Mathem. Appl. 97, 2004, 165-211.
[5] D. Bourn, N. Martins-Ferreira, A. Montoli, M. Sobral, *Schreier split epimorphisms in monoids and in semirings*, Textos de Matemática (Série B), Departamento de Matemática da Universidade de Coimbra, vol. 45 (2013).
[6] D. Bourn, N. Martins-Ferreira, A. Montoli, M. Sobral, *Monoids and pointed S-protomodular categories*, accepted for publication in Homology, Homotopy and Applications, 2015.
[7] S. Burris, H.P. Sankappannavar, *A course in universal algebra*, Graduate Texts in Mathematics 78, Springer-Verlag (1981).
[8] A. Carboni, G.M. Kelly, M.C. Pedicchio, *Some remarks on Maltsev and Goursat categories*, Appl. Categ. Struct. 1 (1993) 385-421.
[9] V. Even, M. Gran, *On factorization systems for surjective quandle homomorphisms*, J. Knot Theory Ramifications, 23, 11, (2014) 1450060.
[10] V. Even, M. Gran, *Closure operators in the category of quandles*, preprint, arXiv:1412.8626, 2014.
[11] T. Everaert, *Higher central extensions in Mal'tsev categories*, Appl. Categ. Structures 22 (2014) 961-979.
[12] R. Freese, R. Mc Kenzie, *Commutator theory for congruence modular varieties*, London Math. Soc. Lect. Note Series 125, Cambridge Univ. press, 1987.
[13] M. Gran, *Central extensions and internal groupoids in Maltsev categories*, J.Pure Appl. Algebra 155 (2001) 139-166.
[14] M. Gran, *Central extensions in modular varieties*, J. Pure Appl. Algebra 174 (2002) 249-261.
[15] G. Janelidze, *Pure Galois Theory in Categories*, J. Algebra 132, (1991) 270-286.
[16] G. Janelidze, G. M. Kelly, *Galois theory and a general notion of central extension*, J. Pure Appl. Algebra 97 (1994) 135-161.
[17] G. Janelidze, G.M. Kelly, *Central extensions in universal algebra: a unifcation of three notions*, Algebra Universalis 44 (2000) 123-128.
[18] P.T. Johnstone, *Affine categories and naturally Mal'cev categories*, J. Pure Appl. Algebra 61 (1989) 251-256.
[19] D. Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra 23 (1982) 37-65.
[20] J.D.H. Smith, *Mal'cev Varieties*, Lecture Notes in Math. 554 (1976).

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A CHARACTERIZATION OF CENTRAL EXTENSIONS IN THE VARIETY OF QUANDLES

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Abstract: The category of symmetric quandles is a Mal'tsev variety whose subvariety of abelian symmetric quandles is the category of abelian algebras. We give an algebraic description of the quandle extensions that are central for the adjunction between the variety of quandles and its subvariety of abelian symmetric quandles.

Keywords: Quandle, symmetric quandle, abelian object, Mal'tsev variety, central extension, categorical Galois theory.
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1. Introduction

A *quandle* [19] is a set A equipped with two binary operations $\lhd$ and $\lhd^{-1}$ such that the following identities hold (for all $a,\ b,\ c \in A$):

(A1) $a \lhd a = a = a \lhd^{-1} a$ (idempotency);
(A2) $(a \lhd b) \lhd^{-1} b = a = (a \lhd^{-1} b) \lhd b$ (right invertibility);
(A3) $(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$ and $(a \lhd^{-1} b) \lhd^{-1} c = (a \lhd^{-1} c) \lhd^{-1} (b \lhd^{-1} c)$ (self-distributivity).

This structure is of interest in knot theory, since the three axioms above correspond to the Reidemeister moves on oriented link diagrams. From a purely algebraic viewpoint, quandles capture the properties of group conjugation: given a group $(G, \cdot, 1)$, by defining the operations $a \lhd b = b \cdot a \cdot b^{-1}$ and $a \lhd^{-1} b = b^{-1} \cdot a \cdot b$ on the underlying set G one gets a quandle structure.

Quandles and quandle homomorphisms form a category denoted Qnd. This category, being a variety in the sense of universal algebra [7], is an exact category (in the sense of Barr [1]). The variety Qnd has some interesting categorical properties, as recently observed in [9, 10, 2]. The present work continues

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this line of research, by investigating the properties of the adjunction between the variety of quandles and its subvariety AbSymQnd of abelian symmetric quandles, in particular from the viewpoint of the categorical theory of central extensions [16].

The variety AbSymQnd of abelian symmetric quandles is the subvariety of Qnd determined by the two additional identities

$$a \lhd b = b \lhd a$$

and

$$(a \lhd b) \lhd (c \lhd d) = (a \lhd c) \lhd (b \lhd d).$$

AbSymQnd is a Mal'tsev variety (actually even a naturally Mal'tsev one [18], see Section 2), and it turns out to be an *admissible* subvariety of Qnd: this fact guarantees the validity of a Galois theorem of classification of the corresponding central extensions (see [15, 16]).

This is particularly interesting by keeping in mind that the variety Qnd is not congruence modular, since it contains the variety of sets as a subvariety. However, the subvariety AbSymQnd of abelian symmetric quandles yields an adjunction

$$\mathsf{Qnd} \underset{H}{\overset{I}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{AbSymQnd} \tag{1}$$

that is similar to the classical one

$$\mathcal{V} \underset{U}{\overset{I}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{V}_{ab} \tag{2}$$

where $\mathcal{V}$ is any modular variety and $\mathcal{V}_{ab}$ its subvariety of abelian algebras in the sense of commutator theory [12]. Many interesting results in the categorical theory of central extensions discovered in the last years actually concern subvarieties of Mal'tsev varieties (see [11], for instance, and the references therein). The example investigated in the present paper is then of a rather different nature, and will be useful to establish some new connections between algebraic quandle theory and categorical algebra.

To explain the main result of this paper more precisely, let us briefly recall how the categorical notions of trivial extension and of central extension are defined in any variety $\mathcal{V}$ with respect to a chosen subvariety $\mathcal{X}$ of $\mathcal{V}$. A surjective homomorphism $f\colon A \to B$ in $\mathcal{V}$ is a *trivial extension* if the commutative

where k_c is the unique element such that $b \lhd k_c = c$ as above. Moreover, we have

$$p_f(a,b,b) = (a \lhd^{-1} b) \lhd b = a$$

for $(a,b,b) \in A \times \mathsf{Eq}(f)$ and

$$p_f(a,a,b) = (a \lhd^{-1} a) \lhd k_b = a \lhd k_b = b$$

for $(a,a,b) \in A \times \mathsf{Eq}(f)$, so p_f is a connector by Lemma 3.6. ■

Before stating our main result, we recall that a surjective homomorphism $f\colon A \to B$ is a *normal extension* when the homomorphism f_1 in the pullback of f along itself is a trivial extension:

$$\begin{array}{ccc} \mathsf{Eq}(f) & \xrightarrow{f_2} & A \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[f]{} & B \end{array}$$

Theorem 3.13. *Given a surjective homomorphism $f\colon A \to B$ in* Qnd*, the following conditions are equivalent:*

(1) *f is an algebraically central extension with abelian symmetric fibers;*
(2) *f is a normal extension;*
(3) *f is a central extension.*

Proof: Let $f\colon A \to B$ be an algebraically central extension with abelian symmetric fibers, then its kernel pair $\mathsf{Eq}(f)$ is isomorphic to a product $Q \times A$ with Q an abelian symmetric quandle by Lemma 3.7. Corollary 3.11 shows that f is then a normal extension.

Every normal extension is a central extension.

Let $f\colon A \to B$ be a central extension. Then there is a surjective homomorphism $p\colon E \to B$ such that the first projection $\pi_1\colon E \times_A B \to E$ in the pullback (7) is a trivial extension. Then $f\colon A \to B$ is an algebraically central extension with abelian symmetric fibers by Lemma 3.12, because π_1 is the pullback of a morphism lying in AbSymQnd. ■

References

[1] M. Barr, *Exact Categories*, in: Lecture Notes in Math. 236, Springer, 1-120 (1971).
[2] D. Bourn, *A structural aspect of the category of quandles*, preprint, 2014.
[3] D. Bourn, M. Gran *Centrality and connectors in Maltsev categories*, Algebra Universalis, 48 (3), 2002, 309-331.
[4] D. Bourn, M. Gran, *Regular, protomodular and abelian categories*, in Categorical Foundations -Special Topics in Order, Topology, Algebra and Sheaf Theory, Cambridge Univ. Press, Encycl.

Define the quandle homomorphism $p_{\pi_1}\colon (E \times_B A) \times \mathsf{Eq}(\pi_1) \to E \times_B A$ as $p_{\pi_1}((e,a),(e',b),(e',c)) = (e, p_f(a,b,c))$. We have

$$p_{\pi_1}((e,a),(e',b),(e',b)) = (e, p_f(a,b,b)) = (e,a)$$

and

$$p_{\pi_1}((e,a),(e,a),(e,b)) = (e, p_f(a,a,b)) = (e,b).$$

It is then a connector by Lemma 3.6.

Now let $\pi_1\colon E \times_B A \to E$ be an algebraically central extension with abelian symmetric fibers. Write $p_{\pi_1}\colon (E \times_B A) \times \mathsf{Eq}(\pi_1) \to E \times_B A$ for the connector between $(E \times_B A) \times (E \times_B A)$ and $\mathsf{Eq}(\pi_1)$. The surjectivity of $p\colon E \to B$ implies the surjectivity of the homomorphism $\widehat{\pi_2}\colon (E \times_B A) \times \mathsf{Eq}(\pi_1) \to A \times \mathsf{Eq}(f)$ defined by

$$\widehat{\pi_2}((e,a),(e',b),(e',c)) = (a,b,c).$$

First let us show that $\mathsf{Eq}(\widehat{\pi_2}) \subset \mathsf{Eq}(\pi_2 \circ p_{\pi_1})$. Let

$$(((e_0,a),(e'_0,b),(e'_0,c)),((e_1,a),(e'_1,b),(e'_1,c))) \in \mathsf{Eq}(\widehat{\pi_2}).$$

Since f has abelian symmetric fibers by Lemma 3.3, it is Σ-special by Proposition 3.8. This means that the split epimorphism $\mathsf{Eq}(f) \underset{f_1}{\overset{\delta_f}{\rightleftarrows}} A$ is in Σ. In other terms, for all $b \in A$ and all $(b,c) \in f_1^{-1}(b)$ there exists $k_{(b,c)} \in f_1^{-1}(b)$, where $k_{(b,c)} = (b,k_c)$, such that $(b,b) \lhd k_{(b,c)} = (b,c)$. Such a $k_{(b,c)} = (b,k_c)$ is unique by Remark 3.9: it follows that, for any $(b,c) \in \mathsf{Eq}(f)$, the element $k_c \in A$ such that $f(k_c) = f(b) = f(c)$ and $b \lhd k_c = c$ is unique. Then, for $i \in \{0,1\}$, we have

$$((e_i,a),(e'_i,b),(e'_i,c)) = ((e_i,a) \lhd^{-1} (e'_i,b),(e'_i,b),(e'_i,b)) \lhd ((e'_i,b),(e'_i,b),(e'_i,k_c)).$$

Consequently we remark that

$$\begin{aligned}
&\pi_2 \circ p_{\pi_1}((e_i,a),(e'_i,b),(e'_i,c)) \\
&= \pi_2 \circ p_{\pi_1}(((e_i,a) \lhd^{-1} (e'_i,b),(e'_i,b),(e'_i,b)) \lhd ((e'_i,b),(e'_i,b),(e'_i,k_c))) \\
&= \pi_2(p_{\pi_1}((e_i,a) \lhd^{-1} (e'_i,b),(e'_i,b),(e'_i,b)) \lhd p_{\pi_1}((e'_i,b),(e'_i,b),(e'_i,k_c)) \\
&= \pi_2(((e_i,a) \lhd^{-1} (e'_i,b)) \lhd (e'_i,k_c)) \\
&= \pi_2((e_i \lhd^{-1} e'_i) \lhd e'_i, (a \lhd^{-1} b) \lhd k_c) = (a \lhd^{-1} b) \lhd k_c
\end{aligned}$$

for both $i \in \{0,1\}$. This implies the existence of a quandle homomorphism $p_f\colon A \times \mathsf{Eq}(f) \to A$ such that $p_f \circ \widehat{\pi_2} = \pi_2 \circ p_{\pi_1}$, i.e. $p_f(a,b,c) = (a \lhd^{-1} b) \lhd k_c$

square induced by the units of the reflection

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & HI(A) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle HI(f)} \\ B & \xrightarrow[\eta_B]{} & HI(B) \end{array}$$

is a pullback. A surjective homomorphism $f\colon A \to B$ is a *central extension* when there exists a surjective homomorphism $p\colon E \to B$ such that the extension $\pi_1\colon E \times_B A \to E$ in the pullback

$$\begin{array}{ccc} E \times_B A & \xrightarrow{\pi_2} & A \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle f} \\ E & \xrightarrow[p]{} & B \end{array}$$

of f along p is a trivial extension. In any modular variety $\mathcal{V}$ the central extensions defined in this way, relatively to the adjunction (2), are precisely the surjective homomorphisms $f\colon A \to B$ whose kernel congruence $\mathsf{Eq}(f) = \{(a_1,a_2) \in A \times A \mid f(a_1) = f(a_2)\}$ is central in the sense of commutator theory: $[\mathsf{Eq}(f), A \times A] = \Delta_A$, where Δ_A is the smallest congruence on A (see [14, 17]). In the present paper we characterize the central extensions corresponding to the adjunction (1) as those surjective quandle homomorphisms $f\colon A \to B$ such that (a condition equivalent to) $[\mathsf{Eq}(f), A \times A] = \Delta_A$ holds and, moreover, each fiber $f^{-1}(b) = \{a \in A \mid f(a) = b\}$ is an abelian symmetric quandle, for any $b \in B$ (Theorem (3.13)).

2. Symmetric quandles and abelian symmetric quandles

A quandle A is symmetric if it satisfies the additional identity:

$$a \lhd b = b \lhd a, \tag{3}$$

for all $a,\ b \in A$. We write SymQnd for the corresponding category of symmetric quandles, which is then a subvariety of the variety Qnd of all quandles. Here below we observe that the category SymQnd is a Mal'tsev variety [20], which will be shown to be an admissible subcategory of Qnd for the categorical theory of central extensions [16].

Proposition 2.1. [2] *The category* SymQnd *is a Mal'tsev variety.*

Proof: Let p be the ternary term defined by

$$p(a,b,c) = (a \lhd c) \lhd^{-1} b.$$

We then have the identities

$$p(a,a,b) = (a \lhd b) \lhd^{-1} a = (b \lhd a) \lhd^{-1} a = b,$$
$$p(a,b,b) = (a \lhd b) \lhd^{-1} b = a.$$

■

Recall that a quandle A is *abelian* [19] if it satisfies the additional axiom

$$(a \lhd b) \lhd (c \lhd d) = (a \lhd c) \lhd (b \lhd d)$$

for all $a,\ b,\ c,\ d \in A$. Note that this axiom is equivalent to the following one:

$$(a \lhd b) \lhd^{-1} (c \lhd d) = (a \lhd^{-1} c) \lhd (b \lhd^{-1} d). \tag{4}$$

Remark 2.2. *Not all abelian quandles are symmetric. Indeed, recall that a quandle A is* trivial *if $a \lhd b = a = a \lhd^{-1} b$ for all $a, b \in A$. Any trivial quandle is abelian, but it is not symmetric (as long as it has at least two elements).*

Let us write $\mathsf{AbSymQnd}$ for the category of abelian symmetric quandles, $U\colon \mathsf{AbSymQnd} \to \mathsf{SymQnd}$ and $V\colon \mathsf{SymQnd} \to \mathsf{Qnd}$ for the inclusion functors. Since $\mathsf{AbSymQnd}$ is a subvariety of SymQnd and SymQnd is a subvariety of Qnd, both these functors have left adjoints, denoted by $ab\colon \mathsf{SymQnd} \to \mathsf{AbSymQnd}$ and $sym\colon \mathsf{Qnd} \to \mathsf{SymQnd}$, respectively:

$$\mathsf{Qnd} \underset{V}{\overset{sym}{\rightleftarrows}} \mathsf{SymQnd} \underset{U}{\overset{ab}{\rightleftarrows}} \mathsf{AbSymQnd}.$$

We are now going to show that abelian symmetric quandles are the internal Mal'tsev algebras in SymQnd.

Definition 2.3. *An internal Mal'tsev algebra in a variety $\mathcal{V}$ is an algebra $A \in \mathcal{V}$ with a homomorphism $p_A\colon A \times A \times A \to A$ such that $p_A(a,a,b) = b$ and $p_A(a,b,b) = a$.*

Let us write $\mathsf{Mal}(\mathcal{V})$ for the category of internal Mal'tsev algebras in $\mathcal{C}$. In a Mal'tsev category, thus in particular in the category SymQnd, any morphism preserves the Mal'tsev operation (see Corollary 4.1 in [13], for instance): this means that the subcategory $\mathsf{Mal}(\mathsf{SymQnd})$ is full in SymQnd. The following observation has been found independently by Bourn [2]:

The quandle homomorphism p_1 is Σ-special by Lemma 3.3 since ϕ has abelian symmetric fibers, thus the homomorphism γ is surjective by Corollary 2.9. The fact that $\pi_1 \circ \gamma = p_1$ and $HI(p_2) \circ \pi_2 \circ \gamma = p_2$ implies that γ is also injective. Indeed, this latter property follows from the fact that the pullback projections p_1 and p_2 are jointly monomorphic. Accordingly, the arrow γ is bijective, thus an isomorphism. Since η_A is a surjective homomorphism it follows that the right face of the diagram is a pullback (see Proposition 2.7 in [16], for instance), and the pullback 6 is preserved by the functor I, as desired. ■

Corollary 3.11. *The functor I preserves products of the type $A \times Q$ where Q is an abelian symmetric quandle and A is any quandle.*

Proof: Remark that $A \times Q$ is the following pullback

$$\begin{array}{ccc} A \times Q & \xrightarrow{p_2} & Q \\ {\scriptstyle p_1}\downarrow & & \downarrow \\ A & \longrightarrow & 1 \end{array}$$

where 1 is the terminal object in Qnd, i.e. the trivial quandle with one element. ■

Lemma 3.12. *Consider the following pullback*

$$\begin{array}{ccc} E \times_B A & \xrightarrow{\pi_2} & A \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle f} \\ E & \xrightarrow[p]{} & B. \end{array} \tag{7}$$

If f is an algebraically central extension with abelian symmetric fibers, then π_1 is an algebraically central extension with abelian symmetric fibers.

Moreover, if $p\colon E \to B$ is a surjective homomorphism, then f is an algebraically central extension with abelian symmetric fibers if π_1 is an algebraically central extension with abelian symmetric fibers.

Proof: First remark that we already know that the property of having abelian symmetric fibers is preserved and reflected by pullbacks along surjective homomorphisms by Lemma 3.3.

Let $f\colon A \to B$ be an algebraically central extension with abelian symmetric fibers. Write $p_f\colon A \times \mathsf{Eq}(f) \to A$ for the connector between $A \times A$ and $\mathsf{Eq}(f)$.

central and normal extensions in Qnd with respect to the adjunction between the category of quandles and the category of abelian symmetric quandles:

$$\mathsf{Qnd} \underset{H}{\overset{I}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{AbSymQnd}$$

The following theorem shows that the functor I preserves a certain type of pullbacks. This is equivalent to the admissibility condition of the subvariety AbSymQnd of Qnd.

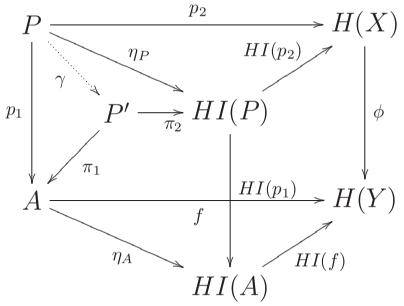
Theorem 3.10. *In the previous adjunction, the reflector* $I\colon \mathsf{Qnd} \to \mathsf{AbSymQnd}$ *preserves all pullbacks in* Qnd *of the form*

$$\begin{array}{ccc} P & \xrightarrow{p_2} & H(X) \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \phi} \\ A & \xrightarrow[f]{} & H(Y) \end{array} \tag{6}$$

where $\phi\colon H(X) \to H(Y)$ *is a surjective homomorphism lying in the subcategory* AbSymQnd *and* $f\colon A \to H(Y)$ *is a surjective homomorphism.*

Proof: Consider the following commutative diagram where:

- the square on the back is the given pullback, where $\phi\colon H(X) \to H(Y)$ is a surjective homomorphism in the subcategory AbSymQnd;
- the universal property of the unit $\eta_P\colon P \to HI(P)$ induces a unique arrow $HI(p_2)\colon HI(P) \to H(X)$ with $HI(p_2) \circ \eta_P = p_2$;
- the universal property of the unit $\eta_A\colon A \to HI(A)$ induces a unique arrow $HI(f)\colon HI(A) \to H(Y)$ with $HI(f) \circ \eta_A = f$;
- (P', π_1, π_2) is the pullback of $HI(p_1)$ along η_A.



Theorem 2.4.

$$\mathsf{AbSymQnd} = \mathsf{Mal}(\mathsf{SymQnd}).$$

Proof: Let $A \in \mathsf{AbSymQnd}$, and let $p_A\colon A \times A \times A \to A$ be the Mal'tsev operation on A defined by $p_A(a,b,c) = (a \lhd c) \lhd^{-1} b$. We have to check that it is a quandle homomorphism. For any $a, b, c, x, y, z \in A$ we have

$$\begin{aligned} p_A((a,b,c) \lhd (x,y,z)) &= p_A(a \lhd x, b \lhd y, c \lhd z) \\ &= ((a \lhd x) \lhd (c \lhd z)) \lhd^{-1} (b \lhd y) \\ &= ((a \lhd c) \lhd (x \lhd z)) \lhd^{-1} (b \lhd y) \\ &= \big((a \lhd c) \lhd^{-1} b\big) \lhd ((x \lhd z) \lhd^{-1} y) \\ &= p_A(a,b,c) \lhd p_A(x,y,z). \end{aligned}$$

This shows that A belongs to Mal(SymQnd).

Conversely, when $A \in \mathsf{Mal}(\mathsf{SymQnd})$, the unique internal Mal'tsev operation on A is necessarily given by (any of) the Mal'tsev operations of the theory of the variety SymQnd. Accordingly, it is defined by $p_A(a,b,c) = (a \lhd c) \lhd^{-1} b$, and it is such that $p_A(a,b,a) = a \lhd^{-1} b$. Moreover, $p_A\colon A \times A \times A \to A$ preserves the binary operation $\lhd$, so that the equality

$$p_A((a,b,a) \lhd (x,y,x)) = p_A(a,b,a) \lhd p_A(x,y,x)$$

gives

$$(a \lhd x) \lhd^{-1} (b \lhd y) = (a \lhd^{-1} b) \lhd (x \lhd^{-1} y).$$

This is precisely the identity (4), and the quandle A belongs to AbSymQnd. ■

We now recall the definition of two classes of morphisms in Qnd, first investigated by Bourn, that will be important for our work:

Definition 2.5. [2] *Let* Σ *be the class of split epimorphisms* $f\colon A \to B$ *with a given section* $s\colon B \to A$ *(i.e.* $f \circ s = 1_B$*) in the category* Qnd *such that the map* $s(b) \lhd -\colon f^{-1}(b) \to f^{-1}(b)$ *is surjective, for any* $b \in B$*.*

In other words, the split epimorphism f with section s is in Σ if, for any $b \in B$ and $a \in f^{-1}(b)$, there is a $k_a \in f^{-1}(b)$ such that $s(b) \lhd k_a = a$.

Remark 2.6. *This element* k_a *also depends on* b*, so that one should write* $k_{b,a}$*, instead. We shall simply write* k_a*, however, to simplify the notations.*

Given an internal equivalence relation (R, r_1, r_2) on A, i.e. a congruence on A, we write $\delta_R \colon A \to R$ for the homomorphism defined by $\delta_R(a) = (a, a)$. An equivalence relation (R, r_1, r_2) is said to be a Σ-*equivalence relation* if the split epimorphism $r_1 \colon R \to A$ with section $\delta_R \colon A \to R$ belongs to the class Σ.

Given a quandle homomorphism $f \colon A \to B$, we write $(\mathsf{Eq}(f), f_1, f_2)$ for the kernel pair of f, where $f_1 \colon \mathsf{Eq}(f) \to A$ and $f_2 \colon \mathsf{Eq}(f) \to A$ are the canonical projections: in a variety of universal algebras $\mathsf{Eq}(f)$ is simply the kernel congruence on A defined by $\mathsf{Eq}(f) = \{(a_1, a_2) \in A \times A \mid f(a_1) = f(a_2)\}$.

Definition 2.7. [5, 6, 2] *A morphism $f \colon A \to B$ in* Qnd *is* Σ-special *if $(\mathsf{Eq}(f), f_1, f_2)$ is a Σ-equivalence relation.*

The following result is a direct consequence of Theorem 3.9 in [2], and will be useful later on:

Theorem 2.8. *Let $f \colon A \to B$ be a Σ-special homomorphism in* Qnd*. Then any congruence R on A permutes with $\mathsf{Eq}(f)$ in the sense of the composition of relations:*

$$R \circ \mathsf{Eq}(f) = \mathsf{Eq}(f) \circ R.$$

Corollary 2.9. *Given a pushout of surjective homomorphisms*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow[l]{} & D \end{array}$$

where f is Σ-special, the induced homomorphism $A \xrightarrow{(g,f)} C \times_D B$ to the pullback is surjective.

Proof: The proof is essentially the same as the one given in [9], Lemma 1.7 (which is adapted from [8]). ■

3. Central extensions in the category of quandles

If $\mathcal{C}$ is a finitely complete category, a double equivalence relation C in $\mathcal{C}$ is an equivalence relation internal in the category of equivalence relations in $\mathcal{C}$.

Proof: Let C be the centralizing relation on $\mathsf{Eq}(f)$ and $A \times A$; consider the following diagram

$$\begin{array}{ccccc} C & \rightrightarrows & A \times A & & \\ {\scriptstyle c_1}\downarrow\downarrow{\scriptstyle c_2} & & \downarrow\downarrow & & \\ \mathsf{Eq}(f) & \rightrightarrows & A & \xrightarrow{f} & B \\ {\scriptstyle q}\downarrow & & \downarrow & & \\ Q & \rightrightarrows & 1 & & \end{array}$$

where q is the coequalizer of c_1 and c_2. By the Barr-Kock theorem [1, 4], the lower squares are pullbacks. By Lemma 3.3, the homomorphisms $Q \to 1$ have abelian symmetric fibers, hence Q is an abelian symmetric quandle. ■

As a consequence, any algebraically central extension $f \colon A \to B$ has its kernel pair $\mathsf{Eq}(f)$ isomorphic to a product of an abelian algebra and A.

Proposition 3.8. *If $f \colon A \to B$ has symmetric fibers, then it is Σ-special.*

Proof: Consider the kernel pair of f

$$\begin{array}{ccc} \mathsf{Eq}(f) & \underset{\delta_f}{\overset{f_2}{\rightleftarrows}} & A \\ {\scriptstyle f_1}\downarrow\uparrow{\scriptstyle \delta_f} & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[f]{} & B \end{array}$$

One has to check that (f_1, δ_f) is in Σ. Let $a \in A$ and $(a, a') \in f_1^{-1}(a)$, then in particular $f(a) = f(a')$, so that $a' \lhd^{-1} a$ is such that $f(a) = f(a' \lhd^{-1} a)$. It follows that $(a, a' \lhd^{-1} a) \in \mathsf{Eq}(f)$, and then

$$(a, a) \lhd (a, a' \lhd^{-1} a) = (a \lhd a, a \lhd (a' \lhd^{-1} a)) = (a, (a' \lhd^{-1} a) \lhd a) = (a, a').$$

■

Remark 3.9. *Observe that when a split epimorphism $f \colon A \to B$ with section $s \colon B \to A$ has symmetric fibers, then $s(b) \lhd - \colon f^{-1}(b) \to f^{-1}(b)$ is always injective: if $x \in f^{-1}(b)$ and $y \in f^{-1}(b)$ are such that $s(b) \lhd x = s(b) \lhd y$, since $s(b) \in f^{-1}(b)$, we get $x \lhd s(b) = y \lhd s(b)$, and hence $x = y$ by right invertibility.*

The results in [2] will be useful to show that the category of abelian symmetric quandles is admissible with respect to surjective homomorphisms in the category of quandles. In the following we shall characterize categorically

By assumption we have that $p \circ i_S = s_2$ and $p \circ i_R = r_1$. To see that $(x, p(x,y,z)) \in S$, we have to prove that $s \circ p = s \circ r_1 \circ \pi_1$. The equalities

$$\begin{aligned} s \circ p \circ i_S &= s \circ s_2 \\ &= s \circ s_1 \\ &= s \circ s_1 \circ p_1 \circ i_S && (p_1 \circ i_S = 1_S) \\ &= s \circ r_1 \circ \pi_1 \circ i_S && (s_1 \circ p_1 = r_1 \circ \pi_1) \end{aligned}$$

and

$$s \circ p \circ i_R = s \circ r_1 = s \circ r_1 \circ \pi_1 \circ i_R,$$

imply that $s \circ p = s \circ r_1 \circ \pi_1$ by Lemma 3.4. A similar argument shows that $(z, p(x,y,z)) \in R$.

Now, to see that

$$p(x, y, p(y, u, v)) = p(x, u, v),$$

let us consider $(a,b,c,d) \in R \times_A R \times_A S$ and write $\phi(a,b,c,d) = p(a,b,p(b,c,d))$ and $\psi(a,b,c,d) = p(a,c,d)$. Observe that

$$\phi(a,b,c,c) = p(a,b,p(b,c,c)) = p(a,b,b) = a = p(a,c,c) = \psi(a,b,c,c)$$

for all $(a,b,c,c) \in R \times_A R \times_A S$, and

$$\phi(e,e,e,f) = p(e,e,p(e,e,f)) = p(e,e,f) = \psi(e,e,e,f)$$

for all $(e,e,e,f) \in R \times_A R \times_A S$. Now, let $(x,y,u,v) \in R \times_A R \times_A S$: since the split epimorphism $s_1 \colon S \to A$ with section $\delta_S \colon A \to S$ is in Σ, there exists $k_{(u,v)} = (u, k_v) \in s_1^{-1}(u)$ such that $(u,v) = (u,u) \lhd (u, k_v)$. Then one can write

$$(x,y,u,v) = (x \lhd^{-1} u, y \lhd^{-1} u, u, u) \lhd (u,u,u,k_v)$$

for all $(x,y,u,v) \in R \times_A R \times_A S$. It follows that $\phi(x,y,u,v) = \psi(x,y,u,v)$. A similar argument shows that $p(p(x,y,u),u,v) = p(x,u,v)$. ∎

Lemma 3.7. *Let $f \colon A \to B$ be an algebraically central extension with abelian symmetric fibers, then $\mathsf{Eq}(f)$ is isomorphic to a product $Q \times A$, where Q is an abelian symmetric quandle.*

It can be represented by a diagram

$$\begin{array}{ccc} C & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & S \\ {\scriptstyle \pi_1}\downarrow\downarrow & {\scriptstyle \pi_2} & {\scriptstyle s_1}\downarrow\downarrow \\ R & \overset{r_1}{\underset{r_2}{\rightrightarrows}} & A, \end{array} \tag{5}$$

where $r_1 \circ \pi_1 = s_1 \circ p_1$, $r_1 \circ \pi_2 = s_2 \circ p_1$, $r_2 \circ \pi_1 = s_1 \circ p_2$ and $r_2 \circ \pi_2 = s_2 \circ p_2$. In this case one usually says that C is a double equivalence relation on the equivalence relations R and S.

Definition 3.1. *Given equivalence relations R and S on A, a double equivalence relation C on R and S (as in (5)) is called a* centralizing relation *when the square*

$$\begin{array}{ccc} C & \xrightarrow{p_2} & S \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle s_1} \\ R & \xrightarrow[r_2]{} & A \end{array}$$

is a pullback.

Definition 3.2. *A* connector *between R and S is an arrow $p \colon R \times_A S \to A$ such that*

1. $p(x,x,y) = y$	*1'.* $p(x,y,y) = x$
2. $xSp(x,y,z)$	*2'.* $zRp(x,y,z)$
3. $p(x,y,p(y,u,v)) = p(x,u,v)$	*3'.* $p(p(x,y,u),u,v) = p(x,u,v)$

In the Mal'tsev context [3] the existence of a connector between R and S is already guaranteed by the existence of a *partial Mal'tsev operation* $p \colon R \times_A S \to A$, i.e. when the identities $p(x,x,y) = y$ and $p(x,y,y) = x$ in Definition 3.2 are satisfied. Accordingly, in a Mal'tsev category the existence of a double centralizing relation on R and S is equivalent to the existence of a partial Mal'tsev operation. Moreover, a connector is unique, when it exists: accordingly, for two given equivalence relations, having a connector becomes a property.

In a Mal'tsev variety a congruence R on an algebra A is called *algebraically central* if there is a centralizing double relation on R and $A \times A$, this latter being the largest equivalence relation on A. In terms of commutators, this fact is expressed by the condition $[R, A \times A] = \Delta_A$.

Also, in the variety Qnd of quandles we shall say that a surjective homomorphism $f\colon A \to B$ in Qnd is an *algebraically central extension* if its kernel congruence $\mathsf{Eq}(f)$ is algebraically central: there is a connector between $\mathsf{Eq}(f)$ and $A \times A$.

Given a homomorphism $f\colon A \to B$ in Qnd, each fiber $f^{-1}(b)$ (for $b \in B$) is a subquandle of A. We shall say that f *has abelian symmetric fibers* if $f^{-1}(b) \in \mathsf{AbSymQnd}$.

Lemma 3.3. *Consider the following pullback*

$$\begin{array}{ccc} E \times_B A & \xrightarrow{\pi_2} & A \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle f} \\ E & \xrightarrow[p]{} & B. \end{array}$$

If $f\colon A \to B$ has abelian symmetric fibers then so does $\pi_1\colon E \times_B A \to E$. Moreover, if $p\colon E \to B$ is a surjective homomorphism, then $f\colon A \to B$ has abelian symmetric fibers if $\pi_1\colon E \times_B A \to E$ has abelian symmetric fibers.

Proof: The first assertion follows from the fact that if $(e, a) \in E \times_B A$ then the fibers $\pi_1^{-1}(e)$ and $f^{-1}(f(a))$ are isomorphic. The proof of the second assertion is similar, the surjectivity of p guaranteeing that, for any $a \in A$, there exists $e \in E$ such that $(e, a) \in E \times_B A$. ■

Lemma 3.4. [2] *Let $f\colon A \to B$ be a split epimorphism, with section $s\colon B \to A$, in Σ. Consider the following pullback of f along a split epimorphism $p\colon E \to B$*

$$\begin{array}{ccc} E \times_B A & \underset{\pi_2}{\overset{(t\circ f, 1_A)}{\leftrightarrows}} & A \\ {\scriptstyle \pi_1}\downarrow\uparrow{\scriptstyle (1_E, s\circ p)} \quad {\scriptstyle t} & & {\scriptstyle f}\downarrow\uparrow{\scriptstyle s} \\ E & \underset{p}{\overset{}{\leftrightarrows}} & B. \end{array}$$

Then $(1_E, s \circ p)$ and $(t \circ f, 1_A)$ are jointly epimorphic.

Proof: Let $(e, a) \in E \times_B A$; we shall show that (e, a) can be rewritten as a product of two elements in the images of $(1_E, s\circ p)$ and $(t\circ f, 1_A)$, respectively. Since the split epimorphism f is in Σ, there exists an element $k_a \in f^{-1}(f(a))$ such that $sf(a) \lhd k_a = a$. Also, we always have $e = (e \lhd^{-1} tp(e)) \lhd tp(e)$. Accordingly, by using the fact that $f(a) = f(k_a)$ and $p(e) = f(a)$, we see

that

$$\begin{aligned} (e, a) &= ((e \lhd^{-1} tp(e)) \lhd tp(e), sf(a) \lhd k_a) \\ &= (e \lhd^{-1} tp(e), sf(a)) \lhd (tp(e), k_a) \\ &= (e \lhd^{-1} tp(e), sp(e)) \lhd (tf(k_a), k_a) \\ &= (e \lhd^{-1} tp(e), sp(e \lhd^{-1} tp(e))) \lhd (tf(k_a), k_a) \\ &= (1_E, s \circ p)(e \lhd^{-1} tp(e)) \lhd (t \circ f, 1_A)(k_a). \end{aligned}$$

■

Corollary 3.5. *Let R be an equivalence relation and S be a Σ-equivalence relation on the same quandle A in Qnd. If there is a connector on R and S, then it is unique.*

Proof: This follows directly from Lemma 3.4. ■

Lemma 3.6. *Let R be an equivalence relation and S be a Σ-equivalence relation on the same quandle A. For a homomorphism $p\colon R \times_A S \to A$, the following conditions are equivalent :*

(1) *p is a partial Mal'tsev operation: $p(x, y, y) = x$ and $p(x, x, y) = y$;*
(2) *p is a connector between R and S.*

Proof: We only have to prove that 1. implies 2. Remark that in any variety, in particular in Qnd, the equivalence relation R is the kernel pair of the canonical quotient $r\colon A \to A/R$, and S the kernel pair of $s\colon A \to A/S$.

$$\begin{array}{ccc} R \times_A S & \underset{p_2}{\overset{p_1}{\leftrightarrows}} & S \\ {\scriptstyle \pi_1}\downarrow\uparrow{\scriptstyle i_R}\ {\scriptstyle \pi_2} \quad {\scriptstyle p} \quad {\scriptstyle s_1}\downarrow\uparrow{\scriptstyle \delta_S}\ {\scriptstyle s_2} & & \\ R & \underset{r_2}{\overset{r_1}{\leftrightarrows}} & A. \end{array}$$