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A NEW OPTIMAL TRANSPORT DISTANCE ON THE SPACE OF FINITE RADON MEASURES

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Abstract: We introduce a new optimal transport distance between nonnegative finite Radon measures with possibly different masses. The construction is based on non-conservative continuity equations and a corresponding modified Benamou–Brenier formula. We establish various topological and geometrical properties of the resulting metric space, derive some formal Riemannian structure, and develop differential calculus following F. Otto's approach. Finally, we apply these ideas to identify an ideal free distribution model of population dynamics as a gradient flow and obtain new long-time convergence results.

Keywords: Optimal transport, variable mass, Otto calculus, geodesic space, ideal free distribution, entropy-entropy production inequality.
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1. Introduction

In the last decades the theory of Monge-Kantorovich optimal transportation problems has seen spectacular developments and provided new powerful tools, deep insights, and numerous applications in functional analysis, partial differential equations, and geometry. The list of references on this topic is steadily growing and we refer to [21] for an extended bibliography. Central ingredients to the theory are the Kantorovich-Rubinstein-Wasserstein distances and their variants, usually defined between probability measures. In their seminal paper [15], Jordan, Kinderlehrer and Otto showed that certain diffusion equations can be interpreted as gradient flows with respect to the Wasserstein metric structure. This was later pushed further by Otto [17], who showed that the space of probability measures can in fact be endowed with a formal Riemannian structure induced by the (quadratic) Wasserstein distance. This differential structure then led to new connections between long time convergence of measure dynamics and functional inequalities via, e.g., entropy dissipation methods, Talagrand and logarithmic Sobolev inequalities, Bakry-Émery strategies. See [6, 20] for a review, and also Section 3 for a brief discussion.

One of the main restrictions of the Wasserstein distances is that they are limited to measures with fixed identical masses, and the theory requires uniform tightness (e.g. some control on the p-moments). Recently, some efforts [13, 18, 19] were made to construct new optimal transport distances between measures with different masses, and some non-mass preserving reaction-diffusion systems were interpreted as gradient flows [8, 16].

In this paper we introduce a new distance on the set of nonnegative finite Radon measures by means of a modified Benamou-Brenier formula. Our approach allows for mass variations and does not require tightness or decay conditions such as finite moments. The classical Benamou-Brenier formula [1, 2, 20] has an interpretation that the squared quadratic Wasserstein distance $\mathcal{W}_2^2(\rho_0,\rho_1)$ between two probability measures ρ_0 and ρ_1 (with finite second moments) is the minimum of the Lagrangian action of the kinetic energy during all possible ways of transporting the original distribution ρ_0 of moving particles to the target one ρ_1 via *continuity equations* $\partial_t\rho + \mathrm{div}(\rho v) = 0$. In a similar spirit, our distance has two interpretations: a mechanical one through motion of charged particles (described later on in Section 3.3), and a biological one through fitness-driven dispersal of organisms described below. For both points of view the crucial role is played by the *non-conservative continuity equation*

$$\partial_t\rho + \mathrm{div}(\rho\nabla u) = \rho u,$$

which allows for mass variations through the reaction term ρu appearing in the right-hand side. Biologically, $\rho(t,x)$ can be viewed as the time evolution of the spatial density of living organisms, and $u(t,x)$ as an intrinsic characteristic of population called *fitness* (cf. [9, 10]). The fitness manifests itself as a growth rate, and simultaneously affects the dispersal, as the species move along the gradient of fitness towards the most favorable environment. The equilibrium $u \equiv 0$ is called the *ideal free distribution*, since no net movement of individuals occurs in this case. Our (squared) distance is the minimum of the Lagrangian action of the total energy, which is the sum of the kinetic energy $\rho|\nabla u^2|$ and of the potential energy $\rho|u|^2$ representing deviation from the ideal free distribution.

The advantages of our distance are that it possesses a rich underlying geometric structure and is easy to handle heuristically by the optimal transport intuition. It metrizes the narrow convergence of measures, and is lower-semicontinuous with respect to the weak-$*$ convergence. We prove that the metric space under consideration is a complete geodesic space. The compactly supported measures are dense in this space. The Lipschitz curves and, in particular, the geodesics in this space have a clear characterization. Our distance endows the space of finite Radon measures with a formal Riemannian structure, and we are able to introduce a first- and second-order calculus à la Otto.

The fitness-driven dispersal model investigated in [11] turns out to be a gradient flow in our formalism. This allows us to show that the solutions to this problem exponentially converge to the ideal free distribution (the convergence itself was proven in [11] by contradiction and thus without any rate). In a forthcoming paper, we study an ecological model for interacting animal populations by observing that it is also a gradient flow with respect to our structure.

The paper is organized as follows: The new distance is defined in Section 2, where we establish the aforementioned topological properties and characterize Lipschitz curves and geodesics. Section 3 is devoted to the Riemannian formalism and differential calculus à la Otto. We introduce the notion of *trajectory geodesics* allowing us to develop the second-order calculus via some Hamilton-Jacobi equations, and also present some explicit and illustrative computations for one-point measures. In Section 4 we exploit this Riemannian formalism to identify the ideal-free distribution model [11] as a gradient flow and retrieve new long-time convergence results. Finally, we opted for moving several auxiliary results, including a new entropy-entropy production inequality, to the Appendix (Section 5).

Notations and conventions.

- Throughout the whole paper and unless otherwise specified we shall always denote by
$$\mathcal{M}^+ = \mathcal{M}^+_b(\mathbb{R}^d)$$
the set of nonnegative finite Radon measures in $\mathbb{R}^d$.
- We use the following notation for sets of functions:

$\mathcal{C}_b$:	bounded continuous with $\|\phi\|_\infty = \sup \|\phi\|$;
$\mathcal{C}^1_b$:	bounded $\mathcal{C}^1$ with bounded first derivatives;
$\mathcal{C}^\infty_c$:	smooth compactly supported;
$\mathcal{C}_0$:	continuous and decaying at infinity;
Lip :	bounded and Lipschitz continuous with $\|\phi\|_{\mathrm{Lip}} = \|\nabla\phi\|_\infty + \|\phi\|_\infty$.

- Given a sequence $\{\rho^k\}_{k\in\mathbb{N}} \subset \mathcal{M}^+$ and $\rho \in \mathcal{M}^+$ we say that:
  (i) ρ^k *converges narrowly* to ρ if there holds
$$\forall\, \phi \in \mathcal{C}_b(\mathbb{R}^d): \qquad \lim_{k\to\infty} \int_{\mathbb{R}^d} \phi(x)\mathrm{d}\rho^k(x) = \int_{\mathbb{R}^d} \phi(x)\mathrm{d}\rho(x).$$
  (ii) ρ^k *converges weakly-*$*$ to ρ if there holds
$$\forall\, \phi \in \mathcal{C}_0(\mathbb{R}^d): \qquad \lim_{k\to\infty} \int_{\mathbb{R}^d} \phi(x)\mathrm{d}\rho^k(x) = \int_{\mathbb{R}^d} \phi(x)\mathrm{d}\rho(x).$$

- Given a measure $\rho_0 \in \mathcal{M}^+$ and a continuous function $F : \mathbb{R}^d \to \mathbb{R}^d$, the measure $F\#\rho_0$ is the pushforward of ρ_0 by F, determined by the relation $\int_{\mathbb{R}^d} \phi \, \mathrm{d}(F\#\rho_0) = \int_{\mathbb{R}^d} \phi \circ F \, \mathrm{d}\rho_0$ for all test functions $\phi(x)$.
- For curves $t \in [0,1] \mapsto \rho_t \in \mathcal{M}^+$ we write

$$\rho \in \mathcal{C}_w([0,1]; \mathcal{M}^+)$$

  for the continuity with respect to the narrow topology.
- Given a nontrivial measure $\rho \in \mathcal{M}^+$ we shall denote by $H^1(\mathrm{d}\rho)$ the Hilbert space obtained by completion of the quotient by the seminorm kernel of the space $\mathcal{C}^1_b(\mathbb{R}^d)$ equipped with the Hilbert seminorm

$$\|\phi\|^2_{H^1(\mathrm{d}\rho)} = \int_{\mathbb{R}^d} \left(|\nabla\phi(x)|^2 + |\phi(x)|^2\right) \mathrm{d}\rho(x).$$

- Similarly, given a narrowly continuous curve

$$\rho \in \mathcal{C}_w([0,1]; \mathcal{M}^+)$$

  we shall denote by $L^2(0,1; H^1(\mathrm{d}\rho_t))$ the Hilbert space obtained by completion of the quotient by the seminorm kernel of the space $\mathcal{C}^1_b((0,1) \times \mathbb{R}^d)$ equipped with the Hilbert seminorm

$$\|\phi\|^2_{L^2(0,1;H^1(\mathrm{d}\rho_t))} = \int_0^1 \left(\int_{\mathbb{R}^d} \left(|\nabla\phi(t,x)|^2 + |\phi(t,x)|^2\right) \mathrm{d}\rho_t(x) \right) \mathrm{d}t.$$

  It is easy to check that if $u \in L^2(0,1; H^1(\mathrm{d}\rho_t))$ then $u_t \in H^1(\mathrm{d}\rho_t)$ is well-defined for a.e. $t \in (0,1)$ as an element of $H^1(\mathrm{d}\rho_t)$, and

$$\|u\|^2_{L^2(0,1;H^1(\mathrm{d}\rho_t))} = \int_0^1 \|u_t\|^2_{H^1(\mathrm{d}\rho_t)} \mathrm{d}t.$$

- Given a narrowly continuous curve $\rho \in \mathcal{C}_w([0,1]; \mathcal{M}^+)$ and

$$u \in L^2(0,1; H^1(\mathrm{d}\rho_t))$$

  we say that

$$\partial_t \rho_t + \operatorname{div}(\rho_t \nabla u_t) = \rho_t u_t \tag{1.1}$$

is satisfied in the distributional sense $\mathcal{D}'((0,1)\times\mathbb{R}^d)$ if

$$\forall \phi \in \mathcal{C}_c^\infty((0,1)\times\mathbb{R}^d):$$
$$-\int_0^1\int_{\mathbb{R}^d}\partial_t\phi\,\mathrm{d}\rho_t(x)\mathrm{d}t = \int_0^1\int_{\mathbb{R}^d}(\nabla\phi\cdot\nabla u_t + \phi u_t)\mathrm{d}\rho_t(x)\mathrm{d}t.$$

We will often refer to (1.1) as the *non-conservative continuity equation.* For any such curve $\rho\in\mathcal{C}_w([0,1];\mathcal{M}^+)$, an easy approximation argument shows that in fact

$$\forall \phi\in\mathcal{C}_b^1(\mathbb{R}^d):$$
$$\int_{\mathbb{R}^d}\phi(\mathrm{d}\rho_t-\mathrm{d}\rho_s) = \int_s^t\int_{\mathbb{R}^d}(\nabla\phi\cdot\nabla u_\tau+\phi u_\tau)\mathrm{d}\rho_\tau(x)\mathrm{d}\tau \quad (1.2)$$

holds for all fixed $0\le s\le t\le 1$.

- The *bounded-Lipschitz distance* between two measures $\rho_0,\rho_1\in\mathcal{M}^+$ is
$$d_{BL}(\rho_0,\rho_1) = \sup_{\|\phi\|_{\mathrm{Lip}}\le 1}\left|\int_{\mathbb{R}^d}\phi(\mathrm{d}\rho_1-\mathrm{d}\rho_0)\right|.$$
Useful properties are that the metric space $(\mathcal{M}^+, d_{BL})$ is complete and that d_{BL} metrizes the narrow convergence on $\mathcal{M}^+$. These facts are well-known [12] if we replace $\mathcal{M}^+$ by the space of probability measures. The reduction of the general case to the measures of unit mass is not involved. We remark that these properties may also be indirectly deduced from the results of [18] and [19]. Another useful observation is that the supremum can be restricted to smooth compactly supported functions. This follows from the tightness of a set consisting of two finite Radon measures.
- Finally, B_R are open balls of radius $R>0$ in $\mathbb{R}^d$ centered at zero, and C is a generic positive constant.

2. The metric space $(\mathcal{M}^+, d)$

2.1. Definition and first properties.

Definition 2.1. Given two finite Radon measures $\rho_0,\rho_1\in\mathcal{M}^+(\mathbb{R}^d)$ we define

$$d^2(\rho_0,\rho_1) = \inf_{\mathcal{A}(\rho_0,\rho_1)}\int_0^1\left(\int_{\mathbb{R}^d}(|\nabla u_t|^2+|u_t|^2)\mathrm{d}\rho_t\right)\mathrm{d}t, \qquad (2.1)$$

where the admissible set $\mathcal{A}(\rho_0, \rho_1)$ consists of all couples $(\rho_t, u_t)_{t\in[0,1]}$ such that

$$\begin{cases} \rho \in \mathcal{C}_w([0,1]; \mathcal{M}^+), \\ \rho|_{t=0} = \rho_0; \quad \rho|_{t=1} = \rho_1, \\ u \in L^2(0,T; H^1(\mathrm{d}\rho_t)), \\ \partial_t \rho_t + \mathrm{div}(\rho_t \nabla u_t) = \rho_t u_t \quad \text{in the distributional sense.} \end{cases}$$

This gives of course a metric space:

Theorem 1. *d is a distance on $\mathcal{M}^+(\mathbb{R}^d)$.*

We first need a preliminary result, which we shall use repeatedly in the sequel:

Lemma 2.2 (Bounded-Lipschitz and mass estimate)**.** *Let $\rho \in \mathcal{C}_w([0,1]; \mathcal{M}^+)$ be a narrowly continuous curve, and assume that the non-conservative continuity equation $\partial_t \rho_t + \mathrm{div}(\rho_t \nabla u_t) = \rho_t u_t$ is satisfied in the distributional sense for some potential $u \in L^2(0,T; H^1(\mathrm{d}\rho_t))$ with finite energy*

$$E = E[\rho; u] = \|u\|^2_{L^2(0,T;H^1(\mathrm{d}\rho_t))} = \int_0^1 \int_{\mathbb{R}^d} (|\nabla u_t|^2 + |u_t|^2) \mathrm{d}\rho_t(x) \mathrm{d}t.$$

Then the masses are bounded uniformly in time, $m_t = \mathrm{d}\rho_t(\mathbb{R}^d) \leq M$ for some M depending only on E and $m_0 = \mathrm{d}\rho_0(\mathbb{R}^d)$, and

$$\forall \phi \in \mathcal{C}^1_b(\mathbb{R}^d): \quad \left| \int_{\mathbb{R}^d} \phi(\mathrm{d}\rho_t - \mathrm{d}\rho_s) \right| \leq (\|\nabla \phi\|_\infty + \|\phi\|_\infty) \sqrt{ME} |t-s|^{1/2} \tag{2.2}$$

for all $0 \leq s \leq t \leq 1$.

Proof: The uniform bound on the mass $M = \sup\limits_{t\in[0,1]} m_t < \infty$ immediately follows by our assumption that $t \mapsto \rho_t$ is narrowly continuous (take $\phi(x) \equiv 1$ in the definition of weak continuity), and we shall check later that M depends only on m_0 and E. Using the Cauchy-Schwarz inequality in (1.2) (first in space and then in time) gives

$$
\begin{aligned}
\left|\int_{\mathbb{R}^d}\phi(\mathrm{d}\rho_t-\mathrm{d}\rho_s)\right| &= \left|\int_s^t\int_{\mathbb{R}^d}(\nabla\phi\cdot\nabla u_\tau+\phi u_\tau)\mathrm{d}\rho_\tau\mathrm{d}\tau\right| \\
&\le \textstyle\int_s^t \left(\int_{\mathbb{R}^d}(|\nabla\phi|^2+|\phi|^2)\mathrm{d}\rho_\tau\right)^{1/2}\left(\int_{\mathbb{R}^d}(|\nabla u_\tau|^2+|u_\tau|^2)\mathrm{d}\rho_\tau\right)^{1/2}\mathrm{d}\tau \\
&\le (\|\nabla\phi\|_\infty+\|\phi\|_\infty)\sqrt{M}\cdot|t-s|^{1/2} \\
&\quad\cdot\left(\int_0^1\int_{\mathbb{R}^d}(|\nabla u_\tau|^2+|u_\tau|^2)\mathrm{d}\rho_\tau\mathrm{d}\tau\right)^{1/2} \\
&\le (\|\nabla\phi\|_\infty+\|\phi\|_\infty)\sqrt{ME}|t-s|^{1/2}
\end{aligned}
$$

as desired.

Choosing $\phi\equiv 1$ we obtain in particular $|m_t-m_s|\le\sqrt{ME}|t-s|^{1/2}$. Let $t_0\in[0,1]$ be any time when the supremum is attained, i-e $m_{t_0}=M$: choosing $t=t_0$ and $s=0$ we immediately get $M\le m_0+\sqrt{ME}|t_0-0|^{1/2}$, which finally bounds M in terms of m_0 and E only. ■

Remark 2.3. The paths and energies can easily be scaled in time as follows: if $(\rho_t,u_t)_{t\in[0,1]}$ connects ρ_0 to ρ_1 with energy $E[\rho;u]$, then for any $T>0$ the path $(\overline{\rho}_s,\overline{u}_s)=\left(\rho_{\frac{s}{T}},\frac{1}{T}u_{\frac{s}{T}}\right)$ connects ρ_0 to ρ_1 in time $s\in[0,T]$ with energy

$$
\begin{aligned}
E[\overline{\rho};\overline{u}] &= \int_0^T\left(\int_{\mathbb{R}^d}(|\nabla\overline{u}_s|^2+|\overline{u}_s|)\mathrm{d}\overline{\rho}_s\right)\mathrm{d}s \\
&= \frac{1}{T}\int_0^1\left(\int_{\mathbb{R}^d}(|\nabla u_t|^2+|u_t|)\mathrm{d}\rho_t\right)\mathrm{d}t=\frac{1}{T}E[\rho;u].
\end{aligned}
$$

of Theorem 1: Let us first show that $d(\rho_0,\rho_1)$ is always finite for any $\rho_0,\rho_1\in\mathcal{M}^+$. Indeed for any finite measure $\nu_0\in\mathcal{M}^+$ it is easy to see that $\nu_t=(1-t)^2\nu_0$ and $u_t=-\frac{2}{1-t}$ give a narrowly continuous curve $t\mapsto\nu_t\in\mathcal{M}^+$ connecting ν_0 to zero, and an easy computation shows that this path has finite energy $E[\nu;u]=4\mathrm{d}\nu_0(\mathbb{R}^d)<\infty$ (this curve is actually the geodesic between ν_0 and 0, see later on the proof of Proposition 2.6 for the details). Rescaling time as in remark 2.3, it is then easy to connect any two measures $\rho_0,\rho_1\in\mathcal{M}^+$ in time $t\in[0,1]$ by first connecting ρ_0 to 0 in time $t\in[0,1/2]$ and then connecting 0 to ρ_1 in time $t\in[1/2,1]$ with cost exactly $E=2(4\mathrm{d}\rho_0(\mathbb{R}^d)+4\mathrm{d}\rho_1(\mathbb{R}^d))<\infty$.

In order to show that d is really a distance, observe first that the symmetry $d(\rho_0,\rho_1)=d(\rho_1,\rho_0)$ is obvious by definition.

For the indiscernability, assume that $\rho_0, \rho_1 \in \mathcal{M}^+$ are such that $d(\rho_0, \rho_1) = 0$. Let $\left(\rho_t^k, u_t^k\right)_{t\in[0,1]}$ be any minimizing sequence in (2.1), i-e $\lim\limits_{k\to\infty} E[\rho^k; u^k] = d^2(\rho_0, \rho_1) = 0$. By Lemma 2.2 we see that the masses $m_t^k = \mathrm{d}\rho_t^k(\mathbb{R}^d)$ are uniformly bounded, $\sup\limits_{t\in[0,1],\, k\in\mathbb{N}} m_t^k \leq M$. For any fixed $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ the fundamental estimate (2.2) gives

$$\left|\int_{\mathbb{R}^d} \phi(\mathrm{d}\rho_1 - \mathrm{d}\rho_0)\right| \leq (\|\nabla\phi\|_\infty + \|\phi\|_\infty)\sqrt{M} \cdot \sqrt{E[\rho^k; u^k]}.$$

Since $\lim\limits_{k\to\infty} E[\rho^k; u^k] = 0$ we conclude that $\int_{\mathbb{R}^d} \phi(\mathrm{d}\rho_1 - \mathrm{d}\rho_0) = 0$ for all $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, thus $\rho_1 = \rho_0$ as desired.

As for the triangular inequality, fix any $\rho_0, \rho_1, \nu \in \mathcal{M}^+$ and let us prove that $d(\rho_0, \rho_1) \leq d(\rho_0, \nu) + d(\nu, \rho_1)$. We can assume that all three distances are nonzero, otherwise the triangular inequality trivially holds by the previous point. Let now $(\underline{\rho}_t^k, \underline{u}_t^k)_{t\in[0,1]}$ be a minimizing sequence in the definition of $d^2(\rho_0, \nu) = \lim\limits_{k\to\infty} E[\underline{\rho}^k; \underline{u}^k]$, and let similarly $(\overline{\rho}_t^k, \overline{u}_t^k)_{t\in[0,1]}$ be such that $d^2(\nu, \rho_1) = \lim\limits_{k\to\infty} E[\overline{\rho}^k; \overline{u}^k]$. For fixed $\tau \in (0, 1)$ let (ρ_t, u_t) be the continuous path obtained by first following $\left(\underline{\rho}^k, \frac{1}{\tau}\underline{u}^k\right)$ from ρ_0 to ν in time τ, and then following $\left(\overline{\rho}^k, \frac{1}{1-\tau}\overline{u}^k\right)$ from ν to ρ_1 in time $1 - \tau$. Then $\left(\rho_t^k, u_t^k\right)_{t\in[0,1]}$ is an admissible path connecting ρ_0 to ρ_1, hence by definition of our distance and the explicit time scaling in Remark 2.3 we get that

$$d^2(\rho_0, \rho_1) \leq E[\rho^k; u^k] = \frac{1}{\tau} E[\overline{\rho}^k; \overline{u}^k] + \frac{1}{1-\tau} E[\underline{\rho}^k; \underline{u}^k].$$

Letting $k \to \infty$ we obtain for any fixed $\tau \in (0, 1)$

$$d^2(\rho_0, \rho_1) \leq \frac{1}{\tau} d^2(\rho_0, \nu) + \frac{1}{1-\tau} d^2(\nu, \rho_1).$$

Finally choosing $\tau = \frac{d(\rho_0,\nu)}{d(\rho_0,\nu)+d(\nu,\rho_1)} \in (0, 1)$ yields

$$d^2(\rho_0, \rho_1) \leq \frac{1}{\tau} d^2(\rho_0, \nu) + \frac{1}{1-\tau} d^2(\nu, \rho_1) = (d(\rho_0, \nu) + d(\nu, \rho_1))^2$$

and the proof is complete. ■

As an immediate consequence of Lemma 2.4 we get

Corollary 2.4. *The elements of a bounded set in $(\mathcal{M}^+, d)$ have uniformly bounded mass.*

The converse statement is also true, see Corollary 2.7 below. Another property, easily following from Remark 2.3, is

Lemma 2.5. *If $(\rho_t, u_t)_{t\in[0,1]}$ is a narrowly continuous curve with total energy E then $t \mapsto \rho_t$ is 1/2-Hölder continuous w.r.t. d, and more precisely*

$$\forall\, t_0, t_1 \in [0,1] : \qquad d(\rho_{t_0}, \rho_{t_1}) \leq \sqrt{E}|t_0 - t_1|^{1/2}.$$

Proof: Rescaling in time and connecting ρ_{t_0} to ρ_{t_1} by the path $(\rho_s, (t_1 - t_0)u_s)_{s\in[0,1]}$ with $t = t_0 + (t_1 - t_0)s$, the resulting energy scales as $d^2(\rho_{t_0}, \rho_{t_1}) \leq E[\overline{\rho}; \overline{u}] \leq E|t_0 - t_1|$. ■

Before proceeding with the study of the topological properties of our metric we need some more results.

Proposition 2.6. *For any $\rho_0 \in \mathcal{M}^+$ there holds $d(\rho_0, 0) = 2\sqrt{m_0}$ with $m_0 = \mathrm{d}\rho_0(\mathbb{R}^d)$, and the geodesic is explicitly given by $\rho_t = (1-t)^2\rho_0$ and $u_t = -\frac{2}{1-t}$.*

Note in particular that $\nabla u_t \equiv 0$, which means that the optimal strategy to send a measure ρ_0 to zero is always to "squeeze it down" without any transport.

Proof: We start by showing that in the minimization problem in the definition of $d^2(\rho_0, 0)$ one can restrict to paths of the form $\rho_t = \lambda(t)\rho_0$, and then we compute the optimal $\lambda(t)$ from which we recover (ρ_t, u_t).
Step 1: no transport is involved. Let $(\rho_t, u_t)_{t\in[0,1]}$ be any admissible path connecting ρ_0 to $\rho_1 = 0$ with finite energy. We claim that

$$\tilde{\rho}_t := \frac{m_t}{m_0}\rho_0, \qquad \tilde{u}_t := \langle u_t \rangle_{\mathrm{d}\rho_t} = \frac{1}{m_t}\int_{\mathbb{R}^d} u_t \mathrm{d}\rho_t$$

always gives a path with lesser energy, where $m_t = \mathrm{d}\rho_t(\mathbb{R}^d)$ denotes the mass at time t as before. Note that because the initial path ρ_t is narrowly continuous we have in particular that $t \mapsto m_t$ is continuous, and we can thus assume that $m_t > 0$ in $[0,1)$ (otherwise $\rho_{t_0} = 0$ is attained for some time $t_0 \in (0,1)$ so whatever happens after t_0 can only cost an unnecessary extra energy, and scaling in time $t = st_0$ with $s \in (0,1)$ decreases the total cost). This continuity also shows that $\tilde{\rho}_t$ connects ρ_0 to 0, since $m_t \to 0$ implies narrow convergence $\tilde{\rho}_t \to 0$ when $t \to 1$. Since $\tilde{u}_t = \langle u_t \rangle_{d\rho_t}$ is constant in space we have by Jensen's inequality

$$E[\tilde{\rho};\tilde{u}] = \int_0^1 \Big((0 + \langle u_t\rangle^2_{\mathrm{d}\rho_t})m_t\Big)\,\mathrm{d}t$$
$$\leq \int_0^1 \left\langle |u_t|^2\right\rangle_{d\rho_t} m_t\,\mathrm{d}t = \int_0^1 \left(\int_{\mathbb{R}^d} |u_t|^2 \mathrm{d}\rho_t\right)\mathrm{d}t \leq E[\rho;u],$$

which shows that $(\tilde{\rho}_t, \tilde{u}_t)$ has lesser energy.

It remains to show that this path solves the non-conservative continuity equation. Taking $\phi(x) \equiv 1$ in the weak formulation of the non-conservative continuity equation satisfied by (ρ_t, u_t) and exploiting $\sup_t m_t \leq M$ it is easy to check that $\frac{d}{dt}m_t = \int_{\mathbb{R}^d} u_t \mathrm{d}\rho_t \in L^2(0,1)$. Then in the sense of distributions $\mathcal{D}'((0,1) \times \mathbb{R}^d)$ we have

$$\partial_t \tilde{\rho}_t = \partial_t \left(\frac{m_t}{m_0}\rho_0\right) = \frac{dm_t}{dt}\cdot\frac{\rho_0}{m_0} = \left(\int_{\mathbb{R}^d} u_t \mathrm{d}\rho_t\right)\frac{\rho_0}{m_0}$$
$$= \left(\frac{1}{m_t}\int_{\mathbb{R}^d} u_t \mathrm{d}\rho_t\right)\frac{m_t}{m_0}\rho_0 = \tilde{u}_t\tilde{\rho}_t,$$

and we conclude recalling that by construction $\nabla \tilde{u}_t \equiv 0$ in the advection term.
Step 2: computing the geodesic. By the previous step it is enough to minimize over all couples (ρ_t, u_t) such that $\rho_t = \lambda(t)\rho_0$ with $\lambda(t) > 0$ in $[0,1)$, and potentials $\nabla u_t \equiv 0$. For such paths it is easy to realize that necessarily $u_t = \frac{\lambda'(t)}{\lambda(t)}$, thus we only have to solve the minimization problem

$$\inf\left\{\int_0^1 \left(\int_{\mathbb{R}^d}\left(0 + \left|\frac{\lambda'}{\lambda}\right|^2\right)\mathrm{d}(\lambda\rho_0)\right)\mathrm{d}t : \ \lambda(0) = 1,\ \lambda(1) = 0\right\}.$$

It is a simple exercise in the calculus of variations to check that the unique minimizer is

$$\lambda(t) = (1-t)^2,\ u(t) = \frac{\lambda'(t)}{\lambda(t)} = -\frac{2}{1-t},\ \rho_t = \lambda(t)\rho_0 = (1-t)^2\rho_0$$

and the explicit computation finally gives

$$d^2(\rho_0, 0) = \int_0^1 \left(\int_{\mathbb{R}^d} |u(t)|^2\,\mathrm{d}(\lambda(t)\rho_0)\right)\mathrm{d}t$$
$$= \int_0^1 \left(\int_{\mathbb{R}^d}\left|\frac{2}{1-t}\right|^2 d((1-t)^2\rho_0)\right)dt = 4m_0$$

as claimed. ■

As an easy consequence of the previous Proposition 2.6 we obtain

Corollary 2.7. *Subsets of $\mathcal{M}^+$ with uniformly bounded mass are bounded in $(\mathcal{M}^+, d)$ as* $\mathrm{d}\rho(\mathbb{R}^d) \leq M \Rightarrow d(\rho, 0) \leq 2\sqrt{M}$.

Proposition 2.8 (Scaling properties). *For any $\lambda \in \mathbb{R}^+$ and measures $\rho_0, \rho_1 \in \mathcal{M}^+$ we have*

$$d(\rho_0, \lambda\rho_0) = 2\sqrt{m_0}\left|1 - \sqrt{\lambda}\right| \tag{2.3}$$

and

$$d(\lambda\rho_0, \lambda\rho_1) = \sqrt{\lambda}d(\rho_0, \rho_1). \tag{2.4}$$

Remark that (2.3) can be rephrased in a more intrinsic way as: "if $\rho_0, \rho_1 \in \mathcal{M}^+$ are proportional measures then $d(\rho_0, \rho_1) = 2|\sqrt{m_0} - \sqrt{m_1}|$", and also that the previous Proposition 2.6 is a particular case with $\lambda = 0$.

Proof: For the first part it is easy to argue as in the proof of Proposition 2.6 to see that it is enough to minimize over all paths $\rho_t = \lambda(t)\rho_0$, with $u_t = \frac{\lambda'(t)}{\lambda(t)}, \nabla u_t \equiv 0$ (i-e averaging in space decreases the energy) and the constraints $\lambda(0) = 1$, $\lambda(1) = \lambda$. Finding the optimal $\lambda(t)$ is then a simple exercise in the calculus of variations, and the explicit computation leads to (2.3) (the minimizer $\lambda(t)$ is again a second order polynomial).

For (2.4), note that our statement is trivial if $\lambda = 0$ so we can assume that $\lambda > 0$. We denote below $\mu_0 = \lambda\rho_0, \mu_1 = \lambda\rho_1$. Let $\left(\rho^k_t, u^k_t\right)_{t\in[0,1]}$ be a minimizing sequence in the definition of $d^2(\rho_0, \rho_1)$. Because both the non-conservative continuity equation and the energy are linear in ρ we see that $(\lambda\rho^k_t, u^k_t)$ is an admissible path connecting μ_0 to μ_1, and

$$d^2(\mu_0, \mu_1) \leq E[\lambda\rho^k; u^k] = \lambda E[\rho^k; u^k] \underset{k\to\infty}{\to} \lambda d^2(\rho_0, \rho_1).$$

The other inequality is obtained similarly: if (μ^k_t, v^k_t) is a minimizing sequence for $d^2(\mu_0, \mu_1)$ then $(\frac{1}{\lambda}\mu^k_t, v^k_t)$ connects ρ_0 to ρ_1, thus

$$d^2(\rho_0, \rho_1) \leq E[\mu^k/\lambda; v^k] = \frac{1}{\lambda}E[\mu^k; v^k] \underset{k\to\infty}{\to} \frac{1}{\lambda}d^2(\mu_0, \mu_1)$$

and the proof is complete. ■

2.2. Topological properties.

Theorem 2. *The compactly supported measures are dense in* $(\mathcal{M}^+, d)$*: for any* $\rho \in \mathcal{M}^+$ *and* $\varepsilon > 0$ *there exists* $\rho' \in \mathcal{M}^+$ *compactly supported such that* $d(\rho, \rho') \leq \varepsilon$*.*

Proof: Observe that ρ has arbitrarily small mass outside of B_R for large R. The argument goes in two steps: first we create an annular gap around $|x| = R$ with arbitrarily small cost, i-e construct a measure $\tilde{\rho}$ which has support in $B_R \cup (\mathbb{R}^d \setminus B_{R+\delta})$ for some small $\delta > 0$ such that $d(\rho, \tilde{\rho}) \leq \varepsilon/2$ and the mass of $\tilde{\rho}$ outside of $B_{R+\delta}$ is still small. The second step consists in sending all the exterior mass $\mathrm{d}\tilde{\rho}(\mathbb{R}^d \setminus B_{R+\delta})$ to zero while keeping the interior part $\tilde{\rho}|_{B_R}$ unchanged (the fact that we can do both simultaneously relies on the gap of size $\delta > 0$). In fact we shall do so without modifying the original measure ρ inside B_R, so that really in the end we shall take $\rho' = \rho|_{B_R}$ with $d(\rho, \rho') \leq \varepsilon$.
Step 1: creating the gap. For fixed $R > 0$ let us first decompose

$$\rho = \underline{\rho} + \overline{\rho} \qquad \text{with } \underline{\rho} := \rho \mathbf{1}_{B_R} \text{ and } \overline{\rho} := \rho \mathbf{1}_{\mathbb{R}^d \setminus B_R}.$$

Let also $U(x) = U_R(x) \in \mathcal{C}^\infty(\mathbb{R}^d)$ be any smooth function such that

$$U(x) = \begin{cases} |x| - R & \text{if } R \leq |x| \leq R+1 \\ 2 & \text{if } |x| > R+2 \end{cases} \qquad \text{and} \qquad |U|, |\nabla U| \leq 2.$$

By Lemma 5.1 we can solve

$$\begin{cases} \partial_t \overline{\rho}_t + \operatorname{div}(\overline{\rho}_t \nabla U) = \overline{\rho}_t U \\ \overline{\rho}|_{t=0} = \overline{\rho}, \end{cases}$$

and $t \mapsto \overline{\rho}_t$ is narrowly continuous. Moreover by construction the initial datum $\overline{\rho}$ has support in $\mathbb{R}^d \setminus B_R$, so for $t \in [0,1]$ it is easy to see that the measure $\overline{\rho}_t$ has support in $\mathbb{R}^d \setminus B_{R+t}$ (the characteristics $\frac{dx}{dt} = \nabla U(x)$ diverge radially away from B_R, with constant speed $|\nabla U| = 1$ for $|x| \gtrsim R$). Denoting the mass

$$\overline{m}_t := \int_{\mathbb{R}^d} \mathrm{d}\overline{\rho}_t = \int_{\mathbb{R}^d \setminus B_R} \mathrm{d}\overline{\rho}_t = \int_{\mathbb{R}^d \setminus B_{R+t}} \mathrm{d}\overline{\rho}_t$$

it is easy to check from the non-conservation continuity equation that

$$\left| \frac{d}{dt} \overline{m}_t \right| = \left| \int_{\mathbb{R}^d} U \mathrm{d}\overline{\rho}_t \right| \leq \int_{\mathbb{R}^d} |U| \mathrm{d}\overline{\rho}_t \leq 2 \overline{m}_t,$$

so that

$$\overline{m}_t \leq e^{2t} \overline{m}_0$$

for $t \in [0,1]$. Define now

$$\rho_t := \underline{\rho} + \overline{\rho}_t \qquad \text{and} \qquad u_t(x) = \begin{cases} 0 & \text{if } |x| < R \\ U(x) & \text{if } |x| \geq R \end{cases}$$

(observe that u_t is constant in time, continuous in space, and uniformly bounded by $|u_t(x)| \leq 2$). By construction for $t > 0$ the measure ρ_t splits into a measure $\underline{\rho}$ with support in B_R and a measure $\overline{\rho}_t$ with support in $\mathbb{R}^d \setminus B_{R+t}$. Exploiting the fact that these supports stay at distance $t > 0$ away from each other it is easy to check that (ρ_t, u_t) solves the non-conservative continuity equation in the sense of distributions (for $t > 0$ the "matching" at $|x| = R$ is never seen!). The positive gap moreover allows to compute for a.e. $t \in (0,1)$

$$\begin{aligned}
&\int_{\mathbb{R}^d} (|\nabla u_t|^2 + |u_t|^2) \mathrm{d}\rho_t \\
&\qquad = \int_{B_R} (|\nabla u_t|^2 + |u_t|^2) \mathrm{d}\underline{\rho} + \int_{\mathbb{R}^d \setminus B_{R+t}} (|\nabla u_t|^2 + |u_t|^2) \mathrm{d}\overline{\rho}_t \\
&\qquad \leq 0 + (\|\nabla U\|_\infty^2 + \|U\|_\infty^2) \int_{\mathbb{R}^d \setminus B_{R+t}} \mathrm{d}\overline{\rho}_t \\
&\qquad \leq C \overline{m}_t \leq C e^{2t} \overline{m}_0,
\end{aligned}$$

which shows that this path has finite energy. In particular by Lemma 2.5 $t \mapsto \rho_t$ is continuous with respect to our metric d, and $d(\rho, \rho_t)$ is small for small $t > 0$. Now for fixed $\varepsilon > 0$ there exists $R > 0$ such that $\overline{m}_0 = \mathrm{d}\rho(\mathbb{R}^d \setminus B_R) \leq \varepsilon$, so choosing $t = \delta > 0$ small enough we get that the measure $\tilde{\rho} := \overline{\rho}_{t=\delta}$ satisfies

$$\operatorname{supp}(\tilde{\rho}) \subset B_R \cup (\mathbb{R}^d \setminus B_{R+\delta}), \ \ \tilde{\rho}|_{B_R} = \rho|_{B_R},$$
$$\mathrm{d}\tilde{\rho}(\mathbb{R}^d \setminus B_{R+\delta}) = \overline{m}_\delta \leq e^{2\delta} \overline{m}_0 \leq 2\varepsilon$$

and

$$d(\rho, \tilde{\rho}) \leq \varepsilon.$$

Step 2: sending the exterior mass to zero. Choose now any smooth function $\tilde{u}(x)$ such that

$$\tilde{u}(x) = \begin{cases} 0 & \text{if } |x| \leq R \\ 1 & \text{if } |x| \geq R + \delta \end{cases}$$

and let

$$\tilde{u}_t(x) := -\frac{2}{1-t} \tilde{u}(x) \quad \text{and} \quad \tilde{\rho}_t := \tilde{\rho} \mathbf{1}_{B_R} + (1-t)^2 \tilde{\rho} \mathbf{1}_{\mathbb{R}^d \setminus B_{R+\delta}}.$$

Since $\tilde{\rho}_t$ always has a gap between $|x| < R$ and $|x| \geq R+\delta$ it is straightforward to check that $(\tilde{\rho}_t, \tilde{u}_t)$ is an admissible curve connecting $\tilde{\rho}$ to $\rho' = \rho \mathbf{1}_{B_R} = \tilde{\rho}\mathbf{1}_{B_R}$ (in particular $\tilde{\rho}_t$ is narrowly continuous and solves the non-conservative continuity equation), with cost exactly

$$\int_0^1 \left(\int_{\mathbb{R}^d} (|\nabla \tilde{u}_t|^2 + |\tilde{u}_t|^2) \mathrm{d}\tilde{\rho}_t \right) \mathrm{d}t$$
$$= 0 + \int_0^1 \left(\int_{\mathbb{R}^d \setminus B_{R+\delta}} (|\nabla \tilde{u}_t|^2 + |\tilde{u}_t|^2) \mathrm{d}\tilde{\rho}_t \right) \mathrm{d}t = 4\overline{m}_\delta \leq 8\varepsilon.$$

Indeed $\tilde{u}_t \equiv 0$ in $\overline{B}_R$, the measure $\tilde{\rho}_t$ does not charge the annulus $R < |x| < R + \delta$, and by construction outside of $B_{R+\delta}$ the potential $\tilde{u}_t$ it is exactly the geodesic between ρ' to zero, see the proof of Proposition 2.6. By the triangular inequality we finally get

$$d(\rho, \rho') \leq d(\rho, \tilde{\rho}) + d(\tilde{\rho}, \rho') \leq \varepsilon + \sqrt{8\varepsilon},$$

and, since $\varepsilon > 0$ was arbitrary and ρ' is compactly supported, the conclusion follows. ■

Proposition 2.9. *Let $\mathcal{P}_2(\mathbb{R}^d) \subset \mathcal{M}_b^+(\mathbb{R}^d)$ be the set of Borel probability measures with finite second moments and $\mathcal{W}_2$ the quadratic Kantorovich-Rubinstein-Wasserstein distance defined on $\mathcal{P}_2$. Then for any $\rho_0, \rho_1 \in \mathcal{P}_2$ there holds*

$$d(\rho_0, \rho_1) \leq \mathcal{W}_2(\rho_0, \rho_1).$$

Proof: It is well known [20, 21] that $\left(\mathcal{P}_2(\mathbb{R}^d), \mathcal{W}_2\right)$ is a geodesic space, so we can connect ρ_0 and ρ_1 by a constant-speed geodesic path ρ_t in this Wasserstein space. By [21, Theorem 13.8] there exists a velocity field $v_t \in L^\infty(0, 1; L^2(\mathrm{d}\rho_t))$ such that

$$\partial_t \rho_t + \operatorname{div}(\rho_t v_t) = 0 \tag{2.5}$$

holds in the distributional sense, and $t \mapsto \rho_t$ is narrowly continuous (since it is continuous with values in the stronger Wasserstein metric topology).

For almost every $t \in (0, 1)$ the distribution $\zeta_t = -\operatorname{div}(\rho_t v_t) \in \mathcal{D}'(\mathbb{R}^d)$ defines by duality a continuous linear form on $H^1(\mathrm{d}\rho_t)$ with norm $\|\zeta_t\|_{H^{-1}(\mathrm{d}\rho_t)} \leq \|v_t\|_{L^2(\mathrm{d}\rho_t)}$, thus by the Riesz representation theorem we can find a unique $u_t \in H^1(\mathrm{d}\rho_t)$ such that

$$(u_t, .)_{H^1(\mathrm{d}\rho_t)} = \zeta_t \quad \text{and} \quad \|u_t\|_{H^1(\mathrm{d}\rho_t)} = \|\zeta_t\|_{H^{-1}(\mathrm{d}\rho_t)} \leq \|v_t\|_{L^2(\mathrm{d}\rho_t)}.$$

In particular by definition of $(.,.)_{H^1(\mathrm{d}\rho_t)}$ for a.e. $t\in(0,1)$ there holds

$$-\operatorname{div}(\rho_t\nabla u_t)+\rho_t u_t=-\operatorname{div}(\rho_t v_t)\qquad\text{in }\mathcal{D}'(\mathbb{R}^d),$$

and by (2.5) it is easy to check that $\partial_t\rho_t+\operatorname{div}(\rho_t\nabla u_t)=\rho_t u_t$ in the sense of distributions. Since $t\mapsto\rho_t$ is continuous for the narrow topology we see that (ρ_t,u_t) is an admissible path in the definition of our distance, and in particular

$$d^2(\rho_0,\rho_1)\le E[\rho;u]=\int_0^1\|u_t\|^2_{H^1(\mathrm{d}\rho_t)}\mathrm{d}t\le\int_0^1\|v_t\|^2_{L^2(\mathrm{d}\rho_t)}\mathrm{d}t.$$

By the Benamou-Brenier formula [1, 2], the right-hand side coincides with the squared Wasserstein distance $\mathcal{W}_2^2(\rho_0,\rho_1)$ and the proof is achieved. ■

Theorem 3. *The metric d is topologically equivalent to the bounded-Lipschitz metric d_{BL}, i-e $d(\rho^k,\rho)\to0$ if and only if $d_{BL}(\rho^k,\rho)\to0$. Moreover, $(\mathcal{M}^+,d)$ is a complete metric space and d metrizes the narrow convergence of measures on $\mathcal{M}^+(\mathbb{R}^d)$.*

Proof of Theorem 3: We proceed in two steps: in step 1 we first prove that convergence in d_{BL} implies convergence in d, and then show in step 2 that any Cauchy sequence for d is Cauchy for d_{BL}. Recalling that $(\mathcal{M}^+,d_{BL})$ is complete, we see by step 2 that any Cauchy sequence in $(\mathcal{M}^+,d)$ is Cauchy in $(\mathcal{M}^+,d_{BL})$ and therefore converges in $(\mathcal{M}^+,d_{BL})$, so by step 1 it converges in $(\mathcal{M}^+,d)$ as well. Also, since any converging sequence is Cauchy this immediately implies that any d-converging sequence is also d_{BL}-converging. Finally, since d_{BL} metrizes the narrow convergence clearly so does d.

For convenience we split the first step into 1a and 1b: the former essentially reduces the problem to compactly supported (sequences of) measures, and the latter concludes by renormalizing to unit masses and comparing d with the Wasserstein distance $\mathcal{W}_2$.

Step 1a. Take any converging sequence $\rho^k\to\rho$ in $(\mathcal{M}^+,d_{BL})$. As the bounded-Lipschitz distance metrizes the narrow convergence, the masses converge $m^k\to m$, and the sequence $\{\rho^k\}$ is tight. We want to prove that it converges in $(\mathcal{M}^+,d)$.

Since only a countable number of concentric spheres in $\mathbb{R}^d$ can have a positive Radon measure, there exists a sequence $R_n\to\infty$ such that $\mathrm{d}\rho(\partial B_{R_n})=0$. By [14, Proposition 1.203], $\mathrm{d}\rho(A)\le\liminf_{k\to\infty}\mathrm{d}\rho^k(A)$ for any open set $A\subset\mathbb{R}^d$, and, moreover, $\mathrm{d}\rho(B_{R_n})=\lim_{k\to\infty}\mathrm{d}\rho^k(B_{R_n})$. So, $\mathrm{d}\rho|_{B_{R_n}}(A)\le\liminf_{k\to\infty}\mathrm{d}\rho^k|_{B_{R_n}}(A)$ for any open $A \subset \mathbb{R}^d$, and $\mathrm{d}\rho|_{B_{R_n}}(\mathbb{R}^d) = \lim_{k\to\infty} \mathrm{d}\rho^k|_{B_{R_n}}(\mathbb{R}^d)$. Consequently, by [14, Proposition 1.206],

$$\forall\, n \in \mathbb{N}: \quad \rho^k|_{B_{R_n}} \underset{k\to\infty}{\longrightarrow} \rho|_{B_{R_n}} \text{ narrowly.} \tag{2.6}$$

Owing to the tightness of $\{\rho^k\}_{k\in\mathbb{N}}$ and using the same construction as in the proof of Theorem 2 one easily deduces that

$$d(\rho|_{B_{R_n}}, \rho) + \sup_k d(\rho^k|_{B_{R_n}}, \rho^k) \underset{n\to\infty}{\longrightarrow} 0,$$

thus by triangular inequality

$$d(\rho^k, \rho) \leq d(\rho^k, \rho^k|_{B_{R_n}}) + d(\rho^k|_{B_{R_n}}, \rho|_{B_{R_n}}) + d(\rho|_{B_{R_n}}, \rho)$$

it suffices to prove that

$$\forall\, n \in \mathbb{N}: \qquad \rho^k|_{B_{R_n}} \underset{k\to\infty}{\longrightarrow} \rho|_{B_{R_n}} \text{ in } (\mathcal{M}^+, d). \tag{2.7}$$

Step 1b. From now on we argue for fixed n, and denote for simplicity $\rho^k_n = \rho^k|_{B_{R_n}}$ and $\rho_n = \rho|_{B_{R_n}}$. Letting $m^k_n = \mathrm{d}\rho^k(B_{R_n})$ and $m_n = \mathrm{d}\rho(B_{R_n})$ we see by (2.6) that $m^k_n \to m_n$ when $k \to \infty$. If the limit $\mathrm{d}\rho_n(\mathbb{R}^d) = m_n = 0$ then $\rho_n = 0$ in $\mathcal{M}^+$, so by Proposition 2.6 we have $d(\rho^k_n, \rho_n) = d(\rho^k_n, 0) = 2\sqrt{m^k_n} \to 0$ and (2.7) clearly holds. Otherwise using $m^n_k \to m_n > 0$ we have by the scaling Proposition 2.8

$$\begin{aligned} d(\rho^k_n, \rho_n) &\leq d\left(\rho^k_n, \frac{m_n}{m^k_n}\rho^k_n\right) + d\left(\frac{m_n}{m^k_n}\rho^k_n, \rho_n\right) \\ &= 2\left|\sqrt{m^k_n} - \sqrt{m_n}\right| + \sqrt{m_n}\, d\left(\frac{\rho^k_n}{m^k_n}, \frac{\rho_n}{m_n}\right), \end{aligned}$$

and it suffices to prove that the sequence of renormalized *probability* measures $\overline{\rho}^k_n = \frac{\rho^k_n}{m^k_n}$ converges to $\overline{\rho}_n = \frac{\rho_n}{m_n}$. Note that by construction $\overline{\rho}^k_n, \overline{\rho}_n$ are supported in a fixed ball B_{R_n}, so in particular they have uniformly bounded second moments. Applying Proposition 2.9 we see that $d(\overline{\rho}^k_n, \overline{\rho}_n) \leq \mathcal{W}_2(\overline{\rho}^k_n, \overline{\rho}_n)$, and recalling that $m^k_n \to m_n > 0$ and that $\rho^k_n \to \rho_n$ narrowly it is easy to see that $\overline{\rho}^k_n \to \overline{\rho}_n$ narrowly as well. Applying [20, Theorem 7.12] we conclude that $\mathcal{W}_2(\overline{\rho}^k_n, \overline{\rho}_n) \to 0$, whence $d(\overline{\rho}^k_n, \overline{\rho}_n) \to 0$ and (2.7) holds as desired.

Step 2. Let ρ_p, ρ_q be any two elements of a fixed Cauchy sequence $\{\rho_k\}$ in $(\mathcal{M}^+, d)$. Then they can be joined by a path $(\rho_t, u_t)_{t\in[0,1]}$ such that the corresponding energy

$$E_{p,q} \leq 2d^2(\rho_p, \rho_q).$$

Since Cauchy sequences are always bounded, the energies $E_{p,q}$ of such curves are bounded uniformly in $p, q \in \mathbb{N}$. Moreover by Corollary 2.4 the masses of these elements are bounded uniformly in $p, q \in \mathbb{N}$ and $t \in [0,1]$ by some M depending only on the sequence $\{\rho_k\}$. Now we infer from the fundamental estimate (2.2) that

$$\forall\, \phi \in \mathcal{C}^1_b(\mathbb{R}^d)\colon \left| \int_{\mathbb{R}^d} \phi(\mathrm{d}\rho_p - \mathrm{d}\rho_q) \right| \leq \sqrt{M}(\|\nabla\phi\|_\infty + \|\phi\|_\infty)\sqrt{E_{p,q}}.$$

By approximation it is easy to see that the same holds for any bounded and Lipschitz function, whence taking the supremum over ϕ we get

$$d_{BL}(\rho_p, \rho_q) \leq C d(\rho_p, \rho_q)$$

with $C = \sqrt{2M}$ depending on the sequence $\{\rho_k\}$ but not on the ranks p, q. This implies at once that $\{\rho_k\}_{k\in\mathbb{N}}$ is Cauchy in $(\mathcal{M}^+, d_{BL})$ and concludes the proof. ∎

Remark 2.10. Observe that we did not prove that the distances d and d_{BL} are *Lipschitz equivalent*, in the sense that the identity map $id : (\mathcal{M}^+, d) \to (\mathcal{M}^+, d_{BL})$ may not be globally bi-Lipschitz. This is actually impossible due to the different scalings: we know that $d(\lambda\rho_0, \lambda\rho_1) = \sqrt{\lambda} d(\rho_0, \rho_1)$ (Proposition 2.8), but it is easy to check that $d_{BL}(\lambda\rho_0, \lambda\rho_1) = \lambda d(\rho_0, \rho_1)$. However, an argument similar to that in step 2 of the proof of Theorem 3 above shows that $id : (\mathcal{M}^+, d) \to (\mathcal{M}^+, d_{BL})$ is Lipschitz on bounded sets.

2.3. Characterization of Lipschitz curves.

Theorem 4. *Let $\{\rho_t\}_{t\in[0,1]}$ be a L-Lipschitz curve w.r.t. our metric d. Then there exists a potential u_t such that*

$$\partial_t\rho_t + \operatorname{div}(\rho_t\nabla u_t) = \rho_t u_t \qquad \text{in } \mathcal{D}'((0,1)\times\mathbb{R}^d),$$

and

$$\|u_t\|_{H^1(\mathrm{d}\rho_t)} \leq L \qquad a.e.\ t \in (0,1).$$

Conversely if $t \mapsto \rho_t \in \mathcal{M}^+$ is a narrowly continuous curve such that $(\rho_t, u_t)_{t\in[0,1]}$ solves the non-conservative continuity equation and $\|u_t\|_{H^1(\mathrm{d}\rho_t)} \leq L$ for a.e. $t \in (0,1)$, then $t \mapsto \rho_t$ is L-Lipschitz with respect to the distance d.

Before proceeding with the proof we shall need the following technical lemma:

Lemma 2.11. *For any $\rho_0, \rho_1 \in \mathcal{M}^+$ and $\varepsilon > 0$ there exists a narrowly continuous curve $\rho_t \in \mathcal{C}_w([0,1], \mathcal{M}^+)$ connecting ρ_0 to ρ_1 and a potential $u_t(x)$, both depending on ε, solving the non-conservative continuity equation and such that*

$$\forall \tau \in [0,1]: \qquad d(\rho_0, \rho_\tau) \leq (1+\varepsilon) d(\rho_0, \rho_1) \tag{2.8}$$

and

$$d^2(\rho_0, \rho_1) \leq E[\rho; u] = \int_0^1 \left(\int_{\mathbb{R}^d} (|\nabla u_t|^2 + |u_t|^2) \mathrm{d}\rho_t \right) \mathrm{d}t \\ \leq (1+\varepsilon)^2 d^2(\rho_0, \rho_1). \tag{2.9}$$

Proof: If $\{\rho_t^k, u_t^k\}_{k\in\mathbb{N}}$ is any minimizing sequence in the definition of $d^2(\rho_0, \rho_1)$ then (2.9) is obviously satisfied for large $k \geq k_0$ and we only have to check that (2.8) holds as well if k is large enough. Note that for any $k \in \mathbb{N}$ and fixed $\tau \in [0,1]$, a simple time reparametrization $s = \tau t$ gives an admissible curve $(\overline{\rho}_s, \overline{u}_s)_{s\in[0,1]} := (\rho^k_{\tau s}, \tau u^k_{\tau s})_{s\in[0,1]}$ connecting ρ_0 to ρ_τ in time $s \in [0,1]$. By definition of our distance, changing variables, and because $\tau \leq 1$, we get

$$\begin{aligned} d^2(\rho_0, \rho_\tau) &\leq \int_0^1 \left(\int_{\mathbb{R}^d} (|\nabla \overline{u}_s|^2 + |\overline{u}_s|^2) \mathrm{d}\overline{\rho}_s \right) \mathrm{d}s \\ &= \tau \int_0^\tau \left(\int_{\mathbb{R}^d} (|\nabla u_t^k|^2 + |u_t^k|^2) \mathrm{d}\rho_t^k \right) \mathrm{d}t \\ &\leq \int_0^1 \left(\int_{\mathbb{R}^d} (|\nabla u_t^k|^2 + |u_t^k|^2) \mathrm{d}\rho_t \right) \mathrm{d}t. \end{aligned}$$

Then we see that for k large enough the last term above converges to $d^2(\rho_0, \rho_1)$ and the conclusion follows. ∎

Proof of Theorem 4: The argument is similar to [21, Theorem 13.8] and [1, Theorem 8.3.1]. Fix any $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. Since we assumed that

$$\forall\, t, s \in [0,1]: \qquad d(\rho_t, \rho_s) \leq L|t-s|$$

and because locally our distance is Lipschitz-stronger than the bounded-Lipschitz one (see Remark 2.10), we see that

$$t \mapsto \Psi(t) := \int_{\mathbb{R}^d} \psi \, \mathrm{d}\rho_t$$

is locally Lipschitz thus differentiable almost everywhere. Fix any time $t_0 \in [0,1]$ where Ψ is differentiable, and let $h_n \to 0$ with $t_0 + h_n \in [0,1]$. Taking $\varepsilon = \frac{1}{n}$, $\tilde\rho_0 = \rho_{t_0}$, and $\tilde\rho_1 = \rho_{t_0+h_n}$, let $(\tilde\rho^n_t, \tilde u^n_t)_{t\in[0,1]}$ be any curve from Lemma 2.11 connecting ρ_{t_0} to $\rho_{t_0+h_n}$ in time $t \in [0,1]$ while solving the non-conservative continuity equation and satisfying (2.8)(2.9). Using the Cauchy-Schwarz inequality (first in space and then in time) gives that

$$\begin{aligned}\left|\frac{\Psi(t_0+h_n)-\Psi(t_0)}{h_n}\right| &= \left|\frac{1}{h_n}\int_{\mathbb{R}^d}\psi(\mathrm{d}\rho_{t_0+h_n}-\mathrm{d}\rho_{t_0})\right| \\ &= \left|\frac{1}{h_n}\int_0^1\left(\int_{\mathbb{R}^d}(\nabla\psi\cdot\nabla\tilde u^n_t+\psi\tilde u^n_t)\mathrm{d}\tilde\rho^n_t\right)\mathrm{d}t\right| \\ &\le \underbrace{\left(\int_0^1\left(\int_{\mathbb{R}^d}(|\nabla\psi|^2+|\psi|^2)\mathrm{d}\tilde\rho^n_t\right)\mathrm{d}t\right)^{\frac12}}_{:=A_n} \\ &\qquad\cdot\underbrace{\frac{1}{h_n}\left(\int_0^1\left(\int_{\mathbb{R}^d}(|\nabla\tilde u^n_t|^2+|\tilde u^n_t|^2)\mathrm{d}\tilde\rho^n_t\right)\mathrm{d}t\right)^{\frac12}}_{:=B_n}.\end{aligned}$$

From the previous Lemma 2.11 and the definition of $\tilde\rho^n$ we first estimate, with $\varepsilon = \frac{1}{n}$ in (2.9),

$$\limsup_{n\to\infty} B_n \le \limsup_{n\to\infty}(1+1/n)\frac{d(\rho_{t_0},\rho_{t_0+h_n})}{h_n} \le L.$$

In order to handle the term A_n, let us define by duality the measures $\mu^n(t,x) \in \mathcal{M}^+((0,1)\times\mathbb{R}^d)$ as

$$\forall\phi\in\mathcal{C}_b((0,1)\times\mathbb{R}^d):\qquad \iint\limits_{(0,1)\times\mathbb{R}^d}\phi(t,x)\mathrm{d}\mu^n(t,x) := \int_0^1\left(\int_{\mathbb{R}^d}\phi(t,x)\mathrm{d}\tilde\rho^n_t(x)\right)\mathrm{d}t.$$

Using again Lemma 2.11, and in particular (2.8) with $\varepsilon = 1/n$, we see that for all fixed $t\in[0,1]$ there holds

$$d(\rho_{t_0},\tilde\rho^n_t) \le (1+1/n)d(\rho_{t_0},\rho_{t_0+h_n}) \to 0$$

when $n\to\infty$ (because the path ρ_t is Lipschitz). By Theorem 3 we get

$$\forall\, t\in[0,1]:\qquad \tilde\rho^n_t \to \rho_{t_0}\quad\text{narrowly.}$$

A simple application of Lebesgue's dominated convergence shows that $\mu^n(t,x)$ converges in the narrow topology to the measure μ similarly defined by

$$\forall \phi \in \mathcal{C}_b((0,1)\times\mathbb{R}^d):\qquad \iint\limits_{(0,1)\times\mathbb{R}^d} \phi(t,x)\mathrm{d}\mu(t,x) := \int_0^1 \left(\int_{\mathbb{R}^d} \phi(t,x)\mathrm{d}\rho_{t_0}(x)\right)\mathrm{d}t.$$

Since $\psi = \psi(x)$ and the limit μ actually does not act in time we get as a consequence

$$A_n \to \left(\int_0^1 \left(\int_{\mathbb{R}^d} (|\nabla\psi|^2 + |\psi|^2)\mathrm{d}\rho_{t_0}\right)\mathrm{d}t\right)^{\frac{1}{2}} = \left(\int_{\mathbb{R}^d} (|\nabla\psi|^2 + |\psi|^2)\mathrm{d}\rho_{t_0}\right)^{1/2} = \|\psi\|_{H^1(\mathrm{d}\rho_{t_0})}.$$

Thus for fixed $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ and at any point of differentiability $t_0 \in [0,1]$ we get that

$$\left|\frac{d\Psi}{dt}(t_0)\right| \leq L\|\psi\|_{H^1(\mathrm{d}\rho_{t_0})}.$$

The distribution $\zeta(t_0) \in \mathcal{D}'(\mathbb{R}^d)$ defined by $\langle \zeta(t_0), \psi\rangle = \frac{d\Psi}{dt}(t_0)$ is thus a continuous linear form on $H^1(\mathrm{d}\rho_{t_0})$ with $\|\zeta(t_0)\|_{H^{-1}(\mathrm{d}\rho_{t_0})} \leq L$. By the Riesz representation theorem in $H^1(\mathrm{d}\rho_{t_0})$ we can then find a unique potential $u_{t_0} \in H^1(\mathrm{d}\rho_{t_0})$ such that

$$\zeta(t_0) = (u_{t_0}, \cdot)_{H^1(\mathrm{d}\rho_{t_0})} \Rightarrow -\operatorname{div}(\rho_{t_0}\nabla u_{t_0}) + \rho_{t_0}u_{t_0} = \zeta(t_0) \quad \text{in } \mathcal{D}'(\mathbb{R}^d), \tag{2.10}$$

and

$$\|u_{t_0}\|_{H^1(\mathrm{d}\rho_{t_0})} = \|\zeta(t_0)\|_{H^{-1}(\mathrm{d}\rho_{t_0})} \leq L$$

(if $\rho_{t_0} = 0$ in $\mathcal{M}^+(\mathbb{R}^d)$ we simply define $u_{t_0} \equiv 0$).

Recalling that $\Psi(t) = \int_{\mathbb{R}^d} \psi\mathrm{d}\rho_t$ is a.e. differentiable, we see by (2.10) that

$$\frac{d}{dt}\int_{\mathbb{R}^d} \psi\mathrm{d}\rho_t = \int_{\mathbb{R}^d} (\nabla\psi\cdot\nabla u_t + \psi u_t)\mathrm{d}\rho_t \qquad \text{a.e. } t \in (0,1). \tag{2.11}$$

A subtle issue is that the "almost every $t \in (0,1)$" set of differentiability of Ψ might depend here on the choice of ψ. Arguing by density and separability as in [21, Theorem 13.8] we can conclude nonetheless that (2.11) holds for any $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, which is of course an admissible weak formulation for $\partial_t\rho_t +$

$\mathrm{div}(\rho_t \nabla u_t) = \rho_t u_t$. Moreover by construction we have that both $\nabla u_t(x)$ and $u_t(x)$ are $\mathrm{d}\rho_t(x)\mathrm{d}t$ measurable with

$$\|u_t\|_{H^1(\mathrm{d}\rho_t)} \leq L \qquad \text{a.e. } t \in (0,1).$$

Conversely, assume that $(\rho_t, u_t)_{t\in[0,1]}$ solves the non-conservative continuity equation with $\|u_t\|_{H^1(\mathrm{d}\rho_t)} \leq L$ for a.e. $t \in (0,1)$ and that $\rho \in \mathcal{C}_w([0,T]; \mathcal{M}^+)$. Then clearly the total energy $E \leq L^2$. For any $0 \leq t_0 \leq t_1$ it is easy to scale in time and connect ρ_{t_0}, ρ_{t_1} with cost $(t_1 - t_0) \int_{t_0}^{t_1} \|u_t\|^2_{H^1(\mathrm{d}\rho_t)} \mathrm{d}t$, thus $d^2(\rho_{t_0}, \rho_{t_1}) \leq L^2|t_1 - t_0|^2$ and the proof is complete. ■

2.4. Lower semicontinuity of the metric with respect to the weak-∗ topology.

Definition 2.12. Let (X, ϱ) be a metric space, σ be a locally compact Hausdorff topology on X. We say that the distance ϱ is lower semicontinuous with respect to σ if for all σ-converging sequences

$$x_k \overset{\sigma}{\to} x, \qquad y_k \overset{\sigma}{\to} y$$

one has

$$\varrho(x,y) \leq \liminf_{k\to\infty} \varrho(x_k, y_k).$$

Theorem 5. *The distance d is lower semicontinuous with respect to the weak-∗ topology on $\mathcal{M}^+$.*

Proof: Consider any two converging sequences

$$\rho_0^k \underset{k\to\infty}{\to} \rho_0, \qquad \rho_1^k \underset{k\to\infty}{\to} \rho_1 \qquad \text{weakly-}*$$

of finite Radon measures from $\mathcal{M}^+(\mathbb{R}^d)$. For each k, the endpoints ρ_0^k and ρ_1^k can be joined by an admissible narrowly continuous path $(\rho_t^k, u_t^k)_{t\in[0,1]}$ with energy

$$E[\rho^k; u^k] \leq d^2(\rho_0^k, \rho_1^k) + k^{-1}.$$

Due to weak-∗ compactness, the masses $m_0^k = d\rho_0^k(\mathbb{R}^d)$ and $m_1^k = d\rho_1^k(\mathbb{R}^d)$ are bounded uniformly in $k \in \mathbb{N}$. By Corollary 2.7 the set $\cup_{k\in\mathbb{N}}\{\rho_0^k, \rho_1^k\}$ is bounded in $(\mathcal{M}^+, d)$, thus the energies $E[\rho^k; u^k]$ and the masses $m_t^k = \mathrm{d}\rho_t^k(\mathbb{R}^d)$ are bounded uniformly in $k \in \mathbb{N}$ and $t \in [0,1]$

$$m_t^k \leq M \quad \text{and} \quad E[\rho^k; u^k] \leq \overline{E}.$$

By the (classical) Banach-Alaoglu theorem with $\mathcal{M}^+ \subset (\mathcal{C}_0)^*$, all the curves $(\rho_t^k)_{t\in[0,1]}$ lie in a fixed weak-∗ sequentially relatively compact set $\mathcal{K}_M = \{\rho \in$

$\mathcal{M}^+ : \mathrm{d}\rho(\mathbb{R}^d) \leq M\}$ uniformly in k, t. By the fundamental estimate (2.2) we get

$$\left| \int_{\mathbb{R}^d} \phi(\mathrm{d}\rho_t^k - \mathrm{d}\rho_s^k) \right| \leq \sqrt{M\overline{E}}|t-s|^{1/2}(\|\nabla\phi\|_\infty + \|\phi\|_\infty)$$

for all $\phi \in \mathcal{C}_b^1$, which implies

$$\forall\, t, s \in [0,1],\, \forall k \in \mathbb{N} : \qquad d_{BL}(\rho_s^k, \rho_t^k) \leq C|t-s|^{1/2}.$$

Invoking the completeness of $(\mathcal{M}^+(\mathbb{R}^d), d_{BL})$, the above uniform 1/2-Hölder continuity w.r.t. d_{BL}, the lower semicontinuity of d_{BL} with respect to the weak-$*$ convergence (Lemma 5.2 in the Appendix), and the fact that $\rho_t^k \in \mathcal{K}_M$, we conclude by a refined version of Arzelà-Ascoli theorem [1, Proposition 3.3.1] that there exists a d_{BL} (thus narrowly) continuous curve $(\rho_t)_{t\in[0,1]}$ connecting ρ_0 and ρ_1 such that

$$\forall t \in [0,1] : \qquad \rho_t^k \to \rho_t \quad \text{weakly-}* \tag{2.12}$$

along some subsequence $k \to \infty$ (not relabeled here). Let $Q := (0,1) \times \mathbb{R}^d$ and $\mu^k(t,x)$ be the measure on Q defined by duality as

$$\forall\, \phi \in \mathcal{C}_c(Q) : \qquad \int_Q \phi(t,x)\mathrm{d}\mu^k(t,x) = \int_0^1 \left(\int_{\mathbb{R}^d} \phi(t,.)\mathrm{d}\rho_t^k \right) \mathrm{d}t.$$

Exploiting the pointwise convergence (2.12) and the uniform bound on the masses $m_t^k \leq M$, a simple application of the Lebesgue dominated convergence guarantees that

$$\mu^k \to \mu^0 \qquad \text{weakly-}* \text{ in } \mathcal{M}^+(Q),$$

where the finite measure $\mu^0(t,x) \in \mathcal{M}^+(Q)$ is defined by duality in terms of the weak-$*$ limit $\rho_t = \lim \rho_t^k$ (as was μ^k in terms of ρ_t^k). Then the energy bound reads

$$\|u^k\|^2_{L^2(0,1;H^1(\mathrm{d}\rho_t^k)} = E[\rho^k; u^k] \leq d^2(\rho_0^k, \rho_1^k) + k^{-1} \leq C.$$

We are going to apply a refined version of the Banach-Alaoglu theorem, Proposition 5.3 in the appendix, in the space

$$X = \mathcal{C}_c^1(Q).$$

Namely, we set

$$\|\phi\| = \|\nabla\phi\|_{L^\infty(Q)} + \|\phi\|_{L^\infty(Q)},$$

$$\|\phi\|_k = \left(\int_Q \left(|\nabla\phi|^2 + |\phi|^2\right)\, \mathrm{d}\mu^k \right)^{1/2}, \qquad k = 0, 1, \ldots,$$

and

$$\varphi_k(\phi) = \int_Q (\nabla u^k \cdot \nabla \phi + u^k \phi)\, \mathrm{d}\mu^k, \qquad k = 1, 2, \ldots .$$

The separability of $\mathcal{C}_c^1(Q)$, the weak-$*$ convergence of μ^k, uniform boundedness of the masses of $\mu^k(Q) \leq M$, and the Cauchy-Schwarz inequality imply that the hypotheses of Proposition 5.3 are met with

$$c_k \leq \|u^k\|_k = \sqrt{E[\rho^k; u^k]} \leq \sqrt{d^2(\rho_0^k, \rho_1^k) + k^{-1}}.$$

Consequently, there exists a continuous functional φ_0 on the space $(X, \|\cdot\|_0)$ such that up to a subsequence

$$\forall \phi \in \mathcal{C}_c^1(Q)\colon \int_0^1 \left(\int_{\mathbb{R}^d} \left\{ \nabla u_t^k \cdot \nabla \phi(t,.) + u_t^k \phi(t,.) \right\} \mathrm{d}\rho_t^k \right) \mathrm{d}t \underset{k\to\infty}{\to} \varphi_0(\phi)$$

with moreover

$$\|\varphi_0\|_{(X,\|\cdot\|_0)^*} \leq \liminf_{k\to\infty} d(\rho_0^k, \rho_1^k). \tag{2.13}$$

Let $N_0 \subset X$ be the kernel of the seminorm $\|\cdot\|_0$. By the Riesz representation theorem, the dual $(X, \|\cdot\|_0)^* = (X/N_0, \|\cdot\|_0)^*$can be isometrically identified with the completion $\overline{X/N_0}$ of X/N_0 with respect to $\|\cdot\|_0$, which is exactly $L^2(0,1; H^1(\mathrm{d}\rho_t))$. As a consequence there exists $u \in L^2(0,T; H^1(\mathrm{d}\rho_t))$ such that

$$\varphi_0(\phi) = \int_Q (\nabla u \cdot \nabla \phi + u\phi)\ \mathrm{d}\mu^0 = \int_0^1 \left(\int_{\mathbb{R}^d} (\nabla u \cdot \nabla \phi + u\phi)\, \mathrm{d}\rho_t \right) \mathrm{d}t$$

and

$$\|u\|_{L^2(0,1;H^1(\mathrm{d}\rho_t))} = \|\varphi_0\|_{(X,\|\cdot\|_0)^*},$$

and it is straightforward to check that (ρ, u) is an admissible curve joining ρ^0, ρ^1 (the above convergence is enough to take the limit in the weak formulation of the non-conservative continuity equation). Recalling (2.13), it remains to take into account that by the definition of our distance

$$d^2(\rho_0, \rho_1) \leq E[\rho; u] = \|u\|^2_{L^2(0,1;H^1(\mathrm{d}\rho_t))} = \|\varphi_0\|^2_{(X,\|\cdot\|_0)^*} \leq \liminf_{k\to\infty} d^2(\rho_0^k, \rho_1^k).$$

$\blacksquare$

2.5. Existence of geodesics. We are now in position of proving an important result, namely the existence and characterization of geodesics for our metric structure. We give two proofs: one is more constructive and inspired by the optimal transport theory, and another one is shorter and more abstract.

Theorem 6. $(\mathcal{M}^+, d)$ *is a geodesic space, and for all* $\rho_0, \rho_1 \in \mathcal{M}^+$ *the infimum in* (2.1) *is always a minimum. Moreover this minimum is attained for a (narrowly continuous) curve* ρ *such that* $d(\rho_t, \rho_s) = |t-s| d(\rho_0, \rho_1)$ *and a potential* $u \in L^2(0,1; H^1(\mathrm{d}\rho_t))$ *such that* $\|u_t\|_{H^1(\mathrm{d}\rho_t)} = \mathrm{cst} = d(\rho_0, \rho_1)$ *for a.e.* $t \in [0,1]$.

In Section 3 we will show that $(\mathcal{M}^+, d)$ can be viewed as a formal Riemannian manifold, with tangent plane $T_\rho \mathcal{M}^+ = \{\frac{d\rho}{dt} = -\operatorname{div}(\rho \nabla u) + \rho u : u \in H^1(\mathrm{d}\rho)\}$ and $\left\| \frac{d\rho}{dt} \right\|_{T_\rho \mathcal{M}^+} = \|u\|_{H^1(\mathrm{d}\rho)}$. In this perspective Theorem 6 can be interpreted as $\left\| \frac{d\rho_t}{dt} \right\|_{T_{\rho_t} \mathcal{M}^+} = \mathrm{cst} = d(\rho_0, \rho_1)$, which should be expected along constant-speed geodesics $(\rho_t)_{t\in[0,1]}$ connecting ρ_0, ρ_1.

Proof 1 via time reparametrization: Fix any $\rho_0, \rho_1 \in \mathcal{M}^+$ and let $\left(\rho^k_t, u^k_t\right)_{t\in[0,1]}$ be a minimizing sequence in

$$E[\rho^k; u^k] = \int_0^1 \|u^k_t\|^2_{H^1(\mathrm{d}\rho^k_t)} \mathrm{d}t \underset{k\to\infty}{\to} d^2(\rho_0, \rho_1).$$

We first claim that we can assume without loss of generality that

$$\forall\, 0 \le t_0 \le t_1 \le 1 : \qquad d^2(\rho^k_{t_0}, \rho^k_{t_1}) \le |t_0 - t_1|^2 E[\rho^k; u^k]. \tag{2.14}$$

Indeed using a simple arclength reparametrization (Lemma 5.4 in the Appendix) we can assume that $\|u^k_t\|^2_{H^1(\mathrm{d}\rho^k_t)} = \mathrm{cst} = E[\rho^k; u^k]$ is constant in time for all $k \in \mathbb{N}$. Scaling in time $t = t_0 + (t_1 - t_0)s$ as before, we get by definition of our distance and Remark 2.3 that $d^2(\rho^k_{t_0}, \rho^k_{t_1}) \le |t_1 - t_0| \int_{t_0}^{t_1} \|u^k_t\|^2_{H^1(\mathrm{d}\rho^k_t)} \mathrm{d}t = (t_1 - t_0)^2 E[\rho^k; u^k]$.
Now because the energies $E[\rho^k; u^k]$ are bounded (2.14) shows that the sequence of curves $\{t \mapsto \rho^k_t\}$ are uniformly 1/2-Hölder continuous with respect to our metric d, and arguing as before the masses $m^k_t \le M$ are bounded uniformly in $t \in [0,1]$ and $k \in \mathbb{N}$, thus ρ^k_t lie in a fixed weakly-$*$ relatively sequentially compact set $\mathcal{K}_M = \{\rho : \mathrm{d}\rho(\mathbb{R}^d) \le M\}$. By Theorem 5 we know that d is lower semi-continuous with respect to weak-$*$ convergence, and applying again

the refined Arzelà-Ascoli theorem [1, Proposition 3.3.1] we conclude that there exists a d-continuous (thus narrowly continuous) curve ρ such that

$$\forall\, t \in [0,1]: \qquad \rho^k_t \to \rho_t \quad \text{weakly-}*\,.$$

From (2.14) this also implies

$$\forall\, 0 \leq t_0 \leq t_1 \leq 1: \\ d^2(\rho_{t_0}, \rho_{t_1}) \leq \liminf_{k\to\infty} |t_0 - t_1|^2 E[\rho^k; u^k] = |t_1 - t_0|^2 d^2(\rho_0, \rho_1).$$

By the triangular inequality we conclude that in fact

$$\forall\, t_0, t_1 \in [0,1]: \qquad\qquad d(\rho_{t_0}, \rho_{t_1}) = |t_1 - t_0| d(\rho_0, \rho_1),$$

and in particular the curve $t \mapsto \rho_t$ is L-Lipschitz with $L = d(\rho_0, \rho_1)$.

Applying Theorem 4 we see that there exists a potential u_t solving the non-conservative continuity equation such that

$$\|u_t\|_{H^1(\mathrm{d}\rho_t)} \leq L \text{ a. e. } t \in (0,1),$$

and because

$$L^2 = d^2(\rho_0, \rho_1) \leq E[\rho; u] = \int_0^1 \|u_t\|^2_{H^1(\mathrm{d}\rho_t)} \mathrm{d}t \leq L^2$$

we conclude that in fact $\|u_t\|_{H^1(\mathrm{d}\rho_t)} = L = d(\rho_0, \rho_1)$ a.e. $t \in (0,1)$. ∎

2 via midpoints: We first observe from the definition of our distance that $(\mathcal{M}^+, d)$ is a length space (e.g. by an application of the almost midpoint criterion, [5, Theorem 2.4.16(2)].). By Corollary 2.4 and the (classical) Banach-Alaoglu theorem, the weak-$*$ topology is d-boundedly compact. Now Lemma 5.5 (variant of the Hopf-Rinow theorem in the Appendix) together with Theorem 3 and Theorem 5 imply that $(\mathcal{M}^+, d)$ is a geodesic space. The existence and claimed properties of the minimizer follow from Theorem 4 exactly as in Proof 1. ∎

Remark 2.13. The geodesics can be non-unique, see Section 3.5.

3. Underlying geometry

We show here that our distance d endows $\mathcal{M}^+(\mathbb{R}^d)$ with a formal Riemannian structure. We closely follow Otto's approach [17], which was originally developed for the optimal transport of probability measures in the Wasserstein metric space $(\mathcal{P}_2, \mathcal{W}_2)$. Refer the reader to the particularly clear exposition in [20, 21], see also the classical monograph [1] for an extension to abstract metric spaces.

3.1. Otto's Riemannian formalism. Let us recall that if $\mathcal{M}$ is a smooth differential manifold then the tangent plane $T_x\mathcal{M}$ at a point $x \in M$ can be viewed as the vector space of tangent vectors $\frac{dx_t}{dt}\big|_{t=0}$ of all $\mathcal{C}^1$ curves $t \mapsto x_t \in \mathcal{M}$ passing through $x(0) = x$.

By Theorem 4 and Theorem 6 we know that, given any fixed endpoint $\rho \in \mathcal{M}^+$, there always exists a constant-speed geodesic ρ_t connecting $\rho|_{t=0} = \rho$ to arbitrary $\nu \in \mathcal{M}^+$ and parametrized as

$$\begin{cases} \partial_t \rho_t = -\operatorname{div}(\rho_t \nabla u_t) + \rho_t u_t & \text{in } \mathcal{D}'((0,1) \times \mathbb{R}^d) \\ \|u_t\|_{H^1(\mathrm{d}\rho_t)} = \mathrm{cst} = d(\rho, \nu) & \text{a.e. } t \in (0,1). \end{cases}$$

Since $t \mapsto \rho_t$ is a constant-speed geodesic in $(\mathcal{M}^+, d)$ it should be a $\mathcal{C}^1$ curve, and, with a slight abuse of notations, we naturally identify the $\partial_t \rho_t$ distributional term above with a tangent vector $\zeta_t = \frac{d\rho_t}{dt} \in T_{\rho_t}\mathcal{M}^+$. According to $\zeta_t = -\operatorname{div}(\rho_t \nabla u_t) + \rho u_t$ we see that the tangent vector $\zeta_t = \frac{d\rho_t}{dt}$ corresponds to a potential $u_t \in H^1(\mathrm{d}\rho_t)$. Assuming for simplicity that $\frac{d\rho_t}{dt}, u_t$ can be somehow evaluated at $t = 0^+$, we see that any geodesic passing through the fixed endpoint $\rho|_{t=0} = \rho$ gives a tangent vector $\frac{d\rho_t}{dt}\Big|_{t=0}$, which thus corresponds to some $u|_{t=0} \in H^1(\mathrm{d}\rho)$. This strongly suggests to define the tangent space in terms of potentials as

$$T_\rho \mathcal{M}^+ := \{\zeta = -\operatorname{div}(\rho \nabla u) + \rho u : \ u \in H^1(\mathrm{d}\rho)\}$$

and

$$\|\zeta\|_{T_\rho \mathcal{M}^+} := \|u\|_{H^1(\mathrm{d}\rho)} = \left(\int_{\mathbb{R}^d} (|\nabla u|^2 + |u|^2) \mathrm{d}\rho \right)^{1/2}.$$

Observe that, given ζ, the elliptic PDE

$$-\operatorname{div}(\rho \nabla u) + \rho u = \zeta \tag{3.1}$$

is coercive in $H^1(\mathrm{d}\rho)$, so the correspondence between tangent vectors ζ and potentials u is at least formally uniquely defined. By polarization this automatically defines the Riemannian metric on $T_\rho \mathcal{M}^+$

$$\left\langle \frac{d\rho}{dt_1}, \frac{d\rho}{dt_2} \right\rangle_\rho := \langle u_1, u_2 \rangle_{H^1(\mathrm{d}\rho)} = \int_{\mathbb{R}^d} (\nabla u_1 \cdot \nabla u_2 + u_1 u_2) \mathrm{d}\rho,$$

where the subscript ρ in the left-hand side highlights the dependence on the base point $\rho \in \mathcal{M}^+$ in the weighted $H^1(\mathrm{d}\rho)$ scalar product. A formal but useful observation is that

$$\left\langle \frac{d\rho}{dt_1}, \frac{d\rho}{dt_2} \right\rangle_\rho = \int_{\mathbb{R}^d} (\nabla u_1 \cdot \nabla u_2 + u_1 u_2) \mathrm{d}\rho$$
$$= \int_{\mathbb{R}^d} u_2 \zeta_1 \mathrm{d}x = \int_{\mathbb{R}^d} u_2 \partial_{t_1} \rho \, \mathrm{d}x = \int_{\mathbb{R}^d} u_1 \zeta_2 \mathrm{d}x = \int_{\mathbb{R}^d} u_1 \partial_{t_2} \rho \, \mathrm{d}x. \quad (3.2)$$

Observe that with this formalism our definition (2.1) simply reads

$$d^2(\rho_0, \rho_1) = \inf \left\{ \int_0^1 \left\| \frac{d\rho_t}{dt} \right\|^2_{T_{\rho_t}\mathcal{M}^+} \mathrm{d}t \right\},$$

where the infimum is taken over all admissible curves connecting ρ_0, ρ_1. This is nothing but the classical Lagrangian formulation of (squared) distances in a reasonably smooth Riemannian manifold.

Remark 3.1. It is worth pointing out that this formal analysis is meaningless if $\rho = 0 \in \mathcal{M}^+$. In other words $\rho = 0$ is a degenerate extremal point in our pseudo-manifold $\mathcal{M}^+$, and we cannot identify a reasonable tangent space $T_0\mathcal{M}^+$.

3.2. Differential calculus in $(\mathcal{M}^+, d)$ and induced dynamics. Now that we defined a Riemannian structure on $\mathcal{M}^+$ in terms of the distance d, one would like to differentiate functionals $\mathcal{F} : \mathcal{M}^+ \to \mathbb{R}$, or in other words compute gradients "$\mathrm{grad}_d \mathcal{F}(\rho)$" with respect to this Riemannian structure. This again has already been done in the setting of optimal transport in the Wasserstein space $(\mathcal{P}_2, \mathcal{W}_2)$ (see [17, 20] and the references therein), and later generalized to abstract metric spaces [1]. The approach is once more very similar, and still formal.

Let us recall that if $(\mathcal{M}, g)$ is a smooth Riemannian manifold and $\mathcal{F} : \mathcal{M} \to \mathbb{R}$, the Riemannian metric tensor g and the Riesz representation theorem allow to pull back the differential $D\mathcal{F}(x) \in T_x^*\mathcal{M}$, which is a cotangent vector, to a unique tangent vector $\mathrm{grad}\,\mathcal{F}(x) \in T_x\mathcal{M}$: For all $\mathcal{C}^1$ curves $t \mapsto x_t$ passing through $x|_{t=0} = x$ with tangent vector $\frac{dx_t}{dt}\big|_{t=0} = \zeta$ the gradient $\mathrm{grad}\,\mathcal{F}$ is defined by duality via the chain rule

$$\left. \frac{d}{dt} \mathcal{F}(x_t) \right|_{t=0} = \langle D\mathcal{F}(x), \zeta \rangle_{T_x^*\mathcal{M}, T_x\mathcal{M}} = g_x(\mathrm{grad}\,\mathcal{F}(x), \zeta).$$

Mimicking this computation and washing out all smoothness issues, it is easy to deduce here that the gradients of functionals $\mathcal{F} : \rho \in \mathcal{M}^+ \to \mathbb{R}$ are at least formally given by

$$\begin{aligned} \operatorname{grad}_d \mathcal{F}(\rho) &= -\operatorname{div}\left(\rho\nabla\tfrac{\delta\mathcal{F}}{\delta\rho}\right) + \rho\tfrac{\delta\mathcal{F}}{\delta\rho}, \\ \|\operatorname{grad}_d \mathcal{F}(\rho)\|_{T_\rho\mathcal{M}^+} &= \left\|\tfrac{\delta\mathcal{F}}{\delta\rho}\right\|_{H^1(\mathrm{d}\rho)}, \end{aligned} \tag{3.3}$$

where $\frac{\delta\mathcal{F}}{\delta\rho}$ denotes the first variation with respect to the usual L^2 Euclidean structure (e.g., if $\mathcal{F}(\rho) = \int U(x)d\rho(x)$, then $\frac{\delta\mathcal{F}}{\delta\rho} = U$).

With gradients available one can obtain induced dynamics on $(\mathcal{M}^+, d)$, the main examples being gradient flows and Hamiltonian flows. More precisely, given some functional $\mathcal{F}(\rho)$ on $\mathcal{M}^+$ the $\mathcal{F}$-gradient flow is defined as

$$\frac{d\rho}{dt} = -\operatorname{grad}_d \mathcal{F}(\rho).$$

Similarly, Hamiltonian systems read

$$\frac{d\rho}{dt} = -J\cdot\operatorname{grad}_d \mathcal{F}(\rho),$$

where J is some Hamiltonian antisymmetric operator (i. e. a closed 2-form on $T_\rho\mathcal{M}^+$). As an application of this Riemannian formalism we shall study in Section 4 a particular gradient flow, and exploit the formal structure to derive new long-time convergence results by means of entropy-entropy dissipation inequalities.

Second order differential calculus can also be developed: the Hessian can be formally defined as

$$\langle \operatorname{Hess}_d \mathcal{F}(\rho)\cdot\zeta, \zeta\rangle_\rho = \left.\frac{d^2}{dt^2}\right|_{\text{geod}} \mathcal{F}(\rho_t), \tag{3.4}$$

where differentiation should be performed along a geodesic path ρ_t passing through ρ with tangent vector ζ at time $t = 0$. In order to exploit this formula one needs to compute the geodesics with prescribed initial position and velocity, and the first issue is thus the very existence of these objects. As for the practical computation itself, consider for example $\mathcal{F}(\rho) = \int_{\mathbb{R}^d}\mathrm{d}\rho$: the second time derivative is

$$\int_{\mathbb{R}^d} \partial^2_{tt}\rho_t = \int_{\mathbb{R}^d} \partial_t(-\nabla\cdot(\rho_t\nabla u_t) + \rho_t u_t) = \int_{\mathbb{R}^d} \partial_t(\rho_t u_t)$$
$$= \int_{\mathbb{R}^d} u_t\partial_t\rho_t + \rho_t\partial_t u_t = \int_{\mathbb{R}^d} \rho_t(|\nabla u_t|^2 + u_t^2 + \partial_t u_t).$$

Thus the second and key issue is to find an equation yielding an expression for $\partial_t u_t$ or $\partial_t(\rho_t u_t)$ in terms of ρ, u and their spatial derivatives. The next two subsections are devoted to these questions. We will see that in the definition of the Hessian it is natural to replace the metric geodesics by some objects which we call *trajectory geodesics*. These possess fine minimization properties, exist for given initial value ρ and velocity ζ, and are described by nice PDEs, which, in particular, provide the required expressions of $\partial_t u_t$ and $\partial_t(\rho_t u_t)$ in terms of ρ and u.

3.3. Trajectory geodesics. In order to introduce the concept of trajectory geodesic, we use a heuristic particle interpretation of the considered dynamics, which better illustrates the underlying ideas.

Think of movement of charged particles in $\mathbb{R}^d$, whose mass is fixed but whose charge can possibly evolve according to some a priori given law which we describe later. Assume first that we have a finite number N of moving particles with position $x_t(i)$ and charge $k_t(i)$. In terms of densities and curves in $\mathcal{M}^+$ this dynamics can be formalized as

$$\rho_t = \sum_{i=1}^{N} k_t(i) r_t(i) \qquad \text{with} \qquad r_t(i) := \delta_{x_t(i)}.$$

Given a reasonably smooth potential $u_t(x)$, the function ρ_t is the solution of the non-conservative continuity equation $\partial_t\rho_t + \operatorname{div}(\rho_t\nabla u_t) = \rho_t u_t$ if and only if

$$\frac{d}{dt}x_t(i) = \nabla u_t(x_t(i))$$

and

$$\frac{d}{dt}(\log k_t(i)) = u_t(x_t(i))$$

along the trajectory of each particle (this claim can be checked, e.g., by employing Lemma 5.1, see however Remark 3.2). Define the energy of each individual particle by

$$E_i = \int_0^1 k_t(i) \left[\left| \frac{d}{dt}(\log k_t(i)) \right|^2 + \left| \frac{d}{dt} x_t(i) \right|^2 \right] dt. \tag{3.5}$$

Then at least formally the total energy sums as

$$E = \int_0^1 \left(\int_{\mathbb{R}^d} (|\nabla u_t|^2 + |u_t|^2) \mathrm{d}\rho_t \right) \mathrm{d}t = \sum_{i=1}^N E_i. \tag{3.6}$$

If the pair $(\rho_t, u_t)_{t\in[0,1]}$ were to be a geodesic (in the sense of Theorem 6), then it should minimize the total energy E for fixed ρ_0 and ρ_1, and thus also minimize each of the E_i's: more precisely,

$$\begin{aligned} E_i = \min \int_0^1 \tilde{k}_t \left[\left| \frac{d}{dt}(\log \tilde{k}_t) \right|^2 + \left| \frac{d}{dt} \tilde{x}_t \right|^2 \right] dt : \\ \tilde{x}_0(i) = x_0(i),\ \tilde{x}_1(i) = x_1(i), \quad \tilde{k}_0(i) = k_0(i), \tilde{k}_1(i) = k_1(i). \end{aligned} \tag{3.7}$$

It turns out to be relevant to replace the condition of being a geodesic by the one of being a minimizer of each E_i in the sense (3.7).

Remark 3.2. The above discussion is legitimate, e.g., if we have strictly N different particles for all times, i-e when the trajectories of the particles do not overlap and hence there is no mass splitting or coalescence. Much more general scenarios allowing for trajectory crossing are certainly possible, provided that the potential u loses spatial smoothness and the non-conservative continuity equation is understood in the weak sense. This is of course a delicate issue, which we shall ignore below by staying at the formal level.

We carry out now a similar formal analysis in a more general framework, considering general measures ρ_t as the superposition of possibly infinitely many indivisible infinitesimal particles.

Let $(\rho_t, u_t)_{t\in[0,1]}$ be a generic admissible path in the sense of Definition 2.1, and assume that the velocity field ∇u_t is smooth enough. Let T_t be t-flow generated by the characteristic ODE $\frac{dy}{dt} = \nabla u_t(y)$, and define $r_t : \mathbb{R}^d \to \mathbb{R}$ as the pushforward

$$r_t = T_t \# \rho_0. \tag{3.8}$$

Then r_t is a curve in $\mathcal{M}^+$ with constant mass $m_t = \mathrm{d}\rho_0(\mathbb{R}^d)$, and

$$\partial_t r_t + \mathrm{div}(r_t \nabla u_t) = 0. \tag{3.9}$$

As in the previous discrete setting, we think of ρ_t as the charge density of moving charged particles and decompose it as a product

$$\rho_t(x) = r_t(x)k_t(x),$$

where r_t defined by (3.8) is the particle density at fixed t (density of particles without thinking of their charge) and $k_t(x)$ is the charge of the particles, initially normalized as $k_0(x) \equiv 1$. When the particles move along their trajectories, they continuously change their charge according to the law

$$\partial_t \rho_t + \operatorname{div}(\rho_t \nabla u_t) = \rho_t u_t. \tag{3.10}$$

Denote the Lagrangian time derivative by

$$\frac{D}{Dt} = \partial_t + \nabla u \cdot \nabla.$$

Multiplying (3.9) by k and subtracting the result from (3.10) it is easy to obtain $r_t \frac{Dk_t}{Dt} = r_t k_t u_t$, thus formally

$$\frac{D}{Dt}(\log k_t) = u_t \tag{3.11}$$

as in the previous discrete setting of finitely many particles.

Now switch to the Lagrangian framework and let $x_t(X) := T_t(X)$ be the trajectory of a particle starting at $x_0(X) = X$. From (3.11) we see that

$$\frac{D}{Dt} x_t = \nabla u_t(x_t) \quad \text{and} \quad \frac{D}{Dt}(\log k_t(x_t)) = u_t(x_t). \tag{3.12}$$

Thus, the infinitesimal energy along the trajectory of a single particle $x_t(X)$ is proportional to

$$E_X = \int_0^1 k_t \left[\left| \frac{D}{Dt}(\log k_t) \right|^2 + \left| \frac{D}{Dt} x_t \right|^2 \right] \mathrm{d}t,$$

which is of course the continuous Lagrangian counterpart of (3.5). For fixed X the functions $x(t) = x_t(X)$ and $k(t) = k_t(x_t(X))$ depend only on time, $\frac{D}{Dt}$ becomes the ordinary time derivative, and the energy reads

$$E_X(x, k) = \int_0^1 \left(\frac{|k'|^2}{k} + k|x'|^2 \right) dt. \tag{3.13}$$

The boundary conditions for x, k are naturally

$$x_0 = X, \; x_1 = T_1(X), \qquad k_0 = 1, \; k_1 = \frac{\rho_1(T_1(X))}{r_1(T_1(X))}. \tag{3.14}$$

We temporarily forget about the origin of the functions x, k and consider the minimization problem

$$E_X(x,k) = \min E_X(\tilde{x},\tilde{k})\colon \tilde{x},\tilde{k} \text{ satisfy (3.14) } \tilde{k}_t > 0 \text{ for } 0 < t < 1. \tag{3.15}$$

Definition 3.3. An admissible curve $(\rho_t, u_t)_{t\in[0,1]}$ is called a trajectory geodesic if the corresponding x, k solve the minimization problem (3.15) for every $X \in \mathbb{R}^d$.

Given a trajectory geodesic and X, we can write the set of Euler-Lagrange equations for (3.15):

$$\frac{2k''}{k} - \frac{|k'|^2}{k^2} = |x'|^2, \tag{3.16}$$

$$(kx')' = 0. \tag{3.17}$$

From (3.17) we see that kx' is conserved along the trajectories, that is, the trajectories are straight lines

$$\left(\frac{x'}{|x'|}\right)' = 0 \tag{3.18}$$

but the speed $|x'|$ is non-constant and is equal to cst/k along the trajectories. Furthermore, (3.16) can be rewritten in the form

$$2(\log k)'' + |(\log k)'|^2 = |x'|^2.$$

Recalling (3.12) we find that $2\frac{D}{Dt}u_t + |u_t|^2 = |\nabla u_t|^2$, or equivalently

$$\partial_t u_t = -\frac{1}{2}(|u_t|^2 + |\nabla u_t|^2). \tag{3.19}$$

Moreover, (3.12) and (3.18) yield $\frac{D}{Dt}\left(\frac{\nabla u_t}{|\nabla u_t|}\right) = 0$, i-e

$$\partial_t\left(\frac{\nabla u_t}{|\nabla u_t|}\right) + \nabla u_t \cdot \nabla\left(\frac{\nabla u_t}{|\nabla u_t|}\right) = 0. \tag{3.20}$$

In order to calculate Hessians as in (3.4), we need a procedure to construct a (trajectory) geodesic starting from any initial measure ρ with prescribed velocity $\zeta \in T_\rho\mathcal{M}^+$. Let u be the corresponding initial potential (formally) determined by $\zeta = -\operatorname{div}(\rho\nabla u) + \rho u$, and consider the solution $(\rho_t, u_t)_{t\in[0,\delta]}$ to the following problem:

$$\begin{cases} \partial_t \rho_t = -\operatorname{div}(\rho_t \nabla u_t) + \rho_t u_t, \\ \partial_t u_t = -\frac{1}{2}(|u_t|^2 + |\nabla u_t|^2), \\ \rho_0 = \rho, \ u_0 = u. \end{cases} \tag{3.21}$$

A crucial and somewhat surprising observation is that u_t satisfies (3.20): the reason is that (3.19) always implies (3.20). Indeed, we first compute

$$\frac{D}{Dt}(\nabla u_t) = -\frac{1}{2}\nabla(|u_t|^2 + |\nabla u_t|^2) + \nabla^2 u_t \cdot \nabla u_t = -u_t \nabla u_t,$$

thus

$$\begin{aligned} \frac{D}{Dt}\left(\frac{\nabla u_t}{|\nabla u_t|}\right) &= \frac{\frac{D}{Dt}(\nabla u_t)}{|\nabla u_t|} - \nabla u_t \frac{(\frac{D}{Dt}\nabla u_t) \cdot \nabla u_t}{|\nabla u_t|^3} \\ &= -\frac{u_t \nabla u_t}{|\nabla u_t|} + \nabla u_t \frac{u_t \nabla u_t \cdot \nabla u_t}{|\nabla u_t|^3} = 0. \end{aligned}$$

Repeating and rearranging the argument above, we deduce that for each X the corresponding pair x, k solves the Euler-Lagrange equations (3.16),(3.17). Observe from (3.13) that the infinitesimal energy $E_X(x,k)$ is convex in its arguments x, k, k': thus at least formally any solution of the Euler-Lagrange equations must be the unique minimizer, which by definition means that (ρ_t, u_t) is a trajectory geodesic.

We have just observed that an admissible curve (ρ_t, u_t) is a trajectory geodesic if and only if it satisfies both the Hamilton-Jacobi equation (3.19) and the non-conservative continuity equation. It sounds plausible that a trajectory geodesic is (at least formally) always a geodesic (since it simultaneously minimizes the energy along every trajectory), but this claim is not necessarily true, see Section 3.5. The underlying obstacles are subtle and fundamental, as similar factors undermine the Riemannian structure of the Wasserstein space $(\mathcal{P}_2, \mathcal{W}_2)$, see Remark 3.4. However, a weaker statement holds true: the trajectory geodesics have constant metric speed in the sense that

$$\left\| \frac{d\rho_t}{dt} \right\|_{T_{\rho_t}\mathcal{M}^+} = \|u_t\|_{H^1(\mathrm{d}\rho_t)} = \text{cst}.$$

Indeed, using (3.10), (3.19) and (3.2) we find at least formally for any trajectory geodesic

$$\frac{d}{dt}\|u_t\|^2_{H^1(\mathrm{d}\rho_t)} = 2\left\langle u_t, \frac{du_t}{dt}\right\rangle_{H^1(\mathrm{d}\rho_t)} + \int_{\mathbb{R}^d}(|\nabla u_t|^2 + |u_t|^2)\partial_t\rho_t \,\mathrm{d}x$$
$$= 2\int_{\mathbb{R}^d}\partial_t u_t \partial_t \rho_t \,\mathrm{d}x + \int_{\mathbb{R}^d}(|\nabla u_t|^2 + |u_t|^2)\partial_t\rho_t \,\mathrm{d}x = 0.$$

Remark 3.4. A straightforward computation shows that the geodesic problem (3.21) implies

$$\begin{cases} \partial_t\rho_t = -\operatorname{div}(\rho_t\nabla u_t) + \rho_t u_t, \\ \partial_t(\rho_t\nabla u_t) = -\operatorname{div}(\rho_t\nabla u_t \otimes \nabla u_t), \\ \rho_0 = \rho, \ u_0 = u. \end{cases}$$

This problem is very similar to the set of geodesic equations for optimal time-dependent transport [20] of probability measures in the Wasserstein framework,

$$\begin{cases} \partial_t\rho_t = -\operatorname{div}(\rho_t\nabla u_t), \\ \partial_t(\rho_t\nabla u_t) = -\operatorname{div}(\rho_t\nabla u_t \otimes \nabla u_t), \\ \rho_0 = \rho, \ u_0 = u, \end{cases} \tag{3.22}$$

which is a particular case of the sticky particles system [4]. A trajectory geodesic formalism similar to the one above can of course be developed in the Wasserstein space $(\mathcal{P}_2, \mathcal{W}_2)$ (with $k \equiv 1$, no charge is considered). The Wasserstein geodesics are trajectory geodesics in that setting. The reason is that the trajectories of the dynamical optimal transport problem have constant velocity [20], thus the respective Euler-Lagrange equality $x'' = 0$ is satisfied. The converse statement is not always true. Note that since the exponential map $\rho_t = \exp_{\rho_0}(t\zeta)$ for the Wasserstein geodesics cannot be properly defined for all velocities, the classical Otto calculus [20, 21, 17] uses variants of the system (3.22) for the calculation of the Hessian. Hence, from the perspective of our formalism, the Otto calculus implicitly employs trajectory geodesics (without defining them) in place of metric geodesics.

3.4. Hessians and λ-convexity. The above formalism suggests the following

Definition 3.5. The Hessian of a functional $\mathcal{F} : \mathcal{M}^+ \to \mathbb{R}$ with respect to our structure is formally defined by the formula

$$\langle \mathrm{Hess}_d\, \mathcal{F}(\rho)\cdot\zeta, \zeta\rangle_\rho = \frac{d^2}{dt^2}\mathcal{F}(\rho_t). \tag{3.23}$$

Here the path ρ_t is determined by (3.21) where the initial potential u is related with the tangent vector ζ via (3.1).

Then we can give

Definition 3.6. A functional $\mathcal{F}$ on $\mathcal{M}^+$ is convex (resp. λ-convex, $\lambda \in \mathbb{R}$) with respect to our structure provided

$$\left\langle \mathrm{Hess}_d\, \mathcal{F}(\rho) \cdot \zeta, \zeta \right\rangle_\rho \geq 0$$
$$\left(\text{resp., } \left\langle \mathrm{Hess}_d\, \mathcal{F}(\rho) \cdot \zeta, \zeta \right\rangle_\rho \geq \lambda \left\langle \zeta, \zeta \right\rangle_\rho = \lambda \|u\|^2_{H^1(\mathrm{d}\rho)}\right)$$

for all ρ, ζ.

Calculations of the Hessians are rather tedious, and as an example we only present here the final result for the *internal energy* [20] determined for absolutely continuous measures by

$$\mathcal{E}(\rho) = \int_{\mathbb{R}^d} E(\rho(x))\mathrm{d}x,$$

where E is a measurable non-negative density of the internal energy. For this functional,

$$\begin{aligned}\left\langle \mathrm{Hess}_d\, \mathcal{E}(\rho) \cdot \zeta, \zeta \right\rangle_\rho = \int_{\mathbb{R}^d} \Bigg\{ & P(\rho)\Gamma_2(u) + P_2(\rho)|\Delta u|^2 \\ & - (2P_2(\rho) + P(\rho))u\Delta u + \left(Q_2(\rho) - \frac{1}{2}Q(\rho) - P_2(\rho)\right)|\nabla u|^2 \\ & + \left(Q_2(\rho) - \frac{1}{2}Q(\rho)\right)|u|^2 \Bigg\}.\end{aligned}$$

Here we have used the notations

$$P(\rho) = E'(\rho)\rho - E(\rho), \qquad P_2(\rho) = P'(\rho)\rho - P(\rho),$$
$$Q(\rho) = E'(\rho)\rho, \qquad Q_2(\rho) = Q'(\rho)\rho,$$
$$\Gamma_2(u) = \sum_{i,j=1}^d |\partial^2_{x_i x_j} u|^2.$$

3.5. Distance and geodesics between two one-point measures. We finish this section by a highly illustrative example, which in particular will show that geodesics may be non-unique and that trajectory geodesics are not necessarily metric ones. Namely, we are going to describe the geodesics and calculate our distance d between two one-point measures $\rho_0 = k_0\delta_{x_0}$ and $\rho_1 = k_1\delta_{x_1}$ with $x_0, x_1 \in \mathbb{R}^d$, $k_0, k_1 > 0$, and $\xi := |x_0 - x_1| > 0$. Observe that the case when one of the masses vanishes fits into Proposition 2.6, and the case when $\xi = 0$ fits into Proposition 2.8.

In order to send ρ_0 to ρ_1, a first natural strategy is to treat the dynamics as the movement of a sole indivisible particle $\rho_t = k(t)\delta_{x(t)}$ moving from x_0 to x_1, as described in Section 3.3. The corresponding energy consumption for this *transport* strategy is

$$E_{tr}(k_0, k_1, \xi) = \min\left\{ \int_0^1 \tilde{k}_t \left[\left| \frac{d}{dt}(\log \tilde{k}_t) \right|^2 + \left| \frac{d}{dt}\, \tilde{x}_t \right|^2 \right] \mathrm{d}t : \begin{array}{c} \tilde{x}_0 = x_0,\ \tilde{x}_1 = x_1, \\ \tilde{k}_0 = k_0, \tilde{k}_1 = k_1, \\ \tilde{k}_t > 0 \text{ for } 0 < t < 1 \end{array} \right\}, \tag{3.24}$$

cf. (3.7). A laborious but rather elementary calculus of the variations shows that this minimization problem has a solution for

$$\xi < 2\pi,$$

which has the form

$$k_t = a(t-b)^2 + c, \tag{3.25}$$

$$x_t = x_0 + 2\frac{x_1 - x_0}{\xi}\left[\arctan((t-b)\sqrt{a/c}) + \arctan(b\sqrt{a/c})\right], \tag{3.26}$$

$$E_{tr}(k_0, k_1, \xi) = 4a, \tag{3.27}$$

where

$$a = k_0 + k_1 - 2\cos(\xi/2)\sqrt{k_0k_1}, \tag{3.28}$$

$$b = \frac{k_0 - \cos(\xi/2)\sqrt{k_0k_1}}{k_0 + k_1 - 2\cos(\xi/2)\sqrt{k_0k_1}}, \tag{3.29}$$

$$c = \frac{k_0k_1\sin^2(\xi/2)}{k_0 + k_1 - 2\cos(\xi/2)\sqrt{k_0k_1}}. \tag{3.30}$$

Observe that the solution (3.26) ceases to exist whenever $\xi = |x_1 - x_0| \geq 2\pi$.

Another natural strategy, made possible by the reaction term $\rho_t u_t$ in the non-conservative continuity equation, is to kill all the mass at x_0 while simultaneously creating mass at x_1. This means that we treat the dynamics as the evolution of two distinct particles $\rho_t = k_t(1)\delta_{x_1} + k_t(2)\delta_{x_2}$ with fixed positions but non-constant charges, with

$$k_0(1) = k_0, k_1(1) = 0 \quad \text{and} \quad k_0(2) = 0, k_1(2) = k_1.$$

The corresponding energy intake for this *stationary* strategy can be calculated using Proposition 2.6, viz.

$$E_{st} = 4(k_0 + k_1). \tag{3.31}$$

Note that this strategy also works for $\xi \geq 2\pi$, contrarily to the previous transport one.

We now make the ansatz that a geodesic joining two one-point measures $\rho_0 = k_0\delta_{x_0}, \rho_1 = k_1\delta_{x_1}$ should use a mixture of these transport and/or stationary strategies, corresponding to an evolution of (at most) three particles: the first one travels between x_0 and x_1; the second one stays in x_0 and its charge goes to zero; the third is always in x_1, and its charge is zero at the initial moment and then it starts growing (this geometry may require a loss of smoothness of the driving potential u, cf. Remark 3.2). That is,

$$\rho_t = \sum_{i=1}^{3} k_t(i) r_t(i),$$
$$r_t(i) = \delta_{x_t(i)},$$
$$\begin{array}{ll} x_0(1) = x_0,\ x_1(1) = x_1, & k_0(1) + k_0(2) = k_0, \\ x_0(2) = x_1(2) = x_0, & k_1(1) + k_1(3) = k_1, \\ x_0(3) = x_1(3) = x_1, & k_1(2) = 0, k_0(3) = 0. \end{array}$$

The heuristics behind this ansatz is that indivisible infinitesimal particles which have non-zero charge at the initial and final times are considered to be in the first bulk, the ones which have zero charge at time $t = 1$ constitute the second bulk, and the remaining ones go to the third bulk. Denoting the independent parameters $k_0(1)$ and $k_1(1)$ by γ_0 and γ_1, resp., we discover that the total energy intake is

$$
\begin{aligned}
E(\gamma_0, \gamma_1) &= E_1 + E_2 + E_3 \\
&= 4[\gamma_0 + \gamma_1 - 2\cos(\xi/2)\sqrt{\gamma_0\gamma_1}] + 4(k_0 - \gamma_0) + 4(k_1 - \gamma_1) \\
&= 4[k_0 + k_1 - 2\cos(\xi/2)\sqrt{\gamma_0\gamma_1}]
\end{aligned}
$$

(here we implicitly restrict to $\xi < 2\pi$ so that both strategies are feasible). For $\xi < \pi$ one can check that the minimum of $E(\gamma_0, \gamma_1)$ is achieved for $\gamma_0 = k_0, \gamma_1 = k_1$, i. e. the pure transport strategy is optimal. For $\xi > \pi$, the energy minimum is achieved at $\gamma_0 = \gamma_1 = 0$, i. e. the geodesic only uses the pure stationary strategy. At the critical value $\xi = \pi$, any combination of the two strategies (i-e any choice of the free parameters $\gamma_0 \in [0, k_0]$ and $\gamma_1 \in [0, k_1]$) does an optimal job, and in particular we see that the geodesics are definitely not unique. Furthermore, for $\pi < \xi < 2\pi$, the pure transport curve determined by (3.25)–(3.30) is a trajectory geodesic by construction, but not a metric geodesic (since the stationary strategy is less expensive in terms of energy).

Let us now consider the particular case of probability measures $k_0 = k_1 = 1$ and compare our (squared) distance $d^2(\delta_{x_0}, \delta_{x_1})$ with the (squared) Wasserstein distance $\mathcal{W}_2^2(\delta_{x_0}, \delta_{x_1})$, which is clearly equal to $\xi^2 = |x_1 - x_0|^2$ in this example. Remember that by Proposition 2.9 our distance does not a priori exceed the Wasserstein one. For $\xi \geq \pi$ this is trivial since $\mathcal{W}_2^2 = \xi^2 \geq \pi^2 > 8 = d^2$. On the other hand, for small $\xi < \pi$ we get $\mathcal{W}_2^2 - d^2 = \xi^2 - (8 - 8\cos(\xi/2)) = o(\xi^3)$, meaning that our distance and the Wasserstein one agree at least at order two, i-e

$$
\mathcal{W}_2(\delta_{x_0}, \delta_{x_1}) - d(\delta_{x_0}, \delta_{x_1}) = o(|x_1 - x_0|^2)
$$

at short range (both $\mathcal{W}_2$ and d being of order $\xi = |x_1 - x_0|$).

Remark 3.7. This threshold effect at $\xi = \pi$ is natural since the cost of the stationary strategy (3.31) is independent of the distance between the supports, while the pure transport strategy(3.25)–(3.30) really does depend on how far we want to transport. This might be relevant in the context of image processing: our distance non-linearly interpolates between "Wasserstein-like/transport" at close distance, and basically total variation for large distances. A similar effect was discovered in [19] for the generalized Wasserstein distance considered therein. The bounded-Lipschitz distance also has such property.

4. Application to a model of population dynamics

As an application of the abstract ideas from the previous sections we consider a problem in spatial population dynamics. As originally proposed in [9], the model describes a population of organisms inhabiting a domain $\Omega \subset \mathbb{R}^d$, whose macroscopic distribution $\rho(t,x)$ evolves according to reproduction and migration. At each point $x \in \Omega$ lies a prescribed quantity of resources $m(x)$, and the local per capita reproduction rate is assumed to be proportional to the *fitness* $m(x) - \rho$. In the absence of displacement of individuals the local population dynamics is thus given by the heterogeneous logistic equation $\frac{d\rho}{dt} = (m(x)-\rho)\rho$. The spatial migration is then dictated by the desire of individuals to maximize their local fitness in order to reproduce as best as possible, corresponding to an advection with local velocity $v = \nabla(m - \rho)$. Assuming that the animals cannot exit through walls and thus imposing a no-flux boundary condition, one obtains the following *fitness-driven* model:

$$\begin{cases} \partial_t \rho = \operatorname{div}(\rho \nabla(\rho - m)) + \rho(m - \rho), & x \in \Omega, t > 0, \\ \rho \frac{\partial \rho}{\partial \nu} = \rho \frac{\partial m}{\partial \nu}, & x \in \partial\Omega,\ t > 0, \\ \rho(0,x) = \rho_0(x), & x \in \Omega. \end{cases} \tag{4.1}$$

Here $\Omega \subset \mathbb{R}^d$ is a bounded, connected, open domain with smooth boundary, ν is the outer normal to $\partial\Omega$, $m(x) \geq 0$ is a given function (assumed to be at least $\mathcal{C}^{2,\alpha}(\overline{\Omega})$), and $\rho(t,x)$ is the unknown density of the population. We consider nonnegative solutions of (4.1), so the initial datum $\rho_0(x) \in \mathcal{C}(\overline{\Omega})$ is supposed to be nonnegative as well.

In the previous sections we constructed our distance d between measures on the whole space $\mathbb{R}^d$ by means of non-conservative continuity equations $\partial_t \rho_t + \operatorname{div}(\rho_t \nabla u_t) = \rho_t u_t$, which allowed to consider mass changes driven by the reaction term $\rho_t u_t$. Working in bounded domains Ω and imposing a no-flux condition on the velocity field ∇u_t, so that the mass can *only* change through the reaction term, the exact same construction formally gives a distance d between measures in Ω (here we ignore all delicate issues related to boundary conditions and remain formal). As a result all the Riemannian formalism in Section 3 remains valid in the case of bounded domain, in particular formula (3.3) allows to compute gradients of functionals $\mathcal{F} : \mathcal{M}_b^+(\Omega) \to \mathbb{R}$. Anticipating that (4.1) enjoys a comparison principle and that solutions to the Cauchy problem remain nonnegative, we rather consider $\rho(t,x)$ as curves $t \mapsto \rho_t \in \mathcal{M}_b^+(\Omega)$. Using (3.3) it is easy to check, again formally, that (4.1) can be written as the gradient flow

$$\frac{d\rho}{dt} = -\operatorname{grad}_d \mathcal{E}(\rho), \qquad \mathcal{E}(\rho) = \frac{1}{2}\int_\Omega |\rho - m|^2 \mathrm{d}x \tag{4.2}$$

with respect to the distance d (just write $\frac{\delta\mathcal{E}}{\delta\rho} = \rho - m$). In the sequel we shall often refer to $\mathcal{E}$ as the *entropy* functional, as is common for gradient flows in metric spaces. In the view of (4.2) it is clear that the system is driven by the fitness $m - \rho$, and it is natural to expect long-time convergence to the (unique) least-entropy stationary solution $\rho(t) \underset{t\to\infty}{\to} m$ (in some sense to be precised shortly). In fact this was proved in [11] by means of comparison principles, but without convergence rate. Our main result in this section states that convergence is exponential in time:

Theorem 7. *Assume that $m(x) > 0$ in $\overline{\Omega}$. Then for any nonnegative initial datum $\rho_0 \in \mathcal{C}(\overline{\Omega})$, $\rho_0 \not\equiv 0$, there is $\gamma \equiv \gamma(\rho_0, \Omega, m) > 0$ such that the weak solution of* (4.1) *satisfies*

$$\forall t \geq 0: \qquad \|\rho(t) - m\|_{L^2(\Omega)} \leq e^{-\gamma t}\|\rho_0 - m\|_{L^2(\Omega)}.$$

The precise dependence of the rate γ on the initial datum ρ_0 will be derived along the proof, see Remark 4.3 below.

Denoting the *entropy dissipation* along trajectories by

$$\mathcal{D}(\rho) := -\frac{d}{dt}\mathcal{E}(\rho) = \|\operatorname{grad}_d \mathcal{E}(\rho)\|^2,$$

a classical way to retrieve exponential convergence $\rho(t) \to \rho_*$ for gradient flows is to obtain an *entropy-entropy production* inequality

$$\mathcal{D}(\rho) \geq \lambda(\mathcal{E}(\rho) - \mathcal{E}(\rho_*)) \tag{4.3}$$

for some $\lambda > 0$. This implies trend to equilibrium in the entropy sense $\mathcal{E}(\rho(t)) \to \mathcal{E}(\rho_*)$ with exponential rate λ, and then using a Csiczár-Kullback inequality one usually deduces that $\rho(t) \to \rho_*$ in some sense (typically $\|\rho(t) - \rho_*\|_{L^1} \to 0$). We refer to [20, Chapter 9] for introductory material on this topic.

Note that the least entropy stationary solution is here $\rho_* = m(x)$ and that $\mathcal{E}(\rho_*) = 0$. Formally multiplying (4.1) by $\rho - m$ and integrating, the dissipation can be formally computed here as

$$\mathcal{D}(\rho) = \|\rho - m\|^2_{H^1(\mathrm{d}\rho)} = \int_\Omega (|\nabla(\rho - m)|^2 + |\rho - m|^2)\rho\, \mathrm{d}x,$$

which is of course consistent with $\mathcal{D} = \|\operatorname{grad}_d \mathcal{E}\|^2$ and the Riemannian formalism $|\zeta|_{T_\rho\mathcal{M}^+} = \|u\|_{H^1(\mathrm{d}\rho)}$ from Section 3. Observe that, due to the very definition of $\mathcal{E}(\rho) = \frac{1}{2}\|\rho - m\|^2_{L^2(\Omega)}$, convergence in the entropy sense $\mathcal{E}(\rho(t)) \to \mathcal{E}(m) = 0$ already implies convergence $\rho(t) \to m$ in the stronger $L^2(\Omega)$ sense. As a consequence we can dispense from Csiczár-Kullback inequalities and we only need an entropy-entropy production one as in (4.3). The latter classically holds (uniformly in the initial data) as soon as the entropy is λ-convex for some $\lambda > 0$ (in the sense of Definition 3.6, see Section 3 for second-order calculus). This can be tested here at least formally using the geodesic equations (3.21). Unfortunately, the specific entropy $\mathcal{E}(\rho) = \int_\Omega |\rho - m|^2$ does not appear to be λ-convex for any $\lambda > 0$. We use instead a generalized Beckner inequality (Theorem 8), and as a consequence we only obtain exponential convergence for rates depending on the initial data as in Theorem 7.

From now on we claim mathematical rigor. Following [11] weak solutions are defined as

Definition 4.1. A nonnegative function

$$\rho \in C([0,+\infty); L^1(\Omega)) \cap L^2_{\mathrm{loc}}([0,+\infty); H^1(\Omega)) \cap L^\infty_{\mathrm{loc}}([0,+\infty) \times \overline{\Omega}) \quad (4.4)$$

is called a *global weak solution* of the Cauchy problem (4.1) if it satisfies the identity

$$\begin{aligned} - \int_0^T \int_\Omega \rho \partial_t \varphi \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \rho(t,x)\varphi(t,x)\,\mathrm{d}x - \int_\Omega \rho_0(x)\varphi(0,x)\,\mathrm{d}x \\ = - \int_0^T \int_\Omega \rho \nabla(\rho - m)\cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t \\ + \int_0^T \int_\Omega \rho(m-\rho)\varphi\,\mathrm{d}x\,\mathrm{d}t \quad (4.5) \end{aligned}$$

for any $T > 0$ and $\varphi \in C^1([0,T] \times \overline{\Omega})$.

The existence as well as uniqueness for weak solutions of problem (4.1) were established in [11], as well as a comparison principle. We start by rigorously deriving the dissipation (entropy production) *equality*:

Lemma 4.2. *Any weak solution actually satisfies*

$$\rho \in \mathcal{C}([0,\infty); L^2(\Omega)).$$

The dissipation

$$\mathcal{D}(t) := \int_\Omega \left(|\nabla(\rho(t) - m)|^2 + |\rho(t) - m|^2 \right) \rho(t) \, \mathrm{d}x$$

belongs to $L^1_{\mathrm{loc}}[0, \infty)$, *and*

$$\frac{d}{dt}\mathcal{E}(t) = -\mathcal{D}(t)$$

in the sense of distributions $\mathcal{D}'(0, \infty)$.

Proof: Since $\rho, m \in L^2_{\mathrm{loc}}([0, T); H^1(\Omega))$ and the function ρ is uniformly bounded, it is straightforward to check that $\mathcal{D} \in L^1_{\mathrm{loc}}[0, \infty)$. By density of $\mathcal{C}^1(\overline{\Omega})$ in $H^1(\Omega)$ and testing $\phi(t, x) = \eta(t)\psi(x)$ with $\eta \in \mathcal{C}^\infty_c(0, T)$ in the definition of weak solutions it is easy to see that $\partial_t \rho \in L^2_{\mathrm{loc}}([0, \infty); (H^1(\Omega))^*)$. By embedding $H^1(\Omega) \subset\subset L^2(\Omega) \subset (H^1(\Omega))^*$ one classically obtains $\rho \in \mathcal{C}([0, T]; L^2(\Omega))$ for all $T > 0$.

Since we know now that

$$\partial_t(\rho - m) = \partial_t \rho \in L^2_{\mathrm{loc}}([0, \infty); (H^1(\Omega))^*),$$

we can legitimately take $\rho - m \in L^2_{\mathrm{loc}}([0, \infty); H^1(\Omega))$ as a test function in (4.1), from which it classically follows that

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(\rho(t)) &= \frac{d}{dt}\left(\frac{1}{2}\|\rho(t) - m\|^2_{L^2(\Omega)}\right) = \langle \partial_t \rho, \rho - m \rangle_{(H^1)^*, H^1} \\ &= -\int_\Omega \left(|\nabla(\rho(t) - m)|^2 + |\rho(t) - m|^2 \right) \rho(t) = -\mathcal{D}(t) \end{aligned}$$

for a.e. $t \in (0, \infty)$ and in $L^1_{\mathrm{loc}}([0, \infty))$ as desired. ∎

Proof of Theorem 7: In order to apply our generalized Beckner inequality and retrieve an entropy-entropy production inequality, we first need to bound the L^1 mass of $\rho(t)$ from below. To this end let $\underline{\rho}_0(x) = \min\{m(x), \rho_0(x)\}$ and define $\underline{\rho}(t, x)$ to be the unique solution of the Cauchy problem with initial datum $\underline{\rho}_0$. Applying the comparison principle [11, Lemma 4.2] we have that

$$\underline{\rho}(t, x) \leq \rho(t, x) \qquad \text{for a.e. } t, x$$

and it suffices to show that $\int_\Omega \underline{\rho}(t) \, \mathrm{d}x \geq c_0 > 0$. Because 0 and $m(x)$ are stationary solutions of (4.1) (thus respectively sub and supersolutions) the comparison principle ensures that $0 \leq \underline{\rho}(t,x) \leq m(x)$. Testing $\phi \equiv 1$ in the weak formulation we get $\frac{d}{dt}\int_\Omega \underline{\rho}(t)\,\mathrm{d}x = \int_\Omega \underline{\rho}(t)(m-\underline{\rho}(t))\,\mathrm{d}x \geq 0$, whence

$$\int_\Omega \rho(t) \geq \int_\Omega \underline{\rho}(t) \geq \int_\Omega \underline{\rho}_0 =: c_0 > 0$$

as desired.

Since we are considering $m(x) > 0$ we can apply the generalized Beckner inequality, Theorem 8 in the appendix, to get

$$\begin{aligned}\Phi\left(c_0\right)\int_\Omega |\rho(t)-m|^2 &\leq \Phi\left(\|\rho(t)\|_{L^1(\Omega)}\right)\cdot\int_\Omega |\rho(t)-m|^2\\ &\leq \int_\Omega \left(|\nabla(\rho(t)-m)|^2+|\rho(t)-m|^2\right)\rho(t).\end{aligned}$$

Wence by Lemma 4.2

$$\frac{d}{dt}\mathcal{E}(t) = -\mathcal{D}(t) \leq -2\gamma\mathcal{E}(t) \qquad \text{a.e. } t>0$$

with $\gamma = \Phi(c_0)$. By standard Grönwall arguments we conclude that

$$\mathcal{E}(t) \leq e^{-2\gamma t}\mathcal{E}(0),$$

which in turn yields $\|\rho(t)-m\|_{L^2(\Omega)} \leq e^{-\gamma t}\|\rho_0 - m\|_{L^2(\Omega)}$ and achieves the proof. ∎

Remark 4.3. From the proof above it is clear that the rate $\gamma > 0$ in Theorem 7 only depends on the initial datum through $c_0 = \int_\Omega \min\{\rho_0, m\}\,\mathrm{d}x$.

5. Appendix

Solution of the non-conservative continuity equation.

Lemma 5.1. *Let $\rho_0 \in \mathcal{M}^+$, let $v_t(x)$ be a uniformly Lipschitz velocity field on $[0,\tau)\times\mathbb{R}^d$, let $(T_t)_{t\in[0,\tau)}$ be the t-flow map on $\mathbb{R}^d$ generated by the characteristics $\frac{dy}{dt} = v_t(y)$, and let $h_t(x)$ be a bounded uniformly continuous function on $[0,\tau)\times\mathbb{R}^d$. Then*

$$\rho_t = \left(\exp\int_0^t h_s(T_s\circ T_t^{-1}(\cdot))\mathrm{d}s\right)\cdot(T_t\#\rho_0)$$

defines a unique solution to

$$\partial_t\rho_t + \operatorname{div}(\rho_t v_t) = \rho_t h_t, \tag{5.1}$$

starting from ρ_0 and belonging to the space $\mathcal{C}_w([0,\tau);\mathcal{M}^+)$.

Proof: Let

$$k_t(x) = \exp\left\{\int_0^t h_s(T_s \circ T_t^{-1}(x))\mathrm{d}s\right\}.$$

Then k is bounded, uniformly continuous, and satisfies

$$\partial_t k_t + v_t \cdot \nabla k_t = k_t h_t.$$

Observe that solutions of (5.1) can be obtained from solutions μ_t of

$$\partial_t \mu_t + \mathrm{div}(\mu_t v_t) = 0$$

multiplying $\rho_t = k_t \mu_t$, and vice-versa. It remains to apply [20, Theorem 5.34]. ∎

Lower-semicontinuity of the bounded-Lipschitz distance.

Lemma 5.2. *The bounded-Lipschitz distance d_{BL} is lower semicontinuous with respect to the weak-$*$ topology.*

Proof: Let

$$\rho_0^k \underset{k\to\infty}{\to} \rho_0, \qquad \rho_1^k \underset{k\to\infty}{\to} \rho_1 \qquad \text{weakly-}*$$

be converging sequences of finite Radon measures from $\mathcal{M}^+(\mathbb{R}^d)$. Remember that the supremum in the definition of d_{BL} would not change if we restrict it to smooth compactly supported functions. Fix $\epsilon > 0$ and find $\phi_* \in C_c^\infty(\mathbb{R}^d)$ such that $\|\phi_*\|_\infty + \|\nabla\phi_*\|_\infty \leq 1$ and

$$d_{BL}(\rho_0, \rho_1) \leq \int_{\mathbb{R}^d} \phi_*(\mathrm{d}\rho_0 - \mathrm{d}\rho_1) + \epsilon.$$

Then

$$\begin{aligned}
&\liminf_{k\to\infty} d_{BL}(\rho_0^k, \rho_1^k) \\
&\qquad = \liminf_{k\to\infty} \sup_{\|\phi\|_\infty + \|\nabla\phi\|_\infty \leq 1} \int_{\mathbb{R}^d} \phi(\mathrm{d}\rho_0^k - \mathrm{d}\rho_1^k) \\
&\qquad \geq \liminf_{k\to\infty} \int_{\mathbb{R}^d} \phi_*(\mathrm{d}\rho_0^k - \mathrm{d}\rho_1^k) = \int_{\mathbb{R}^d} \phi_*(\mathrm{d}\rho_0 - \mathrm{d}\rho_1) \\
&\qquad \geq d_{BL}(\rho_0, \rho_1) - \epsilon,
\end{aligned}$$

and we conclude by letting $\epsilon \to 0$. ∎

A refined version of the Banach-Alaoglu theorem.

Proposition 5.3. *Let $(X, \|\cdot\|)$ be a separable normed vector space. Assume that there exists a sequence of seminorms $\{\|\cdot\|_k\}$ ($k = 0, 1, 2, \dots$) on X such that for every $x \in X$ one has*

$$\|x\|_k \le C\|x\|$$

with a constant C independent of k, x, and

$$\|x\|_k \underset{k\to\infty}{\to} \|x\|_0.$$

*Let φ_k ($k = 1, 2, \dots$) be a uniformly bounded sequence of linear continuous functionals on $(X, \|\cdot\|_k)$, resp., in the sense that**

$$c_k := \|\varphi_k\|_{(X,\|\cdot\|_k)^*} \le C.$$

Then the sequence $\{\varphi_k\}$ admits a converging subsequence $\varphi_{k_n} \to \varphi_0$ in the weak-$*$ topology of X^*, and

$$\|\varphi_0\|_{(X,\|\cdot\|_0)^*} \le c_0 := \liminf_k c_k.$$

Proof: Take any countable dense set $D = \{x_m\}$ in $(X, \|\cdot\|)$. Observe that it is automatically dense in any of the spaces $(X, \|\cdot\|_k)$ since their topologies are weaker. By the Cantor procedure, there is a subsequence $\{\varphi_{k_n}\}$ such that

$$\forall x_m \in D: \qquad \exists \lim_{n\to\infty} \varphi_{k_n}(x_m) =: \varphi_0(x_m)$$

for every fixed m. Without loss of generality and up to extraction of a further sequence we can assume that c_{k_n} converges to $c_0 = \liminf c_k$. In the sequel, we abuse the notation and write k instead of k_n. Passing to the limit $k \to \infty$ in $|\varphi_k(x_m)| \le c_k\|x_m\|_k$ we deduce that

$$\forall x_m \in D: \qquad |\varphi_0(x_m)| \le c_0\|x_m\|_0.$$

By the Hahn-Banach theorem, φ_0 can be extended to a bounded functional on $(X, \|\cdot\|_0)$ with the norm

$$\|\varphi_0\|_{(X,\|\cdot\|_0)^*} \le c_0,$$

and we claim that

$$\forall x \in X: \qquad \lim_{k\to\infty} \phi_k(x) = \phi_0(x).$$

*We recall that the continuous dual of a seminormed space is a Banach space, see [3].

Indeed for any $x \in X$ there is a sequence $\{x_m\} \subset D$ such that $\|x_m - x\| \to 0$, and

$$\begin{aligned}
&|\varphi_k(x) - \varphi_0(x)| \\
&\qquad \leq |\varphi_k(x) - \varphi_k(x_m)| + |\varphi_k(x_m) - \varphi_0(x_m)| + |\varphi_0(x_m) - \varphi_0(x)| \\
&\qquad\quad \leq c_k\|x - x_m\|_k + |\varphi_k(x_m) - \varphi_0(x_m)| + c_0\|x_m - x\|_0 \\
&\qquad\qquad \leq C\|x - x_m\| + |\varphi_k(x_m) - \varphi_0(x_m)|.
\end{aligned}$$

For every $\varepsilon > 0$ there exists $m_0 = m_0(\varepsilon)$ such that $C\|x - x_{m_0}\| \leq \varepsilon/2$. Then we find k_0, depending only on m_0, such that for all $k \geq k_0$ we have $|\varphi_k(x_{m_0}) - \varphi_0(x_{m_0})| \leq \varepsilon/2$, thus $|\varphi_k(x) - \varphi_0(x)| \leq \varepsilon$ for $k \geq k_0(\varepsilon)$, and the proof is complete. ∎

Arc-length reparametrization.

Lemma 5.4. *Let $\rho_0, \rho_1 \in \mathcal{M}^+$ and $(\rho_t, u_t)_{t\in[0,1]}$ be an admissible path connecting ρ_0, ρ_1 with finite energy $E[\rho; u]$. Then there exists an admissible time reparametrization $(\overline{\rho}_t, \overline{u}_t)_{t\in[0,1]}$ with energy $E[\overline{\rho}; \overline{u}] \leq E[\rho; u]$ such that $\|\overline{u}\|^2_{H^1(\mathrm{d}\overline{\rho}_t)} = \mathrm{cst} = E[\overline{\rho}; \overline{u}]$ for a.e. $t \in [0, 1]$.*

Proof: The argument is adapted from [1, Lemma 1.1.4]. Observing that $t \mapsto \|u_t\|_{H^1(\mathrm{d}\rho_t)} \in L^2(0, 1) \subset L^1(0, 1)$ we see that

$$\mathtt{s}(t) = \int_0^t \|u_\tau\|_{H^1(\mathrm{d}\rho_\tau)}\,\mathrm{d}\tau$$

is absolutely continuous, nondecreasing, and

$$\mathtt{s}(0) = 0, \qquad \mathtt{s}(1) = \int_0^1 \|u_\tau\|_{H^1(\mathrm{d}\rho_\tau)}\,\mathrm{d}\tau =: L.$$

The left-continuous inverse

$$s \in [0, L] \mapsto \mathtt{t}(s) := \min\{t \in [0, 1] : \mathtt{s}(t) = s\}$$

is a monotone increasing function, and has therefore countably many jumps. Denoting $\lambda_t := \|u_t\|_{H^1(\mathrm{d}\rho_t)}$ and observing that $\frac{d}{dt}\mathtt{s}(t) = \lambda_t$ we see that the countable set of discontinuities of $\mathtt{t}(s)$ is precisely the image by $\mathtt{s}$ of the critical points $\{t \in [0, 1] : \lambda_t = 0\}$, which by countability has zero $\mathrm{d}s$ measure in $[0, L]$. As a consequence $\lambda_{\mathtt{t}(s)}$ is positive $\mathrm{d}s$ a.e. in $[0, L]$, and

$$s \in [0, L] : \qquad \tilde{\rho}_s := \rho_{\mathtt{t}(s)} \quad \text{and} \quad \tilde{u}_s := \frac{u_{\mathtt{t}(s)}}{\|u_{\mathtt{t}(s)}\|_{H^1(\mathrm{d}\rho_{\mathtt{t}(s)})}}$$

are well-defined and measurable in s with of course

$$\|\tilde{u}_s\|_{H^1(\mathrm{d}\tilde{\rho}_s)} = 1 \quad \text{a.e. } s \in [0, L].$$

Exploiting the narrow continuity of $t \mapsto \rho_t$ and $\mathtt{s}(\mathtt{t}(s)) = s$ it is easy to see that $s \mapsto \tilde{\rho}_s$ is narrowly continuous and connects ρ_0, ρ_t in time $s \in [0, L]$. Furthermore one can check that $\partial_s \tilde{\rho}_s + \operatorname{div}(\tilde{\rho}_s \nabla \tilde{u}_s) = \tilde{\rho}_s \tilde{u}_s$ in the sense of distributions $\mathcal{D}'((0, L) \times \mathbb{R}^d)$, which formally follows by the chain rule

$$\begin{aligned}
\partial_s \tilde{\rho}_s = \partial_s(\rho_{\mathtt{t}(s)}) = \frac{d}{ds}\mathtt{t}(s)\partial_t \rho_{\mathtt{t}(s)} &= \frac{1}{\frac{d}{dt}\mathtt{s}(t)|_{\mathtt{t}(s)}} \partial_t \rho_{\mathtt{t}(s)} \\
&= \frac{1}{\|u_{\mathtt{t}(s)}\|_{H^1(\mathrm{d}\rho_{\mathtt{t}(s)})}} \left(-\operatorname{div}\left(\rho_{\mathtt{t}(s)} \nabla u_{\mathtt{t}(s)}\right) + \rho_{\mathtt{t}(s)} u_{\mathtt{t}(s)}\right) \\
&= -\operatorname{div}\left(\tilde{\rho}_s \nabla \tilde{u}_s\right) + \tilde{\rho}_s \tilde{u}_s
\end{aligned}$$

(this can be made rigorous using a change of variables and the inequality $\|u_{\mathtt{t}(s)}\|_{H^1(d\rho_{\mathtt{t}(s)})} > 0$ a.e. $s \in [0, L]$, see e.g. [1, Lemma 8.1.3]).
As a consequence $(\tilde{\rho}_s, \tilde{u}_s)_{s \in [0,L]}$ is an admissible curve connecting ρ_0, ρ_1 with energy

$$E[\tilde{\rho}; \tilde{u}] = \int_0^L \|\tilde{u}_s\|^2_{H^1(\mathrm{d}\tilde{\rho}_s)} \mathrm{d}s = \int_0^L 1 \, \mathrm{d}s = L.$$

Setting $(\overline{\rho}_t, \overline{u}_t)_{t \in [0,1]} := (\tilde{\rho}_{tL}, L\tilde{u}_{tL})_{t \in [0,1]}$ in order to connect in time $t \in [0, 1]$, the energy finally scales according to Remark 2.3 as

$$\begin{aligned}
E[\overline{\rho}; \overline{u}] &= L.E[\tilde{\rho}; \tilde{u}] = L^2 \\
&= \left(\int_0^1 \|u_t\|_{H^1(\mathrm{d}\rho_t)} \mathrm{d}t\right)^2 \le \int_0^1 \|u_t\|^2_{H^1(\mathrm{d}\rho_t)} \mathrm{d}t = E[\rho; u]
\end{aligned}$$

as desired. By construction we have that $\|\overline{u}_t\|_{H^1(\mathrm{d}\overline{\rho}_t)}$ is constant in time, and because $\int_0^1 \|\overline{u}_t\|^2_{H^1(\mathrm{d}\overline{\rho}_t)} \mathrm{d}t = E[\overline{\rho}; \overline{u}]$ we conclude that $\|\overline{u}_t\|^2_{H^1(\mathrm{d}\overline{\rho}_t)} = E[\overline{\rho}; \overline{u}]$ a.e. $t \in (0, 1)$ and the proof is complete. ■

Lower-semicontinuous translation of the Hopf–Rinow theorem.

Lemma 5.5. *Let a metric space (X, ϱ) be a complete length space. Assume that there exists a ϱ-boundedly compact Hausdorff topology σ on X (i-e ϱ-bounded sequences contain σ-converging subsequences) such that ϱ is lower semicontinuous with respect to σ. Then (X, ϱ) is a geodesic space.*

Proof: Fix any two points $x, y \in X$. By [5, Theorem 2.4.16(1)], it suffices to show that they admit a midpoint, i-e a point z such that

$$\varrho(x, y) = 2\varrho(x, z) = 2\varrho(z, y).$$

By [5, Lemma 2.4.10], there exists a sequence z_k of almost midpoints, i-e

$$|\varrho(x, y) - 2\varrho(x, z_k)| \leq k^{-1}, \quad |\varrho(x, y) - 2\varrho(y, z_k)| \leq k^{-1}.$$

The sequence $\{z_k\}$ is ϱ-bounded, thus without loss of generality it σ-converges to some $z \in X$. Then

$$2\varrho(x, z) \leq \lim_{k\to\infty} 2\varrho(x, z_k) = \varrho(x, y),$$
$$2\varrho(y, z) \leq \lim_{k\to\infty} 2\varrho(y, z_k) = \varrho(x, y).$$

But its is clear from the triangle inequality that the latter inequalities must be equalities. ■

A generalized Beckner inequality.

Theorem 8. *Let Ω in $\mathbb{R}^d$ be a bounded, connected, open domain, satisfying the cone property. Let $m : \Omega \to \mathbb{R}$ be a Lipschitz function such that $\inf_{x\in\Omega} m(x) > 0$. There exists a strictly increasing continuous scalar function Φ (depending merely on Ω and m) such that $\Phi(0) = 0$ and*

$$\Phi\left(\int_\Omega f(x)\,\mathrm{d}x\right)\int_\Omega (f(x) - m(x))^2\,\mathrm{d}x \leq \int_\Omega f(x)(f(x) - m(x))^2\,\mathrm{d}x + \int_\Omega f(x)|\nabla f(x) - \nabla m(x)|^2\,\mathrm{d}x \quad (5.2)$$

for every non-negative $f \in H^1 \cap L^\infty(\Omega)$.

Proof: *Step 1.* Without loss of generality, we may rescale the problem so that Ω has Lebesgue measure 1. Assume first that $m \equiv 1$. Under these assumptions, another generalized Beckner inequality [7, Lemma 4] with $p = 3/2$, $q = 4/3$ implies

$$\|f\|_{L^2(\Omega)}\left[\int_\Omega f^2(x)\,\mathrm{d}x - \left(\int_\Omega f(x)\,\mathrm{d}x\right)^2\right] \leq C_\Omega \int_\Omega f(x)|\nabla f(x)|^2\,\mathrm{d}x \quad (5.3)$$

for every non-negative $f \in H^1 \cap L^\infty(\Omega)$, where C_Ω depends only on Ω. Let

$$\lambda = \int_\Omega f(x)\,\mathrm{d}x.$$

If $\lambda = 0$, then (5.2) is trivial with $\Phi(0) = 0$. If $\lambda = 1$, then $\|f\|_{L^2(\Omega)} \geq 1$, and (5.3) yields

$$\int_\Omega (f(x) - 1)^2 \, \mathrm{d}x \leq C_\Omega \int_\Omega f(x) |\nabla f(x)|^2 \, \mathrm{d}x,$$

which is even stronger than (5.2).

Step 2. We now consider the case of arbitrary $\lambda > 0$. We set

$$\Phi(\lambda) = \Phi_1(\lambda) := \min\left(\lambda, \frac{\lambda^2}{2C_\Omega}\right).$$

Let $u = f/\lambda$. Since $|\Omega| = 1$, by the Hölder inequality we have that

$$\|u\|_{L^3(\Omega)} \geq \|u\|_{L^2(\Omega)} \geq \|u\|_{L^1(\Omega)} = 1,$$

so

$$\|u\|^3_{L^3(\Omega)} \geq \|u\|^2_{L^2(\Omega)}.$$

Then we discover that

$$\begin{aligned}
\int_\Omega (f(x) - 1)^2 \, \mathrm{d}x &= \int_\Omega (\lambda u(x) - 1)^2 \, \mathrm{d}x = \lambda^2 \int_\Omega u^2(x) \, \mathrm{d}x - 2\lambda + 1 \\
&= (\lambda^2 - 2\lambda) \int_\Omega u^2(x) \, \mathrm{d}x + 1 + 2\lambda \int_\Omega (u(x) - 1)^2 \, \mathrm{d}x \\
&\leq \lambda^2 \int_\Omega u^3(x) \, \mathrm{d}x - 2\lambda \int_\Omega u^2(x) \, \mathrm{d}x + 1 + 2\lambda \int_\Omega (u(x) - 1)^2 \, \mathrm{d}x \\
&= \int_\Omega u(x)(\lambda u(x) - 1)^2 \, \mathrm{d}x + 2\lambda \int_\Omega (u(x) - 1)^2 \, \mathrm{d}x \\
&\leq \int_\Omega u(x)(\lambda u(x) - 1)^2 \, \mathrm{d}x + 2\lambda C_\Omega \int_\Omega u(x) |\nabla u(x)|^2 \, \mathrm{d}x \\
&\leq \frac{1}{\Phi(\lambda)} \int_\Omega \lambda u(x)(\lambda u(x) - 1)^2 \, \mathrm{d}x + \frac{\lambda^3}{\Phi(\lambda)} \int_\Omega u(x) |\nabla u(x)|^2 \, \mathrm{d}x \\
&= \frac{1}{\Phi\left(\int_\Omega f(x) \, \mathrm{d}x\right)} \left[\int_\Omega f(x)(f(x) - 1)^2 \, \mathrm{d}x \right. \\
&\qquad \left. + \int_\Omega f(x) |\nabla f(x)|^2 \, \mathrm{d}x \right].
\end{aligned}$$

Step 3. We will now treat the case of generic m by a perturbation argument. Throughout this step we denote by C_m a generic constant depending only on m. Let $v(x) = f(x)/m(x)$. Then in light of the previous step we see that

$$\begin{aligned}\Phi_1\left(\int_\Omega f(x)\,\mathrm{d}x\right)&\int_\Omega (f(x)-m(x))^2\,\mathrm{d}x\\ &= \Phi_1\left(\int_\Omega m(x)v(x)\,\mathrm{d}x\right)\int_\Omega m^2(x)(v(x)-1)^2\,\mathrm{d}x\\ &\le C_m\Phi_1\left(\int_\Omega v(x)\,\mathrm{d}x\right)\int_\Omega (v(x)-1)^2\,\mathrm{d}x\\ &\le C_m\left[\int_\Omega v(x)(v(x)-1)^2\,\mathrm{d}x + \int_\Omega v(x)|\nabla v(x)|^2\,\mathrm{d}x\right].\end{aligned}$$

It suffices to show that the latter sum does not exceed

$$C_m\bigg[\int_\Omega f(x)(f(x)-m(x))^2\,\mathrm{d}x + \int_\Omega f(x)|\nabla f(x)-\nabla m(x)|^2\,\mathrm{d}x\bigg].$$

Let $\alpha \in (0,1)$ be such that

$$2\left(\frac{1}{\alpha}-1\right)\sup_{x\in\Omega} m(x)|\nabla m(x)|^2 = \inf_{x\in\Omega} m^3(x).$$

Then we find that

$$
\begin{aligned}
&\int_\Omega v(x)(v(x)-1)^2\,\mathrm{d}x + \int_\Omega v(x)|\nabla v(x)|^2\,\mathrm{d}x \\
&\qquad \le C_m\left[\frac{1}{2}\int_\Omega m^3(x)v(x)(v(x)-1)^2\,\mathrm{d}x\right. \\
&\qquad\qquad \left. + (1-\alpha)\int_\Omega m^3(x)v(x)|\nabla v(x)|^2\,\mathrm{d}x\right] \\
&\qquad \le C_m\left[\int_\Omega m^3(x)v(x)(v(x)-1)^2\,\mathrm{d}x\right. \\
&\qquad\qquad + (1-\alpha)\int_\Omega m^3(x)v(x)|\nabla v(x)|^2\,\mathrm{d}x \\
&\qquad\qquad \left. + \left(1-\frac{1}{\alpha}\right)\int_\Omega m(x)v(x)(v(x)-1)^2|\nabla m(x)|^2\,\mathrm{d}x\right] \\
&\qquad \le C_m\left[\int_\Omega m^3(x)v(x)(v(x)-1)^2\,\mathrm{d}x + \int_\Omega m^3(x)v(x)|\nabla v(x)|^2\,\mathrm{d}x\right. \\
&\qquad\qquad + 2\int_\Omega m^2(x)v(x)(v(x)-1)\nabla m(x)\cdot\nabla v(x)\,\mathrm{d}x \\
&\qquad\qquad \left. + \int_\Omega m(x)v(x)(v(x)-1)^2|\nabla m(x)|^2\,\mathrm{d}x\right] \\
&\qquad = C_m\left[\int_\Omega m^3(x)v(x)(v(x)-1)^2\,\mathrm{d}x\right. \\
&\qquad\qquad \left. + \int_\Omega m(x)v(x)|m(x)\nabla v(x) + \nabla m(x)(v(x)-1)|^2\,\mathrm{d}x\right] \\
&\qquad = C_m\left[\int_\Omega f(x)(f(x)-m(x))^2\,\mathrm{d}x\right. \\
&\qquad\qquad \left. + \int_\Omega f(x)|\nabla f(x) - \nabla m(x)|^2\,\mathrm{d}x\right].
\end{aligned}
$$

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