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ALGEBRAIC INTEGRABILITY FOR MINIMUM ENERGY CURVES

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Abstract: This paper deals with integrability issues of the Euler-Lagrange equations associated to a variational problem, where the energy function depends on acceleration and drag. Although the motivation came from applications to path planning of underwater robot manipulators, the approach is rather theoretical and the main difficulties result from the fact that the power needed to push an object through a fluid increases as the cube of its speed.

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1. Introduction

This work is motivated by applications to path planning of underwater robot manipulators, where the objective is to find trajectories that minimize acceleration and drag forces, while the manipulator moves from an initial position to a target position, with prescribed initial and final velocities.

Drag is a mechanical force generated by the interaction and contact of a solid body with a fluid. Drag depends on the properties of the fluid and on the size, shape, and speed of the moving body. An underwater vehicle suffers the interaction with the water viscosity much more than an aerial vehicle suffers the air resistance.

Problems dealing with minimal energy trajectories for aerial vehicles typically ignore air resistance and only minimize acceleration. This might result from the fact that air resistance may be neglected, specially when compared with a liquid resistance. Under this assumption, the resulting trajectories are geometric cubic polynomials on the configuration space of the vehicle. These curves, which are generalizations to Riemannian manifolds of the classical and well established cubic polynomials on Euclidean spaces, have been first introduced by Noakes at al. in [NHP89] and further developed, for instance,

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in [CL95] and [CLC01]. These optimization problems are formulated via a variational approach and the corresponding Euler-Lagrange equations have been derived in the general context of manifolds. In spite of that, the resulting curves are far from being completely understood due to challenging questions of geometric integration.

Due to fluid traction, the energy consumption of an underwater manipulator is greater than that of an aerial manipulator, and in extreme environments, such as in deep ocean, it is difficult to supply energy to manipulators. So, it is crucial to determine optimal trajectories of the vehicle that minimize not just the power needed to overcome changes in velocity but also the drag forces. We refer to [KSBB07] and [HSA$^+$12] for some insights related to these problems.

The power needed to push an object through a fluid increases as the cube of its speed. This fact might be another reason for the lack of results when the energy function, besides depending on the norm of acceleration, also depends on the drag forces. Indeed, as it will become clear in this article, the addition of a term corresponding to the drag power substantially increases the complexity of finding solutions even when the geometry of the configuration space is not taken into consideration and the corresponding optimization problem is only formulated in Euclidean space. In the absence of drag, the problem becomes trivial and the Euler-Lagrange equations have a unique solution which is a cubic polynomial whose coefficients are uniquely determined by the boundary conditions.

Our objective here is to study algebraic integrability properties of the Euler-Lagrange equation associated to a variational problem whose solutions are energy curves that minimize acceleration and drag. This problem turns out to be very difficult to solve, but using the theory of Darboux polynomials we have been able to give some partial answers.

This article is organized as follows. In Session 2 we formulate the variational problem, derive the corresponding Euler-Lagrange equations and prove its local integrability. We also show that every solution of these equations is an integral curve of a certain quadratic vector field. In order to find first integrals of this vector field, using the Darboux theory of integrability for polynomial vector fields, we introduce, in Section 3, the essentials of this theory. The main results appear in Section 4, were, in particular, several first integrals of the vector field associated to our problem are identified. The paper ends with a short conclusion.

2. Variational problem

In this section we formulate the variational problem associated to the double objective of minimizing acceleration and drag, and prove local integrability of the corresponding Euler-Lagrange equations.

Let n be any natural number and τ a positive real parameter. Consider the function $\mathcal{L}\colon \mathbb{R}^{3n+1} \to \mathbb{R}$ defined by

$$\mathcal{L}\left(a_1,\ldots,a_n,p_1,\ldots,p_n,x_1,\ldots,x_n,t\right) = \left(a_1^2+\cdots+a_n^2\right)+\tau\left(p_1^2+\cdots+p_n^2\right)^{3/2}.$$

We are interested to study the solution of the variational problem

$$\min_{x\in\Omega}\int_0^T \mathcal{L}\left(\frac{d^2x}{dt},\frac{dx}{dt},x,t\right)dt, \tag{1}$$

where Ω is the set of two-times differentiable functions from $[0,T]\subset\mathbb{R}$ to $\mathbb{R}^n$, such that $x(0)$, $x(T)$, $\frac{dx}{dt}(0)$, and $\frac{dx}{dt}(T)$ are fixed.

This is the situation when the Lagrangian $\mathcal{L}$ is written as

$$\mathcal{L}\left(\frac{d^2x}{dt},\frac{dx}{dt},x,t\right) = \left\langle\frac{d^2x}{dt^2},\frac{d^2x}{dt^2}\right\rangle + \tau\left\langle\frac{dx}{dt},\frac{dx}{dt}\right\rangle^{3/2}. \tag{2}$$

The general theory of calculus of variations tell us that the Euler-Lagrange equations for a minimization problem of type (1) are the following, valid for every $1\le i\le n$:

$$\frac{d^2}{dt^2}\frac{\partial\mathcal{L}}{\partial a_i}\left(\frac{d^2x}{dt^2},\frac{dx}{dt},x,t\right)-\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial p_i}\left(\frac{d^2x}{dt^2},\frac{dx}{dt},x,t\right)+\frac{\partial\mathcal{L}}{\partial x_i}\left(\frac{d^2x}{dt^2},\frac{dx}{dt},x,t\right)=0. \tag{3}$$

In our case, we have

$$\frac{\partial\mathcal{L}}{\partial a_i} = 2a_i$$
$$\frac{\partial\mathcal{L}}{\partial p_i} = 3\tau p_i\left(p_1^2+\cdots+p_n^2\right)^{1/2}$$
$$\frac{\partial\mathcal{L}}{\partial x_i} = 0.$$

Therefore, (3) becomes

$$\frac{d}{dt}\left(2\frac{d^2u_i}{dt^2}-3\tau u_i\left(u_1^2+\cdots+u_n^2\right)^{1/2}\right)=0,\ 1\le i\le n, \tag{4}$$

where $u_i = \frac{dx_i}{dt}$. Let $u_{i+n} := \frac{du_i}{dt}$, $1 \le i \le n$. Then we get the Euler-Lagrange equations associated to our problem as the system of ordinary differential equations

$$\begin{cases} \frac{du_i}{dt} = u_{i+n}, & 1 \le i \le n \\ \frac{du_{i+n}}{dt} = \frac{3}{2}\tau u_i \left(u_1^2 + \cdots + u_n^2\right)^{1/2} + c_i, & 1 \le i \le n \end{cases} \tag{5}$$

where c_i are constants of integration of (4). Let us define the functions f_i on the space $\mathbb{R}^{2n+1}$ by

$$\begin{cases} f_i\left(c_1, \dots, c_n, p_1, \dots, p_n, a_1, \dots, a_n, t\right) = a_i, & 1 \le i \le n \\ f_{i+n}\left(c_1, \dots, c_n, p_1, \dots, p_n, a_1, \dots, a_n, t\right) = \frac{3}{2}\tau p_i \left(p_1^2 + \cdots + p_n^2\right)^{1/2}, & 1 \le i \le n \end{cases}.$$

The resulting map $f\colon \mathbb{R}^{3n+1} \to R^{2n}$ is of class C^1 on $\mathbb{R}^{3n+1}$. This is obvious for all points outside the hyperplane $p_1 = \cdots = p_n = 0$. We also have

$$\frac{\partial f_i}{\partial a_j} = \delta_{ij}, \qquad \frac{\partial f_i}{\partial p_j} = 0, \qquad \frac{\partial f_{i+n}}{\partial a_j} = 0,$$

for all $1 \le i, j \le n$. Thus it is enough to prove that the functions $\frac{\partial f_{i+n}}{\partial p_j}$, $1 \le i, j \le n$, are continuous at the points $(c_1, \dots, c_n, a_1, \dots, a_n, 0, \dots, 0, t)$. We have

$$\frac{\partial f_{i+n}}{\partial p_j} = \frac{3}{2}\tau \left(\delta_{ij}\left(p_1^2 + \cdots + p_n^2\right)^{1/2} + p_i p_j (p_1^2 + \cdots + p_n^2)^{-1/2}\right)$$

at any point with at least one $p_k \ne 0$, $1 \le k \le n$. As $(p_1^2 + \cdots + p_n^2)^{1/2}$ is a continuous function on $\mathbb{R}^{2n+1}$, we have only to verify that for every pair $1 \le i, j \le n$, the function $p_i p_j \left(p_1^2 + \cdots + p_n^2\right)^{-1/2}$ can be continuously extended at points with $p_1 = \cdots = p_n = 0$. Namely, we will show that it can be extended by zero value at these points. In the points, where $p_i p_j = 0$ and at least one $p_k \ne 0$, we have

$$\left| p_i p_j \left(p_1^2 + \cdots + p_n^2\right)^{-1/2} \right| = 0.$$

Consider the point with $p_i p_j \ne 0$ and $|p_k| < \varepsilon$ for all $1 \le k \le n$. Then

$$\left| p_i p_j \left(p_1^2 + \cdots + p_n^2\right)^{-1/2} \right| \le \left| p_i p_j (p_j^2)^{-1/2} \right| = |p_i| < \varepsilon.$$

This shows that the function

$$(\overline{c}, \overline{p}, \overline{a}, \overline{x}, t) \mapsto \begin{cases} p_i p_j (p_1^2 + \cdots + p_n^2)^{-1/2}, & p_1^2 + \cdots + p_n^2 \ne 0 \\ 0, & p_1 = \cdots = p_n = 0 \end{cases}$$

is continuous.

Theorem 2.1. *The system* (5) *has a unique solution* $\eta\,(t, t_{\mathbf{0}}, u_{\mathbf{0}}, \overline{c})$ *defined in a sufficiently small neighbourhood of* t_0*, for every choice of parameters* $\overline{c} = (c_1, \ldots, c_n)$ *and any choice of the initial conditions* $u(t_{\mathbf{0}}) = u_{\mathbf{0}}$*.*

Proof: Since the map $f_c \colon \mathbb{R}^{2n+1} \to \mathbb{R}$ defined by

$$f_c\,(u_1, \ldots, u_{2n}, t) = f(c_1, \ldots, c_n, u_1, \ldots, u_{2n}, t)$$

is of class C^1 for any choice of parameters $c_1, \ldots, c_n$, we get that they are uniformly Lipschitz continuous with respect to u on any compact subset of $\mathbb{R}^{2n+1}$. Now, choose any compact rectangle $|t - t_{\mathbf{0}}| \leq a$, $|u - u_{\mathbf{0}}| \leq b$ arround $(u_{\mathbf{0}}, t_{\mathbf{0}}) \in \mathbb{R}^{2n+1}$. By the Picard-Lindelöf Theorem (see e.g. [Har64, Theorem 1.1]), there is an α such that (5) with the initial condition $u(t_{\mathbf{0}}) = u_{\mathbf{0}}$ has a unique solution on the interval $|t - t_{\mathbf{0}}| \leq \alpha$. ∎

Let us change notation and rewrite (5) in the form

$$\begin{cases} \frac{du}{dt} = v \\ \frac{dv}{dt} = \frac{3}{2}\tau u \left\| u \right\| + c, \end{cases} \tag{6}$$

where $u, v \colon \mathbb{R} \to \mathbb{R}^n$ are unknown functions and $c \in \mathbb{R}^n$ is a parameter. Then, we have

$$\frac{d \left\| u \right\|^3}{dt} = \frac{d(u_1^2 + \cdots + u_n^2)^{3/2}}{dt} = \frac{3}{2}(u_1^2 + \cdots + u_n^2)^{1/2} \sum_{i=1}^{n} 2u_i v_i = 3 \left\| u \right\| \langle u, v \rangle$$

and

$$\frac{d \langle v, v \rangle}{dt} = 2 \left\langle v, \frac{3}{2}\tau u \left\| u \right\| + c \right\rangle = 3\tau \left\| u \right\| \langle v, u \rangle + 2 \langle v, c \rangle .$$

We also have

$$\frac{d \langle u, c \rangle}{dt} = \langle v, c \rangle .$$

Therefore

$$\frac{d}{dt}\left(\tau \left\| u \right\|^3 + 2 \langle u, c \rangle - \left\| v \right\|^2\right) = 0. \tag{7}$$

We get a system of ordinary differential equations with 5 unknown functions $\langle u,u\rangle$, $\langle v,v\rangle$, $\langle u,v\rangle$, $\langle u,c\rangle$, $\langle v,c\rangle$:

$$\begin{cases} \frac{d\langle u,u\rangle}{dt} = 2\langle u,v\rangle \\ \frac{d\langle u,v\rangle}{dt} = \langle v,v\rangle + \frac{3}{2}\tau\langle u,u\rangle^{3/2} + \langle u,c\rangle \\ \frac{d\langle v,v\rangle}{dt} = 3\tau\langle u,u\rangle^{1/2}\langle u,v\rangle + 2\langle v,c\rangle \\ \frac{d\langle u,c\rangle}{dt} = \langle v,c\rangle \\ \frac{d\langle v,c\rangle}{dt} = \frac{3}{2}\tau\langle u,u\rangle^{1/2}\langle u,c\rangle + \langle c,c\rangle . \end{cases} \tag{8}$$

Note that having a numerical solution of the system (8) one can easily find a solution of (6), as with known function $|u|$ the system (6) breaks up into n independent two dimensional systems

$$\begin{cases} \frac{du_i}{dt} = v_i \\ \frac{dv_i}{dt} = \frac{3}{2}\tau u\,\|u_i\| + c_i \end{cases},$$

for $1 \le i \le n$.

In the rest of the article we investigate algebraic properties of the system (8).

Let us introduce the following notation

$$y_1 = \langle u,u\rangle\,,\ \ y_2 = \langle u,v\rangle\,,\ \ y_3 = \langle v,v\rangle\,,\ \ y_4 = \langle u,c\rangle\,,\ \ y_5 = \langle v,c\rangle\,,\ \ y_6 = \langle c,c\rangle$$

$$y_7 = \tau\langle u,u\rangle^{3/2} + 2\langle u,c\rangle - \langle v,v\rangle\,.$$

Note that y_1 is a first integral for the system (5). We have

$$\tau\langle u,u\rangle^{1/2} = \frac{1}{y_1}(y_7 - 2y_4 + y_3).$$

It will be also useful to denote $\tau\langle u,u\rangle^{3/2}$ by z. Then

$$\begin{aligned} z &= y_7 - 2y_4 + y_3, \\ y_7 &= z + 2y_4 - y_3. \end{aligned}$$

Any solution of (5) produces a curve in the 7-dimensional space, which is an integral curve of the vector field

$$
\begin{aligned}
X = {} & 2y_2\frac{\partial}{\partial y_1} + (y_3+y_4)\frac{\partial}{\partial y_2} + \frac{3}{2}z\frac{\partial}{\partial y_2} + \frac{3y_2}{y_1}z\frac{\partial}{\partial y_3} + 2y_5\frac{\partial}{\partial y_3} + y_5\frac{\partial}{\partial y_4} \\
& + \frac{3y_4 z}{2y_1}\frac{\partial}{\partial y_5} + y_6\frac{\partial}{\partial y_5}
\end{aligned}
$$

Multiplying by the function $2y_1$, we obtain the quadratic vector field

$$
\begin{aligned}
Y \;=\; & 4y_1y_2\tfrac{\partial}{\partial y_1} + (2y_1y_3 + 2y_1y_4 + 3y_1z)\tfrac{\partial}{\partial y_2} + (4y_1y_5 + 6y_2z)\tfrac{\partial}{\partial y_3} \\
& + 2y_1y_5\tfrac{\partial}{\partial y_4} + (2y_1y_6 + 3y_4z)\tfrac{\partial}{\partial y_5}.
\end{aligned} \tag{9}
$$

In the next section we recall the general theory of rational first integrals for homogeneous polynomial vector fields, in order to apply this theory later to the vector field Y, hoping to obtain some insight about solutions of our problem.

3. Homogeneous vector fields and Darboux polynomials

In this section we recall the theory of rational first integrals for homogeneous algebraic systems of differential equations. The detailed account of the theory can be found in Chapter 2 of [Gor01].

By a *polynomial vector field* on $\mathbb{R}^n$ we understand a linear combination of the vector fields $\frac{\partial}{\partial x_i}$ with the coefficients $p_i \in \mathbb{R}[x_1, \ldots, x_n]$. It is straightforward that if F is a polynomial and X is a polynomial vector field then $X(F)$ is also a polynomial.

Definition 3.1. Let $\mathcal{V}$ be a polynomial vector field and F a polynomial function on $\mathbb{R}^n$. We say that F is a *Darboux polynomial* if $\mathcal{V}(F) = pF$ for some polynomial p. The polynomial p is called the *cofactor* of F.

The following proposition is a direct consequence of the definition.

Proposition 3.2. *Let $\mathcal{V}$ be a polynomial vector field on $\mathbb{R}^n$. Suppose F_1 is a Darboux polynomial for $\mathcal{V}$ with a cofactor p_1 and F_2 is a Darboux polynomial with a cofactor p_2. Then*

$$
\mathcal{V}(F_1F_2) = (p_1+p_2)F_1F_2,
$$
$$
\mathcal{V}\left(\frac{F_1}{F_2}\right) = (p_1-p_2)\frac{F_1}{F_2}.
$$

Thus, the product of two Darboux polynomials for $\mathcal{V}$ is again a Darboux polynomial for $\mathcal{V}$. Moreover, is F_1 and F_2 are two Darbox polynomials for $\mathcal{V}$ with the same cofactor p, then $\frac{F_1}{F_2}$ is a (rational) first integral of $\mathcal{V}$. We also have the opposite claims.

Proposition 3.3 ([Gor01, Proposition 2.4]). *Let $\mathcal{V}$ be a polynomial vector field on $\mathbb{R}^n$. Suppose $\frac{P}{Q}$ is a rational first integral for $\mathcal{V}$ such that P and Q are coprime. Then, P and Q are Darboux polynomials for $\mathcal{V}$ with the same cofactor.*

Proposition 3.4 ([Gor01, Proposition 2.5]). *Let $\mathcal{V}$ be a polynomial vector field on $\mathbb{R}^n$. Suppose F is a Darboux polynomial for $\mathcal{V}$. Then, every irreducible factor of F is also a Darboux polynomial for $\mathcal{V}$.*

The above two propositions show that to find all rational first integrals for the polynomial vector field $\mathcal{V}$ on $\mathbb{R}^n$, it is enough to describe all irreducible Darboux polynomials for $\mathcal{V}$. This problem can be simplified if the vector field $\mathcal{V}$ has good properties with respect to some grading on $\mathbb{R}[x_1, \ldots, x_n]$.

Definition 3.5. A grading on the ring of polynomials $\mathbb{R}[x_1, \ldots, x_n]$ is a collection of $\mathbb{R}$-vector subspaces V_k, $k \in \mathbb{Z}$, in $\mathbb{R}[x_1, \ldots, x_n]$, such that

(1) $V_k \cap V_l = \varnothing$, if $k \neq l$;
(2) $\bigoplus_{k\in\mathbb{Z}} V_k = \mathbb{R}[x_1, \ldots, x_n]$;
(3) $V_k V_l \subset V_{k+l}$, for all k, $l \in \mathbb{Z}$.

Given a grading $\{\, V_k \,|\, k \in \mathbb{Z}\}$ on $\mathbb{R}[x_1, \ldots, x_n]$, we say that a polynomial vector field $\mathcal{V}$ is *homogeneous of degree* j, if for all $k \in \mathbb{Z}$ holds $\mathcal{V}(V_k) \subset V_{k+j}$.

The next theorem shows that for homogeneous vector fields, one should consider only homogeneous Darboux polynomials

Theorem 3.6. *Let $\{\, V_k \,|\, k \in \mathbb{Z}\}$ be a grading on $\mathbb{R}[x_1, \ldots, x_n]$ and $\mathcal{V}$ a homogeneous polynomial vector field on $\mathbb{R}[x_1, \ldots, x_n]$ of degree j. Suppose F is a Darboux polynomial for $\mathcal{V}$ with a cofactor p. Let us denote by F_k the projection of F on the subspace V_k. Then, every F_k is a Darboux polynomial for $\mathcal{V}$ with the same cofactor p. Moreover, $p \in V_j$.*

Proof: We first consider the case $p = 0$. Then, F is a first integral for $\mathcal{V}$ and we have equation $\mathcal{V}F = 0$. Projecting on the space V_{k+j}, we get $\mathcal{V}F_k = 0$. This shows that F_k is a first integral for $\mathcal{V}$, for any $k \in \mathbb{Z}$.

Now suppose $p \neq 0$. Let us denote by p_k the projection of p on V_k, $k \in \mathbb{Z}$. Let l be the maximal integer such that $p_l \neq 0$ and m the maximal integer such that $F_m \neq 0$. Suppose $l > j$. Then, from the equation $\mathcal{V}F = pF$, projecting on V_{m+l}, we get $0 = p_l F_m$. This shows that either $p_l = 0$ or $F_m = 0$, which is in contradiction with our assumptions on l and m. This shows that $l \leq j$. By symmetrical consideration, we get that $l \geq j$. Thus $p_k = 0$ for any $k > j$ and any $k < j$, or in other words, $p = p_j \in V_j$. Now, projecting both sides of $\mathcal{V}F = pF$ on V_{k+j}, we get $\mathcal{V}F_k = p_j F_k = pF_k$. This shows that every F_k is a Darboux polynomial with the cofactor p. ■

4. Properties of Darbooux polynomials for Y

Recall that we are interested in specializing the content of the previous section to the vector field given in (9). That is, we study now Darboux polynomials for the quadratic vector field

$$Y = 4y_1y_2\frac{\partial}{\partial y_1} + (2y_1y_3 + 2y_1y_4 + 3y_1z)\frac{\partial}{\partial y_2} + (4y_1y_5 + 6y_2z)\frac{\partial}{\partial y_3} + 2y_1y_5\frac{\partial}{\partial y_4} + (2y_1y_6 + 3y_4z)\frac{\partial}{\partial y_5}$$

on $\mathbb{R}[y_1, \ldots, y_7]$, where $z = y_3 - 2y_4 + y_7$. In particular, we will show that the problem of finding Darboux polynomials for Y can be replaced by a computationally more feasible problem of finding all polynomial first integrals for a certain vector field $\tilde{Y}$ on $\mathbb{R}[y_1, \ldots, y_i]$. For this we will prove that any cofactor of Y is of the form $2ky_2$ for some non-negative integer k.

Let us define

$$V = \begin{vmatrix} y_1 & y_2 & y_4 \\ y_2 & y_3 & y_5 \\ y_4 & y_5 & y_6 \end{vmatrix}.$$

Proposition 4.1. *The polynomials y_6, y_7 and V are first integrals of Y. The polynomials y_1 and z are Darboux polynomials for Y with the cofactors $4y_2$ and $6y_2$, respectively.*

Proof: It is obvious that $Yy_6 = Yy_7 = 0$ and $Yy_1 = 4y_2y_1$. Now

$$Y\begin{vmatrix} y_1 & y_2 & y_4 \\ y_2 & y_3 & y_5 \\ y_4 & y_5 & y_6 \end{vmatrix} = \begin{vmatrix} 4y_1y_2 & y_2 & y_4 \\ 2y_1y_3 + 2y_1y_4 + 3y_1z & y_3 & y_5 \\ 2y_1y_5 & y_5 & y_6 \end{vmatrix}$$

$$+ \begin{vmatrix} y_1 & 2y_1y_3 + 2y_1y_4 + 3y_1z & y_4 \\ y_2 & 4y_1y_5 + 6y_2z & y_5 \\ y_4 & 2y_1y_6 + 3y_4z & y_6 \end{vmatrix} + \begin{vmatrix} y_1 & y_2 & 2y_1y_5 \\ y_2 & y_3 & 2y_1y_6 + 3y_4z \\ y_4 & y_5 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 2y_1y_2 & y_2 & y_4 \\ 2y_1y_4 + 3y_1z & y_3 & y_5 \\ 0 & y_5 & y_6 \end{vmatrix} + \begin{vmatrix} y_1 & 2y_1y_3 & y_4 \\ y_2 & 2y_1y_5 + 3y_2z & y_5 \\ y_4 & 0 & y_6 \end{vmatrix}$$

$$+ \begin{vmatrix} y_1 & y_2 & 2y_1y_5 \\ y_2 & y_3 & 2y_1y_6 + 3y_4z \\ y_4 & y_5 & 0 \end{vmatrix}$$

$$= 2y_1\left(\begin{vmatrix} y_2 & y_2 & y_4 \\ y_4 & y_3 & y_5 \\ 0 & y_5 & y_6 \end{vmatrix} + \begin{vmatrix} y_1 & y_3 & y_4 \\ y_2 & y_5 & y_5 \\ y_4 & 0 & y_6 \end{vmatrix} + \begin{vmatrix} y_1 & y_2 & y_5 \\ y_2 & y_3 & y_6 \\ y_4 & y_5 & 0 \end{vmatrix}\right)$$

$$+ 3z\left(\begin{vmatrix} 0 & y_2 & y_4 \\ y_1 & y_3 & y_5 \\ 0 & y_5 & y_6 \end{vmatrix} + \begin{vmatrix} y_1 & 0 & y_4 \\ y_2 & y_2 & y_5 \\ y_4 & 0 & y_6 \end{vmatrix} + \begin{vmatrix} y_1 & y_2 & 0 \\ y_2 & y_3 & y_4 \\ y_4 & y_5 & 0 \end{vmatrix}\right)$$

$$= 2y_1\left(y_2\begin{vmatrix} y_3 & y_5 \\ y_5 & y_6 \end{vmatrix} - y_4\begin{vmatrix} y_2 & y_4 \\ y_5 & y_6 \end{vmatrix} - y_3\begin{vmatrix} y_2 & y_5 \\ y_4 & y_6 \end{vmatrix} + y_5\begin{vmatrix} y_1 & y_4 \\ y_4 & y_6 \end{vmatrix}\right.$$

$$\left. + y_5\begin{vmatrix} y_2 & y_3 \\ y_4 & y_5 \end{vmatrix} - y_6\begin{vmatrix} y_1 & y_2 \\ y_4 & y_5 \end{vmatrix}\right)$$

$$+ 3z\left(-y_1\begin{vmatrix} y_2 & y_4 \\ y_5 & y_6 \end{vmatrix} + y_2\begin{vmatrix} y_1 & y_4 \\ y_4 & y_6 \end{vmatrix} - y_4\begin{vmatrix} y_1 & y_2 \\ y_4 & y_5 \end{vmatrix}\right)$$

$$= 2y_1\left(\begin{vmatrix} y_2 & y_2 & y_4 \\ y_3 & y_3 & y_5 \\ y_5 & y_5 & y_6 \end{vmatrix} - \begin{vmatrix} y_4 & y_1 & y_4 \\ y_5 & y_2 & y_5 \\ y_6 & y_4 & y_6 \end{vmatrix}\right) - 3z\begin{vmatrix} y_1 & y_1 & y_4 \\ y_2 & y_2 & y_5 \\ y_4 & y_4 & y_6 \end{vmatrix} = 0.$$

Further

$$Yz = Y(y_3 - 2y_4) = 4y_1y_5 + 6y_2z - 2 \cdot 2y_1y_5 = 6y_2z. \qquad \blacksquare$$

To study further properties of Y it is convenient to relate it to an infinitesimal action of sl_3 on $\mathbb{R}^7$. For that, define the the vector fields e_α, e_β, $e_{\alpha+\beta}$, f_α, f_β, $f_{\alpha+\beta}$ by

$$\begin{aligned}
e_\alpha &= 2y_2\frac{\partial}{\partial y_1} + y_3\frac{\partial}{\partial y_2} + y_5\frac{\partial}{\partial y_4}, & f_\alpha &= y_1\frac{\partial}{\partial y_2} + 2y_2\frac{\partial}{\partial y_3} + y_4\frac{\partial}{\partial y_5},\\
e_{\alpha+\beta} &= y_2\frac{\partial}{\partial y_4} + y_3\frac{\partial}{\partial y_5} + 2y_5\frac{\partial}{\partial y_6}, & f_{\alpha+\beta} &= y_4\frac{\partial}{\partial y_2} + 2y_5\frac{\partial}{\partial y_3} + y_6\frac{\partial}{\partial y_5},\\
e_\beta &= y_1\frac{\partial}{\partial y_4} + y_2\frac{\partial}{\partial y_5} + 2y_4\frac{\partial}{\partial y_6}, & f_\beta &= 2y_4\frac{\partial}{\partial y_1} + y_5\frac{\partial}{\partial y_2} + y_6\frac{\partial}{\partial y_4},
\end{aligned}$$

and the vector fields h_α, h_β by

$$\begin{aligned}
h_\alpha &= [e_\alpha, f_\alpha] = -2y_1\frac{\partial}{\partial y_1} + 2y_3\frac{\partial}{\partial y_3} - y_4\frac{\partial}{\partial y_4} + y_5\frac{\partial}{\partial y_5},\\
h_\beta &= [e_\alpha, f_\alpha] = 2y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2} - y_5\frac{\partial}{\partial y_5} - 2y_6\frac{\partial}{\partial y_6}.
\end{aligned}$$

By direct computation one gets the following.

Proposition 4.2. *An $\mathbb{R}$-linear span L of the vector fields e_α, $e_{\alpha+\beta}$, e_β, f_α, $f_{\alpha+\beta}$, f_β, h_α, h_β is the Lie algebra sl_3 with respect to the commutator bracket.*

Now we can write Y in the form

$$\begin{aligned}
Y &= 2y_1e_\alpha + 2y_1f_{\alpha+\beta} + 3zf_\alpha\\
&= 2y_1(e_\alpha + f_{\alpha+\beta}) + 3(y_7 - 2y_4 + y_3)f_\alpha\\
&= (2y_1e_\alpha + 3y_3f_\alpha) + (2y_1f_{\alpha+\beta} - 6y_4f_\alpha) + 3y_7f_\alpha.
\end{aligned}$$

Let us define

$$Z = (2y_1e_\alpha + 3y_3f_\alpha) + (2y_1f_{\alpha+\beta} - 6y_4f_\alpha).$$

The following proposition relates Darboux polynomials for Z with Darboux polynomials for Y.

Proposition 4.3. *Suppose F is an indecomposable Darboux polynomial for Y with a cofactor p. Let us write F in the form*

$$F = F_0 + \cdots + F_dy_7^d, \tag{10}$$

where F_i are polynomials in variables different from y_7 and $F_d \neq 0$, and p in the form

$$p = p_1 y_1 + \cdots + p_7 y_7,$$

where $p_1, \ldots, p_7$ are real numbers. Then

(1) *F_d is a first integral for f_α;*
(2) *$p_7 = 0$;*
(3) *F_0 is a non-zero Darboux polynomial for Z with the cofactor p, where*

$$Z = (2y_1 e_\alpha + 3y_3 f_\alpha) + (2y_1 f_{\alpha+\beta} - 6y_4 f_\alpha).$$

Proof: Comparing coefficients of y_7^{d+1} in the equation

$$YF = pF,$$

we get that $3f_\alpha F_d = p_7 F_d$. Therefore $f_\alpha F_d = \frac{p_7}{3} F_d$. This shows that F_d is an eigenvector for the linear operator f_α, acting on the vector space of homogeneous polynomials of degree d in the variables $y_1, \ldots, y_6$. It is easy to check that f_α is a nilpotent operator, therefore all eigenvalues of f_α are zero. This shows that $p_7 = 0$ and F_d is a first integral for f_α.

Now, suppose $F_0 = 0$. Then F is divisible by y_7, which contradicts our assumption that F is indecomposable. Thus $F_0 \neq 0$. Comparing the terms free of y_7 in $YF = pF$ and using $p_7 = 0$, we get that $ZF_0 = pF_0$. Thus F_0 is a Darboux polynomial for Z with the cofactor p. ■

Note that Z does not involve y_7 and therefore can be considered as a polynomial vector field on $\mathbb{R}^6$. Now we define several different gradings on $\mathbb{R}[y_1, \ldots, y_6]$. We start with the ring homomorphism

$$\phi\colon \mathbb{R}[y_1, \ldots, y_6] \mapsto \mathbb{R}[s_1, s_2, s_3]$$

defined by

$$\begin{aligned} \phi(y_1) &= s_2^2, & \phi(y_2) &= s_1 s_2, & \phi(y_3) &= s_1^2, \\ \phi(y_4) &= s_2 s_3, & \phi(y_5) &= s_1 s_3, & \phi(y_6) &= s_3^2. \end{aligned}$$

For every triple $(k_1, k_2, k_3) \in \mathbb{N}^3$, we define $V(k_1, k_2, k_3)$ to be the ϕ-preimage of the set

$$\left\{ \lambda s_1^{k_1} s_2^{k_2} s_3^{k_3} \,\middle|\, \lambda \in \mathbb{R} \right\}.$$

It is obvious that $\mathbb{R}[y_1, \ldots, y_6] = \bigoplus_{(k_1,k_2,k_3)\in\mathbb{N}^3} V(k_1, k_2, k_3)$. Moreover

$$V(k_1, k_2, k_3)V(l_1, l_2, l_3) \subset V(k_1 + l_1, k_2 + l_2, k_3 + l_3),$$

for any (k_1, k_2, k_3), $(l_1, l_2, l_3) \in \mathbb{N}$.

Proposition 4.4. *For every* (k_1, k_2, k_3)*, we have*

$$\begin{aligned} e_\alpha(V(k_1, k_2, k_3)) &\subset V(k_1 + 1, k_2 - 1, k_3), \\ f_\alpha(V(k_1, k_2, k_3)) &\subset V(k_1 - 1, k_2 + 1, k_3), \\ f_{\alpha+\beta}(V(k_1, k_2, k_3)) &\subset V(k_1 - 1, k_2, k_3 + 1). \end{aligned}$$

Proof: Let $F \in V(k_1, k_2, k_3)$. Then, it follows from the definition of the subspaces $V(k_1, k_2, k_3)$ that

$$\begin{aligned} &\frac{\partial F}{\partial y_1} \in V(k_1, k_2 - 2, k_3), && \frac{\partial F}{\partial y_2} \in V(k_1 - 1, k_2 - 1, k_3), \\ &\frac{\partial F}{\partial y_3} \in V(k_1 - 2, k_2, k_3), && \frac{\partial F}{\partial y_4} \in V(k_1, k_2 - 1, k_3 - 1), \\ &\frac{\partial F}{\partial y_5} \in V(k_1 - 1, k_2, k_3 - 1), && \frac{\partial F}{\partial y_6} \in V(k_1, k_2, k_3 - 2). \end{aligned}$$

Using this and the formulae for e_α, f_α and $f_{\alpha+\beta}$, the result follows. ■

Given any triple of integers $i = (i_1, i_2, i_3)$, we define the grading V_k^i on $\mathbb{R}[y_1, \ldots, y_6]$ by

$$V_k^i = \bigoplus_{i_1k_1+i_2k_2+i_3k_3=k} V(k_1, k_2, k_3).$$

Then, from Proposition 4.4, we get that e_α, f_α and $f_{\alpha+\beta}$ are homogeneous with respect to any grading V_k^i. Let us define

$$Z_1 = 2y_1e_\alpha + 3y_3f_\alpha, \qquad Z_2 = 2y_1f_{\alpha+\beta} - 6y_4f_\alpha.$$

Thus $Z = Z_1 + Z_2$. From Proposition 4.4 we get

$$\begin{aligned} Z_1(V(k_1, k_2, k_3)) &\subset V(k_1 + 1, k_2 + 1, k_3), \\ Z_2(V(k_1, k_2, k_3)) &\subset V(k_1 - 1, k_2 + 2, k_3 + 1). \end{aligned}$$

Thus, one gets that also the vector fields Z_1 and Z_2 are homogeneous with respect to any grading V_k^i, $k \in \mathbb{Z}$ on $\mathbb{R}[y_1, \ldots, y_6]$.

Proposition 4.5. *The vector field Z is homogeneous of degree one for the gradings* $V_k^{1,1,1}$*,* $k \in \mathbb{Z}$*, and* $V_k^{1,2,0}$*,* $k \in \mathbb{Z}$*, of degrees* 2 *and* 3*, respectively.*

Proof: One checks both facts for Z_1 and Z_2, and the result follows from the relation $Z = Z_1 + Z_2$. ∎

Corollary 4.6. *Suppose F is a Darboux polynomial for Z and p is its cofactor. Then $p = cy_2$ for some $c \in \mathbb{R}$.*

Proof: By Theorem 3.6, we can assume that F is homogeneous with respect to the gradings $V_k^{1,1,1}$, $k \in \mathbb{Z}$, and $V_k^{1,2,0}$, $k \in \mathbb{Z}$. Then $ZF = pF$ implies that p is homogeneous of degree 2 with respect to the grading $V_k^{1,1,1}$ and of degree 3 with respect to the grading $V_k^{1,2,0}$. But $V_2^{1,1,1}$ is generated by $y_1, \dots, y_6$ as a vector space. Moreover, every y_i is homogeneous with respect to the grading $V_k^{1,2,0}$ and only $y_2 \in V(1,1,0)$ has degree 3. Thus $V_2^{1,1,1} \cap V_3^{1,2,0}$ is generated by y_2 as a vector space. This shows that $p = cy_2$ for some $c \in \mathbb{R}$. ∎

To show that the constant c in the above Corollary is necessarily a positive even integer, we have to perform further analysis on the properties of the vector fields Z_1 and Z_2. For every polynomial F on $\mathbb{R}^6$, we write F_{k_1,k_2,k_3} for its component with respect to the direct sum decomposition $\mathbb{R}[y_1, \dots, y_6] = \bigoplus_{(k_1,k_2,k_3)\in\mathbb{N}^3} V(k_1,k_2,k_3)$. Define $\operatorname{supp}(F)$ as the set of those integers (k_1,k_2,k_3) such that $F_{k_1,k_2,k_3} \neq 0$. Note that, if $F \in V_d^{1,1,1}$ and $F \in V_k^{1,2,0}$ then

$$\operatorname{supp}(F) = \{\, (k_1,k_2,k_3) \in \Lambda(3) \,|\, k_1 + k_2 + k_3 = d,\ k_1 + 2k_3 = k \}\,.$$

Proposition 4.7. *Let $F \in V_d^{1,1,1} \cap V_k^{1,2,0}$ be a Darboux polynomial for Z with a cofactor cy_2. Let (k_1,k_2,k_3) be the element of $\operatorname{supp}(F)$ with the minimal possible first coordinate and (l_1,l_2,l_3) the element of $\operatorname{supp}(F)$ with the maximal possible first coordinate. Then F_{k_1,k_2,k_3} is a first integral for Z_2, and F_{l_1,l_2,l_3} is a Darboux polynomial for Z_1 with the cofactor cy_2.*

Proof: From the definitions of (l_1,l_2,l_3) and (k_1,k_2,k_3), we get

$$\begin{aligned} (ZF)_{l_1+1,l_2+1,l_3} &= Z_1 F_{l_1,l_2,l_3}, \\ (y_2F)_{l_1+1,l_2+1,l_3} &= y_2 F_{l_1,l_2,l_3}, \\ (ZF)_{k_1-1,k_2+2,k_3+1} &= Z_2 F_{k_1,k_2,k_3}. \end{aligned}$$

Therefore, the equality $ZF = cy_2F$ implies that

$$\begin{aligned} Z_1 F_{l_1,l_2,l_3} &= cy_2 F_{l_1,l_2,l_3}, \\ Z_2 F_{k_1,k_2,k_3} &= 0. \end{aligned}$$

∎

A consequence of Proposition 4.7 is that any cofactor of Z appears among cofactors of Z_1. Let us write Z_1 explicitly as

$$Z_1 = 4y_1y_2\frac{\partial}{\partial y_1} + 5y_1y_3\frac{\partial}{\partial y_2} + 6y_2y_3\frac{\partial}{\partial y_3} + 2y_1y_5\frac{\partial}{\partial y_4} + 3y_3y_4\frac{\partial}{\partial y_5}.$$

Proposition 4.8. *Let F be an irreducible Darboux polynomial for Z_1 with a cofactor*

cy_2. Then, either F is a scalar multiple of y_1 or $F|_{y_1=0}$ is a non-zero Darboux polynomial for $Z_1' := 3y_3\left(2y_2\frac{\partial}{\partial y_3} + y_4\frac{\partial}{\partial y_5}\right)$ with the cofactor cy_2.

Proof: Let us write F in the form

$$F = F_0 + F_1y_1 + \cdots + F_dy_1^d,$$

where $F_i \in \mathbb{R}[y_2, \ldots, y_6]$ and $F_d \neq 0$. Then $F|_{y_1=0} = F_0$. If $F_0 = 0$ then F is divisible by y_1. Since F is irreducible, we get that F is a scalar multiple of y_1. Now assume that $F_0 \neq 0$. Then

$$(Z_1F)|_{y_1=0} = (Z_1F_0)|_{y_1=0} = 6y_2y_3\frac{\partial F_0}{\partial y_3} + 3y_3y_4\frac{\partial F_0}{\partial y_5} = Z_1'F_0,$$

and $(cy_2F)_{y_1=0} = cy_2F_0$. Thus, we get that $Z_1'F_0 = cy_2F_0$. ■

We describe all Darboux polynomials for the vector field Z_1' on $\mathbb{R}[y_2, \ldots, y_6]$ in the next proposition. Define $I = 2y_2y_5 - y_3y_4$. The direct computation shows that I is a first integral for Z_1'.

Proposition 4.9. *Every Darboux polynomial for Z_1' is of the form Fy_3^d with $F \in \mathbb{R}[y_2, y_4, y_6, I]$. In particular, any cofactor of Z_1' is of the form $6dy_2$ with d a non-negative integer.*

Proof: If $F \in \mathbb{R}[y_2, y_4, I]$, then F is a first integral for Z_1' since y_2, y_4, y_6, and I are first integrals for I. Thus, by Propostion 3.2, Fy_3^d is a Darboux polynomial for Z_1'.

Now, suppose that F is a Darboux polynomial for Z_1' with a cofactor p. We consider F as an element of the ring $\mathcal{R} = \mathbb{R}[y_2^{\pm 1}, y_3, y_4, y_5]$. Using the relation $y_5 = \frac{1}{2y_2}(I_2 + y_3y_4)$, every element F in $\mathcal{R}$ can be written in the form $F_0(y_2, y_4, y_6, I_2) + F_1(y_2, y_4, y_6, I_2)y_3 + \cdots + F_d(y_2, y_4, y_6, I_2)y_3^d$, where F_j are elements of $\mathbb{R}[y_2^{\pm 1}, y_4, y_6, I_2]$ and $F_d \neq 0$. Since $Z_1'F_j = 0$ for all j and $Z_1'y_3 = 6y_2y_3$, we get

$$Z_1'F = 6y_2F_1y_3 + 2 \cdot 6y_2F_2y_3^2 + \cdots + d \cdot 6y_2F_dy_3^d.$$

Suppose $p = k_2y_2 + k_3y_3 + k_4y_4 + k_5y_5 + k_6y_6$ is the cofactor of F. Then, comparing the coefficients of y_3^{d+1} in $Z_1'F$ and pF, we see that $k_3 = 0$. Further, comparing the coefficients of y_3^d in $Z_1'F$ and pF, we get $p = 6dy_2$. Comparing the coefficients y_3^j with $j \leq d_1$, we see that $F_j = 0$ for $j \leq d-1$. Therefore $F = F_dy_3^d$, with $F_d \in \mathcal{R}$. Since F is a polynomial and y_3 is not invertible in $\mathcal{R}$, we get that also F_d is a polynomial and this proves the proposition. ■

Now we state the main theorem.

Theorem 4.10. *Suppose F is Darboux polynomial for the vector field Y with a cofactor p. Then $p = 2ky_2$ for some non-negative integer k.*

Proof: By Proposition 3.4 we can assume that F is irreducible. Then by Proposition 4.3 there is a non-zero Darboux polynomial F_0 for Z with the cofactor p. Let $F_0 = \prod_{s=1}^m G_s^{k_s}$ be a prime decomposition of F_0. Then, by Corollary 4.6, every cofactor p_s of G_s is of the form c_sy_2 for some $c \in \mathbb{R}$. By Proposition 3.2, we get $p = (k_1c_1 + \cdots + k_mc_m)y_2$. Thus, it is enough to show that every c_s is a non-negative even integer. By Proposition 4.8, we get that G_s is either a scalar multiple of y_1 or a non-zero Darboux polynomial of Z_1'. In the first case $c_s = 4$. In the second case, $c_s = 6d$ for some non-negative integer d by Proposition 4.9. ■

Let us define the vector field $\widetilde{Y}$ on $\mathbb{R}[y_1, \dots, y_8]$ by

$$\widetilde{Y} = Y - 2y_2y_8\frac{\partial}{\partial y_8}.$$

Note that y_8 is a Darboux polynomial for $\widetilde{Y}$ with the cofactor $-2y_2$. Moreover, if F is a Darboux polynomial for Y with the cofactor $p = 2dy_2$, $d \in \mathbb{N}$, then F is also a Darboux polynomial for $\widetilde{Y}$ with the same cofactor. Therefore, by Proposition 3.2, we get that Fy_8^d is a polynomial first integral of $\widetilde{Y}$.

Corollary 4.11. *To classify Darboux polynomials of Y it is enough to classify polynomial first integrals for $\widetilde{Y}$.*

5. Conclusion

We showed that to find solutions of (1), it is enough to find solutions of the system (8). Note that (8) is five dimensional and does not depend on n. Every solution of (8) is an integral curve of the quadratic vector field Y. We studied the existence of rational integrals for Y. Several first integral of Y were identified in Proposition 4.1. We reduced the problem of finding rational integrals for Y to the problem of finding first integrals for the quadratic vector field $\widetilde{Y}$. We plan to solve this problem using computer algebra systems.

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