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STRATIFYING IDEALS AND TWISTED PRODUCTS

ANA PAULA SANTANA AND IVAN YUDIN

Dedicated to Manuela Sobral

Abstract: We study stratifying ideals for rings in the context of relative homological algebra. Using LU-decompositions, which are a special type of twisted products, we give a sufficient condition for an idempotent ideal to be (relative) stratifying.

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1. Introduction

The notion of stratifying ideal was introduced, almost simultaneously, in several articles, although under different names. The first reference we could find to these ideals is [4], where they are considered, without any proper designation, in the context of quasi-hereditary algebras. Then they were studied in [1] under the name of *strong idempotent ideals*. Almost simultaneously, in [7], the notion of homological epimorphism was introduced. Stratifying ideals are exactly the kernels of surjective homological epimorphisms of rings. The term *stratifying ideal* seems to appear for the first time in [5].

Our interest in stratifying ideals was motivated by the problem of constructing minimal projective resolutions. In fact, let Λ be a finite dimensional algebra over a field. One of the many ways to define a stratifying ideal of Λ is the following. Given an idempotent e in Λ, denote by J the ideal $\Lambda e\Lambda$ and by $\overline{\Lambda}$ the quotient ${}^{\Lambda}\!/_{J}$. The ideal J is stratifying if, for any $\overline{\Lambda}$-module M and projective resolution $P_\bullet \to M$ of M over Λ, the complex $\overline{\Lambda} \otimes_\Lambda P_\bullet \to M$ is a projective resolution of M over $\overline{\Lambda}$. Moreover, if $P_\bullet \to M$ is minimal, then the same is true for $\overline{\Lambda} \otimes_\Lambda P_\bullet \to M$. Therefore one can construct minimal projective resolutions over $\overline{\Lambda}$ by constructing them first over Λ and then applying the functor $\overline{\Lambda} \otimes_\Lambda -$.

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It is usually quite difficult to verify if a given ideal of Λ is stratifying. It is well known that hereditary ideals are stratifying. More generally, idempotent ideals of Λ which are projective left Λ-modules are stratifying.

The aim of this paper is to give a new sufficient condition, Theorem 3.10, for an idempotent ideal $\Lambda e \Lambda$ to be stratifying. This result will be used in our work on homological properties of (quantised) Schur algebras (see [10, 6]).

The paper is organized as follows. In Section 2 we give a short overview of relative homological algebra over rings with identity, and define the bar resolution and *relative* stratifying ideals in this context. We also relate our definition of relative stratifying ideal to the usual definition of stratifying ideal of a finite dimensional algebra over a field.

In the first part of Section 3 we define twisted products for relative pairs. Proposition 3.4 relates twisted products and bar resolutions. The second part of Section 3 is dedicated to LU-decompositions of a ring A with a fixed idempotent e. In Theorem 3.10 we prove that AeA is a relative stratifying ideal if A admits an LU-decomposition.

2. Relative homological algebra

In this section we recall the definitions and results in relative homological theory that we will use in the article. All these notions and results are given in terms of left modules, but they can also be applied to right modules, with appropriate changes in formulae if necessary. By an A-module we mean a left A-module and we write A-mod for the category of left A-modules.

Relative homological theory was originally developed in [8] and we follow the terminology used in this work. A more detailed (and slightly more general) treatment of this topic can be found in Chapter VIII of [9].

Let A be a ring with identity 1 and S a subring of A containing 1. We will refer to (A, S) as a *relative pair*. An exact sequence of A-modules will be called (A, S)-*exact* if the kernel of every differential is an S-direct summand of the corresponding object. Equivalently a complex of A-modules

$$\dots \to M_k \xrightarrow{d_k} M_{k-1} \to \dots$$

is (A, S)-exact if there are S-homomorphisms $s_k\colon M_k \to M_{k+1}$ such that $d_{k+1}s_k + s_{k-1}d_k = \mathrm{id}_{M_k}$ for all meaningful values of k.

We say that an A-module P is (A, S)-*projective* if for every short (A, S)-exact sequence

$$0 \to X \to M \xrightarrow{f} N \to 0 \tag{2.1}$$

and every A-homomorphism $g\colon P \to N$ there is an A-homomorphism $h\colon P \to M$ such that the diagram

$$\begin{tikzcd} & & & P \arrow[dl, "h"'] \arrow[d, "g"] & \\ 0 \arrow[r] & X \arrow[r] & M \arrow[r, "f"] & N \arrow[r] & 0 \end{tikzcd}$$

commutes. In other words, P is (A, S)-projective if for every short (A, S)-exact sequence (2.1), the map

$$\mathrm{Hom}_A(P, f)\colon \mathrm{Hom}_A(P, M) \to \mathrm{Hom}_A(P, N)$$

is surjective.

Remark 2.1. *Obviously every projective A-module is (A, S)-projective. On the other hand, if S is a semisimple ring, then every (A, S)-projective module is projective. In fact, in this case, every exact sequence is automatically (A, S)-exact. Therefore the condition for an A-module to be (A, S)-projective coincides with the condition for it to be projective.*

Example 2.2. Let V be an S-module. Then by Lemma 2 in [8] and subsequent considerations, the A-module $A \otimes_S V$ is (A, S)-projective. Moreover, an A-module M is (A, S)-projective if and only if it is isomorphic to a direct A-module summand of $A \otimes_S V$, for some S-module V. Modules of the form $A \otimes_S V$ will be called (A, S)-*free*.

It is interesting to note that (A, S)-projective modules behave well under change of base rings.

Lemma 2.3. *Let (A, S) and (R, D) be relative pairs, and $\phi\colon A \to R$ a homomorphism of rings such that $\phi(S) \subset D$ and $\phi(1) = 1$. Suppose P is an (A, S)-projective module. Then $R \otimes_A P$ is an (R, D)-projective module.*

Proof: We know from Example 2.2 that P is isomorphic to a direct summand of $A \otimes_S V$, for some S-module V. Since the functor $R \otimes_A -$ is additive, the R-module $R \otimes_A P$ is isomorphic to a direct summand of the free (R, D)-module

$$R \otimes_A A \otimes_S V \cong R \otimes_S V \cong R \otimes_D D \otimes_S V.$$

This shows that $R \otimes_A P$ is (R, D)-projective. ■

An (A, S)-exact sequence of left A-modules

$$\cdots \to P_k \xrightarrow{d_k} P_{k-1} \to \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$$

is called an (A,S)*-projective resolution* of $M \in A$-mod if each P_k is an (A,S)-projective module.

Next we describe the bar resolution $B(A,S,M)$ of $M \in A$-mod. This construction will provide an (A,S)-projective resolution for M. We set

$$B_{-1}(A,S,M) = M \ \text{ and } \ B_k(A,S,M) = A^{\otimes_S^{(k+1)}} \otimes_S M, \ \text{ all } \ k \geq 1,$$

where $A^{\otimes_S^l}$ stands for the lth tensor power of A over S. Now we define A-module homomorphisms $d_{kj}\colon B_k(A,S,M) \to B_{k-1}(A,S,M)$, $0 \leq j \leq k$, and S-module homomorphisms $s_k\colon B_k(A,S,M) \to B_{k+1}(A,S,M)$ by

$$\begin{aligned}
d_{0,0}(a \otimes m) &= am,\\
d_{k,0}(a_0 \otimes a_1 \otimes \cdots \otimes a_k \otimes m) &= a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_k \otimes m,\\
d_{k,j}(a_0 \otimes a_1 \otimes \cdots \otimes a_k \otimes m) &= a_0 \otimes a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_k \otimes m,\\
&\qquad\qquad\qquad\qquad\qquad 1 \leq j \leq k-1,\\
d_{k,k}(a_0 \otimes a_1 \otimes \cdots \otimes a_k \otimes m) &= a_0 \otimes a_1 \otimes \cdots \otimes a_{k-1} \otimes a_k m,\\
s_{-1}(m) &= 1 \otimes m,\\
s_k(a_0 \otimes a_1 \otimes \cdots \otimes a_k \otimes m) &= 1 \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_k \otimes m, \ 0 \leq k.
\end{aligned}$$

Define $d_k\colon B_k(A,S,M) \to B_{k-1}(A,S,M)$ by $d_k = \sum_{t=0}^{k}(-1)^t d_{k,t}$. Then one can verify (cf. [10, Section 3]) that

$$d_0 s_{-1} = \mathrm{id}_{B_{-1}(A,S,M)}, \tag{2.2}$$

$$d_{k+1}s_k + s_{k-1}d_k = \mathrm{id}_{B_k(A,S,M)}, \ k \geq 0, \tag{2.3}$$

$$d_k d_{k+1} = 0, \ k \geq 0. \tag{2.4}$$

We have the following result.

Proposition 2.4. *Let (A,S) be a relative pair and M a left A-module. Then the complex $B(A,S,M) = (B_k(A,S,M), d_k)_{k \geq -1}$ is an (A,S)-projective resolution of M.*

Proof: From Example 2.2, we know that all the modules $B_k(A,S,M)$, for $k \geq 0$, are (A,S)-projective. From (2.4), it follows that $B(A,S,M)$ is a complex. From (2.2) and (2.3), we get that $B(A,S,M)$ is contractible as a complex of left S-modules. ∎

The (A,S)-projective resolution $B(A,S,M)$ is called the *bar resolution* for M. We will write $\beta(A,S,M)$ for the complex obtained from $B(A,S,M)$ by deleting the term $B_{-1}(A,S,M)$.

The bar resolution for a right A-module N is defined in a similar way to the one described above. It will be denoted by $\overline{B}(N,S,A)$. In [9, Corollary IX.8.2], there it is proved that $B(A,S,A) \cong \overline{B}(A,S,A)$ and

$$B(A,S,M) \cong B(A,S,A) \otimes_A M \text{ and } \overline{B}(N,S,A) \cong N \otimes_A B(A,S,A). \quad (2.5)$$

Using the bar resolution, it is possible to define relative Tor-groups (cf. [9, (IX.8.5)]). Suppose we are given a left A-module M and a right A-module N. Then the *relative* Tor-*groups* are defined as

$$\mathrm{Tor}_k^{(A,S)}(N,M) = H_k\left(N \otimes_A \beta(A,S,A) \otimes_A M\right).$$

Suppose $P_\bullet \to M \to 0$ and $Q_\bullet \to N \to 0$ are (A,S)-projective resolutions. Then, by Theorem IX.8.5 in [9], we have

$$\mathrm{Tor}_k^{(A,S)}(N,M) \cong H_k\left(N \otimes_A P_\bullet\right) \cong H_k\left(Q_\bullet \otimes_A M\right). \quad (2.6)$$

Remark 2.5. *In case S is a semisimple ring, we have $\mathrm{Tor}_k^{(A,S)}(N,M) \cong \mathrm{Tor}_k^A(N,M)$ for all $k \geq 0$. In fact, for S semisimple, $P_\bullet \to M \to 0$ is an (A,S)-projective resolution of M if and only if it is a projective resolution of M as A-module (see Remark 2.1).*

It is now possible to introduce the notion of (A,S)-stratifying ideal. Given a relative pair (A,S) and an idempotent $e \in S$, we write $\overline{A} := A/AeA$.

Definition 2.6. The ideal AeA is called (A,S)-*stratifying* if

$$\mathrm{Tor}_k^{(A,S)}(\overline{A},\overline{A}) = 0,$$

for all $k \geq 1$.

This definition of (A,S)-stratifying ideal is closely connected with the definition of stratifying ideal given in [5]. In fact, in many situations they are equivalent.

Proposition 2.7. *Let (A,S) be a relative pair with A a finite dimensional algebra over a field. Suppose that S is a semisimple algebra and $e \in S$ is an idempotent. Then the following conditions are equivalent.*

(1) *The ideal AeA is (A,S)-stratifying;*
(2) *AeA is a strong idempotent ideal in the sense of* [1]*;*
(3) *$A \to \overline{A}$ is a homological epimorphism in the sense of* [7]*;*
(4) *AeA is a stratifying ideal in the sense of* [5].

Proof: If S is semisimple, we know, from Remark 2.5, that $\mathrm{Tor}_k^{(A,S)}(\overline{A},\overline{A}) \cong \mathrm{Tor}_k^{A}(\overline{A},\overline{A}),\ \ k \geq 0$. Therefore, by Proposition 1.3(iv') in [1], the ideal AeA is (A,S)-stratifying if and only if it is strong idempotent in the sense of [1].

Since $A \to \overline{A}$ is an epimorphism of rings, the multiplication map $\overline{A} \otimes_A \overline{A} \to \overline{A}$ is an isomorphism. Then, from Theorem 4.4(1) in [7], it follows that $A \to \overline{A}$ is a homological epimorphism if and only if AeA is (A,S)-stratifying.

Finally, from Theorem 4.4(5') in [7] and Remark 2.1.2(a) in [5], we get that AeA is a stratifying ideal in the sense of [5] if and only if $A \to \overline{A}$ is a homological epimorphism. ■

3. Twisted products

3.1. General definitions and bar resolution. In this section we introduce the notion of a twisted product for relative pairs and discuss bar resolutions in this setting.

Definition 3.1. Let (A,S) be a relative pair and A_1, A_2 subrings of A containing S. We say that (A,S) is a *twisted product* of A_1 and A_2 if the map $\alpha\colon A_1 \otimes_S A_2 \to A$ induced by the multiplication in A is an isomorphism of abelian groups.

If (A,S) is a twisted product of A_1 and A_2, we can define the twisting map $T\colon A_2 \otimes_S A_1 \to A_1 \otimes_S A_2$ as the composition

$$A_2 \otimes_S A_1 \xrightarrow{\mu_A} A \xrightarrow{\alpha^{-1}} A_1 \otimes_S A_2,$$

where μ_A is the multiplication in A. The existence of this map motivated the name "twisted product".

Example 3.2. Let G be a group with identity e_G and H_1, H_2 subgroups of G. Suppose that $H_1 \cap H_2 = \{e_G\}$ and $H_1H_2 = G$. Then one says that G is a *Zappa-Szép product* of H_1 and H_2. Given any commutative ring with identity S, it can be checked that the relative pair (SG,S) is a twisted product of SH_1 and SH_2. The Zappa-Szép product was developed independently by Zappa in [12] and Szép in [11].

Suppose a relative pair (A,S) is a twisted product of subrings A_1 and A_2. Then the endofunctors $A \otimes_S -$, $A_1 \otimes_S -$, and $A_2 \otimes_S -$ on the category of S-mod can be turned into monads using multiplication and units of algebras in the obvious way. Moreover, T induces a natural transformation τ between the functors $A_2 \otimes_S A_1 \otimes_S - \to A_1 \otimes_S A_2 \otimes_S -$. One can check that τ is a distributive law in the sense of [2]. Twisted products of algebras were also studied in [3].

Suppose (A, S) is a twisted product of subrings A_1 and A_2. Then we can consider every A-module as an A_1-module and every A_2-module as an S-module. Thus, we have two functors $A \otimes_{A_2} -$ and $A_1 \otimes_S -$ from A_2-mod to A_1-mod.

Lemma 3.3. *Suppose (A, S) is a twisted product of subrings A_1 and A_2. Then the functors $A \otimes_{A_2} -$, $A_1 \otimes_S -\colon A_2\text{-mod} \to A_1\text{-mod}$ are isomorphic.*

Proof: Given any A_2-module M we define f_M as the composition of the three A_1-isomorphisms natural in M

$$A_1 \otimes_S M \overset{\cong}{\to} A_1 \otimes_S (A_2 \otimes_{A_2} M) \overset{\cong}{\to} (A_1 \otimes_S A_2) \otimes_{A_2} M \xrightarrow{\alpha \otimes_{A_2} M} A \otimes_{A_2} M. \quad (3.1)$$

Then $f := (f_M)_{M \in A_2\text{-mod}}$ is the required isomorphism of fuctors. ■

Proposition 3.4. *Suppose that the relative pair (A, S) is a twisted product of A_1 and A_2. Then for any A_2-module M the complex $A \otimes_{A_2} B(A_2, S, M)$ is an (A, S)-projective resolution of $A \otimes_{A_2} M$.*

Proof: For $k \geq 0$ we have isomorphisms of A-modules

$$A \otimes_{A_2} A_2^{\otimes_S^{(k+1)}} \otimes_S M \cong A \otimes_S A_2^{\otimes_S^k} \otimes_S M.$$

Therefore, by Example 2.2, the modules $A \otimes_{A_2} B_k(A_2, S, M)$ are (A, S)-projective for all $k \geq 0$.

By Lemma 3.3, the functors $A \otimes_{A_2} -$ and $A_1 \otimes_S -$ are isomorphic as functors from the category of A_2-modules to the category of A_1-modules, and so they are also isomorphic as functors to the category of S-modules. Therefore, to show that $A \otimes_{A_2} B(A_2, S, M)$ is splittable as a complex of S-modules, it is enough to show that $A_1 \otimes_S B(A_2, S, M)$ is splittable as a complex of S-modules. Since $A_1 \otimes_S -$ is an additive endofunctor in the category of S-modules, and $B(A_2, S, M)$ is a splittable exact sequence in this category, we get that $A_1 \otimes_S B(A_2, S, M)$ is a splittable exact sequence of S-modules. This shows that $A \otimes_{A_2} B(A_2, S, M)$ is an (A, S)-projective resolution of $A \otimes_{A_2} M$. ■

3.2. LU-twisted products and (A, S)-stratifying ideals. Let (A, S) be a relative pair and $e \in S$ an idempotent. We denote by $\bar{e}$ the idempotent

$1-e$. Given a subring B of A containing S it is convenient to think of B as the matrix ring

$$\begin{pmatrix} eBe & eB\bar{e} \\ \bar{e}Be & \bar{e}B\bar{e} \end{pmatrix}.$$

Note that (eBe, e) and $(\bar{e}B\bar{e}, \bar{e})$ are rings. We will say that B is *upper triangular* if $\bar{e}Be = 0$, *lower triangular* if $eB\bar{e} = 0$, and *diagonal* if $\bar{e}Be = eB\bar{e} = 0$.

We will write $\overline{B}$ for the quotient of B by the ideal BeB.

Proposition 3.5. *Let (A, S) be a relative pair and $e \in S$ an idempotent. Suppose that B is an upper or lower triangular subring of A. Then $\overline{B} \cong \bar{e}B\bar{e}$, where the isomorphism is induced by the inclusion of $\bar{e}B\bar{e}$ into B.*

Proof: We prove the proposition in the case when B is an upper triangular ring. The lower triangular case is similar.

Let $b \in B$. Then $b = (\bar{e} + e)b(\bar{e} + e) = \bar{e}b\bar{e} + ebe + eb\bar{e}$. This shows that $[b] = [\bar{e}b\bar{e}]$ in $\overline{B}$. Thus the map $\phi\colon \bar{e}B\bar{e} \to \overline{B}$, $b \mapsto [b]$ is a surjective ring homomorphism. To check that $\mathrm{Ker}(\phi) = 0$ it is enough to notice that $BeB \cap \bar{e}B\bar{e} \subset \bar{e}BeB\bar{e} = 0$, since $\bar{e}Be = 0$. ■

Definition 3.6. Let (A, S) be a relative pair with S diagonal and $e \in S$ an idempotent. We say that A admits an LU-decomposition if there are subrings L and U of A containing S such that:

(1) (A, S) is a twisted product of L and U;
(2) L is lower triangular;
(3) U is upper triangular.

Before we state and prove the main theorem, we need two technical results. Their proofs use the following proposition, which can be found in [9].

Proposition 3.7. (IX.9.3 [9]) Suppose that the ring R is the direct product of two subrings R_1 and R_2. Given a right R-module N and a left R-module M, there is an isomorphism of abelian groups

$$N \otimes_R M \cong (N \otimes_R R_1) \otimes_{R_1} (R_1 \otimes_R M) \oplus (N \otimes_R R_2) \otimes_{R_2} (R_2 \otimes_R M).$$

Proposition 3.8. *Let (A, S) be a relative pair with S diagonal and $e \in S$ an idempotent. Suppose that (A, S) admits an LU-decomposition with subrings L and U. Then $\overline{A}$ is a twisted product of $\overline{L}$ and $\overline{U}$.*

Proof: To show that $\overline{A}$ is a twisted product of $\overline{L}$ and $\overline{U}$ we first need to prove that $\overline{S}$ can be considered a subring of $\overline{L}$ and $\overline{U}$, and $\overline{L}$ and $\overline{U}$ can be considered as subrings of $\overline{A}$. For this it is enough to verify that $L \cap AeA = LeL$, $U \cap AeA = UeU$ and $S \cap AeA = SeS$.

Using the fact that $eL\bar{e} = \bar{e}Ue = 0$, we have

$$\begin{aligned} AeA = LUeLU &= LeUeLU + L\bar{e}UeLU = LeUeLeU + LeUeL\bar{e}U \subset LeAeU \\ &= LeLUeU = LeLeUeU + LeL\bar{e}UeU = LeLeUeU \subset LeU. \end{aligned}$$

Thus $AeA = LeU$. Since S is diagonal, it is the direct product of the rings (eSe, e) and $(\bar{e}S\bar{e}, \bar{e})$ and we have an isomorphism of abelian groups (cf. Proposition 3.7)

$$\begin{array}{rcccc} \gamma\colon Le \otimes_{eSe} eU \oplus L\bar{e} \otimes_{\bar{e}S\bar{e}} \bar{e}U & \to & L \otimes_S U & \stackrel{\alpha}{\to} & A \\ (a \otimes b, a' \otimes b') & \mapsto & (a+a') \otimes (b+b') & \mapsto & ab + a'b'\,. \end{array} \tag{3.2}$$

Hence $LeU \cap L\bar{e}U = \gamma\,(Le \otimes_{eSe} eU) \cap \gamma\,(L\bar{e} \otimes_{\bar{e}S\bar{e}} \bar{e}U) = 0$. So $Le \cap L\bar{e}U = 0$. Since $L = Le \oplus L\bar{e}$ and $Le \subset LeU$, this implies $L \cap LeU = Le \cap LeU = Le$. Now

$$Le \subset LeL = LeLe \oplus LeL\bar{e} = LeLe \subset Le.$$

Therefore $L \cap AeA = L \cap LeU = Le = LeL$, as required.

In a similar way it can be proved $U \cap AeA = UeU$. To show that $S \cap AeA = SeS$ it is enough to notice that

$$S \cap AeA = S \cap L \cap AeA = S \cap Le = Se \subset SeS = SeSe \oplus SeS\bar{e} = SeSe \subset Se.$$

Thus $S \cap AeA = SeS$.

Next we have to check that the map

$$\begin{aligned} \overline{\alpha}\colon \overline{L} \otimes_{\overline{S}} \overline{U} &\to \overline{A} \\ [l] \otimes [u] &\mapsto [lu] \end{aligned}$$

is an isomorphism. By Proposition 3.5, we know that $\overline{L} \cong \bar{e}L\bar{e}$, $\overline{U} \cong \bar{e}U\bar{e}$ and $\overline{S} \cong \bar{e}S\bar{e}$. Therefore, we can replace $\overline{\alpha}$ by the map

$$\begin{aligned} \beta\colon \bar{e}L\bar{e} \otimes_{\bar{e}S\bar{e}} \bar{e}U\bar{e} &\to \overline{A} \\ l \otimes u &\mapsto [lu]. \end{aligned}$$

Notice that $L\bar{e} = \bar{e}L\bar{e} \oplus eL\bar{e} = \bar{e}L\bar{e}$ and $\bar{e}U = \bar{e}U\bar{e} \oplus \bar{e}Ue = \bar{e}U\bar{e}$. Therefore β can be decomposed in the following way:

$$\begin{tikzcd} L\bar{e} \otimes_{\bar{e}S\bar{e}} \bar{e}U \arrow[r, "\beta"] \arrow[d, "\imath"'] & \overline{A} \\ Le \otimes_{eSe} eU \oplus L\bar{e} \otimes_{\bar{e}S\bar{e}} \bar{e}U \arrow[r, "\cong", "\gamma"'] & A \arrow[u, "\pi"'] \end{tikzcd}$$

Now $\mathrm{Ker}(\pi) = AeA = LeU = \gamma\left(Le \otimes_{eSe} eU\right)$, which implies $\mathrm{Ker}(\pi\gamma) = Le \otimes_{eSe} eU$. Therefore $\mathrm{Ker}(\beta) = 0$ and $\beta\left(L\bar{e} \otimes_{\bar{e}S\bar{e}} \bar{e}U\right) = \overline{A}$. ■

Proposition 3.9. *Let (A, S) be a relative pair with S diagonal and $e \in S$ an idempotent. Suppose that A admits an LU-decomposition with subrings L and U. Then $A \otimes_U \overline{U} \cong \overline{A}$ as A-modules and $\overline{L} \otimes_L A \cong \overline{A}$ as right A-modules.*

Proof: Using Proposition 3.7, we have

$$L \otimes_S \overline{U} \cong Le \otimes_{eSe} e\overline{U} \oplus L\bar{e} \otimes_{\bar{e}S\bar{e}} \bar{e}\overline{U} = L\bar{e} \otimes_{\bar{e}S\bar{e}} \overline{U}.$$

As L is lower triangular, we know that $L\bar{e} = \bar{e}L\bar{e}$. Also $\bar{e}U\bar{e} \to \overline{U}$, $u \mapsto [u]$ is an isomorphism by Proposition 3.5. Therefore, the map

$$\begin{aligned} \phi\colon \bar{e}L\bar{e} \otimes_{\bar{e}S\bar{e}} \bar{e}U\bar{e} &\to L \otimes_S \overline{U} \\ l \otimes u &\mapsto l \otimes [u], \end{aligned}$$

is an isomorphism. We remind the reader that in the proof of Lemma 3.3 we constructed the isomorphism $f_{\overline{U}}\colon L \otimes_S \overline{U} \to A \otimes_U \overline{U}$, given by $l \otimes [u] \mapsto l \otimes [u]$. We write $\psi := f_{\overline{U}}\phi$. Consider the isomorphism $\beta\colon \bar{e}L\bar{e} \otimes_{\bar{e}S\bar{e}} \bar{e}U\bar{e} \to \overline{A}$ constructed in the proof of Proposition 3.8 . Then we have the isomorphisms

$$\begin{tikzcd} & \bar{e}L\bar{e} \otimes_{\bar{e}S\bar{e}} \bar{e}U\bar{e} \arrow[dl, "\beta"'] \arrow[dr, "\psi"] & \\ \overline{A} & & A \otimes_U \overline{U} \end{tikzcd} \qquad \begin{tikzcd} & l \otimes u \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ [lu] & & l \otimes [u] \end{tikzcd}$$

of abelian groups. Write $\tau := \beta\psi^{-1}\colon A \otimes_U \overline{U} \to \overline{A}$. It is our aim to prove that τ is a homomorphism of A-modules. It is obvious that τ is a homomorphism of L-modules. Thus, as A is the twisted product of L and U, to prove that τ is a homomorphism of A-modules it is enough to show that τ is a homomorphism of U-modules. For this, let $u' \in U$ and $l \otimes [u] \in A \otimes_U \overline{U}$.

Then, as $u'l \in A$, we have $u'l = \alpha\left(\sum_{i\in I} l_i \otimes u_i\right) = \sum_{i\in I} l_i u_i$, for some finite set I, $l_i \in L$ and $u_i \in U$. Thus

$$\tau(u'(l\otimes[u])) = \tau\left(\sum_{i\in I} l_i u_i \otimes [u]\right) = \sum_{i\in I} \tau\left(l_i \otimes [u_i u]\right)$$

$$= \sum_{i\in I}[l_i u_i u] = \left[\left(\sum_{i\in I} l_i u_i\right) u\right] = [u'lu] = u'\tau(l\otimes[u]).$$

Therefore $A \otimes_U \overline{U} \cong \overline{A}$ as A-modules. Applying this result to the opposite algebras, we conclude that $\overline{L} \otimes_L A \cong \overline{A}$ as right A-modules. ■

We are now ready to prove the main result of the article.

Theorem 3.10. *Let (A, S) be a relative pair with S diagonal and $e \in S$ an idempotent. Suppose that (A, S) admits an LU-decomposition with subrings L and U. Then AeA is an (A, S)-stratifying ideal.*

Proof: We can consider $\overline{U}$ as a U-module and so, by Proposition 3.4, the complex $A \otimes_U B(U, S, \overline{U})$ is an (A, S)-projective resolution of $A \otimes_U \overline{U}$. We know, from Proposition 3.9, that $A \otimes_U \overline{U} \cong \overline{A}$ as A-modules. Therefore, $A \otimes_U B(U, S, \overline{U})$ gives an (A, S)-projective resolution of $\overline{A}$.

If we show that the complex $\overline{A} \otimes_A A \otimes_U B(U, S, \overline{U})$ is exact, then

$$\operatorname{Tor}_k^{(A,S)}(\overline{A}, \overline{A}) = 0,$$

for all $k \geq 1$, and AeA is an (A, S)-stratifying ideal. We have an obvious isomorphism of complexes

$$\overline{A} \otimes_A A \otimes_U B(U, S, \overline{U}) \cong \overline{A} \otimes_U B(U, S, \overline{U}).$$

Therefore to prove the theorem it is enough to check that $\overline{A} \otimes_U B(U, S, \overline{U})$ is exact. Consider the maps

$$\phi_{-1} \colon \overline{A} \otimes_U \overline{U} \to \overline{A} \otimes_{\overline{U}} \overline{U}$$

$$[a] \otimes [u] \mapsto [a] \otimes [u]$$

$$\phi_k \colon \overline{A} \otimes_U U^{\otimes_S^{(k+1)}} \otimes_S \overline{U} \to \overline{A} \otimes_{\overline{U}} \overline{U}^{\otimes_{\overline{S}}^{(k+1)}} \otimes_{\overline{S}} \overline{U}$$

$$[a] \otimes u_0 \otimes \cdots \otimes u_k \otimes [u] \mapsto [a] \otimes [u_0] \otimes \cdots \otimes [u_k] \otimes [u], \qquad k \geq 0.$$

It is straightforward to verify that $\phi := (\phi_k)_{k\geq -1}$ is a well-defined homomorphism of chain complexes from $\overline{A}\otimes_U B(U,S,\overline{U})$ to $\overline{A}\otimes_{\overline{U}} B(\overline{U},\overline{S},\overline{U})$. By Propositions 3.4 and 3.8, the complex $\overline{A}\otimes_{\overline{U}} B(\overline{U},\overline{S},\overline{U})$ is an $(\overline{A},\overline{S})$-projective resolution of $\overline{A}$, and, in particular, it is exact. To finish the proof we will show that ϕ is an isomorphism, that is that for every $k\geq -1$ the map ϕ_k is an isomorphism.

The map ϕ_{-1} is an isomorphism, since both $\overline{A}$ and $\overline{U}$ get the structure of right and left U-modules, respectively, via the projection $U\to\overline{U}$.

Suppose now that $k\geq 0$. Then

$$\overline{A}\otimes_U U^{\otimes_S^{(k+1)}}\otimes_S \overline{U}\cong \overline{A}\otimes_S U^{\otimes_S^k}\otimes_S\overline{U}$$
$$\overline{A}\otimes_{\overline{U}} \overline{U}^{\otimes_{\overline{S}}^{(k+1)}}\otimes_{\overline{S}} \overline{U}\cong \overline{A}\otimes_{\overline{S}} \overline{U}^{\otimes_{\overline{S}}^k}\otimes_{\overline{S}}\overline{U}.$$

Under these isomorphisms ϕ_k corresponds to

$$\psi_k\colon \overline{A}\otimes_S U^{\otimes_S^k}\otimes_S\overline{U}\to \overline{A}\otimes_{\overline{S}} \overline{U}^{\otimes_{\overline{S}}^k}\otimes_{\overline{S}}\overline{U}.$$

We will write for a moment $e_1=e$, $e_2=\bar{e}$, $S_1=e_1Se_1$ and $S_2=e_2Se_2$. As S is diagonal, using repeatedly Proposition 3.7, we get

$$\overline{A}\otimes_S U^{\otimes_S^k}\otimes_S\overline{U}\cong \bigoplus_{(i_0,\dots,i_k)\in\{1,2\}^{k+1}} \overline{A}e_{i_0}\otimes_{S_{i_0}} e_{i_0}Ue_{i_1}\otimes_{S_{i_1}}\cdots\otimes_{S_{i_{k-1}}} e_{i_{k-1}}Ue_{i_k}\otimes_{S_{i_k}} e_{i_k}\overline{U}.$$

Since $\overline{A}e_1=0$ and $e_2Ue_1=0$, one sees that the only non-zero summand in the above direct sum corresponds to the multi-index $(2,2,\dots,2)$. Therefore, as $\overline{A}e_2=\overline{A}$ and $e_2\overline{U}=\overline{U}$, we have that

$$\psi_k\colon \overline{A}\otimes_{S_2}(\bar{e}U\bar{e})^{\otimes_{S_2}^k}\otimes_{S_2}\overline{U}\to \overline{A}\otimes_{\overline{S}} \overline{U}^{\otimes_{\overline{S}}^k}\otimes_{\overline{S}}\overline{U}$$

$$[a]\otimes u_1\otimes\cdots\otimes u_k\otimes[u]\mapsto [a]\otimes[u_1]\otimes\cdots\otimes[u_k]\otimes[u].$$

But these maps are isomorphisms, since $S_2=\bar{e}S\bar{e}\cong\overline{S}$ and $\bar{e}U\bar{e}\cong\overline{U}$, by Proposition 3.5. ∎

Let $\mathbb{K}$ be a field. Suppose that A is a finite dimensional algebra over $\mathbb{K}$ and $e\in A$ is an idempotent such that AeA is a projective left or right A-module. Then, as we mentioned in the introduction, AeA is a stratifying ideal. Next we give an example of a finite dimensional $\mathbb{K}$-algebra A, with an idempotent e, such that AeA is not projective, although it is a stratifying ideal by Theorem 3.10.

Example 3.11. Given two rings A, B, and an A-B-bimodule M, we have a ring structure on $R := A \oplus M \oplus B$, given by

$$(a_1, m_1, b_1)(a_2, m_2, b_2) := (a_1 a_2, a_1 m_2 + m_1 b_2, b_1 b_2).$$

The ring R is upper triangular with respect the idempotent $(1_A, 0, 0)$,

$$R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix},$$

and is a lower triangular ring with respect the idempotent $(0, 0, 1_B)$,

$$R = \begin{pmatrix} B & 0 \\ M & A \end{pmatrix}.$$

Let $A := \mathbb{K}$, $B := \mathbb{K}[x]\big/\big(x^2\big)$, $M := B/xB$. Note that M is a one-dimensional A-B-bimodule. Let v be a generator of M. Then we get a lower triangular ring

$$L := \begin{pmatrix} \mathbb{K}[x]\big/\big(x^2\big) & 0 \\ \langle v \rangle & \mathbb{K} \end{pmatrix}.$$

Taking $A := \mathbb{K}[y]\big/\big(y^2\big)$, $M := A/Ay$, $B := \mathbb{K}$, once more we have that M is a one-dimensional A-B-bimodule. We denote its generator by w. We get an upper triangular ring

$$U := \begin{pmatrix} \mathbb{K}[y]\big/\big(y^2\big) & \langle w \rangle \\ 0 & \mathbb{K} \end{pmatrix}.$$

Note that both L and U contain the semisimple subring

$$S := \begin{pmatrix} \mathbb{K} & 0 \\ 0 & \mathbb{K} \end{pmatrix}.$$

Let $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in S$ and $\overline{e} = 1_S - e = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then applying Proposition 3.7, we get

$$L \otimes_S U = \begin{pmatrix} eL \otimes_S Ue & eL \otimes_S U\overline{e} \\ \overline{e}L \otimes_S Ue & \overline{e}L \otimes_S U\overline{e} \end{pmatrix} = \begin{pmatrix} \langle e_{11}, x, y, xy \rangle & \langle w, xw \rangle \\ \langle v, vy \rangle & \langle vw, e_{22} \rangle \end{pmatrix}.$$

In the above formula we omitted $\otimes$ between the elements in L and U, wrote e_{ii} for the products $1 \otimes 1$ at position (i,i), and abbreviated $x \otimes 1$ by x, $1 \otimes y$ by y, $1 \otimes w$ by w, $v \otimes 1$ by v.

We will define a multiplication in the vector space $A = L \otimes_S U$, such that

$$l \otimes u = (l \otimes 1_U) \cdot (1_L \otimes u).$$

Then A will be a twisted product of the subalgebras $L \otimes_S S \cong L$ and $S \otimes_S U \cong U$. To define such product it is enough to define the images of the elements of $U \otimes_S L$ under the map

$$\tau \colon U \otimes_S L \xrightarrow{\cong} S \otimes_S U \otimes_S L \otimes_S S \to A \otimes_S A \xrightarrow{\mu_A} A.$$

Moreover, the above map restricted to the subspaces $S \otimes_S L$ and $U \otimes_S S$, should be $s \otimes l \mapsto sl \otimes 1_U$ and $u \otimes s \mapsto 1_L \otimes us$. Applying Proposition 3.7, we get

$$U \otimes_S L = \begin{pmatrix} eU \otimes_S Le & eU \otimes_S L\overline{e} \\ \overline{e}U \otimes_S Le & \overline{e}U \otimes_S L\overline{e} \end{pmatrix}$$

$$= \begin{pmatrix} \langle 1 \otimes 1, y \otimes 1, 1 \otimes x, y \otimes x, w \otimes v \rangle & \langle w \otimes 1 \rangle \\ \langle 1 \otimes v \rangle & \langle 1 \otimes 1 \rangle \end{pmatrix}.$$

So we only need to know $\tau(y \otimes x)$ and $\tau(w \otimes v)$. Define

$$\tau(y \otimes x) = xy, \qquad \tau(w \otimes v) = 0.$$

It is easy to check that the resulting multiplication in $A = L \otimes_S U$ is associative. By construction A is a twisted product of subalgebras $L \otimes_S S$ and $S \otimes_S U$. Thus we can apply Theorem 3.10 to A, and we get that AeA is a stratifying ideal. By direct computation one verifies that

$$\dim(Ae) = 6, \qquad \dim(A\overline{e}) = 4, \qquad \dim(AeA) = 9.$$

Since e and $\overline{e}$ are primitive idempotents, the modules Ae and $A\overline{e}$ are indecomposable projective. Therefore we see that AeA is not projective, as 9 can not be represented as an integral combination of 6 and 4. Thus AeA is an example of a non-projective stratifying ideal.

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Ana Paula Santana
CMUC, Department of Mathematics, University of Coimbra, Coimbra, Portugal
E-mail address: aps@mat.uc.pt

Ivan Yudin
CMUC, Department of Mathematics, University of Coimbra, Coimbra, Portugal
E-mail address: yudin@mat.uc.pt