Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 14–37

EFFECTIVE ÉTALE-DESCENT MORPHISMS IN M-ORDERED SETS

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Abstract: Following the results obtained in **Preord** and in **Cat**, we characterize the effective étale-descent morphisms in M-**Ord**, the category of M-ordered sets for a given monoid M. Furthermore we show that in M-**Ord** every effective descent morphism is effective for étale-descent (while the converse is false), and we generalize it to a more general context of relational algebras.

1. Introduction

In [10] G. Janelidze and M. Sobral gave a complete characterization of the morphisms in the category **Preord** of preordered sets which are effective for descent with respect to the class of étale morphisms, i.e., discrete fibrations. In [14] M. Sobral characterized the effective descent morphisms in the category **Cat** of small categories with respect to the class of discrete fibrations. These two works suggested the study of descent theory for the class of étale morphisms in M-**Ord**, the category of M-ordered sets for a given monoid M.

In this paper we give a complete characterization of the effective étale-descent morphisms in M-**Ord**. This characterization allows us to state that an effective descent morphism in M-**Ord** is effective for étale-descent. The converse is false and we give an appropriate counter-example in the surjective case. Being M-**Ord** presented as a category of relational algebras, we conclude investigating the relation between effective descent and effective étale-descent morphisms in **RelAlg**($\mathbb{T}$), the category of relational algebras for a suitable monad $\mathbb{T}$.

Received October 29, 2014.

Research supported by the Centro de Matemática da Universidade de Coimbra (CMUC), funded by the European Regional Development Fund through the program COMPETE and by the Portuguese Government through the FCT-Fundação para a Ciência e a Tecnologia under the project PEst-C/MAT/UI0324/2013 and grant number SFRH/BD/85837/2012. This work has been realized during my PhD program. I wish to thank my supervisor Maria Manuel Clementino for her precious help and support.

2. Basic notions and results

Given a monoid $(M, \circ, e_M)$, consider the monad

$$\mathbb{M} = (M \times (-), \mu, \eta)$$

on **Set**, with $\mu_X : M \times M \times X \to M \times X$ defined by $(m, n, x) \mapsto (m \circ n, x)$ and $\eta_X : X \to M \times X$ by $x \mapsto (e_M, x)$, for each set X. The Barr extension [1] $\overline{\mathbb{M}} : \mathbf{Rel} \to \mathbf{Rel}$ is an extension of the monad $\mathbb{M}$, and it is given by

$$(m, x)(\overline{\mathbb{M}}r)(n, y) \iff m = n \quad \text{and} \quad x(r)y,$$

where $r : X \to Y$ is a relation, $x \in X$, $y \in Y$, and $m, n \in M$. The category $\mathbf{Alg}(\overline{\mathbb{M}}, 2)$ of reflexive and transitive $(\overline{\mathbb{M}}, 2)$-algebras is the category M-**Ord** of *M-ordered sets* and *equivariant maps*. For a relation $a : M \times X \to X$ one writes $x \xrightarrow{m} y$ instead of $(m, x)(a)y$, that is x is related to y with weight m. As remarked in [2, Section V.1.4], this arrow notation for the structure of an $(\overline{\mathbb{M}}, 2)$-category (X, a) emphasizes that X is actually the object set of a small category, denoted again by X, with hom-sets

$$X(x, y) = \{(x, m, y) \mid m \in M \quad \text{and} \quad x \xrightarrow{m} y\}$$

for $x, y \in X$; moreover this small category comes equipped with a faithful functor

$$p : X \to M, \quad (x, m, y) \mapsto m,$$

with M considered as a one-object category. In this direction identity morphisms and composition in an M-ordered set X are given by

$$x \xrightarrow{e_M} x \quad \text{and} \quad (x \xrightarrow{m} y \quad \& \quad y \xrightarrow{n} z \Longrightarrow x \xrightarrow{n \circ m} z),$$

while a morphism $f : X \to Y$ must satisfy

$$x \xrightarrow{m} y \Longrightarrow f(x) \xrightarrow{m} f(y)$$

for all $x, y \in X$ and $m \in M$. Defining a norm to be a functor from the small category X to the monoid M, [2, Proposition 1.4.2] actually reveals that M-ordered sets can be identified as those small categories over M whose norm is faithful.

Now let $\mathbb{E}$ be the class of étale morphisms in M-**Ord**. As introduced in [5], the étale morphisms in the context of relational algebras are defined as the pullback stable discrete fibrations. For a cartesian monad, as $\mathbb{M}$ is, discrete fibrations are pullback stable, hence étale morphisms and discrete fibrations

References

[1] M. Barr, Relational Algebras, in: Reports of the Midwest Category Seminar, IV, pp 39-55, *Lecture Notes in Mathematics* **137**, Springer, Berlin. (1970).

[2] M.M. Clementino, E. Colebunders and W. Tholen, Lax algebras as spaces, in: Monoidal Topology, pp 375-466, *Encyclopedia Math. Appl.* **153**, *Cambridge Univ. Press, Cambridge* (2014).

[3] M.M. Clementino and D. Hofmann, Descent morphisms and a van Kampen Theorem in categories of lax algebras, *Topology Appl.* **159** (2012), 2310-2319.

[4] M.M. Clementino and D. Hofmann, Topological features of lax algebras, *Appl. Categ. Structures* **11** (2003), 267-286.

[5] M.M. Clementino, D. Hofmann and G. Janelidze, On exponentiability of étale algebraic homomorphisms, *J. Pure Appl. Algebra* **217** (2013), 1195-1207.

[6] M.M. Clementino, D. Hofmann and A. Montoli, Covering Morphisms in Categories of Relational Algebras, *Appl. Categ. Structures*, to appear; DOI: 10.1007/s10485-013-9349-0.

[7] M.M. Clementino and W. Tholen, Metric, topology and multicategory - a common approach, *J. Pure Appl. Algebra* **179** (2003), 13-47.

[8] P.J. Higgins, Categories and Groupoids, *Reprints in Theory and Appl. Categ.* **7** (1971).

[9] G. Janelidze and M. Sobral, Finite preorders and topological descent I, *J. Pure Appl. Algebra* **175** (2002), 187-205.

[10] G. Janelidze and M. Sobral, Finite preorders and topological descent II: étale descent, *J. Pure Appl. Algebra* **174** (2002), 303-309.

[11] G. Janelidze and W. Tholen, Facets of Descent I, *Appl. Categ. Structures* **2** (1994), 1-37.

[12] G. Janelidze and W. Tholen, Facets of Descent II, *Appl. Categ. Structures* **5** (1997), 229-248.

[13] S. Mac Lane and R. Paré, Coherence for bicategories and indexed categories, *J. Pure Appl. Algebra* **37** (1985), 59-80.

[14] M. Sobral, Descent for Discrete (Co)fibrations, *Appl. Categ. Structures* **12** (2004), 527-535.

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coincide. Using the arrow notation, an equivariant map $f : X \to Y$ in M-**Ord** is then an étale morphism if and only if:

$$\forall x_0 \in X, \quad \forall y_1 \in Y, \quad \forall m \in M : y_1 \xrightarrow{m} f(x_0) \Longrightarrow \exists! x_1 \in f^{-1}(y_1) : x_1 \xrightarrow{m} x_0.$$

The problem concerning the characterization of effective étale-descent morphisms in M-**Ord** can be stated as follows: given an equivariant map $p : E \to B$ of M-ordered sets, denote by $\mathbb{E}(B)$ the slice category of étale morphisms over B and by $p^* : \mathbb{E}(B) \to \mathbb{E}(E)$ the pullback functor along p. We have then a commutative (up to isomorphism) diagram

$$\begin{array}{ccc} \mathbb{E}(B) & \xrightarrow{K^p} & \mathrm{Des}_{\mathbb{E}}(p) \\ & {\scriptstyle p^*}\searrow \quad \swarrow{\scriptstyle U^p} & \\ & \mathbb{E}(E) & \end{array}$$

where $\mathrm{Des}_{\mathbb{E}}(p)$ is the category of $\mathbb{E}$-descent data for p, and U^p and K^p are the forgetful and the comparison functor, respectively. By definition, the equivariant map $p : E \to B$ is an *(effective) étale-descent morphism* if the comparison functor K^p is full and faithful (an equivalence of categories). For a more accurate understanding of descent theory we refer the reader to the papers [11] and [12]. Following the results given in [9] and [10] for **Preord** and in [14] for **Cat**, we are going to give a complete characterization of effective étale-descent morphisms in M-**Ord**.

3. The equivalence of (indexed) categories

The first step to reach such a characterization is to generalize to M-**Ord** the standard equivalence of categories $\mathbb{E}(B) \simeq \mathbf{Set}^{B^{\mathrm{op}}}$, given in [8, Chapter 13, Proposition 30], for $\mathbb{E}$ the class of discrete fibrations in **Graph**, **Cat** or **Grpd**.

Proposition 3.1. *Let $\mathbb{E}$ be the class of étale morphisms in M-**Ord**, and let B be an M-ordered set. Then the categories $\mathbb{E}(B)$ and $\mathbf{Set}^{B^{\mathrm{op}}}$ are equivalent.*

Proof. We start by defining a functor $F : \mathbb{E}(B) \to \mathbf{Set}^{B^{\mathrm{op}}}$. Given an étale morphism $f : A \to B$, the functor $F(f) : B^{\mathrm{op}} \to \mathbf{Set}$ is described as follows: each element $b \in B$ is mapped to

$$F(f)(b) = f^{-1}(b) = \{a \in A \mid f(a) = b\}$$

and, given an element $a \in F(f)(b)$, if $b' \xrightarrow{m} b$ in B then, by definition of étale morphism, there exists a unique $a' \in f^{-1}(b')$ such that $a' \xrightarrow{m} a$. Hence the assignment $a \mapsto a'$ defines the function from the set $F(f)(b)$ to the set $F(f)(b')$. Remark that we may have $F(f)(b) = \emptyset$ for some $b \in B$, in which case every $b' \xrightarrow{m} b$ is represented by the empty function $\emptyset \to F(f)(b')$. If $g : C \to B$ is another étale morphism over A with corresponding functor $F(g)$, and if $h : A \to C$ is an equivariant map such that $g \circ h = f$, then h induces functions $h_b : F(f)(b) \to F(g)(b)$; the family $\{h_b\}_{b\in B}$ defines actually a natural transformation $F(f) \Rightarrow F(g)$.
Conversely, we define now a functor $G : \mathbf{Set}^{B^{\mathrm{op}}} \to \mathbb{E}(B)$. Given $\alpha : B^{\mathrm{op}} \to \mathbf{Set}$, define an étale morphism over B in the following way: the objects of the domain $\widetilde{B}_\alpha$ are the pairs (b, v), where $b \in B$ and v in an element of the set $\alpha(b)$, while there is a morphism from (b, v) to (b', v'), say $(m, \alpha(m))$, if $b \xrightarrow{m} b'$ in B for $m \in M$ and $\alpha(m) : \alpha(b') \to \alpha(b)$ maps $v' \mapsto v$. The domain $\widetilde{B}_\alpha$ comes equipped with an M-valued norm, i.e., a functor $\widetilde{B}_\alpha \to M$, defined via

$$(b, v) \xrightarrow{(m,\alpha(m))} (b', v') \mapsto (\bullet \xrightarrow{m} \bullet).$$

This functor is faithful making $\widetilde{B}_\alpha$ an M-ordered set. Then the morphism $G(\alpha) : \widetilde{B}_\alpha \to B$ defined via

$$(b, v) \xrightarrow{(m,\alpha(m))} (b', v') \mapsto (b \xrightarrow{m} b')$$

gives rise to an étale morphism. If $\alpha' : B^{\mathrm{op}} \to \mathbf{Set}$ is another functor and $\beta : \alpha \Rightarrow \alpha'$ is a natural transformation between them, given by functions $\{\beta_b : \alpha(b) \to \alpha'(b)\}_{b\in B}$, we get an equivariant map $G(\beta) : \widetilde{B}_\alpha \to \widetilde{B}_{\alpha'}$ by mapping

$$(b, v) \xrightarrow{(m,\alpha(m))} (b', v') \mapsto (b, \beta_b(v)) \xrightarrow{(m,\alpha'(m))} (b', \beta_{b'}(v')).$$

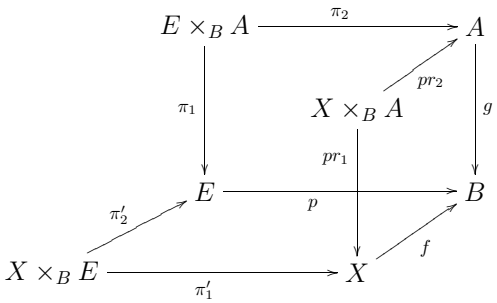
This $G(\beta)$ defines a morphism in $\mathbb{E}(B)$, giving then the claimed functor $G : \mathbf{Set}^{B^{\mathrm{op}}} \to \mathbb{E}(B)$.
It is straightforward to verify that $F \circ G$ is naturally isomorphic to $1_{\mathbf{Set}^{B^{\mathrm{op}}}}$ and that $G \circ F$ is naturally isomorphic to $1_{\mathbb{E}(B)}$. □

In [12] the general framework of descent theory given in Section 2 is generalized to a context of $\mathbf{C}$-indexed categories, i.e., pseudo-functors $\mathbb{A} : \mathbf{C}^{\mathrm{op}} \to$

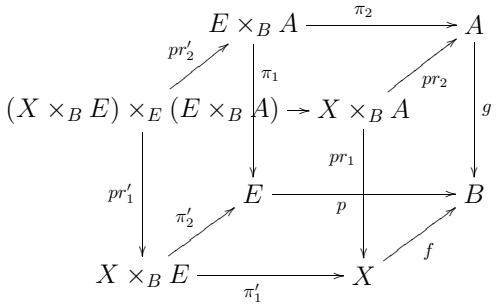
is defined by

$$\omega \to (e, a) \Longleftrightarrow T\pi_1(\omega) \to e \quad \text{and} \quad T\pi_2(\omega) \to a,$$

for any $\omega \in T(E \times_B A)$ and $(e, a) \in E \times_B A$. We want to prove that g is an étale morphism as well.
By Theorem 6.1 and Theorem 6.2(a), g is a discrete fibration. To prove that every pullback of g is a discrete fibration we consider the following diagram



where the three faces are pullbacks. We want to prove that pr_1 is a discrete fibration. First of all observe that since effective descent morphisms are pullback stable π'_1 is an effective descent morphism. Building the pullback on the left-side, i.e., the pullback of π_1 along π'_2, by universality we get a cube such that all faces are pullbacks.



Now, since π_1 is an étale morphism, pr'_1 is a discrete fibration and, using the same argument that we used to prove that g is a discrete fibration, we conclude that pr_1 is a discrete fibration as well. □

to relations with finite fibres is a (BC)-square. Then the following conditions are equivalent, for a morphism $f : (X, a) \to (Y, b)$ in **RelAlg**$(\mathbb{T})$*:*

(i) *f is final;*
(ii) *f is a pullback stable regular epimorphism in* **RelAlg**$(\mathbb{T})$*;*
(iii) *f is a descent morphism in* **RelAlg**$(\mathbb{T})$*.*

Theorem 6.2. [6, Proposition 5.2] *Let $\mathbb{T} = (T, \mu, \eta)$ be a monad on* **Set** *and $\overline{\mathbb{T}}$ its Barr extension. Given a pullback diagram in* **RelAlg**$(\mathbb{T})$

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{\pi_2} & Z \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y, \end{array}$$

with f a final morphism:

(a) *If π_1 is a discrete (co)fibration, then g is a discrete (co)fibration.*
(b) *If π_1 has closed image, then g has closed image.*
(c) *If π_1 has separated fibres, then g has separated fibres.*

Theorem 6.3. *Let $\mathbb{T} = (T, \mu, \eta)$ be a monad on* **Set** *and $\overline{\mathbb{T}}$ its Barr extension. Let* **RelAlg**$(\mathbb{T})$ *the category of relational $\mathbb{T}$-algebras (or lax algebras, or $(\mathbb{T}, \mathbf{2})$-categories, see* [4] *and* [7]*). If $\mathbb{T}$ satisfies the following conditions:*

(1) *$\mathbb{T}$ has the (BC) property,*
(2) *η has (BC) for relations with finite fibres,*

an effective descent morphism $p : E \to B$ in **RelAlg**$(\mathbb{T})$ *is effective for étale-descent.*

Proof. Let $p : E \to B$ be an effective descent morphism in **RelAlg**$(\mathbb{T})$. We use [11, Proposition 2.6] to show that p is also effective with respect to the class of étale morphisms, i.e., pullback stable discrete fibrations. Consider then a pullback diagram in **RelAlg**$(\mathbb{T})$

$$\begin{array}{ccc} E \times_B A & \xrightarrow{\pi_2} & A \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle g} \\ E & \xrightarrow[p]{} & B, \end{array}$$

where π_1 is an étale morphism. The relational structure on

$$E \times_B A = \{(e, a) \in E \times A \mid p(e) = \alpha(a)\}$$

CAT, for a given category **C** with pullbacks. Given then a **C**-indexed category $\mathbb{A} : \mathbf{C}^{\mathrm{op}} \to \mathbf{CAT}$ described by

$$\begin{array}{ccc} X & \longmapsto & \mathbb{A}^X \\ {\scriptstyle f}\uparrow & & \downarrow{\scriptstyle f^*} \\ Y & \longmapsto & \mathbb{A}^Y, \end{array}$$

let $p : E \to B$ be a morphism in **C** and let $Eq(p)$ be the internal category in **C** induced by the kernel pair of p

$$E \times_B E \times_B E \underset{\pi_{12}}{\overset{\pi_{23}}{\overset{\pi_{13}}{\rightrightarrows}}} E \times_B E \underset{\pi_1}{\overset{\pi_2}{\overset{e}{\leftrightarrows}}} E,$$

with $e =< 1_E, 1_E >$. The factorization of p through $Eq(p)$

$$\begin{array}{ccccc} & & Eq(p) & & \\ & {\scriptstyle \delta}\nearrow & & \searrow{\scriptstyle \overline{p}} & \\ E & & \xrightarrow[p]{} & & B, \end{array}$$

where δ and $\overline{p}$ are internal functors given respectively by $\delta_0 = 1_E$ and $\delta_1 = e$, $\overline{p}_0 = p$ and $\overline{p}_1 = p\pi_1 = p\pi_2$, gives rise to a commutative (up to natural isomorphism) diagram in **CAT**

$$\begin{array}{ccccc} & & \mathrm{Des}_{\mathbb{A}}(p) & & \\ & {\scriptstyle \delta^*}\swarrow & & \nwarrow{\scriptstyle K^p = \overline{p}^*} & \\ \mathbb{A}^E & & \xleftarrow[p^*]{} & & \mathbb{A}^B, \end{array}$$

where

$$\mathrm{Des}_{\mathbb{A}}(p) := \mathbb{A}^{Eq(p)}$$

is the category of $\mathbb{A}$-descent data relative to p. By definition p is called an *(effective) $\mathbb{A}$-descent morphism* if the comparison functor $K^p = \overline{p}^*$ is full and faithful (an equivalence of categories).
The equivalence stated in Proposition 3.1 turns out to be an equivalence of M-**Ord**-indexed categories, which is crucial for our purpose. In fact we prove now a general result which can be applied in our case.

Theorem 3.2. *Let* **C** *be a category with pullbacks and let $\mathbb{A}$ and $\mathbb{B}$ be two* **C***-indexed categories. If $\mathbb{A}$ and $\mathbb{B}$ are equivalent as* **C***-indexed categories,*

then a morphism $p : E \to B$ in $\mathbf{C}$ is (effective) for $\mathbb{A}$-descent if and only if it is (effective) for $\mathbb{B}$-descent.

Before proving the theorem we just recall that by an equivalence of $\mathbf{C}$-indexed categories we mean an indexed functor $F : \mathbb{A} \to \mathbb{B}$, in the sense of [13, Section 3], in which each functor $F_X : \mathbb{A}^X \to \mathbb{B}^X$, for $X \in Ob(\mathbf{C})$, is an equivalence of categories.

Proof. Let $\mathbb{A}, \mathbb{B} : \mathbf{C}^{\mathrm{op}} \to \mathbf{CAT}$ be two $\mathbf{C}$-indexed categories described respectively by

$$\begin{array}{ccc} X & \longmapsto & \mathbb{A}^X \\ {\scriptstyle f}\uparrow & & \downarrow{\scriptstyle f^*_{\mathbb{A}}} \\ Y & \longmapsto & \mathbb{A}^Y \end{array} \qquad \begin{array}{ccc} X & \longmapsto & \mathbb{B}^X \\ {\scriptstyle f}\uparrow & & \downarrow{\scriptstyle f^*_{\mathbb{B}}} \\ Y & \longmapsto & \mathbb{B}^Y \end{array}$$

with corresponding natural isomorphisms

$$i^X_{\mathbb{A}} : Id_{\mathbb{A}^X} \to (1_X)^*_{\mathbb{A}} \qquad i^X_{\mathbb{B}} : Id_{\mathbb{B}^X} \to (1_X)^*_{\mathbb{B}}$$

$$j^{f,g}_{\mathbb{A}} : f^*_{\mathbb{A}} g^*_{\mathbb{A}} \to (gf)^*_{\mathbb{A}} \qquad j^{f,g}_{\mathbb{B}} : f^*_{\mathbb{B}} g^*_{\mathbb{B}} \to (gf)^*_{\mathbb{B}},$$

and let F be an equivalence of $\mathbf{C}$-indexed categories from $\mathbb{A}$ to $\mathbb{B}$ given by the following data:

(a) for each object X in $\mathbf{C}$, an equivalence of categories $F_X : \mathbb{A}^X \to \mathbb{B}^X$,
(b) for each morphism $f : Y \to X$ in $\mathbf{C}$, a natural isomorphism

$$\tau_f : f^*_{\mathbb{B}} \circ F_X \Rightarrow F_Y \circ f^*_{\mathbb{A}},$$

such that, for each pair of composable morphisms $Z \xrightarrow{g} Y \xrightarrow{f} X$, the diagram

$$\begin{array}{ccccc} g^*_{\mathbb{B}} f^*_{\mathbb{B}} F_X & \xrightarrow{g^*_{\mathbb{B}} \tau_f} & g^*_{\mathbb{B}} F_Y f^*_{\mathbb{A}} & \xrightarrow{\tau_g f^*_{\mathbb{A}}} & F_Z g^*_{\mathbb{A}} f^*_{\mathbb{A}} \\ {\scriptstyle j^{g,f}_{\mathbb{B}} F_X}\downarrow & & & & \downarrow{\scriptstyle F_Z j^{g,f}_{\mathbb{A}}} \\ (fg)^*_{\mathbb{B}} F_X & & \xrightarrow[\tau_{fg}]{} & & F_Z (fg)^*_{\mathbb{A}} \end{array}$$

commutes. We show how to construct an equivalence $\Phi : \mathrm{Des}_{\mathbb{A}}(p) \to \mathrm{Des}_{\mathbb{B}}(p)$ between the categories of descent data of $\mathbb{A}$ and $\mathbb{B}$. Given an object (A, ξ) in $\mathrm{Des}_{\mathbb{A}}(p)$, where A is an object in $\mathbb{A}^E$ and $\xi : \pi^*_{2_{\mathbb{A}}} \to \pi^*_{1_{\mathbb{A}}}$ is a morphism in

Effective descent morphisms are necessarily surjective, while there are non-surjective effective étale-descent morphisms in M-**Ord**. Hence the converse of Theorem 5.3 is false. For surjective maps the problem is more interesting, although thr answer is the same. An appropriate counter-example in the surjective case of Theorem 5.3 can be given as follows. Let $p : E \to B$ be the following equivariant map:

$$\begin{array}{ccc} e_0 & & \\ {\scriptstyle m}\downarrow & & \\ e_{11} & e_{12} & \xrightarrow{p} \\ & {\scriptstyle n}\downarrow & \\ & e_2 & \end{array} \qquad \begin{array}{c} b_0 \\ {\scriptstyle m}\downarrow \\ b_1 \\ {\scriptstyle n}\downarrow \\ b_2 \end{array}$$

where $p(e_0) = b_0, p(e_{11}) = p(e_{12}) = b_1$ and $p(e_2) = b_2$. By Corollary 4.2, an easy inspection reveals that p is effective for étale-descent but it is not an effective descent morphism.

6. Effective descent and effective étale-descent morphisms in categories of relational algebras

Theorem 5.3 can be generalized to a larger context of relational algebras, including also the known results given in [11] for **Top** $\cong$ **RelAlg**($\mathbb{U}$), where $\mathbb{U}$ is the ultrafilter monad, and in [10] for **Preord** $\cong$ **RelAlg**($\mathbb{I}$), where $\mathbb{I}$ is the identity monad.

Recall that a functor $T :$ **Set** $\to$ **Set** has the Beck-Chevalley (BC) property (in the sense of [4]), if T preserves (BC)-squares, where a (BC)-square is a diagram

$$\begin{array}{ccc} W & \xrightarrow{k} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[g]{} & Y \end{array}$$

such that $f^\circ \circ g = k \circ h^\circ$, with f° and h° the opposite relations of f and h, respectively.

Theorem 6.1. [3, Theorem 2.4] *Let $\mathbb{T} = (T, \mu, \eta)$ be a monad on* **Set** *and $\overline{\mathbb{T}}$ its Barr extension. Assume that every naturality square of η with respect*

$p : E \to B$ satisfying the following condition: for each $b_0, b_1, b_2 \in B$ such that $b_0 \xrightarrow{m} b_1 \xrightarrow{n} b_2$, for $m, n \in M$, there exist $e_0, e_1, e_2 \in E$ such that $p(e_0) = b_0, p(e_1) = b_1, p(e_2) = b_2$ and

$$e_0 \xrightarrow{m} e_1 \xrightarrow{n} e_2.$$

Lemma 5.1. *If $p : E \to B$ is an effective descent morphism, then every 2-zigzag in $Z(Eq(p))$ is equivalent to a 1-zigzag.*

Proof. Let $[t, e']n[z, y]m[e, x]$ be a 2-zigzag in $Z(Eq(p))$. Take $b_0 = p(e) = p(x), b_1 = p(y) = p(z)$ and $b_2 = p(t) = p(e')$; then

$$b_0 \xrightarrow{m} b_1 \xrightarrow{n} b_2.$$

Since p is effective descent, by definition there exist $e_0, e_1, e_2 \in E$ such that $p(e_0) = b_0, p(e_1) = b_1, p(e_2) = b_2$ and

$$e_0 \xrightarrow{m} e_1, e_1 \xrightarrow{n} e_2.$$

After that we have

$$\begin{aligned} [t, e']n[z, y]m[e, x] &\sim [t, e']n[e_1, z][y, e_1]m[e, x] \\ &\sim [t, e'][e_2, t](n \circ m)[x, e_0][e, x] \\ &\sim [e_2, e'](n \circ m)[e, e_0] \end{aligned}$$

as claimed. □

Lemma 5.2. *The morphism $\varphi : Z(Eq(p)) \to B$ is faithful on 1-zigzags.*

Proof. Let $[y, e']m[e, x]$ and $[v, e']n[e, u]$ be two 1-zigzags in $Z(Eq(p))$ such that $m = n$. Then by construction of the category $Z(Eq(p))$ they are equivalent; in fact

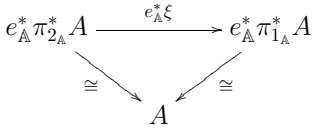
$$[y, e']m[e, x] \sim [v, e'][y, v]m[e, x] \sim [v, e']m[x, u][e, x] \sim [v, e']m[e, u]$$

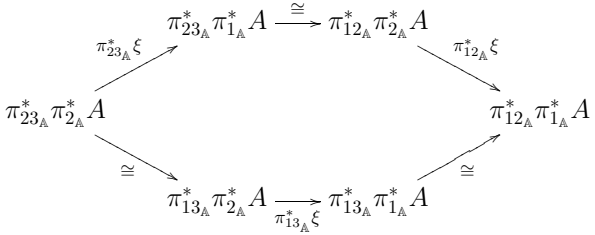
as desired. □

The following theorem is an immediate consequence of Lemma 5.1, Lemma 5.2 and of fact that an n-zigzag in $Z(Eq(p))$ is a composition of n 1-zigzags.

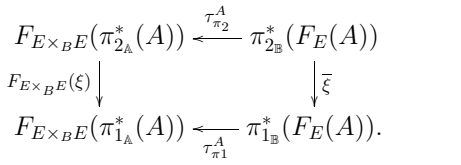
Theorem 5.3. *Every effective descent morphism in M-**Ord** is effective for étale-descent.*

$\mathbb{A}^{E \times_B E}$ such that the diagrams

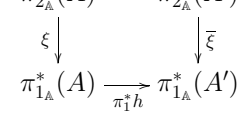




commute respectively in $\mathbb{A}^E$ and in $\mathbb{A}^{E \times_B E \times_B E}$, we define $\Phi(A, \xi) := (F_E(A), \overline{\xi})$, where $\overline{\xi}$ is defined by the following diagram



One can verify that $\overline{\xi}$ is actually a descent data for $F_E(A)$.
We define now Φ on morphisms. Given $h : (A, \xi) \to (A', \xi')$ in $\mathrm{Des}_{\mathbb{A}}(p)$, that is a morphism $h : A \to A'$ in $\mathbb{A}^E$ such that the diagram



commutes in $\mathbb{A}^{E \times_B E}$, define

$$\Phi(h) := F_E(h) : F_E(A) \to F_E(A').$$

One can easily verify that the diagram

$$\begin{array}{ccc} \pi_{2_{\mathbb{B}}}^* F_E(A) & \xrightarrow{\pi_{2_{\mathbb{B}}}^* F_E(h)} & \pi_{2_{\mathbb{B}}}^* F_E(A') \\ \downarrow{\scriptstyle \overline{\xi}} & & \downarrow{\scriptstyle \overline{\xi'}} \\ \pi_{1_{\mathbb{B}}}^* F_E(A) & \xrightarrow[\pi_{1_{\mathbb{B}}}^* F_E(h)]{} & \pi_{1_{\mathbb{B}}}^* F_E(A') \end{array}$$

commutes in $\mathbb{B}^{E\times_B E}$.
The equivalence of categories Φ makes the following diagram commutative up to isomorphism

$$\begin{array}{ccc} \mathbb{A}^B & \xrightarrow{K_{\mathbb{A}}^p} & \mathrm{Des}_{\mathbb{A}}(p) \\ {\scriptstyle F_B}\downarrow & & \downarrow{\scriptstyle \Phi} \\ \mathbb{B}^B & \xrightarrow[K_{\mathbb{B}}^p]{} & \mathrm{Des}_{\mathbb{B}}(p), \end{array}$$

allowing us to conclude that the morphism $p : E \to B$ in the category $\mathbf{C}$ is (effective) for $\mathbb{A}$-descent if and only if it is (effective) for $\mathbb{B}$-descent. □

This general result can be applied in our context of M-ordered sets as claimed. In fact, given $\mathbb{A}, \mathbb{B} : M\text{-}\mathbf{Ord}^{\mathrm{op}} \to \mathbf{CAT}$ defined respectively by

$$\begin{array}{ccc} X & \longmapsto & \mathbb{E}(X) \\ {\scriptstyle f}\uparrow & & \downarrow{\scriptstyle f^*} \\ Y & \longmapsto & \mathbb{E}(Y) \end{array} \qquad \begin{array}{ccc} X & \longmapsto & \mathbf{Set}^{X^{\mathrm{op}}} \\ {\scriptstyle f}\uparrow & & \downarrow{\scriptstyle \mathbf{Set}^{f^{\mathrm{op}}}} \\ Y & \longmapsto & \mathbf{Set}^{Y^{\mathrm{op}}}, \end{array}$$

where $\mathbb{E}$ is the class of étale morphisms in M-$\mathbf{Ord}$, f^* is the pullback functor and $\mathbf{Set}^{f^{\mathrm{op}}}$ is the composition functor, using Proposition 3.1 one can verify that there exists an equivalence of M-$\mathbf{Ord}$-indexed categories from $\mathbb{A}$ to $\mathbb{B}$. Thanks to Theorem 3.2 we can then state that an equivariant map $p : E \to B$ in M-$\mathbf{Ord}$ is an effective $\mathbb{A}$-descent morphism, i.e., it is effective for étale-descent, if and only if it is an effective $\mathbb{B}$-descent morphism.

4. The characterization of effective étale-descent morphisms

The category $\mathrm{Des}_{\mathbb{B}}(p)$ can be described as the category of pairs (X, ξ) where $X : E^{\mathrm{op}} \to \mathbf{Set}$ is a functor and $\xi = (\xi_{e,e'})_{e,e'\in E\times_B E}$ is a family of functions

(commutative) diagram

$$\begin{array}{ccc} Z(Eq(p)) & \xrightarrow{\varphi} & B \\ & {\scriptstyle q\circ\varphi}\searrow \quad \swarrow{\scriptstyle q} & \\ & M & \end{array}$$

where $q : B \to M$ is the faithful M-valued norm of the M-ordered set B. The morphism φ is faithful if and only if the M-valued norm $q \circ \varphi$ for $Z(Eq(p))$ is faithful.

(2) The characterization of the effective étale-descent morphisms leads to the characterization of the morphisms which are effective for descent with respect to the class of discrete op-fibrations $\mathbb{F}$. In fact the dual of Proposition 3.1 states that the slice category $\mathbb{F}(B)$ is equivalent to the category $\mathbf{Set}^B$, for a given M-ordered set B. Being also an equivalence of M-$\mathbf{Ord}$-indexed categories, we conclude that the effective descent morphisms in M-$\mathbf{Ord}$ with respect to the classe of discrete fibrations, i.e., étale maps, and discrete op-fibrations are the same by the fact that φ is a full and faithful lax epimorphism if and only if the same holds for φ^{op}.

(3) Of course we get also a characterization of the étale-descent morphisms, that is the morphisms for which the functor $\mathbf{Set}^{\varphi^{\mathrm{op}}}$ is full and faithful. They are precisely the morphisms for which condition (b) of Corollary 4.2 is satisfied [14, Theorem 1, Corollary 3].

(4) As remarked in [2, Remarks 1.4.3, (2)], in the trivial case $M = 1$ everything collapses to $\mathbf{Preord}$, also identified as the full subcategory of $\mathbf{Cat}$ given by small categories X for which $X \to 1$ is faithful. Because of that, the characterization of (effective) étale-descent morphisms in M-$\mathbf{Ord}$ generalizes the characterization of (effective) étale-descent morphisms in $\mathbf{Preord}$ given in [10].

5. Effective descent and effective étale-descent morphisms in M-Ord

A morphism $p : E \to B$ in a category $\mathbf{C}$ with pullbacks is called an *effective descent morphism* if the pullback functor $p^* : \mathbf{C} \downarrow B \to \mathbf{C} \downarrow E$ is monadic. A complete characterization of the effective descent morphisms in M-$\mathbf{Ord}$ is given in [6, Theorem 1.8]. They are the equivariant maps

an object (X, ξ) of $\mathrm{Des}_{\mathbb{B}}(p)$, i.e., a double functor from $Eq(p)$ to $S(\mathbf{Set})$, is the same as a functor from $Z(Eq(p))$ to $\mathbf{Set}$, and the diagram

$$\begin{array}{ccc} \mathbf{Set}^{B^{\mathrm{op}}} & \xrightarrow{K^p_{\mathbb{B}}} & \mathrm{Des}_{\mathbb{B}}(p) \\ & {\scriptstyle \mathbf{Set}^{p^{\mathrm{op}}}} \searrow \quad \swarrow {\scriptstyle U^p_{\mathbb{B}}} & \\ & \mathbf{Set}^{E^{\mathrm{op}}} & \end{array}$$

can be identified, up to equivalence, to the diagram

$$\begin{array}{ccc} \mathbf{Set}^{B^{\mathrm{op}}} & \xrightarrow{\mathbf{Set}^{\varphi^{\mathrm{op}}}} & \mathbf{Set}^{Z(Eq(p))^{\mathrm{op}}} \\ & {\scriptstyle \mathbf{Set}^{p^{\mathrm{op}}}} \searrow \quad \swarrow {\scriptstyle \mathbf{Set}^{\psi^{\mathrm{op}}}} & \\ & \mathbf{Set}^{E^{\mathrm{op}}}. & \end{array}$$

This allows us to state that a morphism $p : E \to B$ in M-**Ord** is effective for étale-descent if and only the functor $\mathbf{Set}^{\varphi^{\mathrm{op}}} : \mathbf{Set}^{B^{\mathrm{op}}} \to \mathbf{Set}^{Z(Eq(p))^{\mathrm{op}}}$ is an equivalence of categories. The general argument given in [14, Theorem 2] can be applied in our situation.

Theorem 4.1. *An equivariant map $p : E \to B$ is an effective étale-descent morphism in M-**Ord** if and only if $\varphi : Z(Eq(p)) \to B$ is a full and faithful lax epimorphism in **Cat**.*

Corollary 4.2. *An equivariant map $p : E \to B$ is an effective étale-descent morphism in M-**Ord** if and only if*

(a) *for each $p(e) \xrightarrow{k} p(e')$ in B with $k \in M$ there exists a zigzag $z = [e_n, e'_n]m_n \cdots m_1[e_0, e'_0]$ in $Z(Eq(p))$ with $k = m_n \circ \cdots \circ m_1$, and such a zigzag is unique up to equivalence;*

(b) *every point $b \in B$ is in relation to a point of the image via a right-invertible element of the monoid, i.e., for each $b \in B$ there exist $e \in E$, $n, m \in M$ such that $p(e) \xrightarrow{n} b$ and $b \xrightarrow{m} p(e)$ with $n \circ m = e_M$.*

Remarks 4.3.

(1) The uniqueness (up to equivalence) for zigzags in the condition (a) of Corollary 4.2 comes from the faithfulness of the morphism $\varphi : Z(Eq(p)) \to B$ in Theorem 4.1. This condition can be expressed by the fact that $Z(Eq(p))$ is an M-ordered set: consider the following

$\xi_{e,e'} : X(e') \to X(e)$ defined for $e, e' \in E$ with $p(e) = p(e')$ such that

$$\xi_{e,e} = 1_{X(e)} \qquad \xi_{e',e''} \circ \xi_{e,e'} = \xi_{e,e''},$$

for each $e, e', e'' \in E$ with $p(e) = p(e') = p(e'')$, and each diagram

$$\begin{array}{ccc} X(e'_0) & \xrightarrow{\xi_{e_0,e'_0}} & X(e_0) \\ {\scriptstyle X(e'_1 \xrightarrow{m} e'_0)}\downarrow & & \downarrow{\scriptstyle X(e_1 \xrightarrow{m} e_0)} \\ X(e'_1) & \xrightarrow[\xi_{e_1,e'_1}]{} & X(e_1) \end{array}$$

commutes for all $e'_1 \xrightarrow{m} e'_0$ and $e_1 \xrightarrow{m} e_0$ in E with $m \in M$, $p(e_0) = p(e'_0)$ and $p(e_1) = p(e'_1)$. The comparison functor

$$K^p_{\mathbb{B}} : \mathbf{Set}^{B^{\mathrm{op}}} \to \mathrm{Des}_{\mathbb{B}}(p)$$

is given by

$$G \mapsto (G \circ p^{\mathrm{op}}, 1),$$

where 1 is the family of identity morphisms $1_{e,e'} : G(p(e)) \to G(p(e'))$.
The internal category $Eq(p)$ in M-**Ord** can be presented as a double category in the following way:

(1) objects are elements of E;
(2) vertical arrows are morphisms of E, that is $e_0 \xrightarrow{m} e_1$;
(3) horizontal arrows are elements of $E \times_B E$;
(4) squares of 2-cells are morphisms of $E \times_B E$, that is squares of the form

$$\begin{array}{ccc} e_0 & \longrightarrow & e'_0 \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m} \\ e_1 & \longrightarrow & e'_1 \end{array}$$

for e_0, e'_0, e_1, e'_1 in E with $p(e_0) = p(e'_0)$, $p(e_1) = p(e'_1)$ and $e_0 \xrightarrow{m} e_1$, $e'_0 \xrightarrow{m} e'_1$ in E for $m \in M$.

This allows us to present an object (X, ξ) of the category $\mathrm{Des}_{\mathbb{B}}(p)$ as a double functor from the double category $Eq(p)$ to the double category $S(\mathbf{Set})$ of commutative squares in **Set**. The functor

$$S : \mathbf{CAT} \to \mathbf{DoubleCAT},$$

which sends each category $\mathbf{C}$ to the double category of commutative squares of $\mathbf{C}$, has a left adjoint Z described in [10, section 1] as a quotient of a pushout. The category $Z(Eq(p))$ can be then contructed as follows. Consider

- $Eq(p)_0$ to be the discrete category with objects as in $Eq(p)$;
- $Eq(p)_h$ and $Eq(p)_v$ to be categories with the same objects and the morphisms to be, respectively, the horizontal and the vertical arrows of $Eq(p)$;
- $Eq(p)_+$ the pushout in **Cat** of the embeddings $Eq(p)_0 \to Eq(p)_h$ and $Eq(p)_0 \to Eq(p)_v$;

then, for every square of 2-cell in $Eq(p)$

$$\begin{array}{ccc} e_0 & \longrightarrow & e'_0 \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m} \\ e_1 & \longrightarrow & e'_1 \end{array}$$

the pairs

$$\begin{array}{ccc} e_0 & \longrightarrow & e'_0 \\ & & \downarrow{\scriptstyle m} \\ & & e'_1 \end{array} \qquad \begin{array}{ccc} & & e'_0 \\ & & \downarrow{\scriptstyle m} \\ e_1 & \longrightarrow & e'_1 \end{array}$$

become morphisms in $Eq(p)_+$ from e_0 to e'_1, and we construct $Z(Eq(p))$ as the quotient category $Eq(p)_+/\sim$ under the smallest equivalence relation $\sim$ for which

$$\begin{array}{ccc} e_0 & \longrightarrow & e'_0 \\ & & \downarrow{\scriptstyle m} \\ & & e'_1 \end{array} \quad \sim \quad \begin{array}{ccc} e_0 & & \\ \downarrow{\scriptstyle m} & & \\ e_1 & \longrightarrow & e'_1, \end{array}$$

for all such pairs. Observe that a morphism in $Z(Eq(p))$ from a point e_0 to a point e'_n can be then given by an equivalent class of a morphism in $Eq(p)_+$,

say an n-zigzag z

$$\begin{array}{ccccccccc} e_0 & \longrightarrow & e'_0 & & & & & & \\ & & \downarrow{\scriptstyle m_1} & & & & & & \\ & & e_1 & \longrightarrow & e'_1 & & & & \\ & & & & \downarrow{\scriptstyle m_2} & & & & \\ & & & & e_2 & \longrightarrow & e'_2 & & \\ & & & & & & \downarrow{\scriptstyle m_2} & & \\ & & & & & & \vdots & & \\ & & & & & & \downarrow{\scriptstyle m_n} & & \\ & & & & & & e_n & \longrightarrow & e'_n \end{array}$$

where $p(e_i) = p(e'_i)$ for $i = 0, \cdots, n$, and $e'_i \xrightarrow{m_{i+1}} e_{i+1}$ in E for $i = 0, \cdots, n-1$. The notation for such an n-zigzag as above will be $z = [e_n, e'_n]m_n \cdots m_1[e_0, e'_0]$. The equivariant map $p : E \to B$ can be then factorized in **Cat** through the category $Z(Eq(p))$

$$\begin{array}{ccc} E & \xrightarrow{\quad p \quad} & B \\ & {\scriptstyle\psi}\searrow \quad \nearrow{\scriptstyle\varphi} & \\ & Z(Eq(p)) & \end{array}$$

where ψ is defined as the identity on objects and $\psi(e_0 \xrightarrow{m} e_1) = [e_0 \xrightarrow{m} e_1]$ on morphisms, while φ on objects acts as p and the image of an equivalent class of an n-zigzag $z = [e_n, e'_n]m_n \cdots m_1[e_0, e'_0]$ via φ is

$$\varphi([z]) = p(e_0) \xrightarrow{m_1} p(e'_1) \xrightarrow{m_2} \cdots \xrightarrow{m_n} p(e'_n) = p(e_0) \xrightarrow{m_n \circ \cdots \circ m_1} p(e'_n).$$

Thanks to the adjoint situation

$$\mathbf{CAT} \underset{S}{\overset{Z}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{DoubleCAT}$$