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NW-SE EXPANSIONS OF NON-SYMMETRIC CAUCHY KERNELS ON NEAR STAIRCASES AND GROWTH DIAGRAMS

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Abstract: Lascoux has given a triangular version of the Cauchy identity where Schur polynomials are replaced by Demazure characters and Demazure atoms. He has then used the staircase expansion to recover expansions for all Ferrers shapes, where the Demazure characters and Demazure atoms are under the action of Demazure operators specified by the cells above the staircase. The characterisation of the tableau-pairs in these last expansions is less explicit. We give here a bijective proof for expansions over near staircases, where the tableau-pairs are made explicit. Our analysis formulates Mason's RSK analogue, for semi-skylines augmented fillings, in terms of growth diagrams.

Keywords: non-symmetric Cauchy kernel, Demazure character, Demazure operator, RSK analogue, semi-skyline augmented filling, growth diagram.
AMS subject classification [2010]05E05,05E10

1. Introduction

Let λ be a Ferrers shape and $\rho = (n, \dots, 1)$ the biggest staircase inside of λ. Fix a cell in the staircase $(n+1, n, \dots, 1)$ which does not belong to λ. The diagonal passing through this cell cuts the skew diagram λ/ρ into a North-West (NW) and a South-East (SE) parts whose row and column cell indices are encoded as reduced words for $\sigma(\lambda, NW)$ and $\sigma(\lambda, SE)$ in $\mathfrak{S}_n$, respectively. [7] has given the following expansion of the Cauchy kernel $F_\lambda(x,y) = \prod_{(i,j)\in\lambda}(1-x_iy_j)^{-1}$ over the Ferrers shape λ,

$$F_\lambda := F_\lambda(x,y) = \prod_{(i,j)\in\lambda}(1-x_iy_j)^{-1} = \sum_{\nu\in\mathbb{N}^n}(\pi_{\sigma(\lambda,NW)}\widehat{\kappa}_\nu(x))(\pi_{\sigma(\lambda,SE)}\kappa_{\omega\nu}(y)), \quad (1)$$

where $\pi_{\sigma(\lambda,NW)}$ ($\pi_{\sigma(\lambda,SE)}$) is the Demazure operator indexed by $\sigma(\lambda, NW)$ ($\sigma(\lambda, SE)$) acting on the Demazure atom $\widehat{\kappa}_\nu(x)$ (key polynomial or Demazure character $\kappa_{\omega\nu}(y)$). He shows that the staircase ρ expansion, $F_\rho = \sum_{\nu\in\mathbb{N}^n}\widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y)$, allows to recover all the F_λ since divided differences or Demazure operators π_i^x and π_j^y increase the number of poles in the rational function $(1-x_iy_j)^{-1}$: $\pi_i^x(1-x_iy_j)^{-1} = (1-x_iy_j)^{-1}(1-x_{i+1}y_j)^{-1}$ and similarly for

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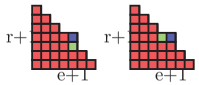
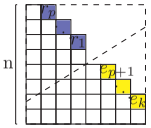
π_j^y. If λ consists of the staircase ρ with the sole cell $(r+1, e+1)$, $e = n - r$, above ρ, the blue box in 

, the rows r and $r+1$ of λ have the same length as well as the columns e and $e+1$. If η and η' are obtained by erasing in ρ the green cells $(r, e+1)$ and $(r+1, e)$ respectively, then F_η is symmetrical in x_r and x_{r+1}, and $F_{\eta'}$ in y_e and y_{e+1}. Demazure operators π_r^x (acting on x_r and x_{r+1}) and π_e^y preserve F_η and $F_{\eta'}$, and reproduce the green cells $(r, e+1)$ and $(r+1, e)$, when acting on F_ρ, by creating both the blue cell $(r+1, e+1)$ above ρ. That is, $\pi_r^x F_\rho = (\pi_r(1 - x_r y_{e+1})^{-1})F_\eta = F_\rho(1 - x_{r+1}y_{e+1})^{-1} = F_\lambda$ and, similarly, $\pi_e^y F_\rho = F_\rho(1 - x_{r+1}y_{e+1})^{-1} = F_\lambda$. Henceforth, $F_\lambda = \sum_{\nu\in\mathbb{N}^n} \pi_r^x \widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y) = \sum_{\nu\in\mathbb{N}^n} \widehat{\kappa}_\nu(x)\pi_e^y\kappa_{\omega\nu}(y)$.

We give here a combinatorial interpretation of this algebraic explanation for (1), when λ is the near staircase of size n: $(\star)$ 

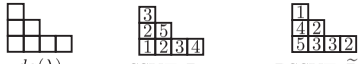
with one layer of k cells, $0 \le p \le k < n$, sited on the stairs of ρ, avoiding the top and the basement. Given the labels $1 \le r_k < \cdots < r_{p+1} < r_1 < \cdots < r_p < n$. The SE labels, $e_{p+1} = n - r_{p+1} < \cdots < e_k = n - r_k$, NW labels, $r_1 < \cdots < r_p$, with respect to the cutting line shown in $(\star)$, indicate the column indices $e_j + 1$, $p+1 \le j \le k$, and the row indices $r_i + 1$, $1 \le i \le p$, counted in French convention. In Section 2, we give the necessary combinatorial tools, being SSAFs the major ones, the detectors of keys in [12]. Section 4 recalls briefly the Bruhat order in $\mathfrak{S}_n$ and orbits. Sections 3 and 5 are devoted to the analysis of the behaviour of the [11]'s RSK analogue under the action of crystal operators, or coplactic operators, e_r, f_r via a growth diagram interpretation: the former delete convex corners of a Ferrers shape of height or width bigger than one, and thus the later inflate a concave to a convex corner. Theorems 1, 2 and 3 detect how the key-pair of a SSYT-pair, of the same shape, change in Bruhat order, when a cell is created in a concave corner of a Ferrers shape. Lastly, in Section 6, the bijective proof of the expansion of F_λ over $(\star)$, as below, is given. As usual, if $m < n$ are in $\mathbb{N}$, put $[m, n] := \{m, \dots, n\}$, and $[n]$ for $m = 1$. For each $(z, t) \in [0, p] \times [0, k-p]$, let $(H_z, M_t) \in \binom{[p]}{z} \times \binom{[p+1,k]}{t}$, and $\mathcal{A}_{z,t}^{H_z,M_t}$ the set of SSAF-pairs with shapes satisfying certain inequalities, in the Bruhat order, as defined in Section 6. Then

$$F_\lambda = \sum_{(F,G)\in\mathcal{A}^{\emptyset,\emptyset}} x^F y^G + \sum_{z=1}^{p} \sum_{H_z\in\binom{[p]}{z}} \sum_{(F,G)\in\mathcal{A}_{z,0}^{H_z,\emptyset}} x^F y^G + \sum_{t=1}^{k-p} \sum_{M_t\in\binom{[p+1,k]}{t}} \sum_{(F,G)\in\mathcal{A}_{0,t}^{\emptyset,M_t}} x^F y^G$$
$$+ \sum_{(z,t)\in[p]\times[k-p]} \sum_{(H_z,M_t)} \sum_{(F,G)\in\mathcal{A}_{z,t}^{H_z,M_t}} x^F y^G = \sum_{\nu\in\mathbb{N}^n} \pi_{r_1}\dots\pi_{r_p}\widehat{\kappa}_\nu(x)\pi_{e_{p+1}}\dots\pi_{e_k}\kappa_{\omega\nu}(y)$$
$$= \sum_{\nu\in\mathbb{N}^n} \widehat{\kappa}_\nu(x)\pi_{n-r_p}\cdots\pi_{n-r_1}\pi_{e_{p+1}}\cdots\pi_{e_k}\kappa_{\omega\nu}(y) = \sum_{\nu\in\mathbb{N}^n} \pi_{n-e_k}\cdots\pi_{n-e_{p+1}}\pi_{r_1}\cdots\pi_{r_p}\widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y).$$

The two last expansions are the SE, NW versions, with respect to the cutting lines through the top of the first column, and the end of the botton row in $(\star)$, respectively. They can be thought as $p = 0$ and $p = k$.

2. SSYTs and RRSYTs. SSAFs detectors of keys.

A weak composition $\gamma = (\gamma_1 \dots, \gamma_n)$ is a vector in $\mathbb{N}^n$. When its entries are in weakly decreasing order, that is, $\gamma_1 \geq \cdots \geq \gamma_n$, it is said to be a partition. Every weak composition γ determines a unique partition λ, arranging its entries in weakly decreasing order. It is the unique partition in the orbit of γ regarding the usual action of symmetric group $\mathfrak{S}_n$ on $\mathbb{N}^n$. A partition $\lambda = (\lambda_1, \dots, \lambda_n)$ is identified with its Young diagram (or Ferrers shape) $dg(\lambda)$ in French convention, an array of left-justified cells (boxes) with λ_i cells in row i from the bottom, for $1 \leq i \leq n$. The cells are located in the diagram $dg(\lambda)$ by their row and column indices (i, j), where $1 \leq i \leq n$ and $1 \leq j \leq \lambda_i$. A filling of shape λ (or a filling of $dg(\lambda)$), in the alphabet $[n]$, is a map $P : dg(\lambda) \to [n]$. A semi-standard Young tableau (SSYT) P of shape $sh(P) = \lambda$, in the alphabet $[n]$, is a filling of $dg(\lambda)$ weakly increasing in each row from left to right and strictly increasing up in each column. The column word of the SSYT P is the word consisting of the entries of each column, read top to bottom and left to right. The content $c(P) = (\alpha_1, \dots, \alpha_n)$ or weight of P is the content or weight of its column word, that is, α_i is the multiplicity of $i \in [n]$ in the column word of P. A key tableau is a SSYT such that the set of entries in the $(j+1)^{th}$ column is a subset of the set of entries in the j^{th} column, for all j. There is a bijection between weak compositions in $\mathbb{N}^n$ and keys in the alphabet $[n]$ given by $\gamma \to key(\gamma)$, where $key(\gamma)$ is the SSYT such that for all j, the first γ_j columns contain the letter j. Any key tableau is of the form $key(\gamma)$ where γ is the content and the shape is the unique partition in its $\mathfrak{S}_n$-orbit. A reverse semi-standard Young tableau (RSSYT), R, of shape $sh(R) = \lambda$, in the alphabet $[n]$, is a filling of $dg(\lambda)$ such that the entries in each row are weakly decreasing from left to right, and strictly decreasing from bottom to top. The reverse Schensted insertion applied to the word $b = b_1 \cdots b_m$, over the alphabet $[n]$, gives a reverse SSYT of the same weight. It is the same as applying the Schensted insertion to the word $b^* = n - b_m + 1 \cdots n - b_1 + 1$ to get a SSYT of reverse content, and then changing i to $n - i + 1$ to obtain the reverse SSYT $\widetilde{P}$ with the same weight as b. See [3], Appendix A.1, and [14]. Thus reverse Schensted insertion applied to the column word of a SSYT, defines a weight and shape preserving bijection between SSYTs and RSSYTs. For instance,

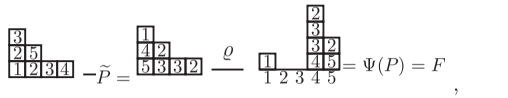


$dg(\lambda)$ SSYT P RSSYT $\widetilde{P}$

are, in this order, the Ferrers diagram of $\lambda = (4, 2, 1)$, a SSYT of shape λ and content $(1, 2, 2, 1, 1)$, and the reverse Schensted insertion of P.

2.1. Weight preserving, shape rearranging bijection between SSYTs and SSAFs.

A weak composition $\gamma = (\gamma_1, \dots, \gamma_n)$ is visualised as a diagram consisting of n columns, with γ_j cells (boxes) in column j, for $1 \leq j \leq n$. The *column diagram* of γ is the set $dg'(\gamma) = \{(i, j) \in \mathbb{N}^2 : 1 \leq j \leq n, 1 \leq i \leq \gamma_j\}$ where the coordinates are in French convention, i indexing the rows, and j indexing the columns. (The prime reminds that the components of γ are the column lengths.) A cell in a column diagram is written (i, j), where i is the row index and j the column index. The *augmented diagram* of γ, $\widehat{dg}(\gamma) = dg'(\gamma) \cup \{(0, j) : 1 \leq j \leq n\}$, is the column diagram with n extra cells adjoined in row 0. This adjoined row is called the *basement* and it always contains the numbers 1 to n in strictly increasing order. The shape of $\widehat{dg}(\gamma)$ is defined to be γ. The empty augmented diagram consists of the basement.

Semi-skyline augmented fillings (SSAFs) are the output of the weight preserving injective map ϱ, [12], acting on RSSYTs as follows. Let $\widetilde{P}$ be a RSSYT in the alphabet $[n]$. Define the empty semi-skyline augmented filling as the empty augmented diagram with basement elements from 1 to n. Pick the first column of $\widetilde{P}$, say, P_1. Put all the elements of the first column P_1 to the top of the same basement elements in the empty semi-skyline augmented filling. The new diagram is called the semi-skyline augmented filling corresponding to the first column of $\widetilde{P}$ and is denoted by SSAF. Assume that the first i columns of $\widetilde{P}$, denoted $P_1, P_2, \ldots, P_i$, have been mapped to a SSAF. Consider the largest element, a_1, in the $(i+1)$-th column P_{i+1}. There exists an element greater than or equal to a_1 in the i-th row of the SSAF. Place a_1 on top of the leftmost such element. Assume that the largest $k-1$ entries in P_{i+1} have been placed into the SSAF. The k-th largest element, a_k, of P_{i+1} is then placed into the SSAF. Place a_k on top of the leftmost entry b in row $k-1$ such that $b \geq a_k$ and the cell immediately above b is empty. Continue this procedure until all entries in P_{i+1} have been mapped into the $(i+1)$-th row and then repeat for the remaining columns of $\widetilde{P}$ to obtain the semi-skyline augmented filling F. The content of the SSAF F is the vector $c(F) := c(\widetilde{P}) \in \mathbb{N}^n$, and the shape of F, $sh(F)$, is the weak composition recording the length of the columns of F, from left to right. Hence a rearranging of $sh(\widetilde{P})$. Thereby, the reverse Schensted insertion composed with ϱ is a weight preserving and shape rearranging bijection, denoted Ψ, between SSYTs and SSAFs. [12] proves that if P is a SSYT, the right key of P, a notion due to [9], is $key(sh(\Psi(P))) = key(sh(\varrho(\widetilde{P})))$. We just say, indistinctly, that $sh(F)$ is the (right) key of F or P. For instance, $P =$  $= \Psi(P) = F$, with $c(F) = (1,2,2,1,1) = c(\widetilde{P}) = c(P)$, $sh(F) = (1,0,0,4,2)$ and $key(1,0,0,4,2)$.

Remark 1. *If, in F, the height of column i is $<$ than the height of column j, $j > i$, the choice of the left most position implies that $a < b \leq c$ in any triple* $\begin{matrix} & & b \\ a & \cdots & c \end{matrix}$ *with b, c in column j, and a in column i.*

3. RSK analogue, growth diagram of RRSK and the key-pair

The two line array $w = \begin{pmatrix} j_1 & j_2 & \cdots & j_l \\ i_1 & i_2 & \cdots & i_l \end{pmatrix}$, such that $j_r < j_{r+1}$, and if $j_r = j_{r+1}$ then $i_r \leq i_{r+1}$, for all $1 \leq r \leq l-1$, where i_r, $j_r \in [n]$, is called a biword in lexicographic order, on the alphabet $[n]$, with respect to the first row. The RSK algorithm is a bijection between the biwords in lexicographic order on the alphabet $[n]$, and the pairs of SSYTs of the same shape on the same alphabet. The reverse RSK (RRSK) algorithm, see [14], is the same as applying RSK to the biword $w^* = \begin{pmatrix} n-j_l+1 & \ldots & n-j_1+1 \\ n-i_l+1 & \cdots & n-i_1+1 \end{pmatrix}$, to get a pair (P,Q) of SSYTs, and then change i to $n-i+1$, in all their entries, to obtain a pair $(\widetilde{P},\widetilde{Q})$ of reverse SSYTs. [11] uses an analogue of Schensted insertion to find an analogue Φ of the RSK to produce pairs of SSAFs. *The map Φ is a bijection between the biwords in lexicographic order in the alphabet*

Identifying $x_i y_j$ with the biletter $\binom{j}{i}$, one has three types of biwords: inside the staircase, consisting only of the extra biletters in the NW part, or the extra biletters in the SE part. Using bijection 3, their concatenation gives, $F_\lambda = F_\rho \prod_{i=1}^{p}(1 - x_{r_i+1}y_{n-r_i+1})^{-1} \prod_{j=p+1}^{k}(1 - x_{r_j+1}y_{e_j+1})^{-1} = \sum_{(F,G)\in\mathcal{A}^{\emptyset,\emptyset}} x^F y^G + \sum_{\substack{(z,t)\in[0,p]\times[0,k-p] \\ (z,t)\neq(0,0)}} \sum_{(H_z,M_t)} \sum_{(F,G)\in\mathcal{A}^{H_z,M_t}_{z,t}} x^F y^G.$ ∎

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$$F_\lambda = \sum_{(F,G)\in\mathcal{A}^{\emptyset,\emptyset}} x^F y^G + \sum_{z=1}^{p} \sum_{H_z\in\binom{[p]}{z}} \sum_{(F,G)\in\mathcal{A}_{z,0}^{H_z,\emptyset}} x^F y^G + \sum_{t=1}^{k-p} \sum_{M_t\in\binom{[p+1,k]}{t}} \sum_{(F,G)\in\mathcal{A}_{0,t}^{\emptyset,M_t}} x^F y^G$$
$$+ \sum_{(z,t)\in[p]\times[k-p]} \sum_{(H_z,M_t)} \sum_{(F,G)\in\mathcal{A}_{z,t}^{H_z,M_t}} x^F y^G = \sum_{\nu\in\mathbb{N}^n} \pi_{r_1}\dots\pi_{r_p}\widehat{\kappa}_\nu(x)\pi_{e_{p+1}}\dots\pi_{e_k}\kappa_{\omega\nu}(y)$$
$$= \sum_{\nu\in\mathbb{N}^n} \widehat{\kappa}_\nu(x)\pi_{n-r_p}\cdots\pi_{n-r_1}\pi_{e_{p+1}}\cdots\pi_{e_k}\kappa_{\omega\nu}(y) = \sum_{\nu\in\mathbb{N}^n} \pi_{n-e_k}\cdots\pi_{n-e_{p+1}}\pi_{r_1}\cdots\pi_{r_p}\widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y).$$

Proof: The proof is by double induction on $p\geq 0$ and $k-p\geq 0$. For $p\geq 0$ and $k-p{=}\,0$, and *vice-versa*, see [2]. Let $p,\,k-p\geq 1$. One has, $\sum_{\nu\in\mathbb{N}^n} \pi_{r_1}\dots\pi_{r_p}\widehat{\kappa}_\nu(x)\pi_{e_{p+1}}\dots\pi_{e_k}\kappa_{\omega\nu}(y) =$

$$\pi_{e_{p+1}}\left(\sum_{\nu\in\mathbb{N}^n} \pi_{r_1}\dots\pi_{r_p}\widehat{\kappa}_\nu(x)\pi_{e_{p+2}}\dots\pi_{e_k}\kappa_{\omega\nu}(y)\right), \text{ by induction,}$$
$$= \pi_{e_{p+1}}\left(\sum_{\substack{0\leq z\leq p\\ 0\leq t\leq k-p-1}} \sum_{(H_z,M_t)} \sum_{(F,G)\in\mathcal{A}_{z,t}^{H_z,M_t}} x^F y^G\right), \quad (H_z,M_t)\in\tbinom{[p]}{z}\times\tbinom{[p+2,k]}{t},$$
$$= \sum_{\substack{0\leq z\leq p\\ 0\leq t\leq k-p-1}} \sum_{(H_z,M_t)} \left(\sum_{\beta\in\mathbb{N}^n} \pi_{e_{p+1}} \sum_{\substack{(F,G)\in\mathcal{A}_{z,t}^{H_z,M_t}\\ sh(G)=\beta}} x^F y^G\right), \quad \text{using (2),}$$
$$= \sum_{\substack{0\leq z\leq p\\ 0\leq t\leq k-p-1}} \sum_{(H_z,M_t)} \left(\sum_{\substack{(F,G)\in\mathcal{A}_{z,t}^{H_z,M_t}\\ sh(G)_{e_{p+1}}\geq sh(G)_{e_{p+1}+1}}} x^F y^G + \sum_{(F,G)\in\mathcal{B}_{z,t}^{H_z,M_t}} x^F y^G\right), \quad \text{from Lemma 3,}$$
$$= \sum_{\substack{0\leq z\leq p\\ 0\leq t\leq k-p-1}} \sum_{(H_z,M_t)} \left(\sum_{\substack{(F,G)\in\mathcal{A}_{z,t}^{H_z,M_t}\\ sh(G)_{e_{p+1}}\geq sh(G)_{e_{p+1}+1}}} x^F y^G + \sum_{\substack{(F,G)\in\mathcal{A}_{z,t}^{H_z,M_t}\\ sh(G)_{e_{p+1}}< sh(G)_{e_{p+1}+1}}} x^F y^G + \sum_{(F,G)\in\mathcal{A}_{z,t+1}^{H_z,M_{t+1}^1}} x^F y^G\right)$$
$$= \sum_{\substack{0\leq z\leq p\\ 0\leq t\leq k-p-1}} \sum_{(H_z,M_t)} \left(\sum_{(F,G)\in\mathcal{A}_{z,t}^{H_z,M_t}} x^F y^G + \sum_{(F,G)\in\mathcal{A}_{z,t+1}^{H_z,M_{t+1}^1}} x^F y^G\right)$$
$$= \sum_{\substack{0\leq z\leq p\\ 0\leq t\leq k-p-1}} \sum_{(H_z,M_t)} \left(\sum_{(F,G)\in\mathcal{A}_{z,t}^{H_z,M_t}} x^F y^G\right) + \sum_{\substack{0\leq z\leq p\\ 0\leq t\leq k-p-1}} \sum_{(H_z,M_t)} \left(\sum_{(F,G)\in\mathcal{A}_{z,t+1}^{H_z,M_{t+1}^1}} x^F y^G\right)$$
$$= \sum_{(F,G)\in\mathcal{A}^{\emptyset,\emptyset}} x^F y^G + \sum_{\substack{(z,t)\in[0,p]\times[0,k-p]\\ (z,t)\neq(0,0)}} \sum_{(H_z,M_t)} \sum_{(F,G)\in\mathcal{A}_{z,t}^{H_z,M_t}} x^F y^G, \text{ with } (H_z,M_t)\in\tbinom{[p]}{z}\times\tbinom{[p+1,k]}{t}.$$

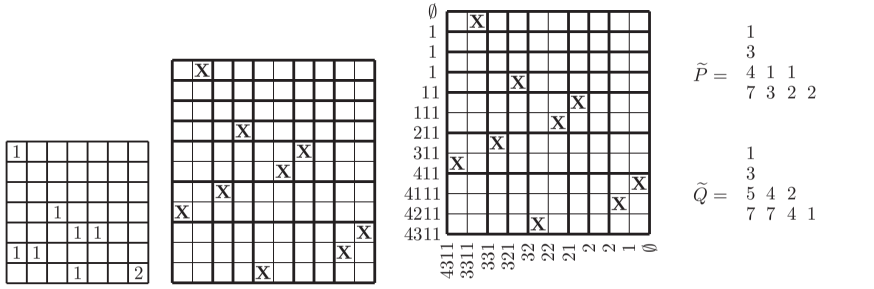
$[n]$, *and the pairs of SSAFs with shapes (keys) in some* $\mathfrak{S}_n$*-orbit.* The bijection Φ applied to a biword w is the same as applying the RRSK to w and then applying ϱ to each reverse SSYT of the output pair $(\widetilde{P},\widetilde{Q})$. That is, $\Phi(w)=(\varrho(\widetilde{P}),\varrho(\widetilde{Q}))$. Equivalently, applying the RSK to w and then Ψ to each SSYT of the output pair (P,Q) gives $\Phi(w)=(\Psi(P),\Psi(Q))$.

The remaining of this section follows closely [5] and [14]. Let w be a biword in the lexicographic order, over the alphabet $[n]$, represented in the $n\times n$ square diagram, by putting the number r in the cell (i,j) of the square grid, whenever the biletter $\binom{j}{i}$ appears $r\geq 1$ times in the biword w. (Rows are counted from bottom to top and columns from left to right.) Scanning the columns, from left to right and bottom to top, the biword w is recovered in lexicographic order. The 01-filling of this diagram has at most one 1 in each row and each column as follows. Construct a rectangle diagram with more rows and columns as follows. The entries which are originally in the same column or in the same row are put in different columns and rows in the larger diagram. An entry m is replaced by m 1's in the new diagram, all of them placed in different rows and columns. The entries in a row are separated from bottom/left to top/right, and the 1's are represented by $\mathbf{X}$'s. If there should be several entries in a column as well, separate entries in a column from bottom/left to top/right. In the cell with entry m, we replace m by a chain of m $\mathbf{X}$'s arranged from bottom/left to top/right. The original n columns and n rows are indicated by thick lines, whereas the newly created columns and rows are indicated by thin lines. To give an interpretation of RRSK in terms of growth diagrams, we start by assigning the empty partition $\emptyset$ to each point of a corner cell on the right column and on the top row of the 01-filling. Then assign partitions to the other corners inductively by applying the following local rules. Consider the cell

$\begin{array}{cc} \nu & \varepsilon \\ & \mu \end{array}$ labeled by the partitions ε,μ,ν, such that $\varepsilon\subseteq\mu$ and $\varepsilon\subseteq\nu$, where the containment means that the Ferrres shapes differ at most by one box. Then β is determined as follows: (i) If $\varepsilon=\mu=\nu$, and if there is no cross in the cell, then $\beta=\varepsilon$. (ii) If $\varepsilon=\mu\neq\nu$, then $\beta=\nu$. If $\varepsilon=\nu\neq\mu$, then $\beta=\mu$. (iii) If ε,μ,ν are pairwise different, then $\beta=\mu\cup\nu$, *i.e*, $\beta_i=max\{\mu_i,\nu_i\}$. (iv) If $\varepsilon\neq\mu=\nu$, then β is formed by adding a box to the $(k+1)$-st row of $\mu=\nu$, given that $\mu=\nu$ and ε differ in the k-th row. (v) If $\varepsilon=\mu=\nu$, and if there is a cross in the cell, then β is formed by adding a box to the first row of $\varepsilon=\mu=\nu$.

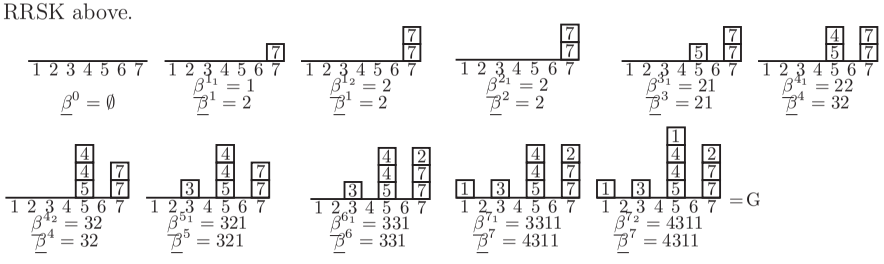
Applying the local rules leads to a pair of nested sequences of partitions on the left column and in the bottom row of the growth diagram. Let β^i be the partition assigned to the i-th thick column, on the bottom row of the growth diagram, when we scan the thick columns from right to left, with the rightmost column being column n. Then the bottom row labelling, assigned to the thick columns of the growth diagram, produces a sequence of partitions $\underline{\beta}^0\supseteq\cdots\supseteq\underline{\beta}^{n-1}\supseteq\underline{\beta}^n=\emptyset$, such that $\underline{\beta}^{i-1}/\underline{\beta}^i$ is a horizontal strip. Let β^i be the partition assigned to the i-th thick row, on the left of the growth diagram, when we scan the thick rows from top to bottom, with the top row being row n. Then the left column labelling, assigned to the thick rows of the growth diagram, produces a sequence of partitions $\emptyset=\beta^n\subseteq\beta^{n-1}\subseteq\cdots\subseteq\beta^0$, such that β^{i-1}/β^i is a horizontal strip. Filling in, with i, the cells of β^{i-1}/β^i and $\underline{\beta}^{i-1}/\underline{\beta}^i$, for $i\geq 1$, produces,

in this order, the pair $(\widetilde{P},\widetilde{Q})$ of RSSYTs of the same shape. This is the same as applying the reverse RSK to the biword w. For instance, if $w = \begin{pmatrix} 1 & 1 & 2 & 3 & 4 & 4 & 5 & 7 & 7 \\ 2 & 7 & 2 & 4 & 1 & 3 & 3 & 1 & 1 \end{pmatrix}$,



$$\widetilde{P} = \begin{array}{cccc} & & & 1 \\ & & & 3 \\ 4 & 1 & 1 & \\ 7 & 3 & 2 & 2 \end{array} \qquad \widetilde{Q} = \begin{array}{cccc} & & & 1 \\ & & & 3 \\ 5 & 4 & 2 & \\ 7 & 7 & 4 & 1 \end{array}$$

with $n = 7$, one has the 7×7 square diagram on the left, the 01-filling in the middle, and the RRSK growth diagram on the right where $(\widetilde{P},\widetilde{Q})$ is the image of w, The map ϱ, defined in Section 2.1, allows to find the pair of SSAFs from the growth diagram of the reverse RSK. Consider the growth diagram bottom labelling, $\underline{\beta}^0 \supseteq \cdots \supseteq \underline{\beta}^{n-1} \supseteq \underline{\beta}^n = \emptyset$, assigned to the thick columns. For each $i = 1,\dots,n$, let $\underline{\beta}^{i_{l_i-1}} \supseteq \cdots \supseteq \underline{\beta}^{i_1}$, with $\underline{\beta}^{i_1-1} := \underline{\beta}^i$, be the bottom sequence of partitions labelling the $l_i - 1$ thin columns, strictly in between the two thick columns $i-1$ and i. *Start with the empty partition $\underline{\beta}^n = \emptyset$ and the empty SSAF with basement $[n]$. Proceed to the left along the growth diagram bottom row labelling. When we arrive to the partition β^{i_j}, we put a cell, filled with i, in the leftmost possible place of the SSAF such that the shape of the new SSAF is a rearrangement of the partition β^{i_j} and the decreasing property on the columns of the SSAF, from the bottom to the top, is preserved. At the end of the scanning of the bottom row labelling, the SSAF G is obtained.* Its shape is a rearrangement of the shape of $\tilde{Q}$. Similarly, consider the left side labelling, $\emptyset = \beta^n \subseteq \beta^{n-1} \subseteq \cdots \subseteq \beta^0$, assigned to the thick rows of the growth diagram. For each $i = 1,\dots,n$, let $\beta^{i_1} \subseteq \cdots \subseteq \beta^{i_{l_i-1}}$, with $\beta^{i_1-1} := \beta^i$, be the sequence of partitions labelling the $l_i - 1$ thin rows, strictly in between the two thick rows $i-1$ and i. At the end of the procedure, when the scanning of the left side labelling is finished, the SSAF F is obtained. Its shape is a rearrangement of the shape of $\tilde{P}$. See the example below corresponding to the RRSK above.



6. A combinatorial Cauchy kernel expansion over near staircases

Let λ be the Ferrers shape as in figure $(\star)$. We are finally equipped for the bijective proof of the F_λ expansion. Before, we still need some definitions and a technical lemma. Let SSYT_n and SSAF_n denote the set of all SSYTs and SSAFs on the alphabet $[n]$, respectively. We assume the reader familiar with the definitions and basic properties of Demazure operators, or divided differences, π_i and $\hat{\pi}_i$, key polynomials, or Demazure characters, κ_ν, and Demazure atoms $\widehat{\kappa}_\nu$, where $\nu \in \mathbb{N}^n$, and refer the reader to [8] and [13]. In particular, recall the action of Demazure operators π_i on the key polynomial κ_ν and on the Demazure atom $\widehat{\kappa}_\nu$: $\pi_i\kappa_\nu = \kappa_{s_i\nu}$, if $\nu_i > \nu_{i+1}$, and else, κ_ν; and $\pi_i\widehat{\kappa}_\nu = \widehat{\kappa}_{s_i\nu} + \widehat{\kappa}_\nu$, if $\nu_i > \nu_{i+1}$, else, $\widehat{\kappa}_\nu$, if $\nu_i = \nu_{i+1}$ and, 0, if $\nu_i < \nu_{i+1}$. Recall also the combinatorial formulas in terms of SSYTs, [9], and SSAFs, [12],

$$\hat{\kappa}_\nu = \sum_{\substack{T\in SSYT_n \\ K_+(T)=key(\nu)}} x^T = \sum_{\substack{F\in SSAF_n \\ sh(F)=\nu}} x^F, \text{ and } \kappa_\nu = \sum_{\substack{T\in SSYT_n \\ K_+(T)\le key(\nu)}} x^T = \sum_{\substack{F\in SSAF_n \\ sh(F)\le\nu}} x^F.$$

Given $0 \le p \le k < n$, let $k-p < r_1 < \cdots < r_p < n$ and $p < e_{p+1} < \cdots < e_k < n$. For each $(z,t) \in [0,p] \times [0,k-p]$, and each $H_z = \{i_1 < \cdots < i_z\} \in \binom{[p]}{z}$, and each $M_t = \{j_1 < \cdots < j_t\} \in \binom{[p+1\ k]}{t}$, define, for $z = t = 0$, $\mathcal{A}^{\emptyset,\emptyset} = \{ {}_{(F,G)\in SSAF_n^2:\ sh(G)\le\omega sh(F)} \}$, and else

$$\mathcal{A}^{H_z,M_t}_{z,t} = \left\{ {}_{(F,G)\in SSAF_n^2:} \begin{array}{c} s_{e_{j_t}}\cdots s_{e_{j_1}} sh(G) \not\le \omega s_{r_{i_z}}\cdots\widehat{s}_{r_{i_m}}\cdots s_{r_{i_1}} sh(F),\ m=1,2,\dots,z \\ s_{e_{j_t}}\cdots\widehat{s}_{e_{j_l}}\cdots s_{e_{j_1}} sh(G) \not\le \omega s_{r_{i_z}}\cdots s_{r_{i_1}} sh(F),\ l=1,2,\dots,t \\ s_{e_{j_t}}\cdots s_{e_{j_1}} sh(G) \le \omega s_{r_{i_z}}\cdots s_{r_{i_1}} sh(F) \end{array} \right\}.$$

Lemma 3. (NW-SE Separation) *Given $0 \le p \le k < n$ and $k-p < r_1 < r_2 < \cdots < r_p < n$ and $p < e_{p+1} < \cdots < e_k < n$. For each $(z,t) \in [0,p] \times [0,k-p]$, and each $H_z = \{i_1 < i_2 < \cdots < i_z\} \in \binom{[p]}{z}$, and each $M_t = \{p+2 \le j_1 < j_2 < \cdots < j_t\} \in \binom{[p+2\ k]}{t}$, let $M^1_{t+1} := \{p+1\} \cup M_t$, and*

$$\mathcal{B}^{H_z,M_t}_{z,t} := \left\{ {}_{(F,G)\in SSAF_n^2:} \begin{array}{c} {\scriptstyle sh(G)_{e_{p+1}} < sh(G)_{e_{p+1}+1}} \\ \begin{array}{c} s_{e_{j_t}}\cdots s_{e_{j_1}} s_{e_{p+1}} sh(G) \not\le \omega s_{r_{i_z}}\cdots\widehat{s}_{r_{i_m}}\cdots s_{r_{i_1}} sh(F),\ m=1,2,\dots,z \\ s_{e_{j_t}}\cdots\widehat{s}_{e_{j_l}}\cdots s_{e_{j_1}} s_{e_{p+1}} sh(G) \not\le \omega s_{r_{i_z}}\cdots s_{r_{i_1}} sh(F),\ l=1,2,\dots,t \\ s_{e_{j_t}}\cdots s_{e_{j_1}} s_{e_{p+1}} sh(G) \le \omega s_{r_{i_z}}\cdots s_{r_{i_1}} sh(F) \end{array} \end{array} \right\}.$$

Then $\mathcal{B}^{H_z,M_t}_{z,t} = \{(F,G) \in \mathcal{A}^{H_z,M_t}_{z,t} : sh(G)_{e_{p+1}} < sh(G)_{e_{p+1}+1}\} \cup \mathcal{A}^{H_z,M^1_{t+1}}_{z,t+1}$.

Let $x = (x_1,\dots,x_n)$ and $y = (y_1,\dots,y_n)$ be two sequences of indeterminates. If $\pi_{e_{p+1}}$ is the Demazure operator with respect to y, one has $\sum_{\beta\in\mathbb{N}^n} \pi_{e_{p+1}} \sum_{\substack{(F,G)\in\mathcal{A}^{H_z,M_t}_{z,t} \\ sh(G)=\beta}} x^F y^G =$

$$\sum_{\substack{(F,G)\in\mathcal{A}^{H_z,M_t}_{z,t} \\ sh(G)_{e_{p+1}} \ge sh(G)_{e_{p+1}+1}}} x^F y^G + \sum_{(F,G)\in\mathcal{B}^{H_z,M_t}_{z,t}} x^F y^G. \quad (2)$$

Theorem 4. *Let $0 \le p \le k < n$. Let $k-p < r_1 < r_2 < \cdots < r_p < n$ and $p < e_{p+1} < \cdots < e_k < n$, with $p+1 < e_{p+1}$, if $p > 0$. For each $(z,t) \in [0,p] \times [0,k-p]$, let $(H_z,M_t) \in \binom{[p]}{z} \times \binom{[p+1,k]}{t}$. Then*

thus still $\nu_r = 0$. In the former case, $\nu_r \geq 1$, and due to the cross $\mathbf{X}$ to the right, the local rules force that, in any stage, we do not have $\nu_{r+1} = \nu_r$ in the SSAF. When we arrive to the thick row $r-1$, one has $\nu_r < \nu_{r+1}$. The contribution of the remain crosses in $\mathcal{G}$, below, will not change the inequality $\nu_r < \nu_{r+1}$, due to $\mathbf{X}$.

(*d*) It follows from (*a*) and (*b*) by considering the biword $\overline{w}$ represented in $\mu := \overline{\lambda}$. ■

5.1. The bijection. Given $\nu \in \mathbb{N}^n$, let be the SSAF where, for all i, the column i has ν_i i's. There exists, then, always a SSAF-pair for a given pair of shapes in the same $\mathfrak{S}_n$-orbit. Let w be a biword in lexicographic order on the alphabet $[n]$, and $\Phi(w) = (F, G)$. The pair $(sh(F), sh(G))$ will be called the key-pair of w. One shows that the biwords whose biletters constitute a multiset of cells in a staircase possibly plus a layer of boxes, as in figure $(\star)$, are characterized by inequalities in the Bruhat order satisfied by its key-pair. Notice that in figure $(\star)$, $1 \leq r_k < \cdots < r_{p+1} < r_1 < \cdots < r_p < n$. We start with the case $k = p$.

Theorem 2. (NW or SE layer) *Let w be a biword in lexicographic order on the alphabet $[n]$, with key-pair (ν, β). Let $0 \leq k < n$, and $1 \leq r_1 < \cdots < r_k < n$. The following conditions are equivalent*

(*a*) *w consists of a multiset of cells in the staircase of size n plus the k cells $\binom{n-r_1+1}{r_1+1}, \ldots, \binom{n-r_k+1}{r_k+1}$, each with multiplicity at least one, as in figure $(\star)$ with $k = p$.*

(*b*) *$\beta \leq \omega s_{r_k} \cdots s_{r_2} s_{r_1} \nu$, and $\beta \not\leq \omega s_{r_k} \cdots \widehat{s}_{r_i} \cdots s_{r_1} \nu$, for $1 \leq i \leq k$, where $\widehat{\ }$ means omission.*

(*c*) *$s_{n-r_1} \cdots s_{n-r_k} \beta \leq \omega\nu$, and $s_{n-r_1} \cdots \widehat{s}_{n-r_i} \cdots s_{n-r_k} \beta \not\leq \omega\nu$, for $1 \leq i \leq k$.*

Proof: By induction on k. For $k = 0$, it is the staircase in [7], and in [1]. For $k > 0$, we use Theorem 1, (c), to prove (*a*) $\Leftrightarrow$ (*b*). The detailed proof can be found in [2]. Once we have proved this, (*a*) $\Leftrightarrow$ (*c*) follows now from Theorem 1, (d). ■

As a consequence of this theorem, if ν, $\beta \in \mathbb{N}^n$ satisfy the inequalities in (*b*), then $\beta \not\leq \omega s_{i_t} \cdots s_{i_1} \nu$, for $0 \leq t < k$, and $i_1 < \cdots < i_t$ any subsequence of $r_1 < \cdots < r_k$. Hence, Lemma 1 implies.

Lemma 2. *Let $0 \leq p \leq k < n$, and $1 \leq r_1 < \cdots < r_k < n$. The three conditions are equivalent*

(*a*) *$\beta \not\leq \omega s_{r_k} \cdots \widehat{s}_i \cdots s_{r_1} \nu$, for $i = 1, \ldots, k$, and $\beta \leq \omega s_{r_k} \cdots s_{r_1} \nu$.*

(*b*) *$s_{n-r_1} \cdots \widehat{s}_{n-r_i} \cdots s_{n-r_k} \beta \not\leq \omega\nu$, for $i = 1, \ldots, k$, and $s_{n-r_1} \cdots s_{n-r_k} \beta \leq \omega\nu$.*

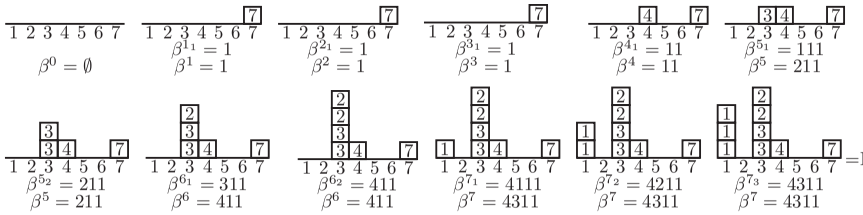
(*c*) *$s_{n-r_{k-p+1}} \cdots s_{n-r_k} \beta \not\leq \omega s_{r_{k-p}} \cdots \widehat{s}_{r_i} \cdots s_{r_1} \nu$, for $i = 1, \ldots, k-p$, $s_{n-r_{k-p+1}} \cdots \widehat{s}_{n-r_i} \cdots s_{n-r_k} \beta \not\leq \omega s_{r_{k-p}} \cdots s_{r_1} \nu$, for $i = k-p+1, \ldots, k$, and $s_{n-r_{k-p+1}} \cdots s_{n-r_k} \beta \leq \omega s_{r_{k-p}} \cdots s_{r_1} \nu$.*

Theorem 2 can now be written according to figure $(\star)$, with the generic cutting line. The condition $r_{p+1} - r_1 > 1$, if $p > 0$, allows to use the commutation relation of $\mathfrak{S}_n$.

Theorem 3. (NW-SE layer) *Let w be a biword in lexicographic order on the alphabet $[n]$, with key-pair (ν, β). Let $0 \leq p \leq k < n$, and $1 \leq r_k < \cdots < r_{p+1} < r_1 < \cdots < r_p < n$, where $r_1 - r_{p+1} > 1$, if $p > 0$. The following conditions are equivalent*

(*a*) *w consists of a multiset of cells in the staircase of size n and the k cells $\binom{e_k+1}{r_k+1}, \ldots, \binom{e_{p+1}+1}{r_{p+1}+1}$, $\binom{n-r_1+1}{r_1+1}, \ldots, \binom{n-r_p+1}{r_p+1}$ above it, each with multiplicity at least one, as shown in figure $(\star)$.*

(*b*) *$s_{e_k} \cdots s_{e_{p+1}} \beta \not\leq \omega s_{r_p} \cdots \widehat{s}_{r_i} \cdots s_{r_1} \nu$, for $i = 1, \ldots, p$, $s_{e_k} \cdots \widehat{s}_{e_i} \cdots s_{e_{p+1}} \beta \not\leq \omega s_{r_p} \cdots s_{r_1} \nu$, for $i = p+1, \ldots, k$, and $s_{e_k} \cdots s_{e_{p+1}} \beta \leq \omega s_{r_p} \cdots s_{r_1} \nu$.*



4. The Bruhat order in $\mathfrak{S}_n$ and orbits.

Let $\theta = \theta_1 \ldots \theta_n \in \mathfrak{S}_n$, written in one line notation. A pair (i, j), with $i < j$, such that $\theta_i > \theta_j$, is said to be an inversion of θ, and $\ell(\theta)$ denotes the number of inversions of θ. The Bruhat order in $\mathfrak{S}_n$ is the partial order in $\mathfrak{S}_n$ defined by the transitive closure of the relations: $\theta < t\theta$, if $\ell(\theta) < \ell(t\theta)$, with t a transposition, $\theta \in \mathfrak{S}_n$. Let $\theta = s_{i_N} \cdots s_{i_1}$ be a decomposition of θ into simple transpositions $s_i = (i\ i+1)$, $1 \leq i < n$. When $N = \ell(\theta)$, the number N in a such decomposition is minimised, and it is said to be a reduced decomposition of θ. The longest permutation of $\mathfrak{S}_n$ is denoted by ω. Let λ be a partition in $\mathbb{N}^n$. The Bruhat ordering of the orbit of λ, $\mathfrak{S}_n\lambda$, is defined by taking the transitive closure of the relations $\alpha < t\alpha$, if $\alpha_i > \alpha_j$, $i < j$, and t the transposition $(i\,j)$, $\alpha \in \mathfrak{S}_n\lambda$. Given $\alpha \in \mathbb{N}^n$, a pair (i, j), with $i < j$, such that $\alpha_i < \alpha_j$, is called an inversion of α, and $\iota(\alpha)$ denotes the number of inversions of α. We may write $\alpha < \beta$ if $\iota(\alpha) < \iota(\beta)$ and $\beta = \tau\alpha$ for some permutation τ in $\mathfrak{S}_n$ that can be written as a product of transpositions each increasing the number of inversions when passing from α to β. Recalling the exchange condition and the property of the Bruhat ordering which says that if $\theta \leq \sigma$ and s is a simple transposition then either $s\theta \leq \sigma$ or $s\theta \leq s\sigma$, [4], one has the useful.

Lemma 1. *Let α, $\beta \in \mathbb{N}^n$ be rearrangements of each other. Let $1 \leq k < n$ and $1 \leq r_1 < r_2 < \ldots < r_k < n$. Then*

(*a*) *$\alpha > s_{r_1}\alpha \Rightarrow s_{r_k} \ldots s_{r_2}\alpha > s_{r_k} \ldots s_{r_2} s_{r_1}\alpha$ and $\alpha < s_{r_1}\alpha \Rightarrow s_{r_k} \ldots s_{r_2}\alpha < s_{r_k} \ldots s_{r_2} s_{r_1}\alpha$.*

(*b*) *$\alpha > s_{r_1}\alpha$ and $s_{r_{k-1}} \ldots s_{r_2}\alpha > s_{r_k} s_{r_{k-1}} \ldots s_{r_2}\alpha \Rightarrow s_{r_{k-1}} \ldots s_{r_2} s_{r_1}\alpha > s_{r_k} s_{r_{k-1}} \ldots s_{r_2} s_{r_1}\alpha$.*

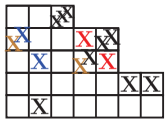
(*c*) *$\beta \not\leq \omega s_{r_k} \cdots \widehat{s}_{r_i} \cdots s_{r_1}\alpha$, $1 \leq i < k$, and $\beta \leq \omega\ s_{r_k} \cdots\ s_{r_1}\ \alpha \Rightarrow s_{r_k} \cdots s_{r_1}\alpha < \cdots < s_{r_1}\alpha < \alpha$.*

(*d*) *$\beta \leq s_k\alpha$ and $\beta \not\leq \alpha$ iff $s_k\beta \leq \alpha$ and $\beta \not\leq \alpha$. Moreover, $\alpha_k > \alpha_{k+1}$ and $\beta_k < \beta_{k+1}$.*

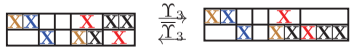
5. Creation and annihilation of corner cells in a Ferrers shape.

Crystal operators or coplactic operations e_r, f_r, $1 \leq r < n$, are defined on any word over the alphabet $[n]$, see [10] and [6]. These operations can be extended to biwords in two ways. Either by considering w in lexicographic order, with respect to the first row, or to the second. Let $w = \binom{\mathbf{u}}{\mathbf{v}}$ be in lexicographic order, with respect to the first row. Write $e_r w := \binom{\mathbf{u}}{e_r\mathbf{v}}$ and, similarly, for $f_r w$. Let $\binom{\mathbf{k}}{\mathbf{l}}$ be the biword w rearranged in lexicographic order, with respect to the second row. Write $\overline{w} := \binom{\mathbf{v}}{\mathbf{u}}$ and $e_r^* w := e_r\overline{w} = \binom{\mathbf{l}}{e_r\mathbf{k}}$ and, similarly, for $f_r^* w$. The resulting biwords are still in lexicographic order with respect to the first row. A biword w

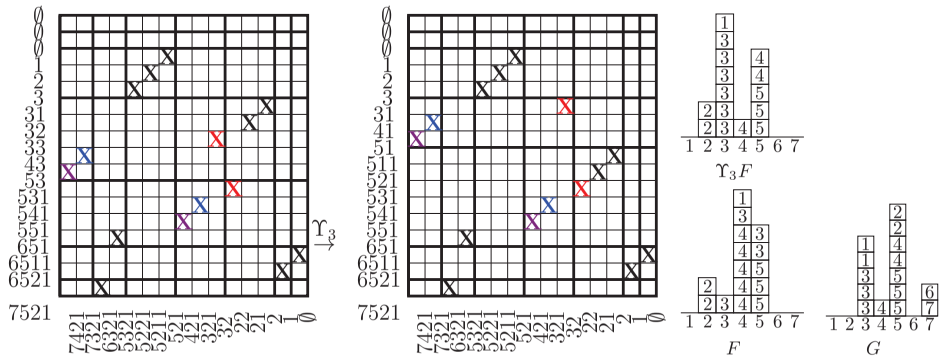
can be seen as a multiset of cells in a Ferrers shape λ, embedded in a $n \times n$ square by putting a cross "$\mathbf{X}$" in the cell (i,j) of λ for each biletter $\binom{j}{i}$ in w. As our running example, consider the biword $w = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & 5 & 5 & 5 & 6 & 7 \\ 4 & 4 & 1 & 3 & 5 & 5 & 5 & 3 & 3 & 4 & 3 & 4 & 4 & 2 & 2 \end{pmatrix}$ in lexicographic order, with respect to the first row, over the alphabet [7], and its representation in the Ferrers shape



. The crystal operator e_3 acts on w through its action on the second row of w. Ignore all entries different from 3 and 4, obtaining the subword 443334344. Match, in the usual way, all 43 (in blue in the example below), remaining the subword 344. Change to 3 the leftmost 4, giving 334. Applying again the operator e_3 means to apply e_3 to the subword 344 which change to 333, obtaining $\begin{pmatrix} 1 & 1 & 2 & 4 & 4 & 4 & 5 & 5 & 5 \\ 4 & 4 & 3 & 3 & 3 & 4 & 3 & 4 & 4 \end{pmatrix} \overset{e_3^2}{\underset{f_3^2}{\rightleftarrows}}$ $\begin{pmatrix} 1 & 1 & 2 & 4 & 4 & 4 & 5 & 5 & 5 \\ 4 & 4 & 3 & 3 & 3 & 4 & 3 & 3 & 3 \end{pmatrix}$. The action of e_r or f_r, as long as it is possible, on the second row of w, translates to the rows r and $r+1$ of the Ferrers shape as a matching of crosses followed by sliding the unmatched crosses in those rows. Write Υ_r ($\overline{\Upsilon}_r$) for applying m times the crystal operator e_r (f_r), to the second row of w, where m is the number of unmatched $r+1$ (r) in the second row of w, then



. Consider now the 01-filling representation of the biwords w and $\Upsilon_r w$ in the Ferrers shape λ embedded in a rectangle shape. Apply the local rules, as defined in Section 3. Notice that in the 01-filling of w, we match a cross in row $r+1$ with a cross to the SE, in row r, such that in these two rows there is no unmatched cross in a column between them. These two growth diagrams have the same bottom sequences of partitions and the left sequences differ only in the partitions assigned to the rows r and $r+1$. Let w_r and $\widetilde{w}_r$ be the biwords obtained from w and $\Upsilon_r w$, after deleting all the biletters with bottom rows different from r and $r+1$. The translation of the movement of the cells in the Ferrers shape to the 01-filling is as follows. In the 01-filling of w_r move up, without changing of columns, the matched crosses of row $r+1$, say s crosses, to the top most s rows such that they form SW chain. Then slide down the remaining unmatched crosses, from row $r+1$ to row r, without changing of columns, such that these crosses and all the crosses of row r form a SW chain. The result is the 01-filling corresponding to $\widetilde{w}_r$. We illustrate with our running example.



Theorem 1. *Let λ be a Ferrers shape and w a biword consisting of a multiset of cells of λ containing the cell $(r+1, \lambda_{r+1})$ with multiplicity at least one. Let $\Phi(w) = (F, G)$ where $sh(F) = \nu$ and $sh(G) = \beta$. The following holds*

(a) If $\nu_r < \nu_{r+1}$, for some $r \geq 1$, then $sh(\Upsilon_r F) = s_r \nu$.

(b) $\Phi(\Upsilon_r w) = (\Upsilon_r F, G)$ and $\Phi(\Upsilon_r \overline{w}) = (\Upsilon_r G, F)$.

(c) If $\lambda_r = \lambda_{r+1} > \lambda_{r+2} \geq 0$, for some $r \geq 1$, then $\nu_r < \nu_{r+1}$ and $sh(\Upsilon_r F) = s_r \nu$. Moreover, $\Upsilon_r w$ fits the Ferrers shape λ with the cell $(r+1, \lambda_{r+1})$ deleted.

(d) If $\mu := \overline{\lambda}$ is the transpose of the Ferrers shape λ , and $\mu_r = \mu_{r+1} > \mu_{r+2} \geq 0$, for some $r \geq 1$, then $\beta_r < \beta_{r+1}$ and $sh(\Upsilon_r G) = s_r \beta$. Moreover, $\Upsilon_r \overline{w}$ fits the Ferrers shape λ with the cell $(\mu_{r+1}, r+1)$ deleted.

Proof: (*a*) This is a consequence of the inductive definition of Demazure crystal in the crystal structure of tableaux. [12] has translated the operator f_r on SSYT's into the operator Θ_r on SSAF's.We use it to show how the $sh(F)$ change under the action of Υ_r. Since $\nu_r < \nu_{r+1}$, from Remark 1, the triples in columns r and $r+1$ of F are so that if r appears k times in the column r, there should be $r+1$ at least $k+1$ times in the column $r+1$. Hence there is at least one unmatched $r+1$ in column $r+1$. The reverse of the operator Θ_r, the analogue of e_r, applied as long as possible, gives $sh(\Upsilon_r F) = s_r \nu$.

(*b*) Consider the 01-filling of the biwords w and $\Upsilon_r w$ and their RRSK growth diagrams. The operations e_r and f_r are coplactic, see [10]. The two growth diagrams have then the same bottom sequence of partitions and $\Phi(\Upsilon_r w) = (\Upsilon_r F, G)$. Transposing the RRSK growth diagram, through the NE-SW diagonal, gives the RRSK growth diagrams of $\overline{w}$ and $\Upsilon_r \overline{w}$, respectively. Thus $\Phi(\Upsilon_r \overline{w}) = (\Upsilon_r G, F)$.

(*c*) One just has to prove that $\nu_r < \nu_{r+1}$. Let $\mathcal{G}$ be the growth diagram of the RRSK of w. Consider its left side. Since $\lambda_{r+1} > \lambda_{r+2}$, the partition, in the thin row below the thick row $r+1$ of $\mathcal{G}$, has one more component. At this level, then the SSAF has $\nu_{r+1} > 0$. When we arrive at the thick row r, the SSAF has $\nu_{r+1} \geq 1$ and $\nu_i = 0$, for $i \leq r$. Since $\lambda_r = \lambda_{r+1}$, $\mathcal{G}$ has at least one cross, say $\mathbf{X}$, contributing with $r+1$, to the right of the rightmost cross between thick rows r and $r-1$. Either a new component is created in the partition, immediately below thick row r, or not. In the last case, when we arrive at thick row $r-1$, we have $\nu_r = 0$, and the crosses below will not sit a box on column r of the SSAF,