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ON THE DISTRIBUTION ESTIMATION OF POWER THRESHOLD GARCH PROCESSES

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Abstract: The aim of this paper is to estimate the probability distribution of power TGARCH processes by establishing bounds for their finite dimensional laws. These bounds only depend on the parameters of the model and on the distribution function of its independent generating process. The application of this study to some particular models allows us to conjecture that this procedure is an adequate alternative to the corresponding estimation using the empirical distribution functions, particularly useful in the development of control charts for this kind of models.

Keywords: Finite dimensional laws, threshold GARCH processes.
AMS Subject Classification (2000): 60G10, 62M10.

1. Introduction

The knowledge of the true theoretical law of conditional heteroskedastic models remains an open question to which it seems difficult to answer. The most part of the analysis undertaken for these models is dedicated to the study of properties or probabilistic summaries of those laws. But the use of these models, for instance within the quality control theory, needs the assessment of the probability of certain regions depending on the process. In order to answer this problem, Pawlak and Schmid [8], Gonçalves and Mendes-Lopes [6] and Gonçalves, Leite and Mendes-Lopes [4] developed studies to find bounds for the finite dimensional laws of certain transformations of ARCH and TGARCH processes. These authors showed for some particular situations the usefulness of this methodology in the evaluation of control charts for conditional heteroskedastic models.

For a real stochastic process $X = (X_t, t \in \mathbb{Z})$ let us define $X_t^+ = \max(X_t, 0)$, $X_t^- = \max(-X_t, 0)$ and $\underline{X}_t$ the sigma field generated by $X_t, X_{t-1}, ...$

The process X follows a power δ generalized threshold autoregressive conditionally heteroskedastic model with orders p and q, denoted δ-TGARCH(p, q), if for real constants $\alpha_0 > 0$, $\alpha_i \geq 0$, $\beta_i \geq 0$, $\gamma_j \geq 0$, $(i = 1, ..., p, j = 1, ..., q)$

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and a sequence of independent and identically distributed real random variables, $(Z_t, t \in \mathbb{Z})$, with zero mean, unit variance and Z_t independent of $\underline{X}_{t-1}$ we have, for every $t \in \mathbb{Z}$,

$$\begin{cases} X_t = \sigma_t Z_t \\ \sigma_t^\delta = \alpha_0 + \sum_{i=1}^{p} [\alpha_i (X_{t-i}^+)^\delta + \beta_i (X_{t-i}^-)^\delta] + \sum_{j=1}^{q} \gamma_j \sigma_{t-j}^\delta, \end{cases}$$

with $\delta \neq 0$, provided the following convention is considered for $\delta < 0$: $(X_t^+)^\delta = 0$ if $X_t < 0$ and $(X_t^-)^\delta = 0$ if $X_t > 0$. The process $Z = (Z_t,\ t \in \mathbb{Z})$ is called the generating process of X. If $\gamma_j = 0$, $j = 1, ..., q$, we say that X follows a δ-TARCH(p) model.

This class of models includes the more significant and useful conditional heteroskedastic models present in literature like GARCH and GTARCH (Gonçalves, Leite and Mendes-Lopes [3] and Pan, Wang and Tong [7]).

We consider in the following that $\sum_{j=1}^{q} \gamma_j < 1$ which is a necessary condition of strict and weak stationary of X and also to its stationarity up to the δ-order ([3]). Moreover, under this condition σ_t is $\underline{X}_{t-1}$-measurable.

The main characteristic of threshold conditionally heteroskedastic models is the fact that they allow to take into account different reactions in the volatility according to the sign of the process values even for values with the same absolute size. So, these models capture the so-called leverage effect very common in financial time series of daily returns (Francq and Zakoian [2]). The introduction of the exponent allows to take into account long memory in the shocks of the conditional variance (Ding, Granger and Engle [1]). A not so very common, but yet still present characteristic in some daily returns series is skewness, positive in some cases and negative in others (Taylor [10]). We note that X has a symmetric marginal distribution (relatively to the origin) if and only if the same happens to its generating process Z.

We point out this relation between the symmetry of the marginal distribution of a δ-TGARCH process and the symmetry of its generating process to stress the importance of studying the distributions of the δ-TGARCH process and not only the distributions of some transformations of these processes as Pawlak and Schmid [8] did for the squared of a GARCH process. The transformation considered by these authors for the GARCH process only produces conclusions for the process itself if the marginal distribution of the generating process is symmetric.

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The bounds obtained for $F_{(X_1,X_2)}(x_1,x_2)$ reveal a very good quality, the upper bound in the third and fourth quadrants needing eventually additional attention.

5. Conclusion

In this paper we estimate the probability distribution of a power TGARCH process, $X=(X_t, t\in\mathbb{Z})$, by establishing bounds for their finite dimensional laws.

These bounds for the distribution function are expressed in terms of the distribution function of its generating process and of the parameters of the model. For $n\geq 2$, they are established by means of a preliminary bound for the $2n$-dimensional vector $\left(X_1^+, X_1^-, ..., X_n^+, X_n^-\right)$.

The overall good quality of these theoretical bounds is illustrated by a simulation study with $n=1$ and $n=2$.

The examples presented show that the procedure here proposed is an alternative to the classical estimation of the finite dimensional laws of a process by the empirical distribution functions.

We point out that this probabilistic methodology will be useful, in particular, to evaluate control charts with symmetric or asymmetric bounds for the general class of conditional heteroskedastic processes considered in this study. For some particular models of this wide class and in the context of symmetrical control charts we have shown in [5] the interest and quality of this methodology.

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In this paper we propose bounds for the finite dimension laws of the process X following a δ-TGARCH(p,q) model with $\delta>0$ and, whenever possible, with $\delta<0$. These bounds are expressed in terms of the distribution function of the independent generating process, Z, and it becomes clear that the marginal law of X is, in certain regions, strongly controlled by that of the process Z. This fact is rather relevant as we know that these laws have in general quite different characteristics; for example, the marginal law of X is leptokurtic even if it doesn't happen with that of the independent generating process.

We remark that the regions where the bounds are valid seem to be larger than those theoretically stated, as suggested by the simulation studies done to evaluate the quality of those bounds.

So far as we know, the study here developed strongly enlarges the results on the subject present on literature. In fact we consider a wide class of general conditionally heteroskedastic models and we establish bounds for the distribution of the finite dimensional laws of the process X.

In Section 2 we study the marginal distribution of X. Section 3 begins with a preliminary bound for the distribution function of the $2n$-dimensional vector $\left(X_1^+, X_1^-, ..., X_n^+, X_n^-\right)$, which reveals useful in the bounding of the joint marginal distributions of X established in the last part of this Section. In Section 4 we illustrate the overall good quality of the theoretical bounds obtained by means of a simulation study with $n=2$.

2. Bounds for the marginal laws

Let $X=(X_t,\ t\in\mathbb{Z})$ be a δ-TGARCH(p,q) process, $\delta\neq 0$, and let us denote by F_{X_t} the distribution function of X_t and by F_Z the distribution function of Z_t.

Defining $\theta = \left[\alpha_0\left(1+\frac{1}{1-\gamma_\tau}\sum_{j=1}^{q}\gamma_j\right)\right]^{\frac{1}{\delta}}$, where $\gamma_\tau = \max_{1\leq j\leq q}\gamma_j$, we have $\sigma_t\geq\theta$, if $\delta>0$, and $\sigma_t\leq\theta$, if $\delta<0$.

In consequence, and taking into account that $F_{X_t}(x)=E\left[F_Z\left(\frac{x}{\sigma_t}\right)\right]$, the following bounds are easily obtained:

a) for $x\geq 0$, we have

- if $\delta > 0$, $F_{X_t}(x) \leq F_Z\left(\frac{x}{\theta}\right)$; if $\delta < 0$, $F_{X_t}(x) \geq F_Z\left(\frac{x}{\theta}\right)$;

b) for $x < 0$,
- if $\delta > 0$, $F_{X_t}(x) \geq F_Z\left(\frac{x}{\theta}\right)$, if $\delta < 0$, $F_{X_t}(x) \leq F_Z\left(\frac{x}{\theta}\right)$.

To complete the bounding of F_{X_t} we must consider some additional assumptions. Namely, we suppose that the generator process is δ-integrable, that is $E\left(|Z_0|^\delta\right) < +\infty$, and

$$S_\delta = \sum_{i=1}^{p} \left(\alpha_i \phi_{1,\delta} + \beta_i \phi_{2,\delta}\right) + \sum_{i=1}^{q} \gamma_i < 1,$$

where $E\left[\left(Z_t^+\right)^\delta\right] = \phi_{1,\delta}$ and $E\left[\left(Z_t^-\right)^\delta\right] = \phi_{2,\delta}$.

Under these assumptions, X is strict and weakly stationary of order δ ([3]); moreover, $E\left(\sigma_t^\delta\right)$ is finite, independent of t and equal to

$$E\left(\sigma_t^\delta\right) = \frac{\alpha_0}{1 - S_\delta}.$$

In addition, we suppose that (Z_t) are absolutely continuous random variables with a differentiable probability density f_Z.

The following result may then be established.

Theorem 1. Supposing X a δ-TGARCH(p, q) under the previous assumptions and, considering the function

$$h_\delta(x) = (1+\delta) f_Z\left(\frac{x}{y^{\frac{1}{\delta}}}\right) + \frac{x}{y^{\frac{1}{\delta}}} f_Z'\left(\frac{x}{y^{\frac{1}{\delta}}}\right),$$

where $y \in [\alpha_0, +\infty[$, we have

a) if $x \geq 0$,

(**a1**) $h_\delta(x) \geq 0, \forall y \in [\alpha_0, +\infty[\Longrightarrow F_{X_t}(x) \geq F_Z\left(\frac{x}{\left[E\left(\sigma_t^\delta\right)\right]^{\frac{1}{\delta}}}\right)$;

(**a2**) $h_\delta(x) \leq 0, \forall y \in [\alpha_0, +\infty[\Longrightarrow F_{X_t}(x) \leq F_Z\left(\frac{x}{\left[E\left(\sigma_t^\delta\right)\right]^{\frac{1}{\delta}}}\right)$;

b) if $x < 0$,

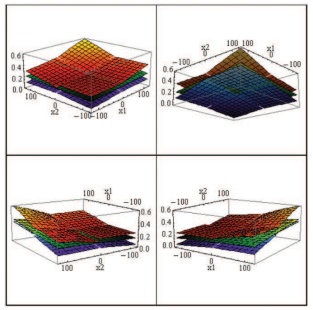

FIGURE 5. Graphical representation of $LS(x_1, x_2)$ (orange), $LI(x_1, x_2)$ (blue) and estimate of $F_{(X_1,X_2)}(x_1, x_2)$ (green), with $X \sim \frac{1}{2}$-TGARCH$(1,1)$ and $Z_t \sim N(0,1)$.

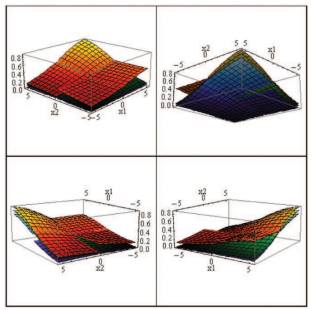

FIGURE 6. Graphical representation of $LS(x_1, x_2)$ (orange), $LI(x_1, x_2)$ (blue) and estimate of $F_{(X_1,X_2)}(x_1, x_2)$ (green), with $X \sim$ 2-TGARCH$(1,1)$ and $Z_t \sim N(0,1)$.

It will be interesting to analyze if the quality of the bounds is related with δ or, alternatively, with other parameters associated to the δ order stationarity, as S_δ, for example, which is equal, in this study, to 0.529, when $\delta = \frac{1}{2}$, and 0.6, when δ=2.

We point out that, by enlarging the region plotted, the bounds seem to be still valid, as happened in the study of the marginal distribution.

and $LI(x_1, x_2) =$

$$= \begin{cases} F_Z\left(\frac{x_1}{\varpi_1}\right) a + \\ + \left[F_Z\left(\frac{x_1}{\varpi_1}\right) - a\right]\left[F_Z\left(\frac{x_2}{\varpi_2(x_1)}\right) - a\right] + \\ + \left[a - F_Z\left(-\frac{y_1}{\varpi_1}\right)\right]\left[F_Z\left(\frac{x_2}{\varpi_3(y_1)}\right) - a\right], & (x_1, x_2) \in 1.^oQ \text{ such that } h_\delta(x_1) \geq 0 \text{ and } h_\delta(x_2) \geq 0 \\ F_Z\left(\frac{x_1}{\theta}\right) a, & (x_1, x_2) \in 2.^oQ \\ F_Z\left(\frac{x_1}{\theta}\right) F_Z\left(\frac{x_2}{\theta}\right), & (x_1, x_2) \in 3.^oQ \\ F_Z\left(\frac{x_1}{\varpi_1}\right) F_Z\left(\frac{x_2}{\theta}\right), & (x_1, x_2) \in 4.^oQ \text{ such that } h_\delta(x_1) \geq 0 \end{cases}$$

In the sequel of the simulation study done for the marginal distribution of X when the marginal distribution of the generator process is Gaussian we consider the δ-TGARCH$(1,1)$ model, with $\alpha_0 = 10$, $\alpha_1 = 0.3$, $\beta_1 = 0.5$, $\gamma_1 = 0.2$, and δ equal to $\frac{1}{2}$ (Figure 5) and 2 (Figure 6). In each figure, the same graphic is presented under four perspectives, with $LS(x_1, x_2)$ in orange, $LI(x_1, x_2)$ in blue and, in green, the empirical estimate of $F_{(X_1,X_2)}(x_1, x_2)$, calculated from 10 000 realizations of the X process. For readability, we only present the region $(x_1, x_2) \in \left[-\alpha_0^{\frac{1}{\delta}}\sqrt{1+\delta}, \alpha_0^{\frac{1}{\delta}}\sqrt{1+\delta}\right]^2$, where it is assured that $h_\delta(x_1) \geq 0$ and $h_\delta(x_2) \geq 0$. Nevertheless, we point out that the theoretical bounds for $F_{(X_1,X_2)}(x_1, x_2)$ by means of $LS(x_1, x_2)$ and $LI(x_1, x_2)$ are valid in a larger region, as we easily see from their expressions. We considered $y_1 = \alpha_0^{\frac{1}{\delta}}\sqrt{1+\delta}$.

From Figures 5 and 6 we note that:

- in the first quadrant, the lower bound presents, in both cases, good quality, contrary to the upper bound that seems to be better with the increase of δ;

- in the second quadrant, we observe a clear increase in the quality with the increase of δ, especially in what concerns the upper bound;

- in the third and fourth quadrants, the quality of the lower bound increases with δ, but it does not happen for the upper bound.

(b1) $h_\delta(x) \geq 0, \forall y \in [\alpha_0, +\infty[\Longrightarrow F_{X_t}(x) \leq F_Z\left(\frac{x}{\left[E\left(\sigma_t^\delta\right)\right]^{\frac{1}{\delta}}}\right)$;

(b2) $h_\delta(x) \leq 0, \forall y \in [\alpha_0, +\infty[\Longrightarrow F_{X_t}(x) \geq F_Z\left(\frac{x}{\left[E\left(\sigma_t^\delta\right)\right]^{\frac{1}{\delta}}}\right)$.

Proof. We restrict ourselves to the proof of the conditions in **a)** as those referred in **b)** are analogously obtained.

For $t \in \mathbb{Z}$ and $x \in \mathbb{R}$, we may write $F_{X_t}(x) = E\left[F_Z\left(\frac{x}{\left(\sigma_t^\delta\right)^{\frac{1}{\delta}}}\right)\right]$. For $x \in \mathbb{R}$ arbitrarily fixed, let us consider the function $R_\delta : [\alpha_0, +\infty[\longrightarrow [0,1]$ such that $R_\delta(y) = F_Z\left(\frac{x}{y^{\frac{1}{\delta}}}\right)$ for which we have

$$\frac{dR_\delta}{dy}(y) = -\frac{x}{\delta y^{\frac{1}{\delta}+1}} f_Z\left(\frac{x}{y^{\frac{1}{\delta}}}\right),$$

and

$$\frac{d^2R_\delta}{dy^2}(y) = \frac{(1+\delta)\,x}{\delta^2 y^{\frac{1}{\delta}+2}} f_Z\left(\frac{x}{y^{\frac{1}{\delta}}}\right) + \frac{x^2}{\delta^2 y^{\frac{2}{\delta}+2}} f'_Z\left(\frac{x}{y^{\frac{1}{\delta}}}\right) = \frac{x}{\delta^2 y^{\frac{1}{\delta}+2}} h_\delta(x).$$

As referred above, under the previous hypotheses, $E\left(\sigma_t^\delta\right)$ exists and is independent of t. As for $x \geq 0$ and $h_\delta(x) \geq 0$, R_δ is a convex function, Jensen's inequality allows to write

$$F_{X_t}(x) = E\left[F_Z\left(\frac{x}{\left(\sigma_t^\delta\right)^{\frac{1}{\delta}}}\right)\right] \geq F_Z\left(\frac{x}{\left[E\left(\sigma_t^\delta\right)\right]^{\frac{1}{\delta}}}\right).$$

Similarly, as R_δ is a concave function when $x > 0$ and $h_\delta(x) \leq 0$, we have

$$F_{X_t}(x) = E\left[F_Z\left(\frac{x}{\left(\sigma_t^\delta\right)^{\frac{1}{\delta}}}\right)\right] \leq F_Z\left(\frac{x}{\left[E\left(\sigma_t^\delta\right)\right]^{\frac{1}{\delta}}}\right). \square$$

From the previous results we conclude that to bound F_{X_t} we have to discuss the sign of the function

$$h_\delta(x) = (1+\delta) f_Z\left(xy^{-\frac{1}{\delta}}\right) + xy^{-\frac{1}{\delta}} f'_Z\left(xy^{-\frac{1}{\delta}}\right), \forall y \in [\alpha_0, +\infty[;$$

namely, we are interested in $h_\delta \leq 0$, when $\delta < 0$, and $h_\delta \geq 0$, when $\delta > 0$. In the following we illustrate this discussion for some distributions of the generator process Z and some values of δ.

Example 1. Let us consider Z_t following a standard normal distribution. For $y \in [\alpha_0, +\infty[$, we have

$$h_\delta(x) = f_Z\left(\frac{x}{y^{\frac{1}{\delta}}}\right)\left[(1+\delta) - \left(\frac{x}{y^{\frac{1}{\delta}}}\right)^2\right],\ x \in \mathbb{R}.$$

So, it is obvious that $\forall x \in \mathbb{R}$, $h_\delta(x) \leq 0$ if $\delta \leq -1$.

Let us consider now $-1 < \delta < 0$. As $h_\delta(x) \leq 0 \Leftrightarrow |x| \geq y^{\frac{1}{\delta}}\sqrt{1+\delta}$ and $y^{\frac{1}{\delta}} \leq \alpha_0^{\frac{1}{\delta}}$, we may assure that $h_\delta(x) \leq 0$ at least for $x \in \left]-\infty, -\alpha_0^{\frac{1}{\delta}}\sqrt{1+\delta}\right] \cup \left[\alpha_0^{\frac{1}{\delta}}\sqrt{1+\delta}, +\infty\right[$.

Analogously we conclude that, if $\delta > 0$, $h_\delta(x) \geq 0$ at least for $x \in \left[-\alpha_0^{\frac{1}{\delta}}\sqrt{1+\delta}, \alpha_0^{\frac{1}{\delta}}\sqrt{1+\delta}\right]$, taking into account that, in this case, $y^{\frac{1}{\delta}} \geq \alpha_0^{\frac{1}{\delta}}$.

Taking into consideration these results, we present in Figure 1 the bounds of the distribution function of a δ-TGARCH$(1,1)$, with $\alpha_0 = 10$, $\alpha_1 = 0.3$, $\beta_1 = 0.5$, $\gamma_1 = 0.2$ and a standard Gaussian generator process, for δ equal to $-\frac{1}{2}$, $-\frac{1}{3}$, $\frac{1}{2}$, 1, $\frac{3}{2}$ and 2, representing, in each case, the bounds $F_Z\left(\frac{x}{\theta}\right)$ and $F_Z\left(x\left[E\left(\sigma_t^\delta\right)\right]^{-\frac{1}{\delta}}\right)$ and the empirical estimation of $F_{X_t}(x)$. This empirical distribution was obtained by a simulation study considering a sample of 10 000 observations of the δ-TGARCH$(1,1)$ process X.

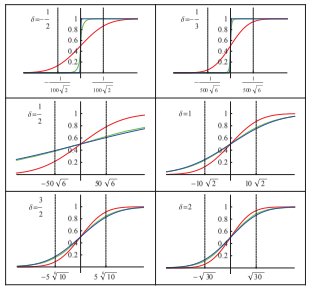

FIGURE 1. Plots of $F_Z\left(\frac{x}{\theta}\right)$ (red), $F_Z\left(x\left[E\left(\sigma_t^\delta\right)\right]^{-\frac{1}{\delta}}\right)$ (blue) and the estimate of $F_{X_t}(x)$ (green), for $X \sim \delta$-TGARCH$(1,1)$ and $Z_t \sim N(0,1)$

$$F_{(X_1,\ldots,X_n)}(x_1,\ldots,x_n) \geq$$
$$\geq F_{(X_1,\ldots,X_{\ddot{\eta}-1})}(x_1,\ldots,x_{\ddot{\eta}-1}) \prod_{t\in\mathcal{I}^-} F_Z\left(\frac{x_t}{\theta}\right) \times [F_Z(0)]^{n-\kappa-(\ddot{\eta}-1)}.$$

4. Simulation study

The theoretical bounds for the finite dimensional distributions of a process X following a δ-TGARCH model are now evaluated by means of a simulation study. This study is devoted to the bounds obtained for $F_{(X_1,X_2)}$ as in this case we are able to graphically compare the results, like we have done in the marginal distributions.

We consider the δ-TGARCH$(1,1)$ model and we firstly resume the expressions of the bounds obtained for $n = 2$. So, for each (x_1, x_2), we have an upper and a lower bound for $F_{(X_1,X_2)}(x_1,x_2)$, denoted by $LS(x_1,x_2)$ and $LI(x_1,x_2)$, respectively. In order to facilitate the presentation, we consider $\mathbb{R}^2$ divided in its four quadrants, including the points $(0,x_2)$ and $(x_1,0)$ in the adequate odd quadrant. Denoting $F_Z(0)$ by a and considering $\theta = \left(\frac{\alpha_0}{1-\gamma_1}\right)^{1/\delta}$, $\varpi_1 = \left[E\left(\sigma_t^\delta\right)\right]^{1/\delta}$, $\varpi_2(x) = \left[\alpha_0 + \alpha_1 x^\delta + \gamma_1 E\left(\sigma_t^\delta\right)\right]^{1/\delta}$ and $\varpi_3(x) = \left[\alpha_0 + \beta_1 x^\delta + \gamma_1 E\left(\sigma_t^\delta\right)\right]^{1/\delta}$ with y_1 chosen as the greatest positive real such that $h_\delta(-y_1) \geq 0$, we get

$$LS(x_1,x_2) = \begin{cases} F_Z\left(\frac{x_1}{\theta}\right)F_Z\left(\frac{x_2}{\theta}\right), & (x_1,x_2) \in 1.^oQ \\ F_Z\left(\frac{x_1}{\varpi_1}\right)F_Z\left(\frac{x_2}{\theta}\right), & (x_1,x_2) \in 2.^oQ \text{ such that } h_\delta(x_1) \geq 0 \\ a^2 - \left[a - F_Z\left(\frac{x_1}{\varpi_1}\right)\right] \times & \\ \times \left[a - F_Z\left(\frac{x_2}{\varpi_3(-x_1)}\right)\right], & (x_1,x_2) \in 3.^oQ \text{ such that } h_\delta(-x_1) \geq 0 \text{ and } h_\delta(-x_2) \geq 0 \\ F_Z\left(\frac{x_1}{\theta}\right)a, & (x_1,x_2) \in 4.^oQ \end{cases}$$

ii) if $x_t > 0$, as $\sigma_t \geq \theta$, then $\frac{x_t}{\sigma_t} \leq \frac{x_t}{\theta}$ and so, $\left\{Z_t \leq \frac{x_t}{\sigma_t}\right\} \subseteq \left\{Z_t \leq \frac{x_t}{\theta}\right\}$.

Consequently

$$F_{(X_1,...,X_n)}(x_1,...,x_n) \leq P(X_1 \leq x_1,...,X_{\breve{\eta}-1} \leq x_{\breve{\eta}-1}, Z_{\breve{\eta}} \leq u_{\breve{\eta}},...,Z_n \leq u_n),$$

where $u_t = \begin{cases} 0, & \text{if } x_t \leq 0 \\ \frac{x_t}{\theta}, & \text{if } x_t > 0 \end{cases}$, for $t = \breve{\eta},...,n$, is non random. As $Z_{\breve{\eta}},...,Z_n$ are independent variables and independent of $\underline{X}_{\breve{\eta}-1}$, then

$$\begin{aligned} &P(X_1 \leq x_1,...,X_{\breve{\eta}-1} \leq x_{\breve{\eta}-1}, Z_{\breve{\eta}} \leq u_{\breve{\eta}},...,Z_n \leq u_n) \\ &\quad = F_{(X_1,...,X_{\breve{\eta}-1})}(x_1,...,x_{\breve{\eta}-1}) \times F_Z(u_{\breve{\eta}}) \times ... \times F_Z(u_n). \end{aligned}$$

Taking into account the u_t definition, we conclude that

$$\begin{aligned} &F_{(X_1,...,X_n)}(x_1,...,x_n) \leq \\ &\quad \leq F_{(X_1,...,X_{\breve{\eta}-1})}(x_1,...,x_{\breve{\eta}-1}) \prod_{t\in\mathcal{I}^+} F_Z\left(\frac{x_t}{\theta}\right) \times [F_Z(0)]^{n-k-(\breve{\eta}-1)}. \ \square \end{aligned}$$

We point out that, in case (a), the dependence of the upper bound on the distribution function of $(X_1,...,X_t)$, with $t < \breve{\eta} - 1$, is addressed taking into consideration the previous studies since all the components of the point $(x_1,...,x_{\breve{\eta}-1})$ where this function is evaluated are non positives which enables the use of the results stated for the region $]-\infty,0]^n \setminus \{(0,...,0)\}$. In case (b), an analogous situation occurs with points belonging now to $[0,+\infty[^n$.

In the next theorem we complete the bound of the distribution function. The proof is omitted due to its similarity with that of the previous one.

Theorem 6. Let X be a δ-TGARCH(p,q) process such that $\delta > 0$. Let $(x_1,...,x_n)$ be any point in $\mathbb{R}^n \setminus (]-\infty,0]^n \cup [0,+\infty[^n)$, for $n \in \mathbb{N}\setminus\{1\}$, with κ negative coordinates, for $1 \leq \kappa < n$. Then,

(a) if $x_1 \leq 0$, we have

$$\begin{aligned} &F_{(X_1,...,X_n)}(x_1,...,x_n) \geq \\ &\quad \geq F_{(X_1,...,X_{\breve{\eta}-1})}(x_1,...,x_{\breve{\eta}-1}) \prod_{t\in\mathcal{I}^{-*}} F_Z\left(\frac{x_t}{\theta}\right) \times [F_Z(0)]^{n-\kappa^*-(\breve{\eta}-1)}, \end{aligned}$$

(b) if $x_1 \geq 0$, we have

According with the previous study these bounds are theoretically validated for x between the lines $x = \pm\alpha_0^{\frac{1}{\delta}}\sqrt{1+\delta}$, when $\delta > 0$, and outside of these lines, when $\delta < 0$. Nevertheless, the previous plots lead us to conjecture that these bounds are still valid for x outside those intervals, which is understandable since our results only establish sufficient conditions for the validity of these bounding; moreover, the quality of the bounds seems to be strongly related to the corresponding value of S_δ (for the chosen increasing values of δ, S_δ is approximately equal to $0.888, 0.739, 0.529, 0.520, 0.544$ and 0.6, respectively). In what concerns the bounding accuracy, we stress the high quality of the bounds related to the function $F_Z\left(x\left[E\left(\sigma_t^\delta\right)\right]^{-\frac{1}{\delta}}\right)$ that depends on all the model parameters.

Example 2. Let us consider random variables Z_t following a centered and reduced distribution based on the Student law with parameter n, $n > 2$, that is, with density $f_Z(x) = \frac{1}{\sqrt{(n-2)\pi}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \left(1+\frac{x^2}{n-2}\right)^{-\frac{n+1}{2}}$, $x \in \mathbb{R}$. So, $f'_Z(x) = -\frac{x(n+1)}{n-2+x^2} f_Z(x)$, $x \in \mathbb{R}$.

As, for $y \in [\alpha_0, +\infty[$, $h_\delta(x) = f_Z\left(\frac{x}{y^{\frac{1}{\delta}}}\right) \frac{(1+\delta)(n-2)y^{\frac{2}{\delta}} + (\delta-n)x^2}{y^{\frac{2}{\delta}}(n-2)+x^2}$, $x \in \mathbb{R}$, we have the following discussion:

i) $h_\delta(x) \leq 0, \forall x \in \mathbb{R}$, if $\delta \leq -1$;

ii) $h_\delta(x) \leq 0$, $\forall x \in \left]-\infty, -\alpha_0^{\frac{1}{\delta}}\sqrt{\frac{(1+\delta)(n-2)}{n-\delta}}\right] \cup \left[\alpha_0^{\frac{1}{\delta}}\sqrt{\frac{(1+\delta)(n-2)}{n-\delta}}, +\infty\right[$, if $-1 < \delta < 0$;

iii) $h_\delta(x) \geq 0$, $\forall x \in \left[-\alpha_0^{\frac{1}{\delta}}\sqrt{\frac{(1+\delta)(n-2)}{n-\delta}}, \alpha_0^{\frac{1}{\delta}}\sqrt{\frac{(1+\delta)(n-2)}{n-\delta}}\right]$, if $0 < \delta < n$;

iv) $h_\delta(x) \geq 0, \forall x \in \mathbb{R}$, if $\delta \geq n$.

As in the Example 1, a simulation study was developed considering four δ-TGARCH models with generator process following the previous distribution with $n = 6$, parameters $\delta = 1$, $\alpha_0 = 10$, $\alpha_1 = 0.3$, $\beta_1 = 0.5$ and $\gamma_1 = 0.2$ and orders $(1,1)$, $(2,1)$, $(1,2)$ and $(2,2)$. We note that the values of S_δ are, respectively, 0.5, 0.688, 0.75 and 0.938. The bounds obtained in each one of these cases are plotted in Figure 2.

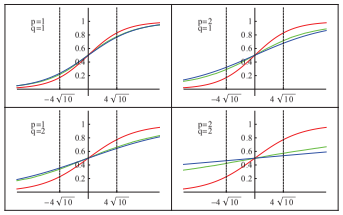

FIGURE 2. Plots of $F_Z\left(\frac{x}{\theta}\right)$ (red), $F_Z\left(x\left[E\left(\sigma_t^\delta\right)\right]^{-\frac{1}{\delta}}\right)$ (blue) and the estimate of $F_{X_t}(x)$ (green), for $X \sim$ 1-TGARCH(p,q) and $Z_t = \sqrt{\frac{2}{3}}W$, with $W \sim t_6$

Despite being in a very different situation, the behaviour of these bounds are similar to those of the previous example; in particular, we highlight that the bounding accuracy seems also to be dependent on the S_δ value.

Example 3. Let us consider a mixture of two Gaussian distributions (not necessarily symmetric), that is

$$f_Z(x) = p_1 f_1(x;\mathfrak{m}_1,\mathfrak{s}_1) + p_2 f_2(x;\mathfrak{m}_2,\mathfrak{s}_2),$$

where $p_1, p_2 \in]0,1[$ with $p_1+p_2=1$, and where $f_k(\cdot;\mathfrak{m}_k,\mathfrak{s}_k)$ is the Gaussian probability density with mean $\mathfrak{m}_k$ and variance $\mathfrak{s}_k^2$.

In order to obtain a centered and reduced generator process, we have the following relations $p_2 = 1-p_1$, $\mathfrak{m}_2 = \frac{p_1\mathfrak{m}_1}{p_1-1}$, and $\mathfrak{v}_2 = \sqrt{\frac{1-p_1(\mathfrak{m}_1^2+\mathfrak{s}_1^2)-(1-p_1)\left(\frac{p_1\mathfrak{m}_1}{p_1-1}\right)^2}{1-p_1}}$.

To analyze the sign of the function h_δ we restrict ourselves to the case $\delta = 1$.

As $f_k'(x;\mathfrak{m}_k,\mathfrak{s}_k) = f_k(x;\mathfrak{m}_k,\mathfrak{s}_k)\left(-\frac{(x-\mathfrak{m}_k)}{\mathfrak{s}_k^2}\right)$, we have, for $x \in \mathbb{R}$,

$$h_1(x) = 2f_Z\left(\frac{x}{y}\right) + \frac{x}{y}f_Z'\left(\frac{x}{y}\right) = \sum_{k=1}^{2} p_k f_k\left(\frac{x}{y};\mathfrak{m}_k,\mathfrak{s}_k\right)\left[2 - \frac{\left(\frac{x}{y}-\mathfrak{m}_k\right)\frac{x}{y}}{\mathfrak{s}_k^2}\right]$$

and so $h_1(x) \geq 0$ if, at least, $\frac{x}{y} \in \bigcap_{k=1}^{2}\left[\frac{\mathfrak{m}_k-\sqrt{\mathfrak{m}_k^2+8\mathfrak{s}_k^2}}{2}, \frac{\mathfrak{m}_k+\sqrt{\mathfrak{m}_k^2+8\mathfrak{s}_k^2}}{2}\right]$, that is $\frac{x}{y} \in [l_1, l_2]$, with $l_1 = \max_{k\in\{1,2\}} \frac{\mathfrak{m}_k-\sqrt{\mathfrak{m}_k^2+8\mathfrak{s}_k^2}}{2}$ and $l_2 = \min_{k\in\{1,2\}} \frac{\mathfrak{m}_k+\sqrt{\mathfrak{m}_k^2+8\mathfrak{s}_k^2}}{2}$.

of indices:

$$\begin{aligned}
&\mathcal{I}^+ = \{t \in \{1,...,n\} : x_t > 0\}, \text{ with } \breve{\eta} = \min \mathcal{I}^+ \text{ and } \#\mathcal{I}^+ = k;\\
&\mathcal{I}^- = \{t \in \{1,...,n\} : x_t < 0\}, \text{ with } \ddot{\eta} = \min \mathcal{I}^-;\\
&\mathcal{I}^{+*} = \{t \in \mathcal{I}^+ : t > \ddot{\eta}\} \text{ such that } \#\mathcal{I}^{+*} = k^*;\\
&\mathcal{I}^{-*} = \{t \in \mathcal{I}^- : t > \breve{\eta}\} \text{ such that } \#\mathcal{I}^{-*} = \kappa^*.
\end{aligned}$$

The following theorem states an upper bound for the distribution function in study.

Theorem 5. Let X be a δ-TGARCH(p,q) process such that $\delta > 0$. Let $(x_1,...,x_n)$ be any element of $\mathbb{R}^n \setminus]-\infty,0]^n \cup [0,+\infty[^n)$, for $n \in \mathbb{N}\setminus\{1\}$, with k positive coordinates, for $1 \leq k < n$. Then,

(a) if $x_1 \leq 0$, we have

$$\begin{aligned}
&F_{(X_1,...,X_n)}(x_1,...,x_n) \leq\\
&\quad \leq F_{(X_1,...,X_{\breve{\eta}-1})}(x_1,...,x_{\breve{\eta}-1}) \prod_{t\in\mathcal{I}^+} F_Z\left(\frac{x_t}{\theta}\right) \times [F_Z(0)]^{n-k-(\breve{\eta}-1)},
\end{aligned}$$

(b) if $x_1 \geq 0$, we have

$$\begin{aligned}
&F_{(X_1,...,X_n)}(x_1,...,x_n) \leq\\
&\quad \leq F_{(X_1,...,X_{\ddot{\eta}-1})}(x_1,...,x_{\ddot{\eta}-1}) \prod_{t\in\mathcal{I}^{+*}} F_Z\left(\frac{x_t}{\theta}\right) \times [F_Z(0)]^{n-k^*-(\ddot{\eta}-1)}.
\end{aligned}$$

Proof. We present only the proof of part **(a)**, as the other is analogous.

Let $(x_1,...,x_n) \in \mathbb{R}^n \setminus]-\infty,0]^n \cup [0,+\infty[^n)$, with n fixed in $\mathbb{N}\setminus\{1\}$, such that k of its coordinates are positive, with $1 \leq k < n$.

If $x_1 \leq 0$, we get

$$\begin{aligned}
F_{(X_1,...,X_n)}(x_1,...,x_n) &= P(X_1 \leq x_1,...,X_{\breve{\eta}-1} \leq x_{\breve{\eta}-1}, X_{\breve{\eta}} \leq x_{\breve{\eta}},...,X_n \leq x_n)\\
&= P\left(X_1 \leq x_1,...,X_{\breve{\eta}-1} \leq x_{\breve{\eta}-1}, Z_{\breve{\eta}} \leq \frac{x_{\breve{\eta}}}{\sigma_{\breve{\eta}}},...,Z_n \leq \frac{x_n}{\sigma_n}\right)
\end{aligned}$$

with $\breve{\eta} \geq 2$. So, for $1 \leq t \leq \breve{\eta}-1$, $x_t \leq 0$ and, for $t = \breve{\eta}$, $x_t > 0$.

For $\breve{\eta} \leq t \leq n$, we upper-bound each coordinate in the following way:

i) if $x_t \leq 0$, then, as $\sigma_t > 0$, we have $\frac{x_t}{\sigma_t} \leq 0$ and so $\left\{Z_t \leq \frac{x_t}{\sigma_t}\right\} \subseteq \{Z_t \leq 0\}$;

Proof. The beginning of the proof is analogous to that of theorem 3, starting with the set $]-\infty,0]^n$.

Let $(x_1,...,x_n) \in]-\infty,0]^n \setminus \{(0,...,0)\}$. Since

$$]-\infty,0]^n = \prod_{t=1}^{n}\left(\mathcal{C}_t^{(+1)} \cup \mathcal{C}_t^{(-1)}\right) = \bigcup_{k=1}^{2^n}\left(\prod_{t=1}^{n}\mathcal{C}_t^{(-1)^{\left\lfloor \frac{k-1}{2^{n-t}}\right\rfloor}}\right),$$

where, for $t = 1,...,n$, $\mathcal{C}_t^{(-1)} =]-\infty, x_t[$ and $\mathcal{C}_t^{(+1)} = [x_t, 0]$, and so $\mathcal{C}_t^{(+1)} \cap \mathcal{C}_t^{(-1)} = \varnothing$, then, $F_{(X_1,...,X_n)}(0,...,0) = [F_Z(0)]^n$ is also equal to

$$F_{(X_1,...,X_n)}(0,...,0) = \sum_{k=1}^{2^n} P\left((X_1,...,X_n) \in \prod_{t=1}^{n}\mathcal{C}_t^{(-1)^{\left\lfloor \frac{k-1}{2^{n-t}}\right\rfloor}}\right).$$

As for $k = 2^n$, $\prod_{t=1}^{n} C_t^{(-1)^{\left\lfloor \frac{k-1}{2^{n-t}}\right\rfloor}} = \prod_{t=1}^{n} C_t^{(-1)}$, the last term of this sum is $F_{(X_1,...,X_n)}(0,...,0) = [F_Z(0)]^n$.

Moreover, the first one, corresponding to $k = 1$, is such that $\prod_{t=1}^{n}\mathcal{C}_t^{(-1)^{\left\lfloor \frac{k-1}{2^{n-t}}\right\rfloor}} = \prod_{t=1}^{n}\mathcal{C}_t^{(+1)}$, and so

$$\begin{aligned} P\left((X_1,...,X_n) \in \prod_{t=1}^{n}\mathcal{C}_t^{(+1)}\right) &= P(-x_t \le X_t \le 0, t = 1,...,n) \\ &= F_{\left(X_1^+,X_1^-,...,X_n^+,X_n^-\right)}(0,-x_1,...,0,-x_n), \end{aligned}$$

Consequently,

$$\begin{aligned} F_{(X_1,...,X_n)}(x_1,...,x_n) &= [F_Z(0)]^n - F_{\left(X_1^+,X_1^-,...,X_n^+,X_n^-\right)}(0,-x_1,...,0,-x_n) + \\ &\quad -\sum_{k=2}^{2^{n-1}} P\left((X_1,...,X_n) \in \prod_{t=1}^{n}\mathcal{C}_t^{(-1)^{\left\lfloor \frac{k-1}{2^{n-t}}\right\rfloor}}\right), \end{aligned}$$

from which we conclude. □

3.2.3. *Region* $\mathrm{R}^n \setminus]-\infty,0]^n \cup [0,+\infty[^n)$.

In the region $\mathrm{R}^n \setminus]-\infty,0]^n \cup [0,+\infty[^n)$, with $n \in \mathbb{N}\setminus\{1\}$, we consider only the δ-TGARCH model with positive power. We use the following sets

As this theoretical study leads us only to a sufficient condition for the positivity of h_1, we may improve this study using numerical and graphical methods.

So, let $(p_1, \mathfrak{m}_1, \mathfrak{s}_1) = (0.65, -0.15, 0.4)$ and $(p_1, \mathfrak{m}_1, \mathfrak{s}_1) = (0.6, 0.3, 0.4)$ be two different set of parameters, denoted A and B respectively, associated to the distribution considered above. In Figure 3 we plot the function h_1 corresponding to each one of these cases.

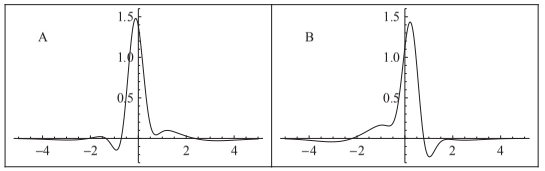

FIGURE 3. Function h_1 when Z_t follows a mixture of Gaussian distributions with parameters A, $(p_1, \mathfrak{m}_1, \mathfrak{s}_1) = (0.65, -0.15, 0.4)$ and B, $(p_1, \mathfrak{m}_1, \mathfrak{s}_1) = (0.6, 0.3, 0.4)$

Analyzing these plots and using numerical methods we conclude, in case A, that $h_1(x) \ge 0$, $\forall y \in [\alpha_0, +\infty[$, if $\frac{x}{y} \in [-0.714, 2.353]$ and also if $x \in [-0.714\alpha_0, 2.353\alpha_0]$. Similarly, in case B, we have the same result if $\frac{x}{y} \in [-2.198, 0.788]$, and also if $x \in [-2.198\alpha_0, 0.788\alpha_0]$.

A simulation study is developed for this kind of generator process distribution, considering the laws associated to the cases of parameter sets A and B. Unlike the previous examples we take now skewed distributions for Z_t. Namely, in cases denoted A1 e A2, we choose $(p_1, \mathfrak{m}_1, \mathfrak{s}_1) = (0.65, -0.15, 0.4)$ (positive skewness); in cases denoted B1 e B2, we take $(p_1, \mathfrak{m}_1, \mathfrak{s}_1) = (0.6, 0.3, 0.4)$ (negative skewness). Regarding the δ-TGARCH process we consider in all the cases $\delta = 1$, $\alpha_0 = 10$, $\alpha_1 = 0.15$, $\beta_1 = 0.5$ and, in A1 and B1, $\gamma_1 = 0.2$; in A2 and B2 we take $\gamma_1 = 0.6$. So, the values for S_δ are equal to 0.416, 0.816, 0.430 and 0.830 in the cases A1, A2, B1 and B2, respectively. The corresponding plots are presented in Figure 4.

Also in this asymmetrical case, the conclusions are similar to the previous ones. Moreover, this example reinforces the conjecture that the bounding accuracy is better for smaller values of S_δ.

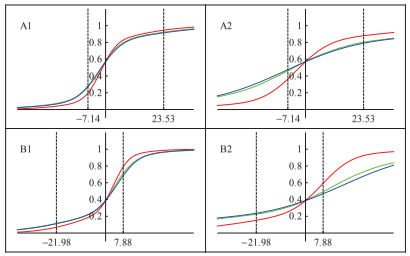

FIGURE 4. Plots of $F_Z\left(\frac{x}{\theta}\right)$ (red), $F_Z\left(x\left[E\left(\sigma_t^\delta\right)\right]^{-\frac{1}{\delta}}\right)$ (blue) and the estimate of $F_{X_t}(x)$ (green), for $X \sim$ 1-TGARCH(1,1) and $Z_t \sim NM$

3. Bounds for the finite dimensional laws

In this section we concentrate our study on δ-TGARCH processes with positive power δ and generator process with diffuse distribution. To develop bounds for the distribution function of the finite dimensional laws of the process, we begin by an auxiliary study on the bounding of the distribution function of $\left(X_1^+, X_1^-, ..., X_n^+, X_n^-\right)$.

3.1. Bounds for the distribution of $\left(X_1^+, X_1^-, ..., X_n^+, X_n^-\right)$.

Let $X = (X_t,\ t \in \mathbb{Z})$ be a δ-TGARCH(p,q) process, $\delta > 0$, for which the law of Z_t is diffuse.

As in Gonçalves, Leite and Mendes-Lopes [4], it is easy to establish the following upper bound for the distribution function of $\left(X_1^+, X_1^-, ..., X_n^+, X_n^-\right)$:

$$\forall\,(x_1, x_1^*, ..., x_n, x_n^*) \in \left(\mathbb{R}_0^+\right)^{2n}, n \in \mathbb{N},$$

$$F_{\left(X_1^+, X_1^-, ..., X_n^+, X_n^-\right)}(x_1, x_1^*, ..., x_n, x_n^*) \leq \prod_{t=1}^{n}\left[F_Z\left(\frac{x_t}{\theta}\right) - F_Z\left(-\frac{x_t^*}{\theta}\right)\right].$$

In fact, taking into account the definitions of X_t^+ and X_t^- and the positivity of σ_t, we have

$$F_{\left(X_1^+, X_1^-, ..., X_n^+, X_n^-\right)}(x_1, x_1^*, ..., x_n, x_n^*) = P\left(-\frac{x_t^*}{\sigma_t} \leq Z_t \leq \frac{x_t}{\sigma_t}, t = 1, ..., n\right).$$

For $t = 1, ..., n$, we have $[-y_t, 0[\subset C_t^{(-1)}$. Denoting $D_t^{(-1)} = [-y_t, 0[$ and $D_t^{(+1)} = C_t^{(+1)}$, we get

$$P\left((X_1, ..., X_n) \in \prod_{t=1}^{n} C_t^{(-1)^{\left\lfloor \frac{k-1}{2^{n-t}} \right\rfloor}}\right) \geq P\left((X_1, ..., X_n) \in \prod_{t=1}^{n} D_t^{(-1)^{\left\lfloor \frac{k-1}{2^{n-t}} \right\rfloor}}\right).$$

Considering now

$$a_{k,t} = \begin{cases} x_t, & \text{if } \left\lfloor \frac{k-1}{2^{n-t}} \right\rfloor \text{ is even} \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad a_{k,t}^* = \begin{cases} 0, & \text{if } \left\lfloor \frac{k-1}{2^{n-t}} \right\rfloor \text{ is even} \\ y_t, & \text{otherwise} \end{cases},$$

we obtain, if $\left\lfloor \frac{k-1}{2^{n-t}} \right\rfloor$ is even, $\left[-a_{k,t}^*, a_{k,t}\right] = [0, x_t] = D_t^{(+1)}$ and, if $\left\lfloor \frac{k-1}{2^{n-t}} \right\rfloor$ is odd, $\left[-a_{k,t}^*, a_{k,t}\right] = [-y_t, 0] = D_t^{(-1)} \cup \{0\}$. So,

$$\begin{aligned} P\left((X_1, ..., X_n) \in \prod_{t=1}^{n} D_t^{(-1)^{\left\lfloor \frac{k-1}{2^{n-t}} \right\rfloor}}\right) &= P\left((X_1, ..., X_n) \in \prod_{t=1}^{n}\left[-a_{k,t}^*, a_{k,t}\right]\right) \\ &= P\left(X_t^+ \leq a_{k,t}, X_t^- \leq a_{k,t}^*, t = 1, ..., n\right) \\ &= F_{\left(X_1^+, X_1^-, ..., X_n^+, X_n^-\right)}\left(a_{k,1}, a_{k,1}^*, ..., a_{k,n}, a_{k,n}^*\right), \end{aligned}$$

as (Z_t), and consequently (X_t), are diffuse random variables.

The conclusions obtained for all the values of k from 1 to 2^n, give the stated lower bound.□

We note that this theorem is valid for any value of δ, positive or negative.

3.2.2. *Region* $]-\infty, 0]^n \setminus \{(0, ..., 0)\}$.

In the region $]-\infty, 0]^n \setminus \{(0, ..., 0)\}$, the lower bound is also a natural generalization of the theorem related to the marginal distribution, that is,

$$F_{(X_1, ..., X_n)}(x_1, ..., x_n) \geq \prod_{t=1}^{n} F_Z\left(\frac{x_t}{\theta}\right).$$

In what concerns the upper bound for the distribution function in this region, the following result is stated using again the distribution function of $\left(X_1^+, X_1^-, ..., X_n^+, X_n^-\right)$ and it does not depend on the sign of δ.

Theorem 4. Let X be a δ-TGARCH(p,q) process such that (Z_t) are diffuse variables. For every $(x_1, ..., x_n) \in]-\infty, 0]^n \setminus \{(0, ..., 0)\}$, $n \in \mathbb{N}$,

$$F_{(X_1, ..., X_n)}(x_1, ..., x_n) \leq \left[F_Z(0)\right]^n - F_{\left(X_1^+, X_1^-, ..., X_n^+, X_n^-\right)}(0, -x_1, ..., 0, -x_n).$$

Taking into account that for $k=2^0$ we have $\prod_{t=1}^{n} C_t^{(-1)^{\left\lfloor \frac{k-1}{2^{n-t}} \right\rfloor}} = \prod_{t=1}^{n} C_t^{(+1)}$, we obtain

$$\begin{aligned} P\left((X_1,...,X_n) \in \prod_{t=1}^{n} C_t^{(+1)}\right) &= P\left(0 \leq X_t \leq x_t, t=1,...,n\right) \\ &= P\left(X_t^+ \leq x_t, X_t^- \leq 0, t=1,...,n\right) \\ &= F_{\left(X_1^+,X_1^-,...,X_n^+,X_n^-\right)}\left(x_1,0,...,x_n,0\right). \end{aligned}$$

Taking $k=2^n$, we have $\prod_{t=1}^{n} C_t^{(-1)^{\left\lfloor \frac{k-1}{2^{n-t}} \right\rfloor}} = \prod_{t=1}^{n} C_t^{(-1)}$, and so

$$\begin{aligned} P\left((X_1,...,X_n) \in \prod_{t=1}^{n} C_t^{(-1)}\right) &= P\left(X_t < 0, t=1,...,n\right) \\ &= P\left(Z_t < 0, t=1,...,n\right) \\ &= \left[F_Z\left(0\right)\right]^n, \end{aligned}$$

since $\sigma_t > 0$, for $t \in \{1,...,n\}$, and $Z_1,...,Z_n$ are i.i.d. diffuse variables.

Finally, let us analyze the case $k=2^{n-j}$, with $j \in \{1,...,n-1\}$. In this case $\prod_{t=1}^{n} C_t^{(-1)^{\left\lfloor \frac{k-1}{2^{n-t}} \right\rfloor}} = \prod_{t=1}^{j} C_t^{(+1)} \times \prod_{t=j+1}^{n} C_t^{(-1)}$, and so

$$P\left((X_1,...,X_n) \in \prod_{t=1}^{j} C_t^{(+1)} \times \prod_{t=j+1}^{n} C_t^{(-1)}\right) =$$

$$\begin{aligned} &= P\left(0 \leq X_t \leq x_t, t=1,...,j, X_t < 0, t=j+1,...,n\right) \\ &= F_{\left(X_1^+,X_1^-,...,X_j^+,X_j^-\right)}\left(x_1,0,...,x_j,0\right) \times \left[F_Z\left(0\right)\right]^{n-j}, \end{aligned}$$

taking into account that Z_t is independent of $\underline{X}_{t-1}$.

For the remaining values of k, that is, for $k \in \{1,...,2^n\}$ such that $k \neq 2^{n-j}$ with j ranging from 0 to n, we obtain now a lower bound for $P\left((X_1,...,X_n) \in \prod_{t=1}^{n} C_t^{(-1)^{\left\lfloor \frac{k-1}{2^{n-t}} \right\rfloor}}\right)$.

Let us fix arbitrarily $(y_1,...,y_n)$ in $]0,+\infty[^n$.

So, as $\sigma_t > \theta$ and Z_t are i.i.d. and diffuse random variables, the announced result follows from the inequality

$$\begin{aligned} P\left(-\frac{x_t^*}{\sigma_t} \leq Z_t \leq \frac{x_t}{\sigma_t}, t=1,...,n\right) &\leq P\left(-\frac{x_t^*}{\theta} \leq Z_t \leq \frac{x_t}{\theta}, t=1,...,n\right) \\ &= \prod_{t=1}^{n}\left[F_Z\left(\frac{x_t}{\theta}\right) - F_Z\left(-\frac{x_t^*}{\theta}\right)\right]. \end{aligned}$$

We note that, when δ is negative, a reversed inequality is obtained using the same technique.

As previously, the methodology used to obtain the other bound is more refined and leads us to a precise approximation of the distribution function under study. However, it is only valid for positive δ.

Let us begin by considering the following result.

Lemma 1. Under the previous conditions, we have

$$P\left(V_t \leq Z_t \leq W_t, t=1,...,k\right) = \prod_{t=1}^{k} E\left[F_Z\left(W_t\right) - F_Z\left(V_t\right)\right],$$

where V_t and W_t, such that $V_t \leq W_t$, are $\underline{X}_0$ - measurable random variables, $t \in \{1,...,k\}$.

Proof. Using expectation and conditional expectation properties and taking into account that $Z_1,...,Z_k$ are i.i.d. diffuse random variables and also that they are independent from $\underline{X}_0$, it is easy to conclude that

$$\begin{aligned} P\left(V_t \leq Z_t \leq W_t, t=1,...,k\right) &= E\left[E\left(\mathbb{1}_{[V_1,W_1]\times...\times[V_k,W_k]}\left(Z_1,...,Z_k\right)\middle|\, \underline{X}_0\right)\right] \\ &= \prod_{t=1}^{k} E\left[F_Z\left(W_t\right) - F_Z\left(V_t\right)\right]. \square \end{aligned}$$

In the next theorem we present the required lower bound. For simplicity, we restrict the presentation of this study to δ-TARCH(p) (that is, $\gamma_1 = ... = \gamma_q = 0$) and δ-TGARCH(1,1) models; more general cases use the same procedure with a more complicated framework.

Theorem 2. Let X be a δ-TGARCH(p,q) process, $\delta > 0$, such that $E\left(|Z_0|^{\delta}\right) < +\infty$ and $S_\delta < 1$. Supposing Z_t absolutely continuous with a differentiable density f_Z, let us define, for each $y \in [\alpha_0, +\infty[$, the function

$$h_\delta(x) = (1+\delta) f_Z\left(\frac{x}{y^{\frac{1}{\delta}}}\right) + \frac{x}{y^{\frac{1}{\delta}}} f_Z'\left(\frac{x}{y^{\frac{1}{\delta}}}\right).$$

For $(x_1, x_1^*, ..., x_n, x_n^*) \in \left(\mathbb{R}_0^+\right)^{2n}$ such that

$$x_t\, h_\delta(x_t) + x_t^*\, h_\delta(-x_t^*) \geq 0, \forall y \in [\alpha_0, +\infty[,$$

where $t = \begin{cases} 1, ..., \min\{p, n\} & if\ q = 0 \\ 1, ..., n, & if\ q = 1 \end{cases}$, we have:

a) if $q = 0$ and

(**a1**) $1 \leq p \leq n$,

$$F_{\left(X_1^+, X_1^-, ..., X_n^+, X_n^-\right)}(x_1, x_1^*, ..., x_n, x_n^*) \geq$$

$$\geq \prod_{t=1}^{p}\left[F_Z\left(\frac{x_t}{u_t^{1/\delta}}\right) - F_Z\left(-\frac{x_t^*}{u_t^{1/\delta}}\right)\right] \prod_{t=p+1}^{n}\left[F_Z\left(\frac{x_t}{v_t^{1/\delta}}\right) - F_Z\left(-\frac{x_t^*}{v_t^{1/\delta}}\right)\right],$$

(**a2**) $p > n \geq 1$

$$F_{\left(X_1^+, X_1^-, ..., X_n^+, X_n^-\right)}(x_1, x_1^*, ..., x_n, x_n^*) \geq \prod_{t=1}^{n}\left[F_Z\left(\frac{x_t}{u_t^{1/\delta}}\right) - F_Z\left(-\frac{x_t^*}{u_t^{1/\delta}}\right)\right],$$

where $u_t = \sum_{i=1}^{t-1}\left[\alpha_i (x_{t-i})^{\delta} + \beta_i \left(x_{t-i}^*\right)^{\delta}\right] + E\left(\sigma_t^{\delta}\right)\left[1 - \sum_{i=1}^{t-1}\left(\alpha_i \phi_{1,\delta} + \beta_i \phi_{2,\delta}\right)\right]$ and $v_t = \alpha_0 + \sum_{i=1}^{p}\left[\alpha_i (x_{t-i})^{\delta} + \beta_i \left(x_{t-i}^*\right)^{\delta}\right]$;

b) if $p = 1$ and $q = 1$,

$$F_{\left(X_1^+, X_1^-, ..., X_n^+, X_n^-\right)}(x_1, x_1^*, ..., x_n, x_n^*) \geq \prod_{t=1}^{n}\left[F_Z\left(\frac{x_t}{w_t^{1/\delta}}\right) - F_Z\left(-\frac{x_t^*}{w_t^{1/\delta}}\right)\right],$$

where $w_t = \alpha_0 \sum_{j=1}^{t-1} \gamma_1^{j-1} + \sum_{j=1}^{t-1} \gamma_1^{j-1}\left[\alpha_1 (x_{t-j})^{\delta} + \beta_1 \left(x_{t-j}^*\right)^{\delta}\right] + \gamma_1^{t-1} E\left(\sigma_t^{\delta}\right)$.

Theorem 3. Let X be a δ-TGARCH(p,q) process such that Z_t are diffuse variables. For every $(x_1, ..., x_n) \in [0, +\infty[^n$, $n \in \mathbb{N}$, we have

$$F_{(X_1,...,X_n)}(x_1, ..., x_n) \geq$$
$$\geq [F_Z(0)]^n + \sum_{k=1}^{n} F_{\left(X_1^+, X_1^-, ..., X_k^+, X_k^-\right)}(x_1, 0, ..., x_k, 0)\,[F_Z(0)]^{n-k} +$$
$$+ \sum_{\substack{k=1 \\ k \neq 2^0, 2^1, ..., 2^n}}^{2^n} F_{\left(X_1^+, X_1^-, ..., X_n^+, X_n^-\right)}\left(a_{k,1}, a_{k,1}^*, ..., a_{k,n}, a_{k,n}^*\right),$$

with $a_{k,t} = \begin{cases} x_t, & \text{if } \left\lfloor \frac{k-1}{2^{n-t}} \right\rfloor \text{ is even} \\ 0, & \text{otherwise} \end{cases}$ and $a_{k,t}^* = \begin{cases} 0, & \text{if } \left\lfloor \frac{k-1}{2^{n-t}} \right\rfloor \text{ is even} \\ y_t, & \text{otherwise} \end{cases}$, and y_t arbitrarily fixed in $[0, +\infty[$.

Proof. Let $(x_1, ..., x_n) \in [0, +\infty[^n$. Using the procedure previously referred, the following equality holds for the set $\prod_{t=1}^{n}]-\infty, x_t]$:

$$\prod_{t=1}^{n}]-\infty, x_t] = \prod_{t=1}^{n}\left(C_t^{(+1)} \cup C_t^{(-1)}\right) = \bigcup_{k=1}^{2^n}\left(\prod_{t=1}^{n} C_t^{(-1)^{\left\lfloor \frac{k-1}{2^{n-t}} \right\rfloor}}\right).$$

As $C_t^{(+1)} \cap C_t^{(-1)} = \varnothing$, $t = 1, ..., n$, then

$$\prod_{t=1}^{n} C_t^{(-1)^{\left\lfloor \frac{k_1-1}{2^{n-t}} \right\rfloor}} \cap \prod_{t=1}^{n} C_t^{(-1)^{\left\lfloor \frac{k_2-1}{2^{n-t}} \right\rfloor}} = \varnothing,$$

for $k_1, k_2 \in \{1, ..., 2^n\}$ such that $k_1 \neq k_2$. So,

$$F_{(X_1,...,X_n)}(x_1, ..., x_n) = \sum_{k=1}^{2^n} P\left((X_1, ..., X_n) \in \prod_{t=1}^{n} C_t^{(-1)^{\left\lfloor \frac{k-1}{2^{n-t}} \right\rfloor}}\right).$$

Let us evaluate now $P\left((X_1, ..., X_n) \in \prod_{t=1}^{n} C_t^{(-1)^{\left\lfloor \frac{k-1}{2^{n-t}} \right\rfloor}}\right)$, for $k = 2^{n-j}$, with j ranging from 0 to n.

zero like, for $n = 2$,

$$F_{(X_1,X_2)}(x_1,x_2) = P(X_1 < 0, X_2 < 0) + P(X_1 < 0, 0 \leq X_2 \leq x_2) +$$
$$+P(0 \leq X_1 \leq x_1, X_2 < 0) + P(0 \leq X_1 \leq x_1, 0 \leq X_2 \leq x_2).$$

In order to facilitate the reading, we introduce some notation and illustrate its use in the case $n = 3$. As $]-\infty, x_t] =]-\infty, 0[\cup [0, x_t]$, let us denote these intervals as $C_t^{(-1)} =]-\infty, 0[$ and $C_t^{(+1)} = [0, x_t]$. We note that the exponent used in this notation is also used as a number, that is, $C_t^{(-1)^2} = C_t^{(+1)}$ and $C_t^{(-1)^3} = C_t^{(-1)}$, for example. In this way, we have

$$\begin{aligned}\prod_{t=1}^{3}]-\infty, x_t] &= \prod_{t=1}^{3} \left(C_t^{(+1)} \cup C_t^{(-1)}\right) \\ &= C_1^{(+1)} \times C_2^{(+1)} \times C_3^{(+1)} \cup C_1^{(+1)} \times C_2^{(+1)} \times C_3^{(-1)} \cup \\ &\quad \cup C_1^{(+1)} \times C_2^{(-1)} \times C_3^{(+1)} \cup C_1^{(+1)} \times C_2^{(-1)} \times C_3^{(-1)} \cup \\ &\quad \cup C_1^{(-1)} \times C_2^{(+1)} \times C_3^{(+1)} \cup C_1^{(-1)} \times C_2^{(+1)} \times C_3^{(-1)} \cup \\ &\quad \cup C_1^{(-1)} \times C_2^{(-1)} \times C_3^{(+1)} \cup C_1^{(-1)} \times C_2^{(-1)} \times C_3^{(-1)}.\end{aligned}$$

This Cartesian product is written as the union of 2^3 Cartesian products, where the exponents are the arrangements with replacement of the numbers (+1) and (-1), three to three. For the condensed representation of this union, we consider the triplet $\left((-1)^{\left\lfloor \frac{k-1}{2^2} \right\rfloor}, (-1)^{\left\lfloor \frac{k-1}{2^1} \right\rfloor}, (-1)^{\left\lfloor \frac{k-1}{2^0} \right\rfloor}\right)$, where $\lfloor x \rfloor$ denotes the integer part of number x. With this triplet and with k ranging from 1 to 2^3, we recover the arrangements with replacement of the numbers (+1) and (-1), three to three, in the same order as that displayed. So,

$$\begin{aligned}\prod_{t=1}^{3}]-\infty, x_t] &= \bigcup_{k=1}^{2^3} \left(C_1^{(-1)^{\left\lfloor \frac{k-1}{2^2} \right\rfloor}} \times C_2^{(-1)^{\left\lfloor \frac{k-1}{2^1} \right\rfloor}} \times C_3^{(-1)^{\left\lfloor \frac{k-1}{2^0} \right\rfloor}}\right) \\ &= \bigcup_{k=1}^{2^3} \left(\prod_{t=1}^{3} C_t^{(-1)^{\left\lfloor \frac{k-1}{2^{3-t}} \right\rfloor}}\right).\end{aligned}$$

The result concerning the lower bound can then be stated.

Proof. Let $(x_1, x_1^*, ..., x_n, x_n^*) \in \left(\mathbb{R}_0^+\right)^{2n}$. We introduce the random variables $\widetilde{U}_t = \alpha_0 + \sum_{i=1}^{t-1} \left[\alpha_i (x_{t-i})^\delta + \beta_i (x_{t-i}^*)^\delta\right] + \sum_{i=t}^{p} \left[\alpha_i \left(X_{t-i}^+\right)^\delta + \beta_i \left(X_{t-i}^-\right)^\delta\right]$, $t \in \{1, ..., n\}$.

(**a1**) Let us begin by considering $q = 0$ and $1 \leq p \leq n$.

As $\delta > 0$, if $X_t^+ \leq x_t$ and $X_t^- \leq x_t^*$, $t \in \{1, ..., n\}$, we have

- if $t \in \{2, ..., p\}$,

$$\begin{aligned}\sigma_t^\delta &= \alpha_0 + \sum_{i=1}^{t-1} \left[\alpha_i \left(X_{t-i}^+\right)^\delta + \beta_i \left(X_{t-i}^-\right)^\delta\right] + \sum_{i=t}^{p} \left[\alpha_i \left(X_{t-i}^+\right)^\delta + \beta_i \left(X_{t-i}^-\right)^\delta\right] \\ &\leq \widetilde{U}_t,\end{aligned}$$

as for $i \in \{1, ..., t-1\}$, $1 \leq t - i \leq p - 1 < n$ and for $i \in \{t, ..., p\}$, $t - i \leq 0$;

- if $t = 1$, $\sigma_1^\delta = \alpha_0 + \sum_{i=1}^{p} \left[\alpha_i \left(X_{1-i}^+\right)^\delta + \beta_i \left(X_{1-i}^-\right)^\delta\right]$, thus $\sigma_1 = \widetilde{U}_1$;

- and, if $t \in \{p+1, ..., n\}$,

$$\begin{aligned}\sigma_t^\delta &= \alpha_0 + \sum_{i=1}^{p} \left[\alpha_i \left(X_{t-i}^+\right)^\delta + \beta_i \left(X_{t-i}^-\right)^\delta\right] \\ &\leq \alpha_0 + \sum_{i=1}^{p} \left[\alpha_i (x_{t-i})^\delta + \beta_i (x_{t-i}^*)^\delta\right] = v_t,\end{aligned}$$

as, for $i \in \{1, ..., p\}$, $1 \leq t - i \leq n - 1$.

Using the definitions of X_t^+ and X_t^-, the previous bounds and lemma, and taking into consideration that $\widetilde{U}_t^{\frac{1}{\delta}}$, $t = 1, ..., p$, and $v_t^{\frac{1}{\delta}}$, $t = p+1, ..., n$, are $\underline{X}_0$-measurable, we are able to write

$$\begin{aligned}&F_{\left(X_1^+, X_1^-, ..., X_n^+, X_n^-\right)}(x_1, x_1^*, ..., x_n, x_n^*) \geq \\ &\geq P\left(-\frac{x_t^*}{\widetilde{U}_t^{\frac{1}{\delta}}} \leq Z_t \leq \frac{x_t}{\widetilde{U}_t^{\frac{1}{\delta}}}, t = 1, ..., p, -\frac{x_t^*}{v_t^{\frac{1}{\delta}}} \leq Z_t \leq \frac{x_t}{v_t^{\frac{1}{\delta}}}, t = p+1, ..., n\right) \\ &= \prod_{t=1}^{p} E\left[F_Z\left(\frac{x_t}{\widetilde{U}_t^{\frac{1}{\delta}}}\right) - F_Z\left(-\frac{x_t^*}{\widetilde{U}_t^{\frac{1}{\delta}}}\right)\right] \prod_{t=p+1}^{n} \left[F_Z\left(\frac{x_t}{v_t^{\frac{1}{\delta}}}\right) - F_Z\left(-\frac{x_t^*}{v_t^{\frac{1}{\delta}}}\right)\right].\end{aligned}$$

For a_1 and a_2 arbitrarily fixed in $\mathbb{R}_0^+$ and $\mathbb{R}_0^-$, respectively, let us consider the function $\mathcal{R}_\delta : [\alpha_0, +\infty[\longrightarrow [-1, 1]$ defined by

$$\mathcal{R}_\delta(y) = F_Z\left(\frac{a_1}{y^{\frac{1}{\delta}}}\right) - F_Z\left(\frac{a_2}{y^{\frac{1}{\delta}}}\right).$$

It is easy to show that

$$\frac{d^2\mathcal{R}_\delta}{dy^2}(y) = \frac{1}{\delta^2 y^{\frac{1}{\delta}+2}}\left[a_1\, h_\delta(a_1) - a_2\, h_\delta(a_2)\right].$$

So, considering for each t, $t \in \{1, ..., p\}$, $a_1 = x_t$ and $a_2 = -x_t^*$, we have the convexity of $\mathcal{R}_\delta$ if $x_t\, h_\delta\left(\frac{x_t}{y}\right) + x_t^*\, h_\delta\left(-\frac{x_t^*}{y}\right) \geq 0$. Thus, applying Jensen inequality, we obtain

$$\prod_{t=1}^{p} E\left[F_Z\left(\frac{x_t}{\widetilde{U}_t^{\frac{1}{\delta}}}\right) - F_Z\left(-\frac{x_t^*}{\widetilde{U}_t^{\frac{1}{\delta}}}\right)\right] \prod_{t=p+1}^{n}\left[F_Z\left(\frac{x_t}{v_t^{\frac{1}{\delta}}}\right) - F_Z\left(-\frac{x_t^*}{v_t^{\frac{1}{\delta}}}\right)\right] \geq$$

$$\geq \prod_{t=1}^{p}\left[F_Z\left(\frac{x_t}{\left[E\left(\widetilde{U}_t\right)\right]^{\frac{1}{\delta}}}\right) - F_Z\left(-\frac{x_t^*}{\left[E\left(\widetilde{U}_t\right)\right]^{\frac{1}{\delta}}}\right)\right] \prod_{t=p+1}^{n}\left[F_Z\left(\frac{x_t}{v_t^{\frac{1}{\delta}}}\right) - F_Z\left(-\frac{x_t^*}{v_t^{\frac{1}{\delta}}}\right)\right].$$

As, under the hypotheses, $E\left(\widetilde{U}_t\right)$ exists and is equal to

$$u_t = \sum_{i=1}^{t-1}\left[\alpha_i\left(x_{t-i}\right)^\delta + \beta_i\left(x_{t-i}^*\right)^\delta\right] + E\left(\sigma_t^\delta\right)\left[1 - \sum_{i=1}^{t-1}\left(\alpha_i\phi_{1,\delta} + \beta_i\phi_{2,\delta}\right)\right],$$

the inequality presented in point (**a1**) is then established.

(**a2**) The inequality corresponding to $q = 0$ and $p > n \geq 1$ is analogously obtained.

In fact, as $\delta > 0$, if $X_t^+ \leq x_t$ and $X_t^- \leq x_t^*,\ t \in \{1, ..., n\}$, we obtain

- if $t \in \{2, ..., n\}$, $\sigma_t^\delta \leq \widetilde{U}_t$, as, for $i \in \{1, ..., t-1\}$, $1 \leq t - i \leq n - 1$ and, for $i \in \{t, ..., p\}$, $t - i \leq 0$;

- if $t = 1$, $\sigma_1^\delta = \alpha_0 + \sum_{i=1}^{p}\left[\alpha_i\left(X_{1-i}^+\right)^\delta + \beta_i\left(X_{1-i}^-\right)^\delta\right]$, and so $\sigma_1 = \widetilde{U}_1$.

In consequence, we have the inequality

$$F_{\left(X_1^+, X_1^-, ..., X_n^+, X_n^-\right)}(x_1, x_1^*, ..., x_n, x_n^*) \geq P\left(-\frac{x_t^*}{\widetilde{U}_t^{\frac{1}{\delta}}} \leq Z_t \leq \frac{x_t}{\widetilde{U}_t^{\frac{1}{\delta}}}, t = 1, ..., n\right),$$

from which the referred condition is similarly deduced.

b) The proof of this last inequality is also analogous to the previous ones. In fact, if $X_t^+ \leq x_t$ and $X_t^- \leq x_t^*,\ t \in \{1, ..., n\}$, and introducing

$$\widetilde{W}_t = \alpha_0 \sum_{j=1}^{t-1}\gamma_1^{j-1} + \sum_{j=1}^{t-1}\gamma_1^{j-1}\left[\alpha_1\left(x_{t-j}\right)^\delta + \beta_1\left(x_{t-j}^*\right)^\delta\right] + \gamma_1^{t-1}\sigma_1^\delta,$$

we may write:

- if $t \in \{2, ..., n\}$, $\sigma_t^\delta \leq \alpha_0 + \alpha_1\left(x_{t-1}\right)^\delta + \beta_1\left(x_{t-1}^*\right)^\delta + \gamma_1\sigma_{t-1}^\delta \leq \widetilde{W}_t$;

- if $t = 1$, we have $\sigma_1 = \widetilde{W}_1$, and so

$$F_{\left(X_1^+, X_1^-, ..., X_n^+, X_n^-\right)}(x_1, x_1^*, ..., x_n, x_n^*) \geq P\left(-\frac{x_t^*}{\widetilde{W}_t^{\frac{1}{\delta}}} \leq Z_t \leq \frac{x_t}{\widetilde{W}_t^{\frac{1}{\delta}}}, t = 1, ..., n\right),$$

with $\widetilde{W}_t^{\frac{1}{\delta}}$ $\underline{X}_0$-measurable ($t = 1, ..., n$). Following the same steps the inequality in **b)** is obtained. □

3.2. Bounds for the distribution function of $(X_1, ..., X_n)$.

We present now bounds for the distribution function of $(X_1, ..., X_n)$ considering the three following regions for $(x_1, ..., x_n) \in \mathbb{R}^n$: $[0, +\infty[^n$, $]-\infty, 0]^n \setminus \{0, ..., 0\}$ and $\mathbb{R}^n \setminus]-\infty, 0]^n \cup [0, +\infty[^n)$.

3.2.1. *Region* $[0, +\infty[^n$.

Let X be a δ-TGARCH(p, q) process with $\delta > 0$. Then, for every $(x_1, ..., x_n) \in [0, +\infty[^n$, $n \in \mathbb{N}$, it is easily concluded that

$$F_{(X_1, ..., X_n)}(x_1, ..., x_n) \leq \prod_{t=1}^{n} F_Z\left(\frac{x_t}{\theta}\right).$$

For the lower bound, we begin by writing the distribution function as a sum of probabilities of sets where the variables are bounded, up or above, by