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HYPERSYMPLECTIC STRUCTURES WITH TORSION ON LIE ALGEBROIDS

P. ANTUNES AND J.M. NUNES DA COSTA

ABSTRACT: Hypersymplectic structures with torsion on Lie algebroids are investigated. We show that each hypersymplectic structure with torsion on a Lie algebroid determines three Nijenhuis morphisms. From a contravariant point of view, these structures are twisted Poisson structures. We prove the existence of a one-to-one correspondence between hypersymplectic structures with torsion and hyperkähler structures with torsion. We show that given a Lie algebroid with a hypersymplectic structure with torsion, the deformation of the Lie algebroid structure by any of the transition morphisms does not affect the hypersymplectic structure with torsion. We also show that if a triplet of 2-forms is a hypersymplectic structure with torsion on a Lie algebroid A, then the triplet of the inverse bivectors is a hypersymplectic structure with torsion for a certain Lie algebroid structure on the dual A^*, and conversely. Examples of hypersymplectic structures with torsion are included.

Introduction

Hypersymplectic structures with torsion on Lie algebroids were introduced in [5], in relation with hypersymplectic structures on Courant algebroids, when these Courant algebroids are doubles of quasi-Lie and proto-Lie bialgebroids. In fact, while looking for examples of hypersymplectic structures on Courant algebroids we found in [5], in a natural way, hypersymplectic structures with torsion on Lie algebroids. A triplet $(\omega_1, \omega_2, \omega_3)$ of non-degenerate 2-forms on a Lie algebroid (A, μ) is a hypersymplectic structure with torsion if the transition morphisms $N_i, i = 1, 2, 3$, satisfy $N_i^2 = -\mathrm{id}_A$ and $N_1 \mathrm{d}\omega_1 = N_2 \mathrm{d}\omega_2 = N_3 \mathrm{d}\omega_3$ (Definition 1.1). These structures can be viewed, in a certain sense, as a generalization of hypersymplectic structures on Lie algebroids, a structure we have studied in [3], but also as being in a one-to-one correspondence with hyperkähler structures with torsion, a notion already known in the literature. Hyperkähler structures with torsion on manifolds, also known as HKT structures, first appear in [10] in relation with sigma models in string theory. Let us briefly recall what a HKT manifold is. Let M be a hyperhermitian manifold, i.e., a manifold equipped with three complex structures N_1, N_2 and N_3 satisfying $N_1N_2 = -N_2N_1 = N_3$

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and a metric g compatible with the three complex structures. If there exists a linear connection ∇ on M such that $\nabla g = 0$, $\nabla N_1 = \nabla N_2 = \nabla N_3 = 0$ and H, defined by $H(X, Y, Z) = g(X, \nabla_Y Z - \nabla_Z Y - [Y, Z])$, is a 3-form on M, then M is a HKT manifold. In [9] it was proved that the condition of H being a 3-form can be substituted by the following equivalent requirement: $N_1 \mathrm{d}\omega_1 = N_2 \mathrm{d}\omega_2 = N_3 \mathrm{d}\omega_3$, where ω_1, ω_2 and ω_3 are the associated Kähler forms. Later, in [7], the authors showed that the assumption of N_1, N_2 and N_3 being Nijenhuis can be removed from the definition of HKT manifold, because the equalities $N_1 \mathrm{d}\omega_1 = N_2 \mathrm{d}\omega_2 = N_3 \mathrm{d}\omega_3$ imply the vanishing of the Nijenhuis torsion of the morphisms N_1, N_2 and N_3. Thus, the definition of a HKT manifold can be simplified, requiring that it is an almost hyperhermitian manifold satisfying $N_1 \mathrm{d}\omega_1 = N_2 \mathrm{d}\omega_2 = N_3 \mathrm{d}\omega_3$. Inspired in the latter definition of HKT manifold, we extend to Lie algebroids the notion of HKT structure. It is worth to mention that our definition of hypersymplectic structure with torsion and HKT structure is more general than the usual one, since we consider the cases of (almost) complex and para-complex morphisms N_i. Besides the relation of hypersymplectic structures with torsion on Lie algebroids with HKT structures, we look at hypersymplectic structures with torsion on Lie algebroids from a different perspective, by presenting an alternative definition that uses bivectors instead of 2-forms (Theorem 2.2). These bivectors are twisted Poisson, also known as Poisson with a 3-form background [20]. Moreover, we prove that the almost complex morphisms N_1, N_2 and N_3, that are constructed out of the 2-forms and the twisted Poisson bivectors, are in fact Nijenhuis morphisms (Theorem 7.1). In other words, a hypersymplectic structure with torsion on a Lie algebroid A determines three Nijenhuis morphisms and three twisted Poisson bivectors. It is well known that if (A, μ) is a Lie algebroid and N is a Nijenhuis morphism, the deformation (in a certain sense) of μ by N yields a new Lie algebroid structure on A, that we denote by μ_N. On the other hand, if a Lie algebroid (A, μ) is equipped with a twisted Poisson bivector π, the dual vector bundle A^* inherits a Lie algebroid structure given by $\mu_\pi + \frac{1}{2}\{\omega, [\pi, \pi]\}$ (Proposition 7.6). So, two natural questions arise.

1. *If $(\omega_1, \omega_2, \omega_3)$ is a hypersymplectic structure with torsion on a Lie algebroid (A, μ), does it remain a hypersymplectic structure with torsion on the Lie algebroid (A, μ_{N_i}), for $i = 1, 2, 3$?*

In Theorem 7.5 we answer this question, actually showing that $(\omega_1, \omega_2, \omega_3)$ is a hypersymplectic structure with torsion on (A, μ) if and only if it is a hypersymplectic structure with torsion on (A, μ_{N_i}), for $i = 1, 2, 3$.

2. *Does a hypersymplectic structure with torsion on a Lie algebroid A induce a hypersymplectic structure with torsion on the Lie algebroid A^*?*

   The answer is given in Theorem 7.8, where we prove that $(\omega_1, \omega_2, \omega_3)$ is a hypersymplectic structure with torsion on (A, μ) if and only if (π_1, π_2, π_3) is a hypersymplectic structure with torsion on $\left(A^*, -\mu_{\pi_i} - \frac{1}{2}\left\{\omega_i, [\pi_i, \pi_i]\right\}\right)$, $i = 1, 2, 3$, where π_i is the inverse of ω_i.

Question 2 was answered in [4] for the case of hypersymplectic structures (without torsion) on Lie algebroids.

As we already mentioned, we have studied hypersymplectic structures on pre-Courant algebroids in our previous paper [5]. The results of [5] turned out to be essential in the current paper, since they are extensively used to prove the theorems in Section 7. The proof of Theorem 7.1 requires some properties of the Nijenhuis torsion on pre-Courant and Lie algebroids. We have obtained several results on this topic which are collected at the end of Section 6.

The paper includes, besides the Introduction, seven sections and one appendix. In Section 1 we define $\boldsymbol{\varepsilon}$-hypersymplectic structures with torsion on a Lie algebroid and present some of their properties. In Section 2 we give the equivalent definition in terms of twisted Poisson bivectors. The structure induced on the base manifold of a Lie algebroid with a hypersymplectic structure with torsion is described in Section 3. In Section 4 we introduce the notion of hyperkähler structure with torsion on a Lie algebroid and prove the existence of a one-to-one correspondence between hypersymplectic structures with torsion and hyperkähler structures with torsion on a Lie algebroid (Theorem 4.5). Section 5 contains three examples of hypersymplectic structures with torsion on $\mathbb{R}^8$, on $\mathfrak{su}(3)$ and on the tangent bundle of $S^3 \times (S^1)^5$, respectively. In Section 6 we start by recalling the definition and main properties of an $\boldsymbol{\varepsilon}$-hypersymplectic structure on a pre-Courant algebroid. Then, we concentrate on pre-Courant structures on the vector bundle $A \oplus A^*$, to study the Nijenhuis torsion of an endomorphism of $A \oplus A^*$ of type $\mathcal{T}_N = N \oplus N^*$, with $N : A \to A$. Namely, we establish some relations between the Nijenhuis torsion of $\mathcal{T}_N$ and the Nijenhuis torsion of N (Propositions 6.8 and 6.10). Section 6 also includes a formula taken from [1], that expresses the Frölicher-Nijenhuis bracket in terms of big bracket (Theorem 6.11). This formula is used to show how the Frölicher-Nijenhuis bracket on a Lie algebroid can be seen in the pre-Courant algebroid setting (Proposition 6.12). Section 7 contains the most important results of the paper. We prove that the transition morphisms N_1, N_2 and N_3 are Nijenhuis (Theorem 7.1) and pairwise compatible with respect to the Frölicher-Nijenhuis bracket (Proposition 7.2), and that the twisted Poisson bivectors are compatible with respect to the Schouten-Nijenhuis bracket of the Lie algebroid (Proposition 7.3). Theorems 7.5 and 7.8, mentioned previously, are also included in this section. Since most of the computations along the paper are done using the big bracket, we include in Appendix A a review of Lie and pre-Courant algebroids in the supergeometric setting.

Notation: We consider $1, 2$ and 3 as the representative elements of the equivalence classes of $\mathbb{Z}_3$, i.e., $\mathbb{Z}_3 := \{[1], [2], [3]\}$. Along the paper, although we omit the brackets, and write i instead of $[i]$, the indices (and corresponding computations) must be thought in $\mathbb{Z}_3 := \mathbb{Z}/3\mathbb{Z}$.

1. Hypersymplectic structures with torsion

Let (A, μ) be a Lie algebroid and take three non-degenerate 2-forms ω_1, ω_2 and $\omega_3 \in \Gamma(\wedge^2 A^*)$ with inverse π_1, π_2 and $\pi_3 \in \Gamma(\wedge^2 A)$, respectively. We define the *transition morphisms* N_1, N_2 and $N_3 : A \to A$, by setting

$$N_i := \pi^\sharp_{i-1} \circ \omega^\flat_{i+1}, \quad i \in \mathbb{Z}_3. \tag{1}$$

In (1), we consider the usual vector bundle maps $\pi^\# : A^* \to A$ and $\omega^\flat : A \to A^*$, associated to a bivector $\pi \in \Gamma(\bigwedge^2 A)$ and a 2-form $\omega \in \Gamma(\bigwedge^2 A^*)$, respectively, which are given by $\langle \beta, \pi^\#(\alpha) \rangle = \pi(\alpha, \beta)$ and $\langle \omega^\flat(X), Y \rangle = \omega(X, Y)$, for all $\alpha, \beta \in \Gamma(A^*)$ and $X, Y \in \Gamma(A)$.

In what follows we shall consider the parameters $\varepsilon_i = \pm 1, i = 1, 2, 3$, and the triplet $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$.

Definition 1.1. A triplet $(\omega_1, \omega_2, \omega_3)$ of non-degenerate 2-forms on a Lie algebroid (A, μ) is an *$\boldsymbol{\varepsilon}$-hypersymplectic structure with torsion* if

$$N_i{}^2 = \varepsilon_i \mathrm{id}_A, \quad i = 1, 2, 3, \quad \text{and} \quad \varepsilon_2 \mathrm{N}_1 \mathrm{d}\omega_1 = \varepsilon_3 \mathrm{N}_2 \mathrm{d}\omega_2 = \varepsilon_1 \mathrm{N}_3 \mathrm{d}\omega_3, \tag{2}$$

where $N_i \mathrm{d}\omega_i(X, Y, Z) = \mathrm{d}\omega_i(N_i X, N_i Y, N_i Z)$, for all $X, Y, Z \in \Gamma(A)$, and N_i is given by (1), $i = 1, 2, 3$.

When the non-degenerate 2-forms ω_1, ω_2 and ω_3 are closed, so that they are symplectic forms and the right hand side of (2) is trivially satisfied, the triplet $(\omega_1, \omega_2, \omega_3)$ is an $\boldsymbol{\varepsilon}$-hypersymplectic structure on (A, μ) [3].

Having an $\boldsymbol{\varepsilon}$-hypersymplectic structure with torsion $(\omega_1, \omega_2, \omega_3)$ on a Lie algebroid A over M, we define a map

$$g : A \times A \longrightarrow \mathbb{R}, \qquad g(X, Y) = \langle g^\flat(X), Y \rangle, \tag{3}$$

where $g^\flat : A \longrightarrow A^*$ is a vector bundle morphism given by

$$g^\flat := \varepsilon_3 \varepsilon_2 \, \omega_3{}^\flat \circ \pi_1{}^\sharp \circ \omega_2{}^\flat. \tag{4}$$

The definition of $g^\flat$ is not affected by a circular permutation of the indices in (4), that is,

$$g^\flat = \varepsilon_{i-1} \varepsilon_{i+1} \, \omega_{i-1}{}^\flat \circ \pi_i{}^\sharp \circ \omega_{i+1}{}^\flat, \quad i \in \mathbb{Z}_3. \tag{5}$$

Moreover,

$$(g^\flat)^* = -\varepsilon_1 \varepsilon_2 \varepsilon_3 \, g^\flat, \tag{6}$$

which means that g is symmetric or skew-symmetric, depending on the sign of $\varepsilon_1 \varepsilon_2 \varepsilon_3$. An important property of g is the following one:

$$g(N_i X, N_i Y) = \varepsilon_{i-1} \varepsilon_{i+1} \, g(X, Y), \quad X, Y \in \Gamma(A). \tag{7}$$

Notice that $g^\flat$ is invertible and, using its inverse, we may define a map $g^{-1} : A^* \times A^* \longrightarrow \mathbb{R}$, by setting

$$g^{-1}(\alpha, \beta) := \langle \beta, (g^\flat)^{-1}(\alpha) \rangle, \tag{8}$$

for all $\alpha, \beta \in \Gamma(A^*)$.

All the algebraic properties of $\boldsymbol{\varepsilon}$-hypersymplectic structures proved in [3] hold in the case of $\boldsymbol{\varepsilon}$-hypersymplectic structures with torsion. We quote some of them in the next proposition.

Proposition 1.2. *Let ω_1, ω_2 and ω_3 be non-degenerate 2-forms on a vector bundle $A \to M$ with inverses π_1, π_2 and π_3, respectively. Let N_1, N_2 and N_3 be the transition morphisms given by (1), such that $N_i^2 = \varepsilon_i \mathrm{id}_A$, and g be the morphism defined by (4). Then,*

i) $\omega_i^\flat \circ N_i = \varepsilon_1\varepsilon_2\varepsilon_3 {N_i}^* \circ \omega_i^\flat = \varepsilon_{i-1} g^\flat$,

ii) $\pi_i^\sharp \circ {N_i}^* = \varepsilon_1\varepsilon_2\varepsilon_3 N_i \circ \pi_i^\sharp = \varepsilon_{i+1} (g^{-1})^\sharp$,

iii) $g^\flat \circ N_i = \varepsilon_1\varepsilon_2\varepsilon_3 {N_i}^* \circ g^\flat = \varepsilon_i\varepsilon_{i-1}\omega_i^\flat$,

for all indices in $\mathbb{Z}_3$. Moreover, for $i, j \in \mathbb{Z}_3, i \neq j$,

iv) $\omega_i^\flat \circ N_j = {N_j}^* \circ \omega_i^\flat = \begin{cases} \omega_{i-1}^\flat & , \; j = i+1 \\ \varepsilon_{i-1}\omega_{i+1}^\flat & , \; j = i-1; \end{cases}$

v) $\pi_i^\sharp \circ {N_j}^* = N_j \circ \pi_i^\sharp = \begin{cases} \varepsilon_{i+1}\pi_{i-1}^\sharp & , \; j = i+1 \\ \pi_{i+1}^\sharp & , \; j = i-1; \end{cases}$

vi) $N_i \circ N_j = \varepsilon_1\varepsilon_2\varepsilon_3 N_j \circ N_i = \begin{cases} \varepsilon_i\varepsilon_{i+1} N_{i-1} & , \; j = i+1 \\ \varepsilon_{i+1} N_{i+1} & , \; j = i-1. \end{cases}$

Proof: See the proofs of Propositions 3.7 and 3.9 in [3]. Notice that these proofs only use algebraic properties of the morphisms and do not use the Lie algebroid structure of $A \to M$. ∎

2. The contravariant perspective

In this section we give a contravariant characterization of an ε-hypersymplectic structure with torsion on a Lie algebroid.

We shall need the next lemma, that can be easily proved.

Lemma 2.1. *Let ω be a non-degenerate 2-form on a Lie algebroid (A, μ), with inverse π. Then,*

$$[\pi, \pi] = 2\mathrm{d}\omega \left(\pi^\sharp(\cdot), \pi^\sharp(\cdot), \pi^\sharp(\cdot)\right) \tag{9}$$

or, equivalently,

$$\mathrm{d}\omega = \frac{1}{2}[\pi, \pi] \left(\omega^\flat(\cdot), \omega^\flat(\cdot), \omega^\flat(\cdot)\right), \tag{10}$$

where $[\cdot, \cdot]$ stands for the Schouten-Nijenhuis bracket of multivectors on (A, μ).

Equation (9) means that π is a *twisted-Poisson* bivector on (A, μ), also known as Poisson bivector with the 3-form background $\mathrm{d}\omega$ [20].

Let $(\omega_1, \omega_2, \omega_3)$ be an ε-hypersymplectic structure with torsion on (A, μ) and let us denote by H the 3-form

$$H := \varepsilon_2 N_1 \mathrm{d}\omega_1 = \varepsilon_3 N_2 \mathrm{d}\omega_2 = \varepsilon_1 N_3 \mathrm{d}\omega_3.$$

So, for all $X, Y, Z \in \Gamma(A)$,

$$H(X, Y, Z) = \varepsilon_{i+1} \mathrm{d}\omega_i(N_i X, N_i Y, N_i Z), \;\; i = 1, 2, 3, \tag{11}$$

or, equivalently,

$$\mathrm{d}\omega_i(X, Y, Z) = \varepsilon_i \varepsilon_{i+1} H(N_i X, N_i Y, N_i Z), \;\; i = 1, 2, 3. \tag{12}$$

Now we prove the main result of this section, that can be seen as a new definition of ε-hypersymplectic structure with torsion on a Lie algebroid.

Theorem 2.2. *A triplet $(\omega_1, \omega_2, \omega_3)$ of non-degenerate 2-forms on a Lie algebroid (A, μ) with inverses π_1, π_2 and π_3, respectively, is an ε-hypersymplectic structure with torsion on A if and only if*

$${N_i}^2 = \varepsilon_i \mathrm{id}_A, \quad i = 1, 2, 3, \quad \text{and} \quad \varepsilon_1[\pi_1, \pi_1] = \varepsilon_2[\pi_2, \pi_2] = \varepsilon_3[\pi_3, \pi_3],$$

with N_i given by (1).

Proof: It is enough to prove that $\varepsilon_2 N_1 \mathrm{d}\omega_1 = \varepsilon_3 N_2 \mathrm{d}\omega_2 = \varepsilon_1 N_3 \mathrm{d}\omega_3$ is equivalent to $\varepsilon_1[\pi_1, \pi_1] = \varepsilon_2[\pi_2, \pi_2] = \varepsilon_3[\pi_3, \pi_3]$.

Assume that $\varepsilon_2 N_1 \mathrm{d}\omega_1 = \varepsilon_3 N_2 \mathrm{d}\omega_2 = \varepsilon_1 N_3 \mathrm{d}\omega_3$. From (9), (12) and Proposition 1.2 ii), we have

$$\begin{aligned}
\varepsilon_i[\pi_i, \pi_i] &= 2\varepsilon_i \mathrm{d}\omega_i \left(\pi_i^\sharp(\cdot), \pi_i^\sharp(\cdot), \pi_i^\sharp(\cdot)\right) \\
&= 2\varepsilon_{i+1} H \left(N_i \circ \pi_i^\sharp(\cdot), N_i \circ \pi_i^\sharp(\cdot), N_i \circ \pi_i^\sharp(\cdot)\right) \\
&= 2\varepsilon_1\varepsilon_2\varepsilon_3 H \left((g^{-1})^\sharp(\cdot), (g^{-1})^\sharp(\cdot), (g^{-1})^\sharp(\cdot)\right),
\end{aligned}$$

for $i \in \mathbb{Z}_3$; thus,

$$\varepsilon_1[\pi_1, \pi_1] = \varepsilon_2[\pi_2, \pi_2] = \varepsilon_3[\pi_3, \pi_3].$$

Conversely, assume that $\varepsilon_1[\pi_1, \pi_1] = \varepsilon_2[\pi_2, \pi_2] = \varepsilon_3[\pi_3, \pi_3]$ and let us set $\psi := \frac{1}{2}\varepsilon_i[\pi_i, \pi_i]$. Then, using (10) and Proposition 1.2 i), we get

$$\begin{aligned}
\varepsilon_{i+1} \mathrm{d}\omega_i \left(N_i(\cdot), N_i(\cdot), N_i(\cdot)\right) &= \frac{1}{2}\varepsilon_{i+1}[\pi_i, \pi_i] \left(\omega_i^\flat \circ N_i(\cdot), \omega_i^\flat \circ N_i(\cdot), \omega_i^\flat \circ N_i(\cdot)\right) \\
&= \frac{1}{2}\varepsilon_{i+1}\varepsilon_{i-1}[\pi_i, \pi_i] \left(g^\flat(\cdot), g^\flat(\cdot), g^\flat(\cdot)\right) \\
&= \varepsilon_1\varepsilon_2\varepsilon_3 \psi \left(g^\flat(\cdot), g^\flat(\cdot), g^\flat(\cdot)\right).
\end{aligned}$$

Therefore, we conclude

$$\varepsilon_2 N_1 \mathrm{d}\omega_1 = \varepsilon_3 N_2 \mathrm{d}\omega_2 = \varepsilon_1 N_3 \mathrm{d}\omega_3.$$

■

Remark 2.3. When we have an ε-hypersymplectic structure with torsion on a Lie algebroid (A, μ), this Lie algebroid is equipped with a triplet (π_1, π_2, π_3) of twisted-Poisson bivectors that share, eventually up to a sign, the same obstruction to be Poisson. This obstruction is denoted by 2ψ in the proof of Theorem 2.2.

3. Structures induced on the base manifold

It is well known that a symplectic structure ω on a Lie algebroid $A \to M$ induces a Poisson structure on the base manifold M. The Poisson bivector $\pi_{_M}$ on M is defined by

$$\pi_{_M}^\sharp = \rho \circ \pi^\sharp \circ \rho^*, \tag{13}$$

where ρ is the anchor map and $\pi \in \Gamma(\wedge^2 A)$ is the Poisson bivector on A which is the inverse of ω. In the case where ω is non-degenerate but not necessarily closed, so that π defines a Poisson structure with the 3-form background $d\omega$ on the Lie algebroid A (see (9)), the base manifold M has an induced Poisson structure with a 3-form background, provided that $d\omega \in \Gamma(\wedge^3 A^*)$ is the pull-back by ρ of a closed 3-form $\phi_{_M} \in \Omega^3(M)$. In fact, if $d\omega = \rho^*(\phi_{_M})$ then, a straightforward computation gives

$$[\pi, \pi] = 2d\omega\left(\pi^\sharp(\cdot), \pi^\sharp(\cdot), \pi^\sharp(\cdot)\right) \Rightarrow [\pi_{_M}, \pi_{_M}]_{_M} = 2\phi_{_M}\left(\pi_{_M}^\sharp(\cdot), \pi_{_M}^\sharp(\cdot), \pi_{_M}^\sharp(\cdot)\right), \tag{14}$$

where $\pi_{_M}$ is given by (13) and $[\cdot,\cdot]_{_M}$ stands for the Schouten-Nijenhuis bracket of multivectors on M.

The next proposition is now immediate.

Proposition 3.1. *Let $(\omega_1, \omega_2, \omega_3)$ be an ε-hypersymplectic structure with torsion on a Lie algebroid (A, μ) over M with inverses π_1, π_2 and π_3, respectively. Let ρ be the anchor map of A. If $d\omega_i = \rho^*(\phi_{_M}^i)$, $i = 1, 2, 3$, with $\phi_{_M}^i$ a closed 3-form on the manifold M, then M is equipped with three Poisson structures $\pi_{_M}^1, \pi_{_M}^2$ and $\pi_{_M}^3$, defined by (13), with the 3-forms background, $\phi_{_M}^1$, $\phi_{_M}^2$ and $\phi_{_M}^3$, respectively. Moreover, the three bivectors on M share, eventually up to a sign, the same obstruction to be Poisson.*

We should stress that if $\omega_i = \rho^*(\omega^i_{_M})$ for some 2-forms $\omega^i_{_M}$ on M, $i = 1,2,3$, then $d\omega_i = \rho^*(\delta\omega^i_{_M})$ with δ the De Rham differential on M. So, the assumptions of Proposition 3.1 are satisfied and

$$[\pi^i_{_M}, \pi^i_{_M}]_M = 2\delta\omega^i_{_M}\left((\pi^i_{_M})^\sharp(\cdot), (\pi^i_{_M})^\sharp(\cdot), (\pi^i_{_M})^\sharp(\cdot)\right),$$

$i = 1,2,3$. However, although the bivectors $\pi^1_{_M}, \pi^2_{_M}$ and $\pi^3_{_M}$ on M satisfy $\varepsilon_1[\pi^1_{_M}, \pi^1_{_M}]_M = \varepsilon_2[\pi^2_{_M}, \pi^2_{_M}]_M = \varepsilon_3[\pi^3_{_M}, \pi^3_{_M}]_M$, in general, the triplet $(\omega^1_{_M}, \omega^2_{_M}, \omega^3_{_M})$ does not define an $\boldsymbol{\varepsilon}$-hypersymplectic structure with torsion on M, because $\omega^i_{_M}$ is not the inverse of $\pi^i_{_M}$.

4. Hypersymplectic structures with torsion versus hyperkähler structures with torsion

In this section we consider $\boldsymbol{\varepsilon}$-hypersymplectic structures with torsion such that $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$ and we prove that these structures are in one-to-one correspondence with (para-)hyperkähler structures with torsion, a notion we shall define later. First, let us consider two different cases of an $\boldsymbol{\varepsilon}$-hypersymplectic structure with torsion such that $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$.

Definition 4.1. Let $(\omega_1, \omega_2, \omega_3)$ be an $\boldsymbol{\varepsilon}$-hypersymplectic structure with torsion on a Lie algebroid (A, μ), such that $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$.

- If $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = -1$, then $(\omega_1, \omega_2, \omega_3)$ is said to be a *hypersymplectic* structure with torsion on A.
- Otherwise, $(\omega_1, \omega_2, \omega_3)$ is said to be a *para-hypersymplectic* structure with torsion on A.

For a (para-)hypersymplectic structure with torsion on a Lie algebroid (A, μ), the morphism $g^\flat$ defined by (4) determines a pseudo-metric on (A, μ) [3].

Hyperkähler structures with torsion on manifolds were introduced in [10] and studied in [9] and in [7]. The definition extends to the Lie algebroid setting in a natural way. Let (A, μ) be a Lie algebroid and consider a map $\mathsf{g} : A \times A \to \mathbb{R}$ and endomorphisms $I_1, I_2, I_3 : A \to A$ such that, for all $i \in \mathbb{Z}_3$

and $X, Y \in \Gamma(A)$,

$$\begin{aligned} &i) \ \mathsf{g} \text{ is a pseudo-metric;} \\ &ii) \ I_i^2 = \varepsilon_i \mathrm{id}_A, \quad \text{where } \varepsilon_i = \pm 1 \text{ and } \varepsilon_1\varepsilon_2\varepsilon_3 = -1; \\ &iii) \ I_3 = \varepsilon_1\varepsilon_2 I_1 \circ I_2; \\ &iv) \ \mathsf{g}(I_iX, I_iY) = \varepsilon_{i-1}\varepsilon_{i+1}\mathsf{g}(X,Y); \\ &v) \ \varepsilon_2 I_1 \mathrm{d}\omega_1 = \varepsilon_3 I_2 \mathrm{d}\omega_2 = \varepsilon_1 I_3 \mathrm{d}\omega_3, \end{aligned} \tag{15}$$

where the 2-forms $\omega_i, i = 1,2,3$, which are called the *Kähler forms*, are defined by

$$\omega_i^\flat = \varepsilon_i\varepsilon_{i-1}\mathsf{g}^\flat \circ I_i. \tag{16}$$

Definition 4.2. A quadruple $(\mathsf{g}, I_1, I_2, I_3)$ satisfying (15)$i) - v)$, on a Lie algebroid (A, μ), is a

- *hyperkähler structure with torsion* on A if $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = -1$;
- *para-hyperkähler structure with torsion* on A, otherwise.

Remark 4.3. Most authors consider that a (para-)hyperkähler structure, with or without torsion, is equipped with a positive definite metric, while for us $\mathsf{g} : A \times A \to \mathbb{R}$ is a pseudo-metric, i.e., it is symmetric and non-degenerate.

Note that, because $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$, on a (para-)hyperkähler structure with torsion $(\mathsf{g}, I_1, I_2, I_3)$, we always have $I_i \circ I_j = -I_j \circ I_i$, for all $i, j \in \{1,2,3\}, i \neq j$.

The next lemma establishes a relation between the pseudo-metric and the Kähler forms of a (para-)hyperkähler structure with torsion on a Lie algebroid. Namely, it is shown that the pseudo-metric satisfies an equation similar to (5).

Lemma 4.4. *Let $(\mathsf{g}, I_1, I_2, I_3)$ be a (para-)hyperkähler structure with torsion on a Lie algebroid with associated Kähler forms ω_1, ω_2 and ω_3. Then,*

$$\mathsf{g}^\flat = \varepsilon_{i-1}\varepsilon_{i+1}{\omega_{i-1}}^\flat \circ {\pi_i}^\sharp \circ {\omega_{i+1}}^\flat, \quad i \in \mathbb{Z}_3,$$

where π_i is the inverse of ω_i.

Proof: It is enough to prove that $\mathsf{g}^\flat = \varepsilon_1\varepsilon_3{\omega_1}^\flat \circ {\pi_2}^\sharp \circ {\omega_3}^\flat$. From $\omega_i^\flat = \varepsilon_i\varepsilon_{i-1}\mathsf{g}^\flat \circ I_i$ in Definition 4.2 we have, on one hand,

$$\mathsf{g}^\flat = \varepsilon_2\omega_3^\flat \circ I_3 = \varepsilon_1\omega_3^\flat \circ I_1 \circ I_2 \tag{17}$$

and, on the other hand,

$$\mathrm{g}^\flat = \varepsilon_1 \omega_2^\flat \circ I_2. \tag{18}$$

From (17) and (18), we get

$$\omega_3^\flat \circ I_1 = \omega_2^\flat \Leftrightarrow I_1 = \pi_3^\sharp \circ \omega_2^\flat \Leftrightarrow I_1 = \varepsilon_1 \pi_2^\sharp \circ \omega_3^\flat,$$

where we used $(I_1)^{-1} = \varepsilon_1 I_1$, and so, $\mathrm{g}^\flat = \varepsilon_3 \omega_1^\flat \circ I_1$ yields

$$\mathrm{g}^\flat = \varepsilon_1 \varepsilon_3 {\omega_1}^\flat \circ {\pi_2}^\sharp \circ {\omega_3}^\flat.$$

■

At this point, we shall see that (para-)hypersymplectic structures with torsion and (para-)hyperkähler structures with torsion on a Lie algebroid are in one-to-one correspondence.

Theorem 4.5. *Let (A, μ) be a Lie algebroid. There exists a one-to-one correspondence between (para-)hypersymplectic structures with torsion and (para-)hyperkähler structures with torsion on (A, μ).*

Proof: Let $(\omega_1, \omega_2, \omega_3)$ be a (para-)hypersymplectic structure with torsion on A and consider the endomorphisms N_1, N_2, N_3 and $g^\flat$ given by (1) and (4), respectively. From Proposition 1.2 vi), the equalities $N_1 \circ N_2 = \varepsilon_1 \varepsilon_2 N_3 = -N_2 \circ N_1$ hold, while Proposition 1.2 i) shows that g satisfies $\omega_i^\flat = \varepsilon_i \varepsilon_{i-1} g^\flat \circ N_i$. Thus, (g, N_1, N_2, N_3) is a (para-)hyperkähler structure with torsion on (A, μ) and its Kähler forms are ω_1, ω_2 and ω_3.

Conversely, let us take a (para-)hyperkähler structure with torsion $(\mathrm{g}, I_1, I_2, I_3)$ on A and consider the associated Kähler forms ω_1, ω_2 and ω_3, given by (16). We claim that $(\omega_1, \omega_2, \omega_3)$ is a (para-)hypersymplectic structure with torsion on A. To prove this, it is enough to show that I_1, I_2 and I_3 are the transition morphisms of $(\omega_1, \omega_2, \omega_3)$, defined by (1). From $\omega_i^\flat = \varepsilon_i \varepsilon_{i-1} \mathrm{g}^\flat \circ I_i$ (definition of the Kähler forms), we get $\mathrm{g}^\flat = \varepsilon_{i-1} \omega_i^\flat \circ I_i$ or, using Lemma 4.4,

$$\varepsilon_i \omega_i^\flat \circ \pi_{i+1}^\sharp \circ \omega_{i-1}^\flat = \omega_i^\flat \circ I_i,$$

which is equivalent to

$$I_i = \varepsilon_i \pi_{i+1}^\sharp \circ \omega_{i-1}^\flat = \pi_{i-1}^\sharp \circ \omega_{i+1}^\flat.$$

■

As a consequence of Theorem 4.5, if we pick a (para-)hypersymplectic structure with torsion $(\omega_1, \omega_2, \omega_3)$ on (A, μ) and consider the (para-)hyperkähler structure with torsion (g, N_1, N_2, N_3) given by Theorem 4.5 then, the (para-)hypersymplectic structure with torsion that corresponds to (g, N_1, N_2, N_3) via Theorem 4.5, is the initial one, i.e., $(\omega_1, \omega_2, \omega_3)$.

5. Examples

We present, in this section, three examples of hypersymplectic structures with torsion. The first two examples are on Lie algebras, viewed as Lie algebroids over a point, and the third is on the tangent Lie algebroid to a manifold.

The first example is inspired from [9]. Let $\{A_1, A_2, B_1, B_2, Z, C_1, C_2, C_3\}$ be a basis for $\mathbb{R}^8$ and let us consider the Lie algebra structure defined by

$$[A_1, B_1] = Z, \quad [A_2, B_2] = -Z$$

and the remaining brackets vanish. We denote by $\{a_1, a_2, b_1, b_2, z, c_1, c_2, c_3\}$ the dual basis and we define a triplet $(\omega_1, \omega_2, \omega_3)$ of 2-forms on $\mathbb{R}^8$ by setting

$$\begin{aligned} \omega_1 &= a_1 b_1 + a_2 b_2 + z c_1 + c_2 c_3; \\ \omega_2 &= -a_1 a_2 + b_1 b_2 - z c_2 + c_1 c_3; \\ \omega_3 &= -a_1 b_2 + a_2 b_1 - z c_3 - c_1 c_2. \end{aligned}$$

Their matrix representation on the basis $\{A_1, A_2, B_1, B_2, Z, C_1, C_2, C_3\}$ and its dual is the following:

$$\mathcal{M}_{\omega_1} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \mathcal{M}_{\omega_2} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

and

$$\mathcal{M}_{\omega_3} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The 2-forms ω_1, ω_2 and ω_3 are non-degenerate and since

$$\mathrm{d}\omega_1 = -a_1b_1c_1+a_2b_2c_1, \quad \mathrm{d}\omega_2 = a_1b_1c_2-a_2b_2c_2, \quad \mathrm{d}\omega_3 = a_1b_1c_3-a_2b_2c_3, \tag{19}$$

they are not closed. The transition morphisms N_1, N_2 and N_3, given by (1), correspond to the following matrices in the considered basis:

$$\mathcal{M}_{N_1} = -\mathcal{M}_{\omega_1}, \quad \mathcal{M}_{N_2} = -\mathcal{M}_{\omega_2}, \quad \mathcal{M}_{N_3} = -\mathcal{M}_{\omega_3}.$$

Using (19), we have

$$\begin{aligned}\mathrm{d}\omega_1(N_1(\cdot), N_1(\cdot), N_1(\cdot)) = \mathrm{d}\omega_2&(N_2(\cdot), N_2(\cdot), N_2(\cdot)) \\ &= \mathrm{d}\omega_3(N_3(\cdot), N_3(\cdot), N_3(\cdot)) = a_1b_1z - a_2b_2z,\end{aligned}$$

so that the triplet $(\omega_1, \omega_2, \omega_3)$ is a hypersymplectic structure with torsion on $\mathbb{R}^8$. The pseudo-metric g determined by $(\omega_1, \omega_2, \omega_3)$, defined by (4), is simply $\mathsf{g} = -\mathrm{id}_{\mathbb{R}^8}$.

Next, we address an explicit example [16] of an hypersymplectic structure with torsion on the Lie algebra $\mathfrak{su}(3)$ of the Lie group $SU(3)$.

We write E_{pq} for the elementary 3×3-matrix with 1 at position (p, q) and consider the basis of $\mathfrak{su}(3)$ consisting of eight complex matrices:

$$\begin{array}{ll} A_1 = i(E_{11} - E_{22}), & A_2 = i(E_{22} - E_{33}), \\ B_{pq} = E_{pq} - E_{qp}, & C_{pq} = i(E_{pq} + E_{qp}), \end{array}$$

where $p, q \in \{1, 2, 3\}$ such that $p < q$. We denote by $\{a_1, a_2, \dots, c_{23}\}$ the dual basis and define a triplet of 2-forms on $SU(3)$ by setting

$$\begin{aligned}\omega_1 &= -\frac{\sqrt{3}}{2}a_1a_2 + b_{12}c_{12} + b_{13}c_{13} - b_{23}c_{23};\\ \omega_2 &= \frac{\sqrt{3}}{2}a_2b_{12} - a_1c_{12} + \frac{1}{2}a_2c_{12} - b_{13}b_{23} + c_{13}c_{23};\\ \omega_3 &= \frac{\sqrt{3}}{2}a_2c_{12} + a_1b_{12} - \frac{1}{2}a_2b_{12} + b_{13}c_{23} + b_{23}c_{13}.\end{aligned}$$

The 2-forms have a matrix representation, on the basis $(A_1, A_2, B_{12}, B_{13}, B_{23}, C_{12}, C_{13}, C_{23})$ and its dual, given by

$$\mathcal{M}_{\omega_1} = \begin{pmatrix} 0 & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \mathcal{M}_{\omega_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$\mathcal{M}_{\omega_3} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\ -1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

These 2-forms are not closed; for example, we have

$$\begin{aligned} \mathrm{d}\omega_1 = -\sqrt{3}a_1(b_{13}c_{13} + b_{23}c_{23}) + \sqrt{3}a_2(b_{12}c_{12} \\ + b_{13}c_{13}) - b_{12}b_{13}c_{23} - b_{12}b_{23}c_{13} - b_{13}b_{23}c_{12} - c_{12}c_{13}c_{23}. \end{aligned}$$

The transition morphisms N_1, N_2 and N_3, given by (1), correspond to the following matrices in the considered basis:

$$\mathcal{M}_{N_1} = \begin{pmatrix} -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \mathcal{M}_{N_2} = \begin{pmatrix} 0 & 0 & -\frac{1}{\sqrt{3}} & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

and

$$\mathcal{M}_{N_3} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & -\frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 \\ 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & \\ 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

An easy computation gives

$$\begin{aligned} &\mathrm{d}\omega_1(N_1(\cdot), N_1(\cdot), N_1(\cdot)) = \mathrm{d}\omega_2(N_2(\cdot), N_2(\cdot), N_2(\cdot)) = \mathrm{d}\omega_3(N_3(\cdot), N_3(\cdot), N_3(\cdot)) \\ &\quad = -a_1b_{13}c_{13} + a_1b_{23}c_{23} - 2a_1b_{12}c_{12} - a_2b_{13}c_{13} - 2a_2b_{23}c_{23} \\ &\qquad + a_2b_{12}c_{12} + b_{23}c_{12}c_{13} + b_{13}c_{12}c_{23} + b_{12}c_{13}c_{23} + b_{12}b_{13}b_{23}, \end{aligned}$$

which shows that the triplet $(\omega_1, \omega_2, \omega_3)$ is a hypersymplectic structure with torsion on $\mathfrak{su}(3)$. Finally, the pseudo-metric is given by

$$\mathcal{M}_{\mathrm{g}} = \begin{pmatrix} -1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

In the third example, which is taken form [7], we describe a hypersymplectic structure on the Lie algebroid tangent to the manifold $M = S^3 \times (S^1)^5$. The sphere S^3 is identified with the Lie group $Sp(1)$. In its Lie algebra $\mathfrak{sp}(1)$ we consider a basis $\{A_2, A_3, A_4\}$ and the brackets

$$[A_2, A_3] = 2A_4, \quad [A_3, A_4] = 2A_2, \quad [A_4, A_2] = 2A_3.$$

Let $\{a_2, a_3, a_4\}$ be the dual basis and let us consider a basis $\{a_1, a_5, a_6, a_7, a_8\}$ of 1-forms on $(S^1)^5$. We define a triplet $(\omega_1, \omega_2, \omega_3)$ of 2-forms on M by setting, on the basis $\{a_2, a_3, a_4, a_1, a_5, a_6, a_7, a_8\}$,

$$\begin{aligned} \omega_1 &= a_2a_1 + a_4a_3 + a_6a_5 + a_8a_7; \\ \omega_2 &= a_3a_1 + a_2a_4 + a_7a_5 + a_6a_8; \\ \omega_3 &= a_4a_1 + a_3a_2 + a_8a_5 + a_7a_6. \end{aligned}$$

Their matrix representation on the considered basis is given by

$$\mathcal{M}_{\omega_1} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \mathcal{M}_{\omega_2} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

and

$$\mathcal{M}_{\omega_3} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The 2-forms are non-degenerate and not closed:

$$\mathrm{d}\omega_1 = -2a_1a_3a_4, \quad \mathrm{d}\omega_2 = -2a_1a_4a_2, \quad \mathrm{d}\omega_3 = -2a_1a_2a_3.$$

The transition morphisms N_1, N_2 and N_3, given by (1), correspond to the following matrices:

$$\mathcal{M}_{N_1} = \mathcal{M}_{\omega_1}, \quad \mathcal{M}_{N_2} = \mathcal{M}_{\omega_2}, \quad \mathcal{M}_{N_3} = \mathcal{M}_{\omega_3}.$$

Moreover,

$$\begin{aligned} \mathrm{d}\omega_1(N_1(\cdot), N_1(\cdot), N_1(\cdot)) &= \mathrm{d}\omega_2(N_2(\cdot), N_2(\cdot), N_2(\cdot)) \\ &= \mathrm{d}\omega_3(N_3(\cdot), N_3(\cdot), N_3(\cdot)) = 2a_2a_3a_4, \end{aligned}$$

which shows that the triplet $(\omega_1, \omega_2, \omega_3)$ is a hypersymplectic structure with torsion on the Lie algebroid TM. Regarding the pseudo-metric, we have $\mathsf{g} = \mathrm{id}_{TM}$.

6. The pre-Courant algebroid case

In this section we firstly recall the notion of ε-*hypersymplectic structure* on a pre-Courant algebroid, introduced in [5], as well as its main properties that we use in the sequel. Then, we prove some results involving the Nijenhuis torsions of morphisms on A and on $A \oplus A^*$.

In order to simplify the notation, when $\mathcal{I}$ and $\mathcal{J}$ are endomorphisms of a pre-Courant algebroid, the composition $\mathcal{I} \circ \mathcal{J}$ will be denoted by $\mathcal{I}\mathcal{J}$.

Definitions and basic properties on pre-Courant algebroids are recalled in Apppendix A.

Definition 6.1. An $\boldsymbol{\varepsilon}$-*hypersymplectic structure* on a pre-Courant algebroid (E, Θ) is a triplet $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ of skew-symmetric endomorphisms $\mathcal{S}_i : E \to E$, $i = 1, 2, 3$, such that

i) ${\mathcal{S}_i}^2 = \varepsilon_i \, \mathrm{id}_E$,
ii) $\mathcal{S}_i\mathcal{S}_j = \varepsilon_1\varepsilon_2\varepsilon_3\mathcal{S}_j\mathcal{S}_i$, $\; i, j \in \{1, 2, 3\}, i \neq j$
iii) $\Theta_{\mathcal{S}_i,\mathcal{S}_i} = \varepsilon_i\Theta$,

where the parameters $\varepsilon_i = \pm 1$ form the triplet $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$. [*]

Proposition 6.2. *Let* $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ *be an* $\boldsymbol{\varepsilon}$*-hypersymplectic structure on a pre-Courant algebroid* (E, Θ)*. Then,* $\mathcal{S}_1, \mathcal{S}_2$ *and* $\mathcal{S}_3$ *are Nijenhuis morphisms.*

Given an $\boldsymbol{\varepsilon}$-hypersymplectic structure $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ on (E, Θ), the *transition morphisms* are the endomorphisms $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$ of E defined as

$$\mathcal{T}_i := \varepsilon_{i-1}\mathcal{S}_{i-1}\mathcal{S}_{i+1}, \qquad i \in \mathbb{Z}_3. \tag{20}$$

The parameter $\varepsilon_1\varepsilon_2\varepsilon_3 = \pm 1$ is determinant for some basic properties of the morphisms $\mathcal{T}_i$, and $\mathcal{S}_j$, $i, j \in \{1, 2, 3\}$, and for the relations between them. We shall now focus on the case $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$.

Definition 6.3. Let $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ be an $\boldsymbol{\varepsilon}$-hypersymplectic structure on a pre-Courant algebroid (E, Θ), such that $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$.

- If $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = -1$, then $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is said to be a *hypersymplectic* structure on (E, Θ).
- Otherwise, $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is said to be a *para-hypersymplectic* structure on (E, Θ).

Theorem 6.4. *Let* $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ *be a (para-)hypersymplectic structure on a pre-Courant algebroid* (E, Θ)*. Then, for each* $i = 1, 2, 3$*, the transition morphism* $\mathcal{T}_i$ *is a Nijenhuis morphism.*

Concomitants of the morphisms $\mathcal{S}_i$ and $\mathcal{T}_i$ associated to a (para-)hypersymplectic structure on a pre-Courant algebroid vanish, as stated in the next proposition.

[*]As it is mentioned in Appendix A, we use the following notation: $\Theta_{\mathcal{I}} = \{\mathcal{I}, \Theta\}$ and $\Theta_{\mathcal{I},\mathcal{J}} = \{\mathcal{J}, \{\mathcal{I}, \Theta\}\}$, with $\mathcal{I}, \mathcal{J}$ skew-symmetric endomorphisms of E.

Proposition 6.5. *Let $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ be a (para-)hypersymplectic structure on a pre-Courant algebroid (E, Θ). Then, $C_\Theta(\mathcal{S}_i, \mathcal{S}_j) = C_\Theta(\mathcal{T}_i, \mathcal{T}_j) = C_\Theta(\mathcal{S}_i, \mathcal{T}_j) = 0$, for all $i, j \in \{1, 2, 3\}, i \neq j$.*

Theorem 6.6. *Let (E, Θ) be a pre-Courant algebroid. The following assertions are equivalent:*

i) $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a (para-)hypersymplectic structure on (E, Θ);
ii) $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a (para-)hypersymplectic structure on $(E, \Theta_{\mathcal{S}_i})$;
iii) $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a (para-)hypersymplectic structure on $(E, \Theta_{\mathcal{T}_j})$,

$i, j \in \{1, 2, 3\}$, where $\mathcal{T}_j$ is defined by (20).

Among the pre-Courant algebroid structures, we shall be interested in those defined on vector bundles of type $A \oplus A^*$, since these can be related to structures on A.

If we take a triplet $(\omega_1, \omega_2, \omega_3)$ of 2-forms and a triplet (π_1, π_2, π_3) of bivectors on A, we may define the skew-symmetric endomorphisms $\mathcal{S}_i : A \oplus A^* \to A \oplus A^*$, $i = 1, 2, 3$,

$$\mathcal{S}_i := \begin{bmatrix} 0 & \varepsilon_i \, \pi_i^\sharp \\ \omega_i^\flat & 0 \end{bmatrix}. \tag{21}$$

In the supergeometric setting (see Appendix A), we have

$$\mathcal{S}_i(X + \alpha) = \{X + \alpha, \omega_i + \varepsilon_i \pi_i\},$$

for all $X + \alpha \in A \oplus A^*$.

The next proposition was proved in [5] for the hypersymplectic case. The para-hypersymplectic case has an analogous proof.

Proposition 6.7. *Let $(\omega_1, \omega_2, \omega_3)$ be a triplet of 2-forms and (π_1, π_2, π_3) be a triplet of bivectors on a Lie algebroid (A, μ). Consider the triplet $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ of endomorphisms of $A \oplus A^*$, with $\mathcal{S}_i$ given by (21). The following assertions are equivalent:*

i) $(\omega_1, \omega_2, \omega_3)$ is a (para-)hypersymplectic structure with torsion on the Lie algebroid (A, μ) and π_i is the inverse of ω_i, $i = 1, 2, 3$;
ii) $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a (para-)hypersymplectic structure on the pre-Courant algebroid $(A \oplus A^*, \mu + \psi)$, for some $\psi \in \Gamma(\wedge^3 A)$.

Notice that in the assertion ii) of Proposition 6.7, since $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a hypersymplectic structure, condition $(\mu + \psi)_{\mathcal{S}_k,\mathcal{S}_k} = \varepsilon_k(\mu + \psi)$ holds and implies that ψ has to be of the form $\frac{\varepsilon_k}{2}[\pi_k, \pi_k]$, for any $k \in \{1, 2, 3\}$.

Under the conditions of Proposition 6.7, the transition morphisms of the (para-)hypersymplectic structure $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ on $(A \oplus A^*, \mu + \psi)$, defined by (20), are given by

$$\mathcal{T}_i = \begin{bmatrix} N_i & 0 \\ 0 & -N_i{}^* \end{bmatrix}, \quad i = 1, 2, 3, \tag{22}$$

where N_i is defined by (1).

Recall that the Nijenhuis torsion $\mathcal{T}_\mu N$ of an endomorphism N on a pre-Lie algebroid[†] (A, μ) is given by

$$\mathcal{T}_\mu N(X, Y) = [NX, NY] - N([NX, Y] + [X, NY] - N[X, Y]), \tag{23}$$

for all $X, Y \in \Gamma(A)$. The next proposition addresses a relation between $\mathcal{T}_\mu N$ and the Nijenhuis torsion of the skew-symmetric morphism $\mathcal{T}_N = N \oplus (-N^*)$ on the pre-Courant algebroid $(A \oplus A^*, \mu)$ (see (45)).

Proposition 6.8. *Let $N : A \to A$ be a bundle morphism and define $\mathcal{T}_N : A \oplus A^* \to A \oplus A^*$ by setting $\mathcal{T}_N = \begin{pmatrix} N & 0 \\ 0 & -N^* \end{pmatrix}$. Let $\mu \in \mathcal{F}^{1,2}(A \oplus A^*)$ be a pre-Courant algebroid structure on $A \oplus A^*$. Then, for all $X + \alpha, Y + \beta \in \Gamma(A \oplus A^*)$,*

$$\begin{aligned} \mathcal{T}_\mu \mathcal{T}_N(X + \alpha, Y + \beta) &= [\![X + \alpha, Y + \beta]\!]_{\mathcal{T}_\mu N} \\ &\quad + (N^*)^2[\![X, \beta]\!] - [\![X, (N^*)^2\beta]\!] + (N^*)^2[\![\alpha, Y]\!] - [\![(N^*)^2\alpha, Y]\!], \end{aligned} \tag{24}$$

where $[\![\cdot, \cdot]\!]$ and $[\![\cdot, \cdot]\!]_{\mathcal{T}_\mu N}$ stand for the Dorfman bracket determined by $\mu \in \mathcal{F}^{1,2}(A \oplus A^*)$ and by $\mathcal{T}_\mu N \in \mathcal{F}^{1,2}(A \oplus A^*)$, respectively, according to (43).

Proof: Using the $\mathbb{R}$-bilinearity of $\mathcal{T}_\mu \mathcal{T}_N$ we have

$$\begin{aligned} \mathcal{T}_\mu \mathcal{T}_N(X + \alpha, Y + \beta) = &\,\mathcal{T}_\mu \mathcal{T}_N(X, Y) + \mathcal{T}_\mu \mathcal{T}_N(X, \beta) \\ &+ \mathcal{T}_\mu \mathcal{T}_N(\alpha, Y) + \mathcal{T}_\mu \mathcal{T}_N(\alpha, \beta), \end{aligned} \tag{25}$$

for all $X, Y \in \Gamma(A)$ and $\alpha, \beta \in \Gamma(A^*)$, where we identify $X + 0 \in \Gamma(A \oplus A^*)$ with X and $0 + \alpha \in \Gamma(A \oplus A^*)$ with α. In what follows, we explicit each of the summands of the r.h.s. of Equation (25). For the first summand, denoting

[†]A pre-Lie algebroid is a pair (A, μ) that satisfies the axioms of the Lie algebroid definition except, eventually, the Jacobi identity. In other words, we may have $\{\mu, \mu\} \neq 0$.

by $[\![\cdot,\cdot]\!]_{\mathcal{T}_N}$ the deformation of the Dorfman bracket $[\![\cdot,\cdot]\!]$ by $\mathcal{T}_N$ (see (44)), we have

$$\begin{aligned}\mathcal{T}_\mu \mathcal{T}_N(X,Y) &= [\![\mathcal{T}_N X, \mathcal{T}_N Y]\!] - \mathcal{T}_N\left([\![X,Y]\!]_{\mathcal{T}_N}\right)\\ &= [\![\mathcal{T}_N X, \mathcal{T}_N Y]\!] - \mathcal{T}_N[\![\mathcal{T}_N X, Y]\!] - \mathcal{T}_N[\![X, \mathcal{T}_N Y]\!] + \mathcal{T}_N^2[\![X,Y]\!]\\ &= [NX, NY] - N[NX,Y] - N[X,NY] + N^2[X,Y]\\ &= \mathcal{T}_\mu N(X,Y), \end{aligned} \tag{26}$$

where we used (45) and (23) and the fact that, when restricted to sections of A, $\mathcal{T}_N$ coincides with N. In the supergeometric setting, (26) may be written as

$$\mathcal{T}_\mu \mathcal{T}_N(X,Y) = \frac{1}{2}\left\{\{X, \mu_{N,N} - \mu_{N^2}\}, Y\right\}. \tag{27}$$

The second summand of the r.h.s. of (25) can be written as (see (45))

$$\mathcal{T}_\mu \mathcal{T}_N(X,\beta) = \frac{1}{2}\left([\![X,\beta]\!]_{\mathcal{T}_N,\mathcal{T}_N} - [\![X,\beta]\!]_{\mathcal{T}_N^2}\right). \tag{28}$$

The key argument of this proof is the relation between morphisms $\mathcal{T}_N^2 = \mathcal{T}_N \circ \mathcal{T}_N$ and $\mathcal{T}_{N^2}$. In fact

$$\mathcal{T}_N^2(X+\alpha) = N^2(X) + {N^*}^2(\alpha), \quad \text{while} \quad \mathcal{T}_{N^2}(X+\alpha) = N^2(X) - {N^*}^2(\alpha).$$

Thus, Equation (28) becomes

$$\begin{aligned}\mathcal{T}_\mu \mathcal{T}_N(X,\beta) &= \frac{1}{2}\left([\![X,\beta]\!]_{\mathcal{T}_N,\mathcal{T}_N} - [\![X,\beta]\!]_{\mathcal{T}_{N^2}}\right) - [\![X, {N^*}^2\beta]\!] + {N^*}^2[\![X,\beta]\!]\\ &= \frac{1}{2}\left\{\{X, \mu_{N,N} - \mu_{N^2}\}, \beta\right\} - [\![X, {N^*}^2\beta]\!] + {N^*}^2[\![X,\beta]\!]. \end{aligned} \tag{29}$$

Analogously, the third summand of the r.h.s. of Equation (25) is given by

$$\mathcal{T}_\mu \mathcal{T}_N(\alpha,Y) = \frac{1}{2}\left\{\{\alpha, \mu_{N,N} - \mu_{N^2}\}, Y\right\} - [\![{N^*}^2\alpha, Y]\!] + {N^*}^2[\![\alpha,Y]\!]. \tag{30}$$

Finally, the fourth summand vanishes because the Dorfman bracket vanishes when restricted to sections of $\Gamma(A^*)$. Thus, using (27), (29) and (30), Equation (25) becomes

$$\begin{aligned}
\mathcal{T}_\mu \mathcal{T}_N(X+\alpha, Y+\beta) &= \left\{\left\{X+\alpha, \frac{1}{2}\left(\mu_{N,N}-\mu_{N^2}\right)\right\}, Y+\beta\right\} \\
&\quad - [\![X, N^{*2}\beta]\!] + N^{*2}[\![X,\beta]\!] - [\![N^{*2}\alpha, Y]\!] + N^{*2}[\![\alpha, Y]\!] \\
&= [\![X+\alpha, Y+\beta]\!]_{\mathcal{T}_\mu N} \\
&\quad - [\![X, N^{*2}\beta]\!] + N^{*2}[\![X,\beta]\!] - [\![N^{*2}\alpha, Y]\!] + N^{*2}[\![\alpha, Y]\!].
\end{aligned}$$

■

Corollary 6.9. *If $\mathcal{T}_N$ is a Nijenhuis morphism on a pre-Courant algebroid $(A \oplus A^*, \mu)$, then N is a Nijenhuis morphism on the pre-Lie algebroid (A, μ).*

Proof: It suffices to evaluate (24) on pairs of type $(X+0, Y+0)$. ■

The next proposition, that already appears in [15], is a direct consequence of (24). The notations used are the same as in Proposition 6.8.

Proposition 6.10. *Let $N : A \to A$ be a bundle morphism such that $N^2 = \lambda \mathrm{id}_A$, for some $\lambda \in \mathbb{R}$. Then, the skew-symmetric morphism $\mathcal{T}_N = N \oplus (-N^*)$ satisfies $\mathcal{T}_N^2 = \lambda \mathrm{id}_{A\oplus A^*}$ and, in this case, $\mathcal{T}_\mu \mathcal{T}_N = 0$ if and only if $\mathcal{T}_\mu N = 0$.*

The next proposition establishes a relation between the Frölicher-Nijenhuis bracket $[I,J]_{FN}$ of two endomorphisms on a Lie algebroid (A,μ) and the concomitant $C_\mu(\mathcal{T}_I, \mathcal{T}_J)$ of the induced morphisms on $(A \oplus A^*, \mu)$. But beforehand we need to recall a result from [1].

Theorem 6.11. [1] *Let (A, μ) be a Lie algebroid. For all vector-valued forms $K \in \Gamma(\bigwedge^k A^* \otimes A)$ and $L \in \Gamma(\bigwedge^l A^* \otimes A)$, we have*

$$[K, L]_{FN} = \{\{K, \mu\}, L\} + (-1)^{k(l+1)}\{i_L K, \mu\}, \tag{31}$$

where $i_L K$ is the interior product of K by L (see section 8 in [11]).

Proposition 6.12. *Let (A,μ) be a Lie algebroid and $I, J : A \to A$ two anticommuting endomorphisms of A. Then, $\mathcal{T}_I := \begin{pmatrix} I & 0 \\ 0 & -I^* \end{pmatrix}$ and $\mathcal{T}_J := \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}$ are anticommuting skew-symmetric endomorphisms of $A \oplus A^*$ and*

$$C_\mu(\mathcal{T}_I, \mathcal{T}_J)(X+0, Y+0) = -2[I,J]_{FN}(X,Y),$$

for all $X, Y \in \Gamma(A)$. In particular, if $C_\mu(\mathcal{T}_I, \mathcal{T}_J)$ vanishes then so does $[I, J]_{FN}$.

Proof: The fact that $\mathcal{T}_I$ and $\mathcal{T}_J$ are anticommuting skew-symmetric endomorphisms of $A \oplus A^*$ is immediate to check. Moreover, using (31), we have

$$[I, J]_{FN} = \{\{I, \mu\}, J\} + \{I \circ J, \mu\}. \tag{32}$$

Because I and J anticommute, $I \circ J = \frac{1}{2}\{J, I\}$ and (32) becomes

$$[I, J]_{FN} = \{\{I, \mu\}, J\} + \frac{1}{2}\{\{J, I\}, \mu\}. \tag{33}$$

Using the Jacobi identity in the last term of (33) we get

$$[I, J]_{FN} = -\frac{1}{2}\big(\{J, \{I, \mu\}\} + \{I, \{J, \mu\}\}\big) = -\frac{1}{2}C_\mu(\mathcal{T}_I, \mathcal{T}_J).$$

■

7. Compatibilities and deformations

It was proved in [7] that, when $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = -1$, condition (15)v) in Definition 4.2 implies that the Nijenhuis torsion of the endomorphisms I_1, I_2 and I_3 vanishes, so that they are in fact complex structures on (A, μ). Taking into account Theorem 4.5, the next theorem can be seen as a generalization of the mentioned result in [7].

Theorem 7.1. *Let $(\omega_1, \omega_2, \omega_3)$ be a (para-)hypersymplectic structure with torsion on a Lie algebroid (A, μ). The endomorphisms N_1, N_2 and N_3 given by (1) are Nijenhuis morphisms.*

Proof: As a consequence of Proposition 6.7, the triplet $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$, with $\mathcal{S}_i$ given by (21), is a (para-)hypersymplectic structure on the pre-Courant algebroid $(A \oplus A^*, \mu + \psi)$, with $\psi = \frac{\varepsilon_k}{2}[\pi_k, \pi_k]$, for any $k \in \{1, 2, 3\}$. By Theorem 6.4, the endomorphisms $\mathcal{T}_i$, $i = 1, 2, 3$, given by (22) are Nijenhuis morphisms on $(A \oplus A^*, \mu + \psi)$. This means that

$$\begin{cases} \mathcal{T}_i^2 = \varepsilon_i \mathrm{id}_{A \oplus A^*} \\ (\mu + \psi)_{\mathcal{T}_i, \mathcal{T}_i} = \varepsilon_i(\mu + \psi). \end{cases}$$

Splitting up the equality $(\mu + \psi)_{\mathcal{T}_i, \mathcal{T}_i} = \varepsilon_i(\mu + \psi)$ in terms of bidegree, we get, on bidegree $(1, 2)$:

$$\mu_{\mathcal{T}_i, \mathcal{T}_i} = \varepsilon_i \mu.$$

Thus, $\mathcal{T}_i$ is Nijenhuis on $(A \oplus A^*, \mu)$ and Corollary 6.9 yields that N_i is Nijenhuis on (A, μ). ■

Next, we prove that a (para-)hypersymplectic structure with torsion on a Lie algebroid (A, μ) determines some compatibility properties among the N_i's, the π_i's and the ω_i's.

Proposition 7.2. *Let $(\omega_1, \omega_2, \omega_3)$ be a (para-)hypersymplectic structure with torsion on (A, μ). The Nijenhuis morphisms N_1, N_2 and N_3 given by (1) are pairwise compatible in the sense that $[N_i, N_j]_{FN} = 0, i, j \in \{1, 2, 3\}$.*

Proof: First, notice that $[N_i, N_i]_{FN} = -2\,\mathcal{T}_\mu N_i$, thus, when $i = j$, the statement was proved in Theorem 7.1. Let us consider now $i, j \in \{1, 2, 3\}$, with $i \neq j$. From Proposition 6.7, the triplet $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a (para-)hypersymplectic structure on the pre-Courant algebroid $(A \oplus A^*, \mu + \psi)$, with $\psi = \frac{\varepsilon_k}{2}[\pi_k, \pi_k]$, for any $k \in \{1, 2, 3\}$. Using Proposition 6.5, we have $C_{\mu+\psi}(\mathcal{T}_i, \mathcal{T}_j) = 0$, with $\mathcal{T}_i$ given by (22), i.e.,

$$\{N_i, \{N_j, \mu + \psi\}\} + \{N_j, \{N_i, \mu + \psi\}\} = 0.$$

Splitting up this equality in terms of bidegrees, we get, on bidegree $(1, 2)$,

$$\{N_i, \{N_j, \mu\}\} + \{N_j, \{N_i, \mu\}\} = 0,$$

which can be written as $C_\mu(\mathcal{T}_i, \mathcal{T}_j) = 0$. Applying Proposition 6.12, the statement is proved. ■

Proposition 7.3. *Let $(\omega_1, \omega_2, \omega_3)$ be a (para-)hypersymplectic structure with torsion on (A, μ) and π_k be the inverse of ω_k, $k = 1, 2, 3$. Then, $[\pi_i, \pi_j] = 0$, $i, j \in \{1, 2, 3\}, i \neq j$.*

Proof: We can assume, without loss of generality, that $j = i - 1$ and prove $[\pi_i, \pi_{i-1}] = 0$, for $i \in \mathbb{Z}_3$. From Proposition 6.7, the triplet $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a (para-)hypersymplectic structure on the pre-Courant algebroid $(A \oplus A^*, \mu + \psi)$, with $\psi = \frac{\varepsilon_k}{2}[\pi_k, \pi_k]$, for any $k \in \{1, 2, 3\}$. Using Proposition 6.5, we have $C_{\mu+\psi}(\mathcal{S}_i, \mathcal{S}_{i-1}) = 0$, i.e.,

$$\{\omega_i + \varepsilon_i\pi_i, \{\omega_{i-1} + \varepsilon_{i-1}\pi_{i-1}, \mu + \psi\}\} + \{\omega_{i-1} + \varepsilon_{i-1}\pi_{i-1}, \{\omega_i + \varepsilon_i\pi_i, \mu + \psi\}\} = 0.$$

Splitting up this equality in terms of bidegrees, we get, on bidegree $(3, 0)$,

$$\varepsilon_i\varepsilon_{i-1}\{\pi_i, \{\pi_{i-1}, \mu\}\} + \varepsilon_i\{\pi_i, \{\omega_{i-1}, \psi\}\} + \circlearrowleft_{i,i-1} = 0,$$

where $\circlearrowleft_{i,i-1}$ stands for the permutation on indices i and $i-1$. Using the Jacobi identity of the big bracket, we get

$$2\varepsilon_i\varepsilon_{i-1}\{\pi_i, \{\pi_{i-1}, \mu\}\} + \varepsilon_i\{\{\pi_i, \omega_{i-1}\}, \psi\} + \varepsilon_{i-1}\{\{\pi_{i-1}, \omega_i\}, \psi\} = 0. \quad (34)$$

Noticing that the transition morphisms N_{i+1} given by (1) can be equivalently defined as $N_{i+1} = -\{\pi_i, \omega_{i-1}\} = -\varepsilon_{i+1}\{\pi_{i-1}, \omega_i\}$ (see [3]), (34) becomes

$$2\varepsilon_i\varepsilon_{i-1}\{\pi_i, \{\pi_{i-1}, \mu\}\} - (\varepsilon_i + \varepsilon_{i-1}\varepsilon_{i+1})\{N_{i+1}, \psi\} = 0. \tag{35}$$

Because $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$, we have $\varepsilon_i + \varepsilon_{i-1}\varepsilon_{i+1} = 0$, for all $i \in \mathbb{Z}_3$, so that (35) simplifies to $\{\pi_i, \{\pi_{i-1}, \mu\}\} = 0$, which is equivalent to $[\pi_i, \pi_{i-1}] = 0$. ∎

Proposition 7.4. *Let $(\omega_1, \omega_2, \omega_3)$ be a (para-)hypersymplectic structure with torsion on (A, μ) and consider the endomorphisms N_1, N_2, N_3 given by (1). Then, $C_\mu(\omega_i, N_j) = 0, i, j \in \{1, 2, 3\}, i \neq j$.*

Proof: The triplet $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is, by Proposition 6.7, a (para-)hypersymplectic structure on $(A \oplus A^*, \mu + \psi)$. Proposition 6.5 yields that $C_{\mu+\psi}(\mathcal{S}_i, \mathcal{T}_j) = 0$, $i \neq j$. Considering the part of bidegree $(0, 3)$ in equation $C_{\mu+\psi}(\mathcal{S}_i, \mathcal{T}_j) = 0$, we get

$$\{\omega_i, \{N_j, \mu\}\} + \{N_j, \{\omega_i, \mu\}\} = 0.$$

∎

Recall [12] that if N is a Nijenhuis morphism on a Lie algebroid (A, μ) then $\mu_N = \{N, \mu\}$ is a Lie algebroid structure on A. The Lie bracket on $\Gamma(A)$ will be denoted by $[\cdot, \cdot]_N$.

Theorem 7.5. *A triplet $(\omega_1, \omega_2, \omega_3)$ is a (para-)hypersymplectic structure with torsion on (A, μ) if and only if $(\omega_1, \omega_2, \omega_3)$ is a (para-)hypersymplectic structure with torsion on (A, μ_{N_i}), $i = 1, 2, 3$.*

Proof: Let $(\omega_1, \omega_2, \omega_3)$ be a (para-)hypersymplectic structure with torsion on (A, μ) and fix $i \in \{1, 2, 3\}$. From Proposition 6.7 and Theorem 6.6 we get that $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a (para-)hypersymplectic structure on the pre-Courant algebroid $(A \oplus A^*, (\mu + \psi)_{\mathcal{T}_i})$, with $\psi = \frac{\varepsilon_k}{2}[\pi_k, \pi_k]$, for any $k \in \{1, 2, 3\}$. In the computation that follows we consider $k = i$. The pre-Courant structure $(\mu + \psi)_{\mathcal{T}_i}$ is given by $\{N_i, \mu + \psi\}$ and we have

$$\begin{aligned}
\{N_i, \mu + \psi\} &= \mu_{N_i} + \left\{N_i, -\frac{\varepsilon_i}{2}\{\pi_i, \{\pi_i, \mu\}\}\right\} \\
&= \mu_{N_i} - \frac{\varepsilon_i}{2}\{\{N_i, \pi_i\}, \{\pi_i, \mu\}\} - \frac{\varepsilon_i}{2}\{\pi_i, \{\{N_i, \pi_i\}, \mu\}\} \\
&\quad - \frac{\varepsilon_i}{2}\{\pi_i, \{\pi_i, \{N_i, \mu\}\}\} \\
&= \mu_{N_i} + \frac{\varepsilon_i}{2}[\pi_i, \pi_i]_{N_i},
\end{aligned}$$

where we used, in the last equality, the fact that $\{N_i, \pi_i\} = 0$ (which is a consequence of Proposition 1.2 ii)). Applying Proposition 6.7 we conclude that $(\omega_1, \omega_2, \omega_3)$ is a (para-)hypersymplectic structure with torsion on (A, μ_{N_i}).

The converse holds because the statements of Theorem 6.6 and Proposition 6.7 are equivalences. ∎

The proof of Theorem 7.5 can be summarized in the diagram:

$$\begin{tikzcd}
\begin{array}{c}(\omega_1,\omega_2,\omega_3) \text{ (para-)HST} \\ \text{on } (A,\mu)\end{array} \arrow[r, Leftrightarrow, "\textit{Thm } 7.5"] \arrow[d, Leftrightarrow, "\textit{Prop } 6.7"'] & \begin{array}{c}(\omega_1,\omega_2,\omega_3) \text{ (para-)HST} \\ \text{on } (A,\mu_{N_i})\end{array} \arrow[d, Leftrightarrow, "\textit{Prop } 6.7"] \\
\begin{array}{c}(\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3) \text{ (para-)HS} \\ \text{on } (A\oplus A^*, \mu+\psi)\end{array} \arrow[r, Leftrightarrow, "\textit{Thm } 6.6"] & \begin{array}{c}(\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3) \text{ (para-)HS} \\ \text{on } (A\oplus A^*, (\mu+\psi)_{\mathcal{T}_i})\end{array}
\end{tikzcd}$$

where we used the abbreviations HS and HST for hypersymplectic and hypersymplectic with torsion, respectively.

Proposition 7.6. *Let (A, μ) be a Lie algebroid and π a non-degenerate bivector on A with inverse ω. Then, $\gamma^\pi := \mu_\pi + \frac{1}{2}\{\omega, [\pi, \pi]\}$ is a Lie algebroid structure on A^*.*

Proof: Aiming to prove that $\{\gamma^\pi, \gamma^\pi\} = 0$, we compute

$$\begin{aligned}
\{\gamma^\pi, \gamma^\pi\} &= \left\{\mu_\pi + \frac{1}{2}\{\omega, [\pi,\pi]\}, \mu_\pi + \frac{1}{2}\{\omega, [\pi,\pi]\}\right\} \\
&= \{\mu_\pi, \mu_\pi\} + \{\mu_\pi, \{\omega, [\pi,\pi]\}\} + \frac{1}{4}\{\{\omega,[\pi,\pi]\}, \{\omega,[\pi,\pi]\}\} \\
&= \{\mu_\pi, \mu_\pi\} + \{\mu_\pi, \{\omega, [\pi,\pi]\}\} + \frac{1}{4}\{\{\{\omega,[\pi,\pi]\}, \omega\}, [\pi,\pi]\} \\
&\qquad + \frac{1}{4}\{\omega, \{\{\omega,[\pi,\pi]\}, [\pi,\pi]\}\},
\end{aligned} \tag{36}$$

where we used the Jacobi identity in the last equality. Using again the Jacobi identity and taking into account the bidegree, we get $\{\{\omega, [\pi, \pi]\}, [\pi, \pi]\} = 0$. Furthermore, a straightforward computation leads to $\{\{\omega, [\pi, \pi]\}, \omega\} = 4\{\pi, \{\omega, \mu\}\}$. Therefore, equality (36) becomes

$$\{\gamma^\pi, \gamma^\pi\} = \{\mu_\pi, \mu_\pi\} + \{\mu_\pi, \{\omega, [\pi,\pi]\}\} + \{\{\pi, \{\omega, \mu\}\}, [\pi,\pi]\}. \tag{37}$$

A simple computation shows that $\{\mu_\pi, \mu_\pi\} = \{\mu, [\pi, \pi]\}$ and the Jacobi identity applied to the last two terms of (37) yields

$$\begin{aligned}\{\gamma^\pi, \gamma^\pi\} = \{\mu, [\pi, \pi]\} + \{\{\mu_\pi, \omega\}, [\pi, \pi]\} + \{\omega, \{\mu_\pi, [\pi, \pi]\}\} \\ + \{\{-\mathrm{id}_A, \mu\}, [\pi, \pi]\} + \{\{\omega, \{\pi, \mu\}\}, [\pi, \pi]\}.\end{aligned}$$

Since $\{\mathrm{id}_A, \mu\} = \mu$ and $\{\mu_\pi, \omega\} = \{\{\pi, \mu\}, \omega\} = -\{\omega, \{\pi, \mu\}\}$, we obtain

$$\{\gamma^\pi, \gamma^\pi\} = \{\omega, \{\mu_\pi, [\pi, \pi]\}\}.$$

Finally, a cumbersome computation shows that $\{\mu_\pi, [\pi, \pi]\} = 0$ and completes the proof. ■

Remark 7.7. In the proof of Proposition 7.6 we only use the properties of the big bracket. However, the proof can be done using the operation of twisting by a bivector and by a 2-form that was introduced in [19]. Let us briefly explain this, using the notation of [19]. The twisting of $(\mu, 0, 0, 0)$ by $-\omega$ yields the quasi-Lie bialgebroid structure $(\mu, 0, 0, \{\mu, \omega\})$ on (A, A^*) and the twisting of $(\mu, 0, 0, \{\mu, \omega\})$ by π gives

$$\left(\mu + \{\pi, \{\mu, \omega\}\},\ \{\mu, \pi\} + \frac{1}{2}\{\omega, [\pi, \pi]\},\ 0,\ \{\mu, \omega\}\right). \tag{38}$$

From Lemma 2.1, the twisted Maurer-Cartan equation

$$[\pi, \pi] = 2d\omega\left(\pi^\sharp(\cdot), \pi^\sharp(\cdot), \pi^\sharp(\cdot)\right)$$

holds, so that (38) is a quasi-Lie bialgebroid structure on (A, A^*), as it is proved in [19]. This, in turn, implies that the term of bidegree $(2, 1)$ in (38),

$$\{\mu, \pi\} + \frac{1}{2}\{\omega, [\pi, \pi]\}, \tag{39}$$

is a Lie algebroid structure on A^*.

In the next theorem we show that there is a one-to-one correspondence between (para-)hypersymplectic structures with torsion on A and on A^*.

Theorem 7.8. *The triplet $(\omega_1, \omega_2, \omega_3)$ is a (para-)hypersymplectic structure with torsion on (A, μ) if and only if (π_1, π_2, π_3) is a (para-)hypersymplectic structure with torsion on $(A^*, \varepsilon_i\gamma^{\pi_i})$, where $\gamma^{\pi_i} := \mu_{\pi_i} + \frac{1}{2}\{\omega_i, [\pi_i, \pi_i]\}, i = 1, 2, 3$, and π_i is the inverse of ω_i.*

Under the conditions of Theorem 7.8, the Lie algebroid structures on A^* can be written as $\varepsilon_i\mu_{\pi_i} + \{\omega_i, \psi\}, i = 1, 2, 3$, with $\psi = \frac{\varepsilon_k}{2}[\pi_k, \pi_k]$, for any $k \in \{1, 2, 3\}$.

Proof of Theorem 7.8: Let $(\omega_1, \omega_2, \omega_3)$ be a (para-)hypersymplectic structure with torsion on (A, μ) and fix $i \in \{1, 2, 3\}$. From Proposition 6.7 and Theorem 6.6 we have that $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a (para-)hypersymplectic structure on the pre-Courant algebroid $(A \oplus A^*, (\mu + \psi)_{\mathcal{S}_i})$, with $\psi = \frac{\varepsilon_k}{2}[\pi_k, \pi_k]$, for any $k \in \{1, 2, 3\}$. In the computation that follows we consider $k = i$. The pre-Courant structure $(\mu + \psi)_{\mathcal{S}_i}$ is given by

$$(\mu + \psi)_{\mathcal{S}_i} = \varepsilon_i \gamma^{\pi_i} + \mu_{\omega_i}.$$

A direct computation shows that $\mu_{\omega_i} = \frac{\varepsilon_i}{2}[\omega_i, \omega_i]_{\gamma^{\pi_i}}$, where $[\cdot, \cdot]_{\gamma^{\pi_i}}$ stands for the Schouten-Nijenhuis bracket of the Lie algebroid structure γ^{π_i} on A^*. Then,

$$(\mu + \psi)_{\mathcal{S}_i} = \varepsilon_i \gamma^{\pi_i} + \frac{\varepsilon_i}{2}[\omega_i, \omega_i]_{\gamma^{\pi_i}} = \varepsilon_i \gamma^{\pi_i} + \frac{1}{2}[\omega_i, \omega_i]_{\varepsilon_i \gamma^{\pi_i}}$$

and, using Proposition 6.7, the result follows.

The converse holds because the statements of Theorem 6.6 and Proposition 6.7 are equivalences. ∎

The proof of Theorem 7.8 can be summarized in the diagram:

$$\begin{tikzcd}
\begin{array}{c}(\omega_1, \omega_2, \omega_3) \text{ (para-)HST} \\ \text{on } (A, \mu)\end{array} \arrow[r, Leftrightarrow, "\mathit{Thm}\ 7.8"] \arrow[d, Leftrightarrow, "\mathit{Prop}\ 6.7"'] & \begin{array}{c}(\pi_1, \pi_2, \pi_3) \text{ (para-)HST} \\ \text{on } (A^*, \varepsilon_i \gamma^{\pi_i})\end{array} \arrow[d, Leftrightarrow, "\mathit{Prop}\ 6.7"] \\
\begin{array}{c}(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3) \text{ (para-)HS} \\ \text{on } (A \oplus A^*, \mu + \psi)\end{array} \arrow[r, Leftrightarrow, "\mathit{Thm}\ 6.6"] & \begin{array}{c}(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3) \text{ (para-)HS} \\ \text{on } (A \oplus A^*, (\mu + \psi)_{\mathcal{S}_i})\end{array}
\end{tikzcd}$$

Appendix A. Preliminaries on pre-Courant algebroids. The supergeometric setting and Nijenhuis torsion.

We introduce the supergeometric setting following the approach in [17, 18, 22]. Given a vector bundle $A \to M$, we denote by $A[n]$ the graded manifold obtained by shifting the degree of coordinates on the fiber by n. The graded manifold $T^*[2]A[1]$ is equipped with a canonical symplectic structure which induces a Poisson bracket on its algebra of functions $\mathcal{F} := C^\infty(T^*[2]A[1])$. This Poisson bracket is called the *big bracket* (see [13], [14]).

In local coordinates $x^i, p_i, \xi^a, \theta_a$, $i \in \{1, \dots, n\}, a \in \{1, \dots, d\}$, in $T^*[2]A[1]$, where x^i, ξ^a are local coordinates on $A[1]$ and p_i, θ_a are the the conjugate coordinates, the Poisson bracket is given by

$$\{p_i, x^i\} = \{\theta_a, \xi^a\} = 1, \quad i = 1, \dots, n, \; a = 1, \dots, d,$$

while the remaining brackets vanish.

The Poisson algebra of functions $\mathcal{F}$ is endowed with an $(\mathbb{N} \times \mathbb{N})$-valued bidegree. We define this bidegree (locally but it is well defined globally, see [23, 17]) as follows: the coordinates on the base manifold M, x^i, $i \in \{1, \dots, n\}$, have bidegree $(0,0)$, while the coordinates on the fibres, ξ^a, $a \in \{1, \dots, d\}$, have bidegree $(0,1)$ and their associated moment coordinates, p_i and θ_a, have bidegrees $(1,1)$ and $(1,0)$, respectively. We denote by $\mathcal{F}^{k,l}$ the space of functions of bidegree (k,l). The *total degree* of a function $f \in \mathcal{F}^{k,l}$ is equal to $k+l$ and the subset of functions of total degree t is denoted by $\mathcal{F}^t$. We can verify that the big bracket has bidegree $(-1,-1)$, i.e.,

$$\{\mathcal{F}^{k_1,l_1}, \mathcal{F}^{k_2,l_2}\} \subset \mathcal{F}^{k_1+k_2-1, l_1+l_2-1}.$$

Thus, the big bracket on functions of lowest degrees, $\{\mathcal{F}^0, \mathcal{F}^0\}$ and $\{\mathcal{F}^0, \mathcal{F}^1\}$, vanish. For $X+\alpha, Y+\beta \in \mathcal{F}^1 = \Gamma(A \oplus A^*)$, $\{X+\alpha, Y+\beta\}$ is an element of $\mathcal{F}^0 = C^\infty(M)$ and is given by

$$\{X+\alpha, Y+\beta\} = \langle X+\alpha, Y+\beta\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the usual fiberwise symmetric bilinear form on $A \oplus A^*$:

$$\langle X+\alpha, Y+\beta\rangle = \alpha(Y) + \beta(X), \; \forall X, Y \in \Gamma(A), \alpha, \beta \in \Gamma(A^*). \tag{40}$$

This construction is a particular case of a more general one [18] in which we consider a vector bundle E equipped with a fibrewise non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$. In this more general setting, we consider the graded symplectic manifold $\mathcal{E} := p^*(T^*[2]E[1])$, which is the pull-back of $T^*[2]E[1]$ by the map $p : E[1] \to E[1] \oplus E^*[1]$ defined by $X \mapsto (X, \frac{1}{2}\langle X, .\rangle)$. We denote by $\mathcal{F}_E$ the graded algebra of functions on $\mathcal{E}$, i.e., $\mathcal{F}_E := C^\infty(\mathcal{E})$. The algebra $\mathcal{F}_E$ is equipped with the canonical Poisson bracket, denoted by $\{\cdot,\cdot\}$, which has degree -2. Notice that $\mathcal{F}_E^0 = C^\infty(M)$ and $\mathcal{F}_E^1 = \Gamma(E)$. Under these identifications, the Poisson bracket of functions of degrees 0 and 1 is given by

$$\{f, g\} = 0, \;\; \{f, X\} = 0 \quad \text{and} \quad \{X, Y\} = \langle X, Y\rangle,$$

for all $X, Y \in \Gamma(E)$ and $f, g \in C^\infty(M)$.

When $E := A \oplus A^*$ (with A a vector bundle over M) and when $\langle \cdot, \cdot \rangle$ is the usual symmetric bilinear form given by (40), the algebras $\mathcal{F} = C^\infty(T^*[2]A[1])$ and $\mathcal{F}_{A\oplus A^*}$ are isomorphic Poisson algebras [18].

Definition A.1. [2] A *pre-Courant* structure on $(E, \langle \cdot, \cdot \rangle)$ is a pair $(\rho, [\![\cdot, \cdot]\!])$, where $\rho : E \to TM$ is a morphism of vector bundles called the *anchor*, and $[\![\cdot, \cdot]\!] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ is a $\mathbb{R}$-bilinear (non necessarily skew-symmetric) bracket, called the *Dorfman bracket*, satisfying the relations

$$\rho(X) \cdot \langle Y, Z \rangle = \langle [\![X, Y]\!], Z \rangle + \langle Y, [\![X, Z]\!] \rangle \tag{41}$$

and

$$\rho(X) \cdot \langle Y, Z \rangle = \langle X, [\![Y, Z]\!] + [\![Z, Y]\!] \rangle, \tag{42}$$

for all $X, Y, Z \in \Gamma(E)$.

From (41) and (42), we obtain the Leibniz rule [14]

$$[\![X, fY]\!] = f[\![X, Y]\!] + (\rho(X).f)Y,$$

for all $X, Y \in \Gamma(E)$ and $f \in C^\infty(M)$.
If a pre-Courant structure $(\rho, [\![\cdot, \cdot]\!])$ satisfies the Jacobi identity,

$$[\![X, [\![Y, Z]\!]]\!] = [\![[\![X, Y]\!], Z]\!] + [\![Y, [\![X, Z]\!]]\!],$$

for all $X, Y, Z \in \Gamma(E)$, then the pair $(\rho, [\![\cdot, \cdot]\!])$ is called a *Courant* structure on $(E, \langle \cdot, \cdot \rangle)$.

There is a one-to-one correspondence between pre-Courant structures on $(E, \langle \cdot, \cdot \rangle)$ and elements in $\mathcal{F}^3_E$. The anchor and Dorfman bracket associated to a given $\Theta \in \mathcal{F}^3_E$ are defined, for all $X, Y \in \Gamma(E)$ and $f \in C^\infty(M)$, by the derived bracket expressions

$$\rho(X) \cdot f = \{\{X, \Theta\}, f\} \quad \text{and} \quad [\![X, Y]\!] = \{\{X, \Theta\}, Y\}. \tag{43}$$

A function $\Theta \in \mathcal{F}^3_E$ determines a Courant structure on $(E, \langle \cdot, \cdot \rangle)$ if and only if $\{\Theta, \Theta\} = 0$ ([18]). If Θ is a (pre-)Courant structure on $(E, \langle \cdot, \cdot \rangle)$, then the triple $(E, \langle \cdot, \cdot \rangle, \Theta)$ is called a *(pre-)Courant algebroid*. For the sake of simplicity, we often denote a (pre-)Courant algebroid by the pair (E, Θ) instead of the triple $(E, \langle \cdot, \cdot \rangle, \Theta)$.

When $E = A \oplus A^*$ and $\langle \cdot, \cdot \rangle$ is the usual symmetric bilinear form (40), a pre-Courant structure $\Theta \in \mathcal{F}^3_E$ can be decomposed using the bidegrees:

$$\Theta = \mu + \gamma + \phi + \psi,$$

with $\mu \in \mathcal{F}^{1,2}_{A\oplus A^*}, \gamma \in \mathcal{F}^{2,1}_{A\oplus A^*}, \phi \in \mathcal{F}^{0,3}_{A\oplus A^*} = \Gamma(\wedge^3 A^*)$ and $\psi \in \mathcal{F}^{3,0}_{A\oplus A^*} = \Gamma(\wedge^3 A)$. We recall from [14] that, when $\gamma = \phi = \psi = 0$, Θ is a Courant structure on $(A\oplus A^*, \langle\cdot,\cdot\rangle)$ *if and only if* (A,μ) is a Lie algebroid; the anchor and the bracket of the Lie algebroid (A,μ) are given by (43), where a section X of A is identified with $X + 0 \in \Gamma(A\oplus A^*)$. When $\phi = \psi = 0$, Θ is a Courant structure on $(A\oplus A^*, \langle\cdot,\cdot\rangle)$ *if and only if* $((A,A^*),\mu,\gamma)$ is a Lie bialgebroid and when $\phi = 0$ (resp. $\psi = 0$), Θ is a Courant structure on $(A\oplus A^*, \langle\cdot,\cdot\rangle)$ *if and only if* $((A,A^*),\mu,\gamma,\psi)$ (resp. $((A,A^*),\mu,\gamma,\phi)$) is a quasi-Lie bialgebroid. In the more general case, $\Theta = \mu+\gamma+\phi+\psi$ is a Courant structure *if and only if* $((A,A^*),\mu,\gamma,\psi,\phi)$ is a proto-Lie bialgebroid.

Let $(E,\langle\cdot,\cdot\rangle,\Theta)$ be a pre-Courant algebroid with anchor and Dorfman bracket defined by (43). Given an endomorphism $\mathcal{I}: E \to E$, we define a *deformed* pre-Courant algebroid structure $(\rho_{\mathcal{I}}, [\![\cdot,\cdot]\!]_{\mathcal{I}})$ on E by setting

$$\begin{cases} \rho_{\mathcal{I}} = \rho\circ\mathcal{I} \\ [\![X,Y]\!]_{\mathcal{I}} = [\![\mathcal{I}X,Y]\!] + [\![X,\mathcal{I}Y]\!] - \mathcal{I}[\![X,Y]\!], \quad \forall X,Y\in\Gamma(E). \end{cases} \tag{44}$$

The deformation of $(\rho_{\mathcal{I}}, [\![\cdot,\cdot]\!]_{\mathcal{I}})$ by an endomorphism $\mathcal{J}$ of E is denoted by $(\rho_{\mathcal{I},\mathcal{J}}, [\![\cdot,\cdot]\!]_{\mathcal{I},\mathcal{J}})$. The *concomitant* $C_\Theta(\mathcal{I},\mathcal{J})$ of two endomorphisms $\mathcal{I}$ and $\mathcal{J}$, on a pre-Courant algebroid $(E,\langle\cdot,\cdot\rangle,\Theta)$, is a $\mathbb{R}$-bilinear map $\Gamma(E)\times\Gamma(E)\to\Gamma(E)$ defined, for all sections X,Y of E, by

$$C_\Theta(\mathcal{I},\mathcal{J})(X,Y) := [\![X,Y]\!]_{\mathcal{I},\mathcal{J}} + [\![X,Y]\!]_{\mathcal{J},\mathcal{I}}.$$

Recall that an endomorphism $\mathcal{I}: E\to E$ on a pre-Courant algebroid $(E,\langle\cdot,\cdot\rangle,\Theta)$ is a *Nijenhuis morphism* if its Nijenhuis torsion $\mathcal{T}_\Theta\mathcal{I}$ vanishes, where

$$\mathcal{T}_\Theta\mathcal{I}(X,Y) = [\![\mathcal{I}X,\mathcal{I}Y]\!] - \mathcal{I}\left([\![X,Y]\!]_{\mathcal{I}}\right) = \frac{1}{2}\left([\![X,Y]\!]_{\mathcal{I},\mathcal{I}} - [\![X,Y]\!]_{\mathcal{I}^2}\right), \tag{45}$$

for all $X,Y\in\Gamma(E)$.

Given an endomorphism $\mathcal{I}: E\to E$, the transpose morphism $\mathcal{I}^*: E^*\simeq E\to E^*\simeq E$ is defined by $\langle\mathcal{I}^*u,v\rangle = \langle u,\mathcal{I}v\rangle$ for all $u,v\in E$. If $\mathcal{I} = -\mathcal{I}^*$ the morphism $\mathcal{I}$ is said to be skew-symmetric and, in this case, the deformed pre-Courant structure $(\rho_{\mathcal{I}}, [\![\cdot,\cdot]\!]_{\mathcal{I}})$ corresponds to the function $\Theta_{\mathcal{I}} := \{\mathcal{I},\Theta\}\in\mathcal{F}^3_E$, (via (43)). The deformation of $\Theta_{\mathcal{I}}$ by a skew-symmetric morphism $\mathcal{J}$ is denoted by $\Theta_{\mathcal{I},\mathcal{J}}$, i.e. $\Theta_{\mathcal{I},\mathcal{J}} = \{\mathcal{J},\{\mathcal{I},\Theta\}\}$. When $\mathcal{I}$ and $\mathcal{J}$ are skew-symmetric endomorphisms of E, the concomitant $C_\Theta(\mathcal{I},\mathcal{J})$ may be defined as an element of $\mathcal{F}_E^3$ by setting [2]:

$$C_\Theta(\mathcal{I}, \mathcal{J}) = \Theta_{\mathcal{I},\mathcal{J}} + \Theta_{\mathcal{J},\mathcal{I}}. \tag{46}$$

When $\mathcal{I}$ is skew-symmetric and satisfies $\mathcal{I}^2 = \lambda \, \mathrm{id}_E$, for some $\lambda \in \mathbb{R}$, we have [8, 1]

$$\mathcal{T}_\Theta \mathcal{I} = \frac{1}{2}(\Theta_{\mathcal{I},\mathcal{I}} - \lambda\Theta). \tag{47}$$

If $\mathcal{I}^2 = -\mathrm{id}_E$ (resp. $\mathcal{I}^2 = \mathrm{id}_E$) then $\mathcal{I}$ is said to be an *almost complex* (resp. *almost para-complex*) *structure*. If moreover $\mathcal{T}_\Theta \mathcal{I} = 0$, then $\mathcal{I}$ is a *complex* (resp. *para-complex*) *structure*.

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P. Antunes
CMUC, Department of Mathematics, University of Coimbra, 3001-454 Coimbra, Portugal
E-mail address: pantunes@mat.uc.pt

J.M. Nunes da Costa
CMUC, Department of Mathematics, University of Coimbra, 3001-454 Coimbra, Portugal
E-mail address: jmcosta@mat.uc.pt