 1. Given an object A and a morphism $h : X \to Y$ in an order-enriched category we say that A is *left Kan-injective with respect to h* if, for every morphism $f : X \to A$,

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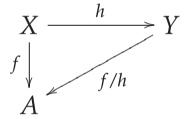
(i) f admits a left Kan-extension along h, that is, the set of all morphisms $s : Y \to A$ such that $f \leq sh$ has an infimum, which we denote by f/h; and
(ii) $(f/h)h = f$.



A morphism $g : A \to B$ is said to be *left Kan-injective with respect to h* if the objects A and B are so, and g satisfies the equality $g(f/h) = (gf)/h$ for every $f : X \to A$.

2. If in (i) we replace "$f \leq sh$" with "$f \geq sh$" and "infimum" with "supremum" we obtain the *right Kan-injectivity* concept.

Of course, if we reverse the order of the hom-sets of an order-enriched category, we get a new order-enriched category, and left Kan-injectivity in the former category becomes right Kan-injectivity in the latter, and vice-versa.

Along the paper we are going to work with left Kan-injectivity and we often omit the word "left".

In an order-enriched category a pair of morphisms $X \underset{l}{\overset{r}{\rightleftarrows}} Y$ is said to be an *adjoint pair*, written $l \dashv r : X \to Y$, if $\mathrm{id}_Y \leq rl$ and $lr \leq \mathrm{id}_X$, being r the *right adjoint* and l the *left adjoint*. (We recall that r and l are mutually uniquely determined.) If, moreover, r is a retraction, then it satisfies the equality $rl = \mathrm{id}_Y$. Indeed, let $s : Y \to X$ be a right inverse of r; then $rl = rlrs \leq rs = \mathrm{id}_Y$, and this, combined with $\mathrm{id}_Y \leq rl$, gives $rl = \mathrm{id}_Y$. When this is the case, we shall call r a *right adjoint retraction*. More generally, we analogously have that, if $l \dashv r : X \to Y$, then:

- l is a right (respectively, left) inverse of r, whenever r has a right (respectively, left) inverse; and
- r is a right (respectively, left) inverse of l, whenever l has a right (respectively, left) inverse.

We shall use the expressions *left adjoint retraction* and *left* (or *right*) *adjoint section* with the obvious meaning.

Remark 2.2. We recall from Kock [21] and Escardó [7, 8] that a KZ-monad over an order-enriched category $\mathcal{X}$ is a monad whose functor $F : \mathcal{X} \to \mathcal{X}$ is

locally monotone and whose unit η fulfils the condition

$$F\eta_X \leq \eta_{FX}, \text{ for all } X \in \mathcal{X}. \tag{1}$$

Moreover, the structure map of each Eilenberg-Moore algebra of a KZ-monad is adjoint to the unit. More precisely, for $X \in \mathcal{X}$ and $\alpha : FX \to X$, the pair (X, α) is an algebra if and only if we have an adjunction $\eta_X \dashv \alpha$ with $\alpha\eta_X = \mathrm{id}_X$, that is, η_X is a left adjoint section; further, if it is the case, $\alpha = \mathrm{id}_X/\eta_X$.

Consequently, every object X of $\mathcal{X}$ is the structure of at most one Eilenberg-Moore algebra of the monad. Thus categories of algebras of these monads can be seen, up to isomorphism of categories, as subcategories of $\mathcal{X}$. These subcategories are said to be *KZ-monadic* ([3]).

In an order-enriched category $\mathcal{X}$, given a subcategory $\mathcal{A}$, we write

$$\mathcal{A}^{\mathrm{KInj}}$$

to denote the class of all morphisms of $\mathcal{X}$ with respect to which all objects and all morphisms of $\mathcal{A}$ are left Kan-injective. And, given a class $\mathcal{H}$ of morphisms of $\mathcal{X}$, taking all objects and morphisms which are left Kan-injective with respect to $\mathcal{H}$ we obtain a subcategory, designated by

$$\mathrm{KInj}(\mathcal{H})$$

and said to be a *Kan-injective* subcategory.

This way, (left) Kan-injectivity gives rise to a Galois connection between subcategories and classes of morphisms of $\mathcal{X}$. Thus, given a subcategory $\mathcal{A}$ of $\mathcal{X}$, the subcategory $\mathrm{KInj}(\mathcal{A}^{\mathrm{KInj}})$ is the smallest Kan-injective subcategory of $\mathcal{X}$ containing $\mathcal{A}$. It will be called the *Kan-injective hull of* $\mathcal{A}$ and denoted by

$$\mathbf{K}(\mathcal{A}).$$

The next theorem and proposition characterize KZ-monadic subcategories $\mathcal{A}$ and the corresponding class $\mathcal{A}^{\mathrm{KInj}}$. First we need the following definitions:

Remark 5.15. The union of all subcategories $\mathcal{A}_n$ is also a category, and it is contained in Sob (since the image of Σ is contained in Sob). Let us denote it by $\mathcal{A}$. Then, in Top_0, denoting the class of all completely flat embeddings by $\mathbb{C}$, we have

$$\mathcal{A}^{\mathrm{KInj}} = \left(\bigcup_{n\in\mathrm{Card}} \mathcal{A}_n\right)^{\mathrm{KInj}} = \bigcap_{n\in\mathrm{Card}} \mathcal{A}_n^{\mathrm{KInj}} = \mathbb{C}.$$

But we know that Sob is the Eilenberg-Moore category of the completely prime filter monad $\bar{\mathbf{T}}$, being the completely flat embeddings the corresponding $\bar{\mathbf{T}}$-embeddings. Consequently, by Proposition 2.7, $\mathsf{Sob} = \mathbf{K}(\mathcal{A})$.

Also the Kan-injective hull of $\mathcal{A}$ in Sob is Sob as we show now. We use the index Sob to specify that we are working in the category Sob; otherwise we are referring to Top_0. Indeed, since Sob is full in Top_0 we have that $\mathrm{KInj}_{\mathsf{Sob}}(\mathcal{A}^{\mathrm{KInj}_{\mathsf{Sob}}}) = \mathrm{KInj}(\mathcal{A}^{\mathrm{KInj}_{\mathsf{Sob}}}) \cap \mathsf{Sob}$. Then, since $\mathbf{K}(\mathcal{A}) = \mathrm{KInj}(\mathcal{A}^{\mathrm{KInj}}) = \mathsf{Sob}$, and $\mathcal{A}^{\mathrm{KInj}_{\mathsf{Sob}}} \subseteq \mathcal{A}^{\mathrm{KInj}}$, we obtain the desired equality:

$$\mathsf{Sob} = \mathbf{K}(\mathcal{A}) \cap \mathsf{Sob} \subseteq \mathrm{KInj}(\mathcal{A}^{\mathrm{KInj}_{\mathsf{Sob}}}) \cap \mathsf{Sob} = \mathrm{KInj}_{\mathsf{Sob}}(\mathcal{A}^{\mathrm{KInj}_{\mathsf{Sob}}}) = \mathbf{K}_{\mathsf{Sob}}(\mathcal{A}).$$

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to $\mathbf{K}(\mathcal{A}_n)$ for some n. Let $f : Z \to W$ be a morphism in **Sob**. Let $n = \max\{\mathrm{card}(\Omega(Z)), \mathrm{card}(\Omega(W))\}$. Then both Z and W belong to $\mathbf{K}(\mathcal{A}_n)$, and λ_Z and λ_W are isomorphisms. Then $f = \eta_Y^{-1}(T_n f)\eta_X^{-1}$ and, since $\mathbf{K}(\mathcal{A}_n)$ is closed under isomorphisms, it contains f. ∎

Remark 5.11. The fact that $\mathbf{K}(\mathcal{A}_1)$ and $\mathbf{K}(\mathcal{A}_2)$ are precisely the categories of Eilenberg-Moore algebras of the proper filter monad and the prime filter monad, respectively, was proved in [5]. However, there the morphism f_0 of $\mathcal{A}_1$, used to characterize the dense embeddings, was different; moreover, this morphism was wrongly missing in the description of the category $\mathcal{A}_2$.

Remark 5.12. Let **ScottDI** be the category of Scott domains and functions which preserve directed suprema and non-empty infima, which is a subcategory of Top_0 via the Scott topology ([12]). This category was proved to be the Eilenberg-Moore category of the proper filter monad by Wyler [33]. Thus $\mathbf{K}(\mathcal{A}_1) =$ **ScottDI**.

The Eilenberg-Moore category of the prime filter monad was characterized by Simmons [28] and Wyler [32], it is the subcategory **SComp** of Top_0 of all stably compact spaces and stably continuous maps. That is, the objects of **SComp** are those spaces which are sober, locally compact and whose family of all saturated compacts is closed under finite intersections, and the morphisms are the continuous maps such that for every pair of open sets U and V and a compact set K with $U \subseteq K \subseteq V$ in Y, there exists some compact K' in X fulfilling $f^{-1}(U) \subseteq K' \subseteq f^{-1}(V)$. From the above theorem, $\mathbf{K}(\mathcal{A}_2) =$ **SComp**.

We also recall from [5] that, for $\mathcal{A}_0$ the category consisting of the Sierpiński space A_0 and its identity map, $\mathbf{K}(\mathcal{A}_0) =$ **ContI**, where **ContI** denotes the category of continuous lattices and maps that preserve directed suprema and arbitrary infima.

Remark 5.13. In [14] Hofmann gave a characterization of the Eilenberg-Moore category for every monad $\mathbf{T}_n$, which generalizes the one of **SComp** described in Remark 5.12.

Remark 5.14. We have been working in Top_0. But all we have said remains true when we restrict ourselves to the full subcategory Sob of sober spaces. That is, the Kan-injective hull of $\mathcal{A}_n$ in Sob coincides with the one in Top_0 and every $\mathbf{K}(\mathcal{A}_n)$ is KZ-monadic in Sob. This follows easily using the fact that Sob is a full KZ-monadic subcategory of Top_0.

Definitions 2.3. 1. A subcategory $\mathcal{A}$ of an order-enriched category $\mathcal{X}$ is said to be *closed under left adjoint retractions* if, for every commutative diagram in $\mathcal{X}$ of the form

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\ X & \xrightarrow[g]{} & Y \end{array}$$

with x and y left adjoint retractions, if $f \in \mathcal{A}$ then also $g \in \mathcal{A}$.

2. Given a locally monotone functor $F : \mathcal{X} \to \mathcal{Y}$, we call a morphism $f : X \to Y$ an *F-embedding* if Ff is a left adjoint section in $\mathcal{Y}$.

Theorem 2.4. ([5]) *Let $\mathcal{A}$ be a subcategory of the order-enriched category $\mathcal{X}$. Then the following statements are equivalent:*

(a) *$\mathcal{A}$ is a KZ-monadic subcategory of $\mathcal{X}$.*
(b) *$\mathcal{A}$ is a reflective subcategory of $\mathcal{X}$ closed under left adjoint retractions, whose corresponding reflector $F : \mathcal{X} \to \mathcal{A}$ is locally monotone, and whose unit η satisfies the inequalities (1).*
(c) *$\mathcal{A}$ is a reflective subcategory of $\mathcal{X}$ closed under left adjoint retractions, whose reflector functor F and unit η satisfy the conditions:*
  (i) $\mathcal{A} = \mathrm{KInj}(\{\eta_X \mid X \in \mathcal{X}\})$, and
  (ii) for every morphism $f : FX \to A$ in $\mathcal{A}$, $(f\eta_X)/\eta_X = f$.

Remark 2.5. The condition (ii) above may be equivalently replaced with the following one: for every pair of morphisms $f, g : FX \to A$ with f in $\mathcal{A}$, $f\eta_X \leq g\eta_X$ implies $f \leq g$.

Remark 2.6. In particular, Theorem 2.4 shows that every KZ-monadic subcategory is a Kan-injective subcategory, that is, of the form $\mathrm{KInj}(\mathcal{H})$, for some class $\mathcal{H}$ of morphisms. In [3] the authors present conditions on the base category under which the converse is true for all sets $\mathcal{H}$. The particular case of ordinary categories, enriched with the order equality, is then the solution for the Orthogonal Subcategory Problem given in [11, 19] (see also [1]).

Proposition 2.7. ([5]) *Let $\mathcal{A}$ be a KZ-monadic subcategory of $\mathcal{X}$ with $F : \mathcal{X} \to \mathcal{A}$ and $I : \mathcal{A} \to \mathcal{X}$ the reflector and inclusion functors, respectively. Then,*

$$\mathcal{A}^{KInj} = \{f \in \mathcal{X} \mid f \text{ is an } F\text{-embedding}\} = \{f \in \mathcal{X} \mid f \text{ is an } IF\text{-embedding}\},$$

and $\mathcal{A} = \mathbf{K}(\mathcal{A})$.

We prove here another property of Kan-injectivity, which will be useful in the last section.

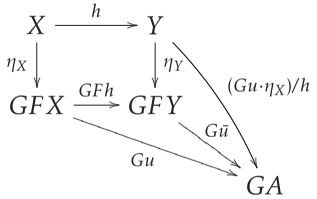
Proposition 2.8. *Let $F \dashv G : \mathcal{A} \to \mathcal{X}$ be an adjunction of locally monotone functors between order-enriched categories. Then, given a morphism h of $\mathcal{X}$, an object A (respectively, a morphism f) of $\mathcal{A}$ is Kan-injective with respect to Fh in $\mathcal{A}$ if and only if GA (respectively, Gf) is Kan-injective with respect to h in $\mathcal{X}$.*

Remark. Before proving the proposition, we point out that, given the adjunction $F \dashv G : \mathcal{A} \to \mathcal{X}$ between order-enriched categories, the local monotonicity of the functors G and F is equivalent to the following condition for all pairs of morphisms $f, g : FX \to A$, where η is the unit:

$$Gf \cdot \eta_X \leq Gg \cdot \eta_X \;\Leftrightarrow\; f \leq g. \tag{2}$$

Indeed, if G and F are locally monotone, one of the implications of (2) is obvious; and the less obvious one follows easily, since, given $f, g : FX \to A$ with $Gf \cdot \eta_X \leq Gg \cdot \eta_X$, then, using the counit ε of the adjunction, we have $\varepsilon_A \cdot FGf \cdot F\eta_X \leq \varepsilon_A \cdot FGg \cdot F\eta_X$, thus $f \cdot \varepsilon_{FX} \cdot F\eta_X \leq g \cdot \varepsilon_{FX} \cdot F\eta_X$, that is, $f \leq g$. Conversely, assuming that condition (2) is satisfied, given $f, g : X \to Y$ in $\mathcal{X}$ with $f \leq g$, we have $\eta_Y f \leq \eta_Y g$, that is, $GFf \cdot \eta_X \leq GFg \cdot \eta_X$, thus $Ff \leq Fg$. Moreover, given $f, g : A \to B$ in $\mathcal{A}$ with $f \leq g$, then $f\varepsilon_A \leq g\varepsilon_A$, and, using condition (2), $Gf \cdot G\varepsilon_A \cdot \eta_{GA} \leq Gg \cdot G\varepsilon_A \cdot \eta_{GA}$, that is, $Gf \leq Gg$.

Proof: Let GA be Kan-injective with respect to $h : X \to Y$. In order to prove that A is Kan-injective with respect to Fh, consider a morphism $u : FX \to A$. By hypothesis, there exists $(Gu \cdot \eta_X)/h$. Let $\bar{u}$ be the unique morphism with $G\bar{u} \cdot \eta_Y = (Gu \cdot \eta_X)/h$. We show that $u/Fh = \bar{u}$.



The equality $\bar{u}Fh = u$ follows from the fact that we have $G(\bar{u} \cdot Fh) \cdot \eta_X = G\bar{u} \cdot \eta_Y \cdot h = ((Gu \cdot \eta_X)/h) \cdot h = Gu \cdot \eta_X$. Let now $s : FY \to A$ be a morphism satisfying the inequality $u \leq sFh$. Then, by (2), $Gu \cdot \eta_X \leq Gs \cdot GFh \cdot \eta_X = Gs \cdot \eta_Y \cdot h$, and thus $(Gu \cdot \eta_X)/h \leq Gs \cdot \eta_Y$, that is, $G\bar{u} \cdot \eta_Y \leq Gs \cdot \eta_Y$; using

In [5] we described dense embeddings and flat embeddings in terms of Kan-injectivity concerning a finite subcategory. In the following we obtain a similar characterization of n-embeddings, for every cardinal n, as a consequence of Theorem 3.9. Moreover, we use this characterization to obtain a tower of KZ-monadic subcategories of Top_0 whose union is the full subcategory of sober spaces.

Theorem 5.10. *In Top_0, for every cardinal $n \geq 1$, the class of n-flat embeddings is precisely $\mathcal{A}_n^{KInj}$, and the Kan-injective hull of $\mathcal{A}_n$ is the Eilenberg-Moore category of the KZ-monad $\mathbf{T}_n$. Moreover, the union of the chain of KZ-monadic subcategories $\mathbf{K}(\mathcal{A}_n)$ is the category Sob.*

Proof: As we have shown in Remark 5.3, the two functors of the adjunction $\mathsf{Lc} \dashv \Sigma : \mathsf{Loc} \to \mathsf{Top}_0$ are locally monotone. Then, from Proposition 2.8, we know that, for every cardinal n, Σf_n is (left) Kan-injective with respect to a morphism $h : X \to Y$ in Top_0 if and only if f_n is Kan-injective with respect to $\mathsf{Lc}(h)$ in Loc. Consequently, it follows from Theorem 3.9 that $\mathcal{A}_n^{\mathrm{KInj}}$ is indeed the class of n-flat embeddings.

But n-flat embeddings coincide also with T_n-embeddings (see Notation 5.7). Then, by Proposition 2.7, the Eilenberg-Moore category of the KZ-monad $\mathbf{T}_n$ is $\mathrm{KInj}(T_n\text{-embeddings}) = \mathrm{KInj}\left(\mathcal{A}_n^{\mathrm{KInj}}\right) = \mathbf{K}(\mathcal{A}_n)$.

It remains to show that the category Sob is the union of the categories $\mathbf{K}(\mathcal{A}_n)$. As observed in [10], Sob is the Eilenberg-Moore category of the KZ-monad $\bar{\mathbf{T}} = \Sigma\mathsf{Lc}$ of completely prime filters; and the $\bar{T}$-embeddings are the completely flat embeddings, i.e. those embeddings for which $\mathsf{Lc}(g) = (g^{-1})_*$ preserves arbitrary unions. Thus, from Proposition 2.7, it follows that $\mathsf{Sob} = \mathrm{KInj}(\mathbb{C})$ for $\mathbb{C}$ the class of all completely flat embeddings. It is clear that every completely flat embedding is n-flat for every n. Then the inclusion of the categories $\mathbf{K}(\mathcal{A}_n)$ in $\mathbf{Sob}$ follows.

In order to show that $\mathbf{Sob} \subseteq \mathbf{K}(\mathcal{A}_n)$ for every n, let X be a sober space, and let ΩX have cardinality n. We show that $X \in \mathbf{K}(\mathcal{A}_n)$. Indeed, given any union $\cup_{i\in I} A_i$ of open sets of X, there is some $J \subseteq I$ such that $\mathrm{card}(J) \leq n$ and, for every $i \in I$, $A_i = A_j$ for some $j \in J$, in particular, $\cup_{j\in J} A_j = \cup_{i\in I} A_i$. Consequently any n-prime filter of X is also completely prime. But then $\Sigma\mathsf{Lc}(X) = T_nX$. Since X is sober, the map $\lambda_X : X \to \Sigma\mathsf{Lc}(X)$, where λ is the unit of the adjunction $\mathsf{Lc} \dashv \Sigma$, is a homeomorphism. Being $\mathbf{K}(\mathcal{A}_n)$ closed under isomorphisms in Top_0, we conclude that $X \in \mathbf{K}(\mathcal{A}_n)$. It remains to show that every continuous map between sober spaces belongs

Remark 5.6. If in the above definition we put $n = 1$ we obtain the definition of proper filter, that is, a filter not containing the empty set: just consider $\mathrm{card}(k) = 0$. If we put $n = 2$ then we obtain the definition of prime filter.

As it was shown in [8, 10], if, in Remark 5.4, we replace "filter" by "proper filter", or by "prime filter", we obtain also KZ-monads, the proper filter monad and the prime filter monad, respectively. Moreover, in the same manner we have the KZ-monad of the completely prime filters.

Notation 5.7. For $n \geq 1$ and $X \in \mathsf{Top}_0$, let T_nX be the set of all n-prime filters on ΩX with the topology generated by the sets $\Box U = \{\psi \in T_nX \mid U \in \psi\}$, $U \in \Omega X$. By defining the functor

$$T_n : \mathsf{Top}_0 \to \mathsf{Top}_0$$

and the natural transformations η_n and μ_n exactly as we defined, in Remark 5.4, T, η and μ, but considering always just n-prime filters instead of filters, we still obtain a KZ-monad. This is figured out in an entirely analogous way to what was done in [10] for $n = 1, 2$, and is used in Theorem 5.10 below. Moreover, by imitating the techniques used in [10] for $n = 1, 2$, we conclude that the corresponding T_n-embeddings are precisely the n-flat embeddings, which we define next. First, let us recall that, as it is easy to see, a morphism $h : X \to Y$ is an embedding of Top_0 iff the morphism $\mathrm{Lc}(h) = \left(h^{-1}\right)_*$, already described in Remark 3.11, is an embedding in Loc (see [17]).

Definition 5.8. In Top_0, a morphism $h : X \to Y$ is said to be an *n-flat embedding* if the morphism $\mathrm{Lc}(h)$ is an n-flat embedding in Loc.

Notation 5.9. For every cardinal n, the image under Σ of the subcategory $\mathcal{D}_n$ described in Notation 3.8 is a subcategory of Top_0, let us denote it by $\mathcal{A}_n$,

$$\mathcal{A}_n = \Sigma[\mathcal{D}_n].$$

In particular, $\mathcal{A}_1$ is, up to isomorphism, the subcategory of Top_0 whose only non identity morphism is the inclusion $g_0 : \{0\} \to \mathrm{S}$, where S is the Sierpiński space $\{0, 1\}$ with $\{1\}$ the only nontrivial open set. We obtain $\mathcal{A}_2$, up to isomorphism, by adding to $\mathcal{A}_0$ the morphism $g_2 : A_2 \to \mathrm{S}$, where $A_2 = \{a, b, 0, 1\}$ with the topology generated by $\{1\}$, $\{a, 1\}$ and $\{b, 1\}$, and g_2 sends all elements of A_2 except 0 to 1.

property (2) again, $\bar{u} \leq s$. Now it is easy to show the property also for morphisms: for $f : A \to B$ with Gf Kan-injective with respect to h, we will have $G(f\bar{u})\eta_Y = Gf \cdot G\bar{u} \cdot \eta_Y = Gf \cdot ((Gu \cdot \eta_X)/h) = (G(fu) \cdot \eta_X)/h) = G\overline{fu} \cdot \eta_Y$, then $f\bar{u} = \overline{fu}$, that is, $f(u/Fh) = (fu)/Fh$.

Conversely, let A be Kan-injective with respect to Fh, consider a morphism $w : X \to GA$, and let $\overline{w} : FX \to A$ be the unique morphism such that $G\overline{w} \cdot \eta_X = w$. We show that $w/h = G(\overline{w}/Fh)\eta_Y$. Indeed, $G(\overline{w}/Fh)\eta_Y h = G(\overline{w}/Fh) \cdot GFh \cdot \eta_X = G\overline{w} \cdot \eta_X = w$; and, if $s : Y \to GA$ satisfies the inequality $w \leq sh$, then, for $s = G\overline{s} \cdot \eta_Y$, we have $G\overline{w} \cdot \eta_X \leq G\overline{s} \cdot \eta_Y \cdot h = G\overline{s} \cdot GFh \cdot \eta_X = G(\overline{s} \cdot Fh) \cdot \eta_X$, and, thus, $\overline{w} \leq \overline{s}Fh$. It follows that $\overline{w}/Fh \leq \overline{s}$, hence $G(\overline{w}/Fh) \cdot \eta_Y \leq G\overline{s} \cdot \eta_Y = s$. Again the property for morphisms easily follows. ∎

Remark 2.9. We will also use the notion of Kan-projectivity, dual of the one of Kan-injectivity. Namely, in an order-enriched category $\mathcal{X}$, an object A is *right Kan-projective* with respect to a morphism $h : X \to Y$ if, for every morphism $f : A \to Y$ there is a (unique) morphism

$$\frac{f}{h} : A \to X$$

such that

(i) if $s : A \to X$ is a morphism with $f \geq hs$, then $\frac{f}{h} \geq s$;
(ii) $h\frac{f}{h} = f$.

And a morphism $g : A \to B$ is said to be *right Kan-projective with respect to h* if the objects A and B are so, and g satisfies the equality $\frac{fg}{h} = \frac{f}{h}g$ for every $f : B \to Y$.

Reversing the order we obtain left Kan-projectivity.

3. n-flat embeddings in Loc

We recall that a frame is a complete lattice satisfying the infinite distributive law

$$a \wedge (\vee B) = \vee\{a \wedge b \mid b \in B\}$$

for all its elements a and subsets B. And a frame homomorphism between frames is a map preserving all joins and all finite meets. We denote the category of frames and frame homomorphisms by Frm. Its opposite category is the category Loc of locales and localic maps. Thus the objects of Loc, known as locales, are just the frames. The morphisms of Loc, the localic

maps, may be seen as infima-preserving maps $f : L \to M$ such that the corresponding left-adjoint $f^* : M \to L$ preserves finite meets ([17], [25]). We recall that the fact that f preserves arbitrary meets implies that it has a left adjoint defined by

$$f^*(y) = \wedge\{x \in L \,|\, y \leq f(x)\}$$

and f^* preserves arbitrary joins. Analogously, every frame homomorphism f has a right adjoint which preserves arbitrary meets.

Let us consider the categories Frm and Loc order-enriched via the pointwise order, that is, in both cases, for $f, g : X \to Y$, $f \leq g$ iff $f(x) \leq g(x)$, $x \in X$. Then, for $f, g : X \to Y$ taken in Frm, we have that

$$f \leq g \text{ in Frm iff } f_* \geq g_* \text{ in Loc} \tag{3}$$

and, analogously, for $f, g : X \to Y$ taken in Loc,

$$f \leq g \text{ in Loc iff } f^* \geq g^* \text{ in Frm.} \tag{4}$$

Indeed, if f and g lie in Loc with $f(x) \leq g(x)$, $x \in X$, then, for every $y \in Y$,

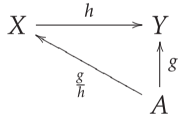
$$g^*(y) = \wedge\{x \in X \,|\, y \leq g(x)\} \leq \wedge\{x \in X \,|\, y \leq f(x)\} = f^*(y).$$

Similarly, we conclude (3). As a consequence, an adjoint pair $f \dashv g$ in Frm gives rise to the adjoint pair $f_* \dashv g_*$ in Loc, and vice-versa.

It is easy to see that left Kan-injectivity in Loc translates into right Kan-projectivity in Frm. More precisely, a frame A is right Kan-projective in Frm with respect to a frame homomorphism $h : X \to Y$ iff A is left Kan-injective in Loc with respect to the localic map $h_* : Y \to X$. And the existence of $\frac{g}{h}$ for $g : A \to Y$ implies that g_*/h_* exists in Loc and is given by the formula

$$g_*/h_* = \left(\frac{g}{h}\right)_*. \tag{5}$$

Indeed, let A be right Kan-projective with respect to $h : X \to Y$ in Frm, and let $g : A \to Y$ be a frame homomorphism. Consider the commutative diagram



in Frm. Thus,

$$\left(\frac{g}{h}\right)_* h_* = \left(h\frac{g}{h}\right)_* = g_*;$$

yields an adjunction with both functors locally monotone. Moreover, as it is well known, this adjunction induces an equivalence of categories between the full subcategory SpLoc of Loc of all spatial spaces and the full subcategory Sob of Top_0 of all sober spaces.

Let us recall that a filter ψ in ΩX is *completely prime* if, for every family $(U_i)_{i\in I}$ of open sets of X, the fact that $\bigcup_{i\in I} U_i \in \psi$ implies that ψ contains some of the sets U_i. We recall that the functor $\Sigma : \mathsf{Loc} \to \mathsf{Top}_0$ may also be defined, up to isomorphism, by looking at ΣL as the space whose points are the completely prime filters in L and whose open sets are the sets Σ_a, $a \in L$, with Σ_a consisting of all completely prime filters containing a.

Remark 5.4. We recall here the definition of the *filter monad* $\mathbf{T} = (T, \eta, \mu)$ over Top_0.

For every object X, TX is the set of all filters in ΩX equipped with the topology generated by the sets

$$\Box U = \{\psi \in TX \mid U \in \psi\}, \;\; U \in \Omega X.$$

For every morphism $f : X \to Y$,

$$Tf(\psi) = \{V \in \Omega Y \mid f^{-1}(V) \in \psi\}, \;\; \psi \in TX.$$

The unit η and the multiplication μ are defined pointwisely by

$$\eta_X(x) = \{U \in \Omega X \mid x \in U\} \qquad \text{and} \qquad \mu_X(\psi) = \{U \in \Omega X \mid \Box U \in \psi\}.$$

The filter monad $\mathbf{T}$ was studied by Day [6] and Wyler [33], who showed that the Eilenberg-Moore category of this monad is the category **ContI** of all continuous lattices and maps preserving all directed suprema and all infima (relative to the specialization order). Escardó [7, 8] observed that the filter monad is indeed a KZ-monad, and the T-embeddings are precisely the embeddings.

Besides the filter monad $\mathbf{T}$, Escardó and Flagg also analysed some submonads of the monad $\mathbf{T}$ ([7, 8, 10]). Namely they considered the proper filter monad and the prime filter monad. Here we are going to take attention to the n-filter monad for every $n \geq 1$, according to the following definition.

Definition 5.5. Given a space X, a filter ψ in ΩX is said to be *n-prime* provided that, for every family $(U_i)_{i\in k}$ of members of ψ with $\mathrm{card}(k) \leq n$, if $\bigcup_{i\in k} U_i$ belongs to ψ then U_j belongs to ψ for some j.

Example 5.2. Recall that a topological space Z is said to be T_1 if for every two points u and v of Z, there is an open set U of Z which contains u but not v. This is equivalent to say that the specialization order is just equality. We show that every T_0 space Kan-injective with respect to a non one-to-one continuous map $f : X \to Y$ must be T_1. Indeed, Let Z be Kan-injective with respect to f. If there are $x, y \in X$ such that $f(x) = f(y)$ with $x \neq y$, without loss of generality we may assume that there is an open set G of X containing y but not x. Let $u, v \in Z$ such that $u \leq v$; we want to show that $u = v$. The map $g : X \to Z$ taking all $z \in G$ to v and the remaining z to u is continuous. Then there is g/f in Top_0. But then $u = g(x) = (g/f)(f(x)) = (g/f)(f(y)) = g(y) = v$.

Remark 5.3. Let

$$\mathrm{Lc} : \mathsf{Top}_0 \to \mathsf{Loc}$$

be the restriction of the functor Lc described in Remark 3.11 to Top_0. Then Lc is locally monotone, because $f \leq g$ in Top_0 implies that $f^{-1} \geq g^{-1}$ in Frm, so $(f^{-1})_* \leq (g^{-1})_*$, that is, $\mathrm{Lc}(f) \leq \mathrm{Lc}(g)$ in Loc. We also recall that its right adjoint

$$\Sigma : \mathsf{Loc} \to \mathsf{Top}_0$$

sends every locale A to the topological space ΣA consisting of the pair $(\mathrm{pt}(A), \varphi)$, where $\mathrm{pt}(A)$ is the set of all frame homomorphisms $p : A \to \mathrm{Lc}(1) \cong F_0$ and φ is the topology formed by the sets $\varphi(a) = \{p \in \mathrm{pt}(A) \,|\, p(a) = 1\}$. The functor Σ is also locally monotone as we show next. For that, recall that the counit ε of the adjunction $\mathrm{Lc} \dashv \Sigma$ behaviours as follows: for every $f : \mathrm{Lc}(X) \to A$, the unique morphism $\bar{f} : X \to \Sigma A$ of Top_0 with $\varepsilon_A \mathrm{Lc}(\bar{f}) = f$ is given by $\bar{f}(x) = (\ A \xrightarrow{f_*} \mathrm{Lc}(X) \xrightarrow{\mathrm{Lc}(x)} \mathrm{Lc}(1)\ )$, where the map $x : 1 \to X$ sends the element of the singleton 1 to the element x of X. Thus, in order to verify that Σ is locally monotone, it suffices to show that for every pair $f, g : \mathrm{Lc}(X) \to A$ with $f \leq g$ in Loc, it holds $\bar{f} \leq \bar{g}$ in Top_0. Let then f and g be under those conditions, that is, $f^* \geq g^*$ in Frm. Hence, for every $x \in X$, $\mathrm{Lc}(x)f^* \geq \mathrm{Lc}(x)g^*$ in Frm. Consequently, for every $a \in A$, the equality $\mathrm{Lc}(x)g^*(a) = 1$ implies $\mathrm{Lc}(x)f^*(a) = 1$. In other words, $\mathrm{Lc}(x)g^* \in \varphi(a)$ implies $\mathrm{Lc}(x)f^* \in \varphi(a)$. Since the topology of ΣA is formed by all sets $\varphi(a)$, we have that, concerning the dual of the specialization order, $\mathrm{Lc}(x)f^* \leq \mathrm{Lc}(x)g^*$ for all $x \in X$. That is, $\bar{f} \leq \bar{g}$, as desired. In conclusion,

$$\mathrm{Lc} \dashv \Sigma : \mathsf{Loc} \to \mathsf{Top}_0$$

moreover, given $s : X \to A$ in Loc such that $sh_* \geq g_*$, we have $(sh_*)^* \leq (g_*)^*$, i.e., $hs^* \leq g$, thus $s^* \leq \frac{g}{h}$ and, consequently, $s \geq \left(\frac{g}{h}\right)_*$. Therefore we have the formula (5).

Analogously, if A is a locale left Kan-injective with respect to a localic map $h : X \to Y$, then A is right Kan-projective with respect to h^* in Frm and it holds the formula

$$\frac{g^*}{h^*} = (g/h)^*. \tag{6}$$

Terminology 3.1. From now on we proceed with just the terminologies *Kan-projectivity* in Frm and *Kan-injectivity* in Loc, referring, in the former case, to right Kan-projectivity, and, in the latter one, to left Kan-injectivity.

A localic map $f : L \to M$ is said to be an *embedding* if it is one-to-one. This is equivalent to its left-adjoint $f^* : M \to L$ being a surjection, and also equivalent to f satisfying the equality $f^* f = id_L$. Embeddings are precisely the extremal monomorphisms of Loc ([17], [25]).

Definition 3.2. For every cardinal $n \geq 1$, an embedding $g : L \to M$ in Loc is said to be *n-flat* provided that it preserves suprema of families of cardinality less or equal than n.

Remark 3.3. A 1-flat embedding is an embedding which preserves the least element, that is, a *dense embedding*. If we consider the above notion of n-flat also for $n = 0$, then clearly 0-flat has the same meaning as 1-flat. The 2-flat embeddings are the usual flat embeddings, that is, those embeddings which preserve finite joins (see [17]).

Remark 3.4. The category Frm is a subcategory of the category SLat of meet-semilattices and homomorphisms, and the forgetful functor from Frm to SLat has a left adjoint

$$D : \mathsf{SLat} \to \mathsf{Frm}.$$

It sends every semilattice L to the frame DL of all lower sets of L ordered by inclusion, that is, the elements of DL are all subsets W of L with

$$W = {\downarrow} W = \{x \in L \,|\, x \leq w \text{ for some } w \in W\};$$

and, given a morphism $f : L \to M$ of SLat, $(Df)(U) = {\downarrow} f[U]$ for all $U \in DL$. For every $L \in \mathsf{SLat}$, the corresponding universal morphism $d_L : L \to DL$ is defined by

$$d_L(x) = {\downarrow} x = \{y \in L \,|\, y \leq x\},\ x \in L;$$

and, given $f : L \to M$ in SLat with M in Frm, the unique frame homomorphism $\bar{f} : DL \to M$ with $\bar{f}d_L = f$ is defined by $\bar{f}(U) = \bigvee f[U]$. Moreover, Frm is a KZ-monadic subcategory of SLat via the reflector functor D. This follows from [4], and can also be easily shown by using (c) of Theorem 2.4. Indeed, (i) and (ii) are immediately seen; in addition, given a commutative square as in Definition 2.3, lying in SLat and with f in Frm, it is easy to see that the semilattice X has suprema given by the formula $\bigvee_{i\in I} a_i = x(\bigvee_{i\in I} x_*(a_i))$, and analogously for Y; using this fact, it follows easily that Frm is closed under left adjoint retractions in SLat.

Also the forgetful functor from SLat to the category Set of sets and maps has a left adjoint $S : \mathsf{Set} \to \mathsf{SLat}$. It sends every set X to the set of all finite subsets of X equipped with the partial order given by $A \leq B$ iff $A \supseteq B$.

Consequently, given a set X, the free frame generated by X may be constructed as follows ([17, 25]):

First we form the free meet-semilattice SX generated by X. The free frame generated by X, denoted FX, is precisely $D(SX)$. The corresponding universal map $\eta_X : X \to FX$ is then defined by

$$\begin{aligned}\eta_X(x) = {\downarrow}\{x\} &= \{W \in SX \mid W \leq_{SX} \{x\}\} = \{W \in SX \mid \{x\} \subseteq W\}\\ &= \{W \subseteq X \mid W \text{ is finite and } x \in W\}.\end{aligned}$$

Notation 3.5. Let n be any cardinal looked just as a set. In the following,

$$F_n$$

denotes the free frame generated by n. In particular, we put $F_0 = (0 < 1)$ and $F_1 = (0 < d < 1)$.

Next we characterize the embeddings of Loc via Kan-injectivity.

Proposition 3.6. *In* Loc, *a localic map h is an embedding if and only if the three-element chain is Kan-injective w.r.t. h, if and only if every finite chain is Kan-injective with respect to h.*

Proof: Let $F_1 = (0 < d < 1)$ be Kan-injective w.r.t. $h : X \to Y$ in Loc. We want to show that h is one-to-one. For every $x \in X$, let $g_x : X \to F_1$ be the localic map given by

$$g_x(z) = \begin{cases} 1, & \text{if } z = 1;\\ d, & \text{if } 0 \neq x \leq z \neq 1;\\ 0, & \text{otherwise.}\end{cases}$$

into account that, by Theorems 4.4 and 2.4, $\mathbb{E}_n = \text{Loc-}\lhd_n^{\text{KInj}}$, it suffices to show that $\mathbb{E}_m \neq \mathbb{E}_n$. For $m = 0 < 1$, take the non dense embedding from the chain $(0 < 1)$ to the chain $(0 < d < 1)$ which sends 0 to d and 1 to 1. For $m = 1 < 2$, take the embedding $h : A \to B$ where A is the diamond consisting of the elements a and b, its meet $a \wedge b = 0$ and its join $a \vee b = 1$, B is the five-element frame with elements a, b, the meet $a \wedge b = 0$, the join $a \vee b = s$ and the greatest element 1, and h is defined by $h(x) = x$ for all $x \in A$. Thus h is dense but not flat. For $m < n$ with n an infinite regular cardinal, let $h : n+1 \to n+2$ be the embedding from $n+1 = \{0, 1, \ldots, n\}$ to $n+2 = \{0, 1, \ldots, n, n+1\}$, both equipped with the usual order, defined by $h(k) = k$ if $k < n$ and $h(n) = n+1$. Then h is clearly m-flat but it is not n-flat, because $h(\bigvee_{k\in n} k) = h(n) = n+1$ but $\bigvee_{k\in n} h(k) = \bigvee_{k\in n} k = n$.

5.Kan-injectivity in Top_0

The study of full reflective subcategories, and the reflective hull of full subcategories, in categories of topological spaces, and in particular in the category Top_0, has taken the attention of several authors. Concerning Top_0, we refer, for instance, to [24, 13, 15, 23, 29]. An isomorphism-closed full reflective subcategory of an ordinary category $\mathcal{X}$ is just a KZ-monadic subcategory of $\mathcal{X}$ when we regard $\mathcal{X}$ as order-enriched via the partial order "=". In the last section we have just seen that Loc is the union of an increasing chain of KZ-monadic subcategories which are Kan-injective hulls, and then also KZ-monadic hulls, of finite subcategories. In this section we will use the results on Loc to show that an analogous situation occurs in the full subcategory Sob of Top_0 of all sober spaces.

We recall that the specialization order is the partial order between the points of a T_0 topological space defined by $x \leq y$ iff every open set which contains x also contains y. We are going to regard the category Top_0 as an order-enriched category by considering the dual of this order pointwisely: given morphisms $f, g : X \to Y$, we put $f \leq g$ if $f^{-1}(G) \supseteq g^{-1}(G)$ for all open sets G of Y.

In the following, when we speak of Kan-injectivity in Top_0, we refer to left Kan-injectivity.

Example 5.1. Let $f : X \to Y$ be an embedding in Top_0 and let S denote the Sierpiński space. For every open set U of X, and $\chi_U : X \to \mathrm{S}$ the characteristic map, we have that $\chi_U / f = \chi_{(f^{-1})_*(U)}$ ([5]).

mf. We know that $\varepsilon_L l = \mathrm{id}$ and $m\varepsilon_M \leq \mathrm{id}$ (see Remark 4.3.2); moreover, since ε is a natural transformation, $\varepsilon_M H_n f = f\varepsilon_L$. Thus, we have that

$$mf = mf\varepsilon_L l = m\varepsilon_M (H_n f)l \leq (H_n f)l.$$

To show that also $(H_n f)l \leq mf$, let $a \in L$. Then, we have that

$$\begin{aligned}(H_n f)(l(a)) &= (H_n f)(\bigvee_{x \lhd_n a} \downarrow x), \text{ by (19)}\\ &= \bigvee_{x \lhd_n a} (H_n f)(\downarrow x), \quad \text{because } H_n f \text{ is a frame homomorphism}\\ &= \bigvee_{x \lhd_n a} \downarrow f(x).\end{aligned}$$

For $x \in L$ satisfying $x \lhd_n a$, we have $f(x) \lhd_n f(a)$, because f preserves the relation $\lhd_n$; thus, $(H_n f)(l(a)) \leq \bigvee_{f(x) \lhd_n f(a)} \downarrow f(x) \leq \bigvee_{y \lhd_n f(a)} \downarrow y$, that is, taking into account the characterization of the coalgebra structure maps given in (19), $(H_n f)(l(a)) \leq m(f(a))$ in $H_n M$. ∎

Remark 4.5. The objects of the category Loc-$\lhd_2$ are precisely the compact and stably locally compact locales and, moreover, they are spatial and coincide with the retracts of coherent locales ([16]). A localic map between coherent locales belongs to Loc-$\lhd_2$ iff it is a coherent morphism in the sense of [17]. From Theorem 2.4, it follows that Loc-$\lhd_2$ is precisely the closure under left adjoint retractions, on objects and on morphisms, of the category of coherent locales and coherent localic maps. Furthermore, by Theorems 4.4 and 2.4, the flat embeddings coincide with Loc-$\lhd_2^{\mathrm{KInj}}$, that is, they form the largest class of localic maps with respect to which Loc-$\lhd_2$ is Kan-injective.

Remark 4.6. The above theorem shows that Loc is the union of an increasing chain of KZ-monadic subcategories that are Kan-injective hulls of finite subcategories. Indeed, we know that Loc-$\lhd_n$ = KInj($\mathbb{E}_n$), where, for every $n \geq 1$, $\mathbb{E}_n$ is the class of n-flat embeddings, and $\mathbb{E}_0$ is the class of embeddings; since $m \leq n$ clearly implies $E_m \supseteq \mathbb{E}_n$, it follows that Loc-$\lhd_m$ = KInj($\mathbb{E}_m$) $\subseteq$ KInj($\mathbb{E}_n$) = Loc-$\lhd_n$.

Recall that a cardinal n is said to be *regular* if it cannot be expressed as the union of a family of sets of cardinality less than n indexed by a set of cardinality also less then n. Thus the only finite regular cardinals are 0, 1 and 2. If n and m are cardinals with $m < n$ and n regular, then the category Loc-$\lhd_m$ is strictly contained in the category Loc-$\lhd_n$. Indeed, taking

Then there exists the localic map $g_x/h : Y \to F_1$. For x and x' in X with $x \neq x'$, assume, without loss of generality, that $x \not\leq x'$. Then $g_x(x) \neq g_x(x')$ and, since $g_x = (g_x/h)h$, we conclude that $h(x) \neq h(x')$.

We prove now that every finite chain is Kan-injective wr.t. all embeddings. We are going to work in the category Frm. Thus we want to show that finite chains are Kan-projective with respect to the frame homomorphisms $h : X \to Y$ for which $h_* : Y \to X$ satisfies the equality $hh_* = id_Y$. Indeed, let A be a finite chain and let $h : X \to Y$ be a surjection in Frm. Let $f : A \to Y$ be a morphism of Frm. We are going to show that the morphism $\frac{f}{h} : A \to X$ exists and is defined by:

$$\frac{f}{h}(a) = (h_* f)(a), \text{ if } a \neq 0, \text{ and } \frac{f}{h}(0) = 0. \tag{7}$$

We recall that the right adjoint of the morphism h is given by $h_*(y) = \bigvee h^{-1}(\downarrow y)$, where $\downarrow y = \{z \in Y \,|\, z \leq y\}$. The function $\frac{f}{h}$ defined as above is monotonous, because

$$\begin{aligned}a \leq a' &\Rightarrow \downarrow f(a) \subseteq \downarrow f(a')\\ &\Rightarrow \vee h^{-1}(\downarrow f(a)) \leq \vee h^{-1}(\downarrow f(a')).\end{aligned}$$

Since A is a finite chain, we conclude that $\frac{f}{h}$ preserves all the meets and non-empty joins; by definition it also preserves the empty joins. So it is a morphism of Frm. It is now easy to see that $\frac{f}{h}$ defined as in (7) satisfies (i) and (ii) of Remark 2.9. ∎

Remark 3.7. Let L be a meet-semilattice and let DL be the frame of its lower sets ordered by inclusion (as described in Remark 3.4). Then, for every lower set $W \in DL$, we have that $W = \downarrow W = \bigcup_{a \in W} \downarrow a$, where $\downarrow a = \{b \in L \,|\, b \leq a\}$. Consider the set

$$\overline{L} = \{\downarrow a \,|\, a \in L\}. \tag{8}$$

Then every $W \in DL$ may be expressed as a supremum in DL as follows: $W = \bigvee\{A \in \overline{L} \,|\, A \leq W\}$. In particular, given a set X, the free frame generated by X has the property that every $W \in FX$ can be expressed as a supremum of elements of $\overline{SX}$, where SX is the meet-semilattice generated by X and $\overline{SX}$ is defined as in (8):

$$W = \bigvee\{A \in \overline{SX} \,|\, A \leq W\}. \tag{9}$$

In order to point out another feature of the free frame FX, observe that, for every $x \in X$, $\{x\}$ is an element of the semilattice SX and the element $\downarrow\{x\}$ of FX is defined by $\downarrow\{x\} = \{A \in SX \mid x \in A\}$. The elements $\{x\}$ of SX are clearly immediate predecessors of the greatest element 1_{SX}, and, consequently, in the frame FX, the supremum $\bigvee_{x\in X} \downarrow\{x\}$ is an immediate predecessor of 1_{FX}, indeed the only one. For every cardinal n, and $X = n$, let us denote this supremum by s:

$$s = \bigvee_{k\in n} \downarrow\{k\}. \tag{10}$$

In $F_0 = (0 < 1)$ this element s is just 0, and in $F_1 = (0 < d < 1)$ it is precisely d.

Notation 3.8. For every cardinal n, we are going to consider the localic map

$$f_n : F_n \to F_1$$

defined by $f_n(1) = 1$, $f_n(s) = d$, and $f_n(x) = 0$ for $x \neq 1, s$. In particular, $f_0(0) = d$ and f_1 is the identity map.

For every cardinal $n \geq 1$, let

$$\mathcal{D}_n$$

denote the category whose objects are F_0, F_1 and F_n, and whose morphisms, besides the identities, are f_0 and f_n. Of course, since f_1 coincides with the identity, $\mathcal{D}_1$ has just a non-identity morphism, which is f_0. We denote by

$$\mathcal{D}_0$$

the category formed just by the object F_1 and its identity map.

In Proposition 3.6 we characterized the embeddings of Loc in terms of Kan-injectivity of the category $\mathcal{D}_0$. In the next theorem we characterize n-flat embeddings by means of Kan-injectivity of the categories $\mathcal{D}_n$, for $n \geq 1$.

Theorem 3.9. *For every cardinal $n \geq 1$, the n-flat embeddings in* Loc *are exactly the localic maps with respect to which f_0 and f_n are Kan-injective, that is, $\mathcal{D}_n^{KInj}$ coincides with the n-flat embeddings.*

Proof: 1. From Proposition 3.6 we already know that the localic maps with respect to which F_0 and F_1 are Kan-injective are precisely the embeddings.

The functor $R_n : \mathcal{K}_n \to \mathsf{Frm}$ is locally monotone, then it preserves adjoint pairs and left inverses, that is, $R_nf^* \dashv R_nf$ in Frm and $R_nf^*R_nf = \mathrm{id}$. This implies that $(R_nf^*)_* \dashv (R_nf)_*$ in Loc and $(R_nf)_*(R_nf^*)_* = \mathrm{id}_{G_nX}$; equivalently, taking into account that $G_nf = (H_nf^*)_* = (R_nf^*)_*$, we have that $G_nf \dashv (R_nf)_*$ in Loc and $(R_nf)_*\, G_nf = \mathrm{id}_{G_nX}$, that is, f is a G_n-embedding.

Conversely, assume that $f : X \to Y$ is a localic map such that $(G_nf)_* \in \mathsf{Loc}$ and $(G_nf)_*\, G_nf = \mathrm{id}_{G_nX}$. Then we conclude that the map $(R_nf^*)_*$, which coincides with G_nf, lies in $\mathcal{K}_n$ and satisfies the equality $R_nf^*(R_nf^*)_* = \mathrm{id}_{R_nX}$. Let η and ε be the unit and counit relative to the reflector functor $R_n : \mathsf{Frm} \to \mathcal{K}_n$. Then, we have, on one hand,

$$f^*(\varepsilon_Y((G_nf)\eta_X) = f^*(\varepsilon_Y(R_nf^*)_*\eta_X) = \varepsilon_X(R_nf^*)(R_nf^*)_*\eta_X = \varepsilon_X\eta_X = \mathrm{id}_X; \tag{21}$$

and, on the other hand,

$$(\varepsilon_Y((G_nf)\eta_X)f^* = (\varepsilon_Y(R_nf^*)_*\eta_X)f^* = \varepsilon_Y(R_nf^*)_*R_nf^*\eta_Y \geq \varepsilon_Y\eta_Y = \mathrm{id}_Y.$$

We conclude that $f^* \dashv \varepsilon_Y(G_nf)\eta_X$, that is, $f = \varepsilon_Y(G_nf)\eta_X$; since the three morphisms of the last composition belong to $\mathcal{K}_n$, so do f, thus it preserves unions of k-indexed families, for $k \leq n$, i.e., f is n-flat. The equality (21) assures that $f^*f = \mathrm{id}_X$, that is, f is an embedding.

In order to complete the proof of the theorem, we have to show that the category Loc-$\lhd_n$ is the Eilenberg-Moore category of the monad $\mathbf{G}_n$. For objects the assertion immediately follows from [4], as we have said in Remark 4.3. It remains to prove that the morphisms of Loc-$\lhd_n$ are precisely the localic maps f such that f^* preserves the binary relation $\lhd_n$. Equivalently, we show that the morphisms of the category of the Eilenberg-Moore coalgebras of the comonad $\mathbf{H}_n$ are the frame homomorphisms that preserve the relation $\lhd_n$.

Let (L, l) and (M, m) be two coalgebras and let $f : L \to M$ be a morphism of coalgebras, i.e., we have the equality $(H_nf)l = mf$. To conclude that f preserves the relation "$\lhd_n$", let $a, b \in L$ be such that $a \lhd_n b$. Then we have:

$a \lhd_n b$	$\Rightarrow \downarrow a \subseteq l(b)$	by (20)
	$\Rightarrow (H_nf)(\downarrow a) \subseteq (H_nf)(l(b))$	because H_nf is a frame homomorphism
	$\Rightarrow \downarrow f(a) \subseteq mf(b)$	since $(H_nf)(\downarrow a) = \bigvee\{\downarrow f(z) \mid z \leq a\} = \downarrow f(a)$
	$\Rightarrow f(a) \lhd_n f(b)$	by (20).

Conversely, suppose that $f : L \to M$ is a frame homomorphism whose right adjoint f_* is a morphism of Loc-$\lhd_n$. Then there are morphisms l and m such that (L, l) and (M, m) are coalgebras; we want to show that $(H_nf)l =$

(i) For every $a \in L$, $a = \bigvee_{x \lhd_n a} x$,
(ii) Given $x, y, a \in L$, if $x \lhd_n a$ and $x \lhd_n b$ then $x \lhd_n a \wedge b$,
(iii) $1_L \lhd_n 1_L$;
and whose morphisms are all the localic maps f with a left adjoint f^* that preserves the relation "$\lhd_n$".

Remark 4.3. 1. From general properties of the reflector $R : \mathcal{K} \to \mathsf{Frm}$ of Remark 4.1 proved in [4], we know that the Eilenberg-Moore coalgebras of the KZ-comonad $\mathbf{H}_n$ are precisely those frames which satisfy the three conditions (i)-(iii) of Notation 4.2.2, and also precisely those frames which are Kan-projective with respect to n-flat surjections (that is, with respect to the frame homomorphisms f whose right adjoint f_* is an n-flat embedding in Loc). In other words, a locale is Kan-injective with respect to the n-flat embeddings iff it is an object of Loc-$\lhd_n$.

In the next theorem we complete the characterization of the Eilenberg-Moore category of the monad $\mathbf{G}_n$ over Loc by describing also its morphisms.

2. We observe that, as it follows from Remark 2.2, the structure map of a coalgebra (L, l) of the comonad $\mathbf{H}_n$ is the left adjoint of ε_L and $\varepsilon_L l = \mathrm{id}_L$, where ε is the counit of the comonad $\mathbf{H}_n$. Moreover, as it was proved in [4], l is defined by

$$l(a) = \bigvee_{x \lhd_n a} \downarrow x,\ a \in L, \tag{19}$$

and the binary relation $\lhd_n$ is characterized by

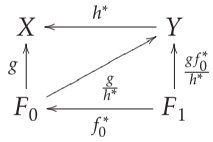
$$a \lhd_n b \Leftrightarrow \downarrow a \subseteq l(b),\ a, b \in L. \tag{20}$$

Theorem 4.4. *For every cardinal n, $G_n : \mathsf{Loc} \to \mathsf{Loc}$ is the functor of a KZ-monad over* Loc *whose Eilenberg-Moore category is* Loc-$\lhd_n$. *Moreover, the G_n-embeddings are precisely the n-flat embeddings, and, consequently,* Loc-$\lhd_n = \mathbf{K}(\mathcal{D}_n)$.

Proof: We already know from Notation 4.2 that G_n yields a KZ-monad. Next we verify that the G_n-embeddings (see Definition 2.3.2 and Proposition 2.7) are precisely the n-flat embeddings.

Let then $f : X \to Y$ be an n-flat embedding in Loc. Since f is an embedding we know that the frame homomorphism f^* satisfies the equality $f^* f = \mathrm{id}_X$. But, since f is n-flat, f is a morphism of $\mathcal{K}_n$ (see Remark 4.1.2).

2. Next we prove that an embedding $h : X \to Y$ is dense iff f_0 is Kan-injective with respect to it. We know that f_0 is Kan-injective with respect to h in Loc iff f_0^* is Kan-projective with respect to h^* in Frm, that is, iff, for every frame homomorphism $g : F_0 \to X$, the lower triangle of the diagram



is commutative. Since we are dealing with frame homomorphisms, which preserve the least and greatest elements, we just need to prove that the equality

$$\frac{gf_0^*}{h^*}(d) = \frac{g}{h^*} f_0^*(d)$$

holds iff h is dense, i.e., iff $h(0) = 0$. Indeed, by using formula (7) of the proof of Proposition 3.6 for morphisms f whose domain is a finite chain, and using the preservation of the least element by the frame homomorphisms g and $\frac{g}{h^*}$, we have that:

$$\frac{gf_0^*}{h^*}(d) = hgf_0^*(d) = hg(\bigwedge f_0^{-1}(\uparrow d)) = h(g(0)) = h(0) \tag{11}$$

and

$$\frac{g}{h^*} f_0^*(d) = \frac{g}{h^*}(0) = 0. \tag{12}$$

Then f_0 is Kan-injective with respect to h if and only if the last members of the equations (11) and (12) are equal, that is, iff h(0)=0.

3. In order to show that F_n is Kan-injective with respect to n-flat embeddings, we work in Frm: we prove that F_n is Kan-projective with respect to frame homomorphisms $h : X \to Y$ whose right adjoint h_* fulfils $hh_* = \mathrm{id}_Y$ and preserves unions of families indexed by $k \leq n$.

Let then $h : X \to Y$ be of that form, and let $g : F_n \to Y$ be a frame homomorphism. We show that $h_* g$ is a frame homomorphism. (Moreover, it will be clear from the proof that the same is true for any $g : F_i \to Y$ with $i < n$.) This will guarantee that

$$\frac{g}{h} = h_* g, \tag{13}$$

since we have that $h(h_* g) = g$, and $hs \leq g \Rightarrow h_* hs \leq h_* g \Rightarrow s \leq h_* g$.

In order to show that h_*g is indeed a frame homomorphism, we observe that it preserves finite meets because h_* is a right adjoint and g is a frame homomorphism. And h_*g preserves the least element because h_* is dense. Moreover, by hypothesis, h_*g preserves suprema of every set of cardinality no greater than n. We are to prove that the same is true for arbitrary suprema. Let then $\{W_t : t \in T\}$ be a subset of F_n, and let us consider the subset of F_n given by $\overline{Sn} = \{\downarrow a \,|\, a \in Sn\}$, which is defined by departing from the semilattice Sn accordingly to formula (8) of Remark 3.7. Following formula (9), we know that, for every $t \in T$, $W_t = \bigvee\{A \in \overline{Sn} : A \leq W_t\}$. Then, we have that

$$\begin{aligned}(h_*g)(\bigvee_{t\in T} W_t) &= (h_*g)(\bigvee_{t\in T}(\bigvee\{A \in \overline{Sn} : A \leq W_t\})) \\ &= (h_*g)(\bigvee\{A \in \overline{Sn} : A \leq W_t \text{ for some } t \in T\}).\end{aligned}$$

The cardinality of $\overline{Sn}$, which is equal to the one of Sn, is finite if n is so, and it is equal to n if n is infinite. Consequently, we know that h_* preserves suprema of $g[M]$, for every $M \subseteq \overline{Sn}$. Therefore, using also the fact that g is a morphism of Frm, we have:

$$\begin{aligned}(h_*g)(\bigvee_{t\in T} W_t) &= h_*(\bigvee\{g(A) : A \in \overline{Sn} \text{ e } A \leq W_t \text{ for some } t \in T\}) \\ &= \bigvee\{(h_*g)(A) : A \in \overline{Sn} \text{ and } A \leq W_t \text{ for some } t \in T\} \\ &= \bigvee(\bigcup_{t\in T}\{(h_*g)(A) : A \in \overline{Sn} \text{ and } A \leq W_t\}) \\ &= \bigvee_{t\in T}(\bigvee\{(h_*g)(A) : A \in \overline{Sn} \text{ and } A \leq W_t\}) \\ &= \bigvee_{t\in T}(h_*g)(\bigvee\{A \in \overline{Sn} : A \leq W_t\}) \\ &= \bigvee_{t\in T}(h_*g)(W_t).\end{aligned}$$

Consequently, given an n-flat embedding $m : X \to Y$ in Loc, and a localic map $l : X \to F_n$ (or a localic map $l : X \to F_i$ with $i < n$), using the formula

The counit ε is given, for every $L \in \mathsf{Frm}$ and $U \in H_nL$, by

$$\varepsilon_L(U) = \vee U.$$

The comultiplication $\delta : H_n \to H_nH_n$ is given, for every $L \in \mathsf{Frm}$ and $U \in H_nU$, by $\delta_L(U) = \bigvee\{\downarrow_{H_nL} (\downarrow_L u) \,|\, u \in U\}$.

Moreover, this comonad is of KZ type, more precisely, it satisfies the inequalities

$$\varepsilon_{H_nX} \leq H_n\varepsilon_X,\ X \in \mathsf{Frm}. \tag{18}$$

The last remark legitimates the following notations.

Notation 4.2. 1. By duality, the comonad $\mathbf{H_n}$ yields a monad in Loc; let

$$G_n : \mathsf{Loc} \to \mathsf{Loc}$$

be its corresponding functor; thus, for every localic map f, $G_nf = (H_nf^*)_*$. As a consequence of the inequalities (18), taking into account that the order of the morphisms is reversed in the passage from Frm to Loc, we obtain the inequalities

$$\varepsilon^{\mathrm{op}}_{H_nX} \geq H_n^{\mathrm{op}}\varepsilon^{\mathrm{op}}_X$$

in Loc, being the unit of the monad $\mathbf{G}_n$ over Loc given by $\varepsilon^{\mathrm{op}}_X = (\varepsilon_X)_*$. Those inequalities mean that this monad is of KZ type (see 2.2.) Accordingly, our G_n-embeddings are going to be those localic maps $f : X \to Y$ such that G_nf is a left adjoint section, i.e., it has a right adjoint $(G_nf)_*$ in Loc with $(G_nf)_*G_nf = \mathrm{id}_{G_nX}$ (see Proposition 2.7).

2. Aiming to give a characterization of the subcategories KInj($\mathbb{E}_n$) of Loc for $\mathbb{E}_0$ the class of embeddings and $\mathbb{E}_n$ (with $n \geq 1$) the class of n-flat embeddings, let us consider, for every cardinal n and every frame L, the binary relation defined in the underlying set of L as follows:

$$x \lhd_n a \Leftrightarrow (\forall U \in H_nL, a \leq \vee U \Rightarrow x \in U).$$

This kind of relation was possibly used for the first time by Raney in [26], and Banaschewski [4] considered a generalization of it which encompasses all comonads induced by a reflector $R : \mathcal{K} \to \mathsf{Frm}$ of the type mentioned in Remark 4.1.

We denote by

$$\mathsf{Loc}\text{-}\lhd_n$$

the subcategory of Loc whose objects are the locales L that satisfy the following conditions:

is then constituted by the objects A of SLat such that for every $P \in SA$, $\vee P$ exists and $a \wedge (\vee P) = \vee\{a \wedge t \,|\, t \in P\}$, and by the morphisms $f : A \to B$ of SLat such that $f(\vee P) = \vee f[P]$, $P \in SA$.

For a subcategory $\mathcal{K}$ obtained this way, Banaschewski proved that Frm is reflective in $\mathcal{K}$ with every reflection $\eta_A : A \to RA$ being given by

$$RA = \{U \in DA \,|\, (P \subseteq U \text{ and } P \in SA) \Rightarrow \vee P \in U\}, \text{ and } \eta_A(a) = {\downarrow} a,\ a \in A.$$

For every morphism $f : A \to L$ in $\mathcal{K}$, with $L \in$ Frm, the unique morphism $\overline{f} : RA \to L$ such that $\overline{f}\eta_A = f$ is defined by $\overline{f}(U) = \vee f[U]$. The counit is thus given by $\varepsilon_L(U) = \vee U$, $U \in RL$.

This reflection induces then a comonad in Frm whose functor part is the composition of the inclusion of Frm into SLat with the reflector functor R, and this comonad is of KZ type ([4]).

2. For every cardinal n, let

$$(S_nA)_{A \in \mathsf{SLat}}$$

be the family defined by:

$$S_0A = \emptyset; \text{ and}$$
$$S_nA = \{P \subseteq A \,|\, \mathrm{card}(P) \leq n\}, \text{ for } n \geq 1.$$

It is easy to see that S_n satisfies the conditions described for S above. Let us denote the corresponding subcategory $\mathcal{K}$ and its reflector in Frm by $\mathcal{K}_n$ and R_n, respectively. For $n = 0$, $\mathcal{K}_0$ is just SLat, and, for $n \geq 1$, it consists of all objects A and morphisms $f : A \to B$ of SLat such that, for all $a \in A$ and $P \subseteq A$ with $\mathrm{card}(P) \leq n$, $\vee P$ exists, $a \wedge (\vee P) = \bigvee\{a \wedge t \,|\, t \in P\}$, and $f[\vee P] = \vee f[P]$. The comonad over Frm induced by the reflector

$$R_n : \mathcal{K}_n \to \mathsf{Frm}$$

is precisely $\mathbf{H_n} = (H_n, \varepsilon, \delta)$, defined as follows:

The functor

$$H_n : \mathsf{Frm} \to \mathsf{Frm}$$

sends each frame L to the frame

$$H_nL = \{U \in DL \,|\, (P \subseteq U, \mathrm{card}(P) \leq n) \Rightarrow \vee P \in U\}$$

ordered by inclusion. Moreover, for each frame homomorphism $f : L \to M$, $H_nf : H_nL \to H_nM$ is defined by

$$H_nf(U) = \bigvee\{{\downarrow} f(u) \,|\, u \in U\},\ U \in H_nL.$$

(5) and the equality (13), we have

$$l/m = \left(\frac{l^*}{m^*}\right)_* = ((m^*)_*l^*)_* = (ml^*)_* = lm_* \tag{14}$$

with lm_* a localic map. In order to verify that f_n is Kan-injective with respect to $m : X \to Y$, it suffices now to use (14) applied to l and f_nl:

$$f_n(l/m) = f_n(lm_*) = (f_nl)m_* = (f_nl)/m.$$

Conversely, let f_n be Kan-injective with respect to the embedding $m : X \to Y$ in Loc; equivalently, f_n^* is Kan-projective with respect to $m^* : Y \to X$ in Frm. We want to prove that m preserves suprema of non-empty n-indexed families. For that, let $\{x_k \,|\, k \in n\}$ be a set of elements of X. Define a map $p : n \to X$ by putting $p(k) = x_k$. Let $\eta_n : n \to F_n$ be the universal map given by $\eta_n(k) = {\downarrow}\{k\}$ (see Remark 3.4). Then there is a unique frame homomorphism $g : F_n \to X$ such that $g\eta_n = p$; in particular, $g({\downarrow}\{k\}) = x_k$ for every $k \in n$, and then, $g(s) = g\left(\bigvee_{k\in n} {\downarrow}\{k\}\right) = \bigvee_{k\in n} x_k$.

By hypothesis, there exist the morphisms $\frac{g}{m^*}$ and $\frac{gf_n^*}{m^*}$ in Frm, and, moreover, they satisfy the equality

$$\frac{gf_n^*}{m^*} = \frac{g}{m^*}f_n^*.$$

In particular,

$$\frac{gf_n^*}{m^*}(d) = \frac{g}{m^*}f_n^*(d). \tag{15}$$

On one hand, taking into account that $f_n^*(d) = \bigwedge\{z \in F_n \,|\, d \leq f_n(z)\} = s$, and the formula (7) obtained in the proof of Proposition 3.6, we have:

$$\frac{gf_n^*}{m^*}(d) = mgf_n^*(d) = mg(s) = m\left(\bigvee_{k\in n} x_k\right). \tag{16}$$

Since $\mathrm{id}_Y \leq mm^*$ and $m^*\frac{g}{m^*} = g$, then $\frac{g}{m^*} \leq mm^*\frac{g}{m^*} = mg$. Thus, using also the fact that $\frac{g}{m^*}$ is a frame homomorphism, we obtain, on the other hand, that

$$\frac{g}{m^*}f_n^*(d) = \frac{g}{m^*}(s) = \frac{g}{m^*}\left(\bigvee_{k\in n} {\downarrow}\{k\}\right) = \bigvee_{k\in n}\frac{g}{m^*}({\downarrow}\{k\}) \leq \bigvee_{k\in n} m(g({\downarrow}\{k\})) = \bigvee_{k\in n} m(x_k). \tag{17}$$

From (15), (16) and (17), we obtain $m\left(\bigvee_{k\in n} x_k\right) \leq \bigvee_{k\in n} m(x_k)$. As the other inequality trivially holds, we conclude that $m\left(\bigvee_{k\in n} x_k\right) = \bigvee_{k\in n} m(x_k)$. ∎

Remark 3.10. In the proof of the above theorem we saw that the locale F_n is Kan-injective with respect to n-flat embeddings. More than that, it is Kan-injective with respect to all embeddings. In other words, every free frame is Kan-projective with respect to quotients in Frm. Indeed, let $D : \mathsf{SLat} \to \mathsf{Frm}$ and $S : \mathsf{Set} \to \mathsf{SLat}$ be the functors described in Remark 3.4. Given a quotient $h : X \to Y$ in Frm and a frame homomorphism $g : F_n \to Y$, it is easy to see that the morphism $\frac{g}{h}$ is the frame homomorphism $\hat{g} : F_n = DSn \to X$ with $h_* g d_{Sn} = \hat{g} d_{Sn}$, where d_{Sn} is the universal morphism from Sn to Frm. Indeed, the equality $h\hat{g} = g$ follows from the equalities $h\hat{g}d_{Sn} = hh_* g d_{Sn} = g d_{Sn}$, and, given a frame homomorphism $s : F_n \to X$ with $hs \leq g$, then we have $s d_{Sn} \leq h_* hs d_{Sn} \leq h_* g d_{Sn} = \hat{g} d_{Sn}$. Thus, since the reflection is of KZ type, as observed in Remark 3.4, it follows that $s \leq \hat{g}$ (see Theorem 2.4 and Remark 2.5).

Remark 3.11. It is well known that there is a contravariant functor Ω from Top to Frm sending every space X to the frame ΩX of its open subsets and every continuous map $f : X \to Y$ to the frame homomorphism $f^{-1} : \Omega Y \to \Omega X$. This functor, seen as a covariant one, may be defined as the functor

$$\mathsf{Lc} : \mathsf{Top} \to \mathsf{Loc}$$

sending every continuous map $f : X \to Y$ to the right-adjoint $(f^{-1})_* : \Omega X \to \Omega Y$ of $f^{-1} : \Omega Y \to \Omega X$, thus $(f^{-1})_*(H) = \bigcup\{V \in \Omega Y : f^{-1}(V) \subseteq H\}$, $H \in \Omega X$. Moreover, a locale is said to be *spatial* if it is isomorphic to the frame ΩX for some space X.

The union of all subcategories $\mathcal{D}_n$ defined in Notation 3.8 is a subcategory

$$\mathcal{D}$$

of Loc. The category $\mathcal{D}$ is made of spatial locales, since the lower sets of a meet-semilattice L form a topology over the underlying set of L. Furthermore, as a consequence of Theorem 3.9, we have the following:

Corollary 3.12. *The Kan-injective hull of $\mathcal{D}$ in* Loc *is the entire category* Loc.

Proof: Let us denote by $\mathbb{E}_n$ the class of all n-flat embeddings, and by $\mathbb{E}_C$ the class of all completely flat embeddings, that is, embeddings that preserve arbitrary suprema. Then, using the fact that the operators $_^{\mathrm{KInj}}$ and $\mathrm{KInj}(-)$ establish a Galois connection between subcategories and classes of morphisms, we have:

$$\mathcal{D}^{\mathrm{KInj}} = \left(\bigcup_{n\geq 1} \mathcal{D}_n\right)^{\mathrm{KInj}} = \bigcap_{n\geq 1} \mathcal{D}_n^{\mathrm{KInj}} = \bigcap_{n\geq 1} \mathbb{E}_n = \mathbb{E}_C.$$

A completely flat morphism $f : X \to Y$ of Loc is a frame homomorphism, thus f_* also belongs to Loc. As a consequence, it follows that, if f is, moreover, an embedding, that is, $f_* f = \mathrm{id}$, then all locales and all localic maps are Kan-injective with respect to f: for every $g : X \to A$ in Loc, $g/f = g f_*$. Therefore,

$$\mathbf{K}(\mathcal{D}) = \mathrm{KInj}\left(\mathcal{D}^{\mathrm{KInj}}\right) = \mathrm{KInj}(\mathbb{E}_C) = \mathsf{Loc}.$$

∎

4. The categories Loc-$\triangleleft_n$

In this section we study the Kan-injective hulls in Loc of the subcategories $\mathcal{D}_n$ of the last section. In particular, we will see that the Kan-injective hull of $\mathcal{D}_n$ is precisely the Eilenberg-Moore category of a KZ-monad $G_n : \mathsf{Loc} \to \mathsf{Loc}$. For that, we are going to use some results of [4], as explained in the following remark.

Remark 4.1. Let $D : \mathsf{SLat} \to \mathsf{Frm}$ be the reflector functor already described in Remark 3.4.

1. Banaschewski [4] considered subcategories $\mathcal{K}$ of the category SLat obtained as follows:

Let

$$(SA)_{A\in\mathsf{SLat}}$$

be a family where, for every object A, SA is a subset of the power-set of A, satisfying the following conditions:

(a) $\{a \wedge t \,|\, t \in P\} \in SA$, for every $a \in A$ and every $P \in SA$.
(b) for every morphism $f : A \to B$ of SLat and every $P \in SA$, $f[P] \in SB$.

The corresponding category

$$\mathcal{K}$$