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Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 15–28

ON BIADJOINT TRIANGLES

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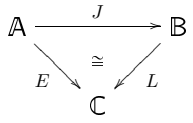
Abstract: We give a 2-dimensional version of the adjoint triangle theorem due to Eduardo Dubuc. We also show that the pseudomonadicity characterization (due to Enrico Vitale, Francisco Marmolejo and Ivan Creurer) is a consequence of the biadjoint triangle theorem. Furthermore, our main theorem can be seen as a generalization of the construction given by Stephen Lack of the left 2-adjoint $\mathbf{Ps\text{-}Alg} \to \mathbf{Alg}_s$ (which is itself a corollary of the strict version of the biadjoint triangle theorem given in this paper). At last, we give two brief applications: we prove a result on lifting biadjunctions and study pseudo-Kan extensions.

Keywords: adjoint triangles, descent objects, Kan extensions, pseudomonads, biadjunctions.
AMS Subject Classification (2010): 18D05, 18A40, 18C15.

Introduction

Assume that $E : \mathbb{A} \to \mathbb{C}$, $J : \mathbb{A} \to \mathbb{B}$, $L : \mathbb{B} \to \mathbb{C}$ are functors such that there is a natural isomorphism



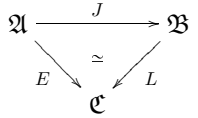
Eduardo Dubuc [2] proved that if $L : \mathbb{B} \to \mathbb{C}$ is precomonadic, $E : \mathbb{A} \to \mathbb{C}$ has a right adjoint and $\mathbb{A}$ has some needed equalizers, then J has a right adjoint. In this paper, we give a 2-dimensional version of this theorem. More precisely, let $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ be 2-categories and assume that

$$E : \mathfrak{A} \to \mathfrak{C}, J : \mathfrak{A} \to \mathfrak{B}, L : \mathfrak{B} \to \mathfrak{C}$$

are pseudofunctors such that L is pseudoprecomonadic and E has a right biadjoint. We prove that, if we have the pseudonatural equivalence below, then

Received June 5, 2015.
This work was supported by CNPq (National Council for Scientific and Technological Development) – Brazil, and by the Centre for Mathematics of the University of Coimbra – UID/MAT/00324/2013, funded by the Portuguese Government through FCT/MEC and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020.

J has a right biadjoint, provided that $\mathfrak{A}$ has some needed descent objects.



We, also, give necessary and sufficient conditions under which the unit and the counit of the obtained biadjunction are pseudonatural equivalences. Moreover, we prove a strict version of this theorem. That is to say, we show that, with some extra hypotheses, it is possible to construct (strict) right 2-adjoints.

Robert Blackwell, G. M. Kelly and John Power [1] had already given several constructions of biadjunctions related to two-dimensional monad theory. Since some of them are particular cases of the biadjoint triangle theorem established here, many applications are already covered by the fundamental article [1].

Also, Theorem 4.1 may be seen as a generalization of the construction given by Stephen Lack [10] of the left biadjoint to the inclusion of the strict algebras into the pseudoalgebras

$$\mathbf{Alg}_s \to \mathbf{Ps\text{-}Alg}$$

Still, the idea of the proof of the biadjoint triangle theorem came from the original adjoint triangle theorem due to Eduardo Dubuc [2, 16].

In Section 1, we recall Eduardo Dubuc's theorem, in its enriched version. This version gives the 2-adjoint triangle theorem for 2-pre(co)monadicity. In Section 2, we change our setting: we recall some definitions and results of the 3-category 2-$\mathbf{CAT}$ of 2-categories, pseudofunctors, pseudonatural transformations and modifications. Most of them can be found in Ross Street's articles [14, 15].

Section 3 gives definitions and results related to descent objects, which is a type of 2-categorical limit presented in [14, 15]. To do so, we employ the concept of pointwise pseudo-Kan extension, claiming that it is a good way of handling the universal properties and exact conditions related to this 2-dimensional limit.

Within our established setting, in Section 4 we prove our main results on biadjoint triangles, while, in Section 5, we give such results in terms of pseudopre(co)monadicity and show that the pseudo(co)monadicity theorem of Ivan Le Creurer, Francisco Marmolejo and Enrico Vitale [11] is a corollary

we get the pseudonatural equivalences below (analogous to the enriched case [7])

$$\begin{aligned}
\left[\dot{\mathcal{S}}, \mathfrak{A}\right]_{PS}(W, \mathrm{Ps}\mathcal{R}an_{\mathrm{h}}\mathcal{D}) &\simeq \int_{\dot{\mathcal{S}}} \mathfrak{A}(Wx, \mathrm{Ps}\mathcal{R}an_{\mathrm{h}}\mathcal{D}x) \\
&\simeq \int_{\dot{\mathcal{S}}} \mathfrak{A}(Wx, \left\{\dot{\mathcal{S}}(x, \mathrm{h}-), \mathcal{D}\right\}_{\mathrm{bi}}) \\
&\simeq \int_{\dot{\mathcal{S}}} [\mathcal{S}, \mathbf{Cat}]_{PS}(\mathfrak{A}(x, \mathrm{h}-), \mathrm{h}(Wx, \mathcal{D}-)) \\
&\simeq \int_{\dot{\mathcal{S}}} \int_{\mathcal{S}} \mathbf{CAT}(\mathfrak{A}(x, \mathrm{h}(a)), \mathrm{h}(Wx, \mathcal{D}a)) \\
&\simeq \int_{\mathcal{S}} \int_{\dot{\mathcal{S}}} \mathbf{CAT}(\mathfrak{A}(x, \mathrm{h}(a)), \mathrm{h}(Wx, \mathcal{D}a)) \\
&\simeq \int_{\mathcal{S}} \left[\dot{\mathcal{S}}^{\mathrm{op}}, \mathbf{CAT}\right]_{PS}(\mathfrak{A}(-, \mathrm{h}(a)), \mathrm{h}(W-, \mathcal{D}a)) \\
&\simeq \int_{\mathcal{S}} \mathfrak{A}(W \circ \mathrm{h}(a), \mathcal{D}a) \\
&\simeq [\mathcal{S}, \mathfrak{A}]_{PS}(W \circ \mathrm{h}, \mathcal{D})
\end{aligned}$$

This completes the proof that if the pointwise right pseudo-Kan extension $\mathrm{Ps}\mathcal{R}an_{\mathrm{h}}$ exists, it is a pseudo-Kan extension. Within this setting and assuming this result, the original argument used to prove Proposition 7.3 gets the construction via descent objects of weighted bilimits originally given in [15].

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Moreover, Corollary 7.2 gives a way of comparing strict Kan extensions with pseudo-Kan extensions, provided that $\mathfrak{A}$ has strict descent objects and the (strict) Kan extensions exist: within this setting, that corollary gives the construction of pseudo-Kan extensions via a strict Kan extension of a flexible diagram [10].

At last, let $\mathfrak{A}$ be a 2-category with all descent objects and T be a pseudocomonad on $\mathfrak{A}$. Recall that, if T preserves all effective descent diagrams, **Ps**-T-Co**Alg** has all descent objects. Therefore, if $\mathrm{h} : \mathcal{S} \to \dot{S}$ is a pseudofunctor, in this setting, the commutative diagram below satisfies the hypotheses of Corollary 7.1 (and, thereby, it can be used to lift pseudo-Kan extensions to pseudocoalgebras).

$$\begin{array}{ccc} [\dot{\mathcal{S}}, \textbf{Ps}\text{-}T\text{-}Co\textbf{Alg}]_{PS} & \longrightarrow & [\mathcal{S}, \textbf{Ps}\text{-}T\text{-}Co\textbf{Alg}]_{PS} \\ \downarrow & & \downarrow \\ [\dot{\mathcal{S}}, \mathfrak{A}]_{PS} & \longrightarrow & [\mathcal{S}, \mathfrak{A}]_{PS} \end{array}$$

Remark 7.4. Assume that $\mathrm{h} : \mathcal{S} \to \dot{\mathcal{S}}$ is a pseudofunctor, in which $\mathcal{S}, \dot{\mathcal{S}}$ are small 2-categories. There is another way of proving Proposition 7.3. Firstly, we define the bilimit version of *end*. That is to say, if $T : \mathcal{S} \times \mathcal{S}^{\mathrm{op}} \to \mathbf{CAT}$ is a pseudofunctor, we define

$$\int_{\mathcal{S}} T := [\mathfrak{A} \times \mathfrak{A}^{\mathrm{op}}, \mathbf{CAT}]_{PS} \left(\mathfrak{A}(-,-), T\right)$$

From this definition, it follows Fubini's theorem (up to equivalence). And, if $\mathfrak{D}, \mathcal{D} : \mathcal{S} \to \mathfrak{A}$ are pseudofunctors, the following equivalence holds:

$$\int_{\mathcal{S}} \mathfrak{A}(\mathfrak{D}a, \mathcal{D}a) \simeq [\mathcal{S}, \mathfrak{A}]_{PS} \left(\mathfrak{D}, \mathcal{D}\right)$$

Therefore, if $\mathrm{h} : \mathcal{S} \to \dot{\mathcal{S}}$ is a pseudofunctor and we define

$$\mathrm{Ps}\mathcal{R}an_{\mathrm{h}}\mathcal{D}x = \left\{\dot{\mathcal{S}}(x, \mathrm{h}-), \mathcal{D}\right\}_{\mathrm{bi}},$$

of the biadjoint triangle theorem. Moreover, in Section 6 we show the theorem of [10] on the inclusion $\mathbf{Alg}_s \to \mathbf{Ps}\text{-}\mathbf{Alg}$ as a consequence of the theorems presented herein. At last, we discuss a straightforward application on lifting biadjunctions in Section 7.

Since our main application in Section 7 is about construction of right biadjoints, we prove the theorem for pseudoprecomonadic functors instead of proving the theorem on pseudopremonadic functors. But, for instance, to apply the results of this work in the original setting of [1], or to get the construction of the left biadjoint given in [10], we should, of course, consider the dual version: the Biadjoint Triangle Theorem 4.3.

This work was realized during my PhD program at University of Coimbra, under supervision of Maria Manuel Clementino.

1. Enriched Adjoint Triangles

Consider a (cocomplete, complete and symmetric monoidal) closed category V. It is well known that the results on (co)monadicity in V-**CAT** are similar to those of the classical context of **CAT** (see, for instance, [1, 3, 12]). And, actually, some of those results of the enriched context can be seen as consequences of the classical theorems because of Ross Street's work [12].

Our main interest is in Beck's theorem for V-precomonadicity. More precisely, it is known that the 2-category V-**CAT** admits construction of coalgebras [12]. Therefore every left V-adjoint $L : \mathcal{B} \to \mathcal{C}$ comes with the corresponding Eilenberg-Moore factorization.

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\phi} & Co\textbf{Alg} \\ & \overset{L}{\searrow} & \downarrow \\ & & \mathcal{C} \end{array}$$

If $V = \mathbf{Set}$, Beck's theorem says that ϕ is fully faithful if and only if every component of the unit $\eta : \mathrm{Id}_{\mathcal{B}} \to UL$ of the adjunction $L \dashv U$ is a regular monomorphism. This is equivalent to say that the diagram below is an equalizer for every object X of $\mathcal{B}$. And, if this happens, we say that L is precomonadic.

$$X \xrightarrow{\eta_X} ULX \underset{UL(\eta_X)}{\overset{\eta_{ULX}}{\rightrightarrows}} ULULX$$

With due adaptations, this theorem also holds for enriched categories. That is to say, ϕ is V-fully faithful if and only if the image of the diagram above by the hom-functors $\mathcal{B}(Y,-)$ are equalizers in V for every object Y and every object X of $\mathcal{B}$ (*i.e.* the diagram above is a V-equalizer for every object X of $\mathcal{B}$). And, therefore, this result gives what we need to prove Proposition 1.1, which is the enriched version for Eduardo Dubuc's theorem [2].

Proposition 1.1 (Enriched Adjoint Triangle Theorem)**.** *Let $E : \mathcal{A} \to \mathcal{C}$, $J : \mathcal{A} \to \mathcal{B}$ and $L : \mathcal{B} \to \mathcal{C}$ be V-functors such that $LJ = E$. Assume that $(L, U, \eta, \varepsilon)$, (E, R, ρ, μ) are V-adjunctions and L is V-precomonadic. The V-functor J has a right V-adjoint G if and only if, for each object Y of $\mathcal{B}$, GY is the V-equalizer of*

$$RLY \underset{RL(\eta_Y)}{\overset{RL(U(\mu_{LY})\eta_{JRLY})\rho_{RLY}}{\rightrightarrows}} RLULY$$

in the V-category $\mathcal{A}$.

Proof: Given an object A of $\mathcal{A}$ and an object Y in $\mathcal{B}$,

$$\mathcal{B}(JA,Y) \xrightarrow{\mathcal{B}(JA,\eta_Y)} \mathcal{B}(JA,ULY) \underset{\mathcal{B}(JA,UL(\eta_Y))}{\overset{\mathcal{B}(JA,\eta_{ULY})}{\rightrightarrows}} \mathcal{B}(JA,ULULY)$$

is, by hypothesis, an equalizer in V. Furthermore, we have isomorphisms

$$\mathcal{B}(JA,ULY) \cong \mathcal{C}(LJA,LY) = \mathcal{C}(EA,LY) \cong \mathcal{A}(A,RLY)$$

and

$$\mathcal{B}(JA,ULULY) \cong \mathcal{A}(A,RLULY),$$

natural in A and Y. Therefore

$$\mathcal{B}(JA,Y) \longrightarrow \mathcal{A}(A,RLY) \rightrightarrows \mathcal{A}(A,RLULY)$$

is an equalizer. And, by the weak Yoneda Lemma, since the parallel morphisms above are natural in A, there are morphisms $q_Y, r_Y : RLY \to RLULY$ such that the images by the hom-functor $\mathcal{A}(A,-)$ are the parallel morphisms of the diagram above. Assuming that the pair (q_Y, r_Y) has a V-equalizer GY in $\mathcal{A}$ for every Y, we have that $\mathcal{A}(A,GY)$ is also an equalizer of $\mathcal{A}(A,q_Y), \mathcal{A}(A,r_Y)$. Therefore we get a V-natural isomorphism

$$\mathcal{A}(-,GY) \cong \mathcal{B}(J-,Y).$$

in which $\prod$ and $\pitchfork$ denote the bilimit versions of the product and cotensor product, respectively. Thereby, by Corollary 7.1, the pseudo-Kan extension Ps-$\mathcal{R}an_{\mathrm{h}}$ can be constructed pointwisely as descent objects of a diagram obtained from the pseudo-Kan extensions above. Namely, Ps-$\mathcal{R}an_{\mathrm{h}}\mathcal{D}x$ is the descent object of a diagram

$$\mathfrak{a}_0 \underset{}{\overset{}{\rightleftarrows}} \mathfrak{a}_1 \underset{}{\overset{}{\rightrightarrows}} \mathfrak{a}_2$$

in which, by Theorem 4.1 and the last observations,

$$\begin{aligned}
\mathfrak{a}_0 &= \prod_{y\in\mathcal{S}_0} \left(\dot{\mathcal{S}}(x,y) \pitchfork \prod_{\mathrm{h}(a)=y} \mathcal{D}a \right) \\
&\simeq \prod_{a\in\mathcal{S}_0} \left(\dot{S}(x,\mathrm{h}(a)) \pitchfork \mathcal{D}a \right) \\
\mathfrak{a}_1 &= \left(\dot{\mathcal{S}}(x,y) \pitchfork \prod_{\mathrm{h}(a)=y} \left(\prod_{b\in\mathcal{S}_0} \mathcal{S}(a,b) \pitchfork \mathcal{D}b \right) \right) \\
&\simeq \prod_{a\in\mathcal{S}_0} \left(\dot{S}(x,\mathrm{h}(a)) \pitchfork \left(\prod_{b\in\mathcal{S}_0} \mathcal{S}(a,b) \pitchfork \mathcal{D}b \right) \right) \\
&\simeq \prod_{(a,b)\in\mathcal{S}_0\times\mathcal{S}_0} \left(\left(\mathcal{S}(a,b) \times \dot{\mathcal{S}}(x,\mathrm{h}(a)) \right) \pitchfork \mathcal{D}b \right) \\
\mathfrak{a}_2 &\simeq \prod_{(a,b,c)\in\mathcal{S}_0\times\mathcal{S}_0\times\mathcal{S}_0} \left(\left(\mathcal{S}(b,c) \times \mathcal{S}(a,b) \times \dot{\mathcal{S}}(x,\mathrm{h}(a)) \right) \pitchfork \mathcal{D}c \right)
\end{aligned}$$

This implies that, indeed, if $\mathfrak{A}$ is bicategorically complete, then Ps-$\mathcal{R}an_{\mathrm{h}}\mathcal{D}$ exists and, once we assume the results of [15] related to the construction of weighted bilimits via descent objects, we conclude that

$$\text{Ps-}\mathcal{R}an_{\mathrm{h}}\mathcal{D}x = \left\{ \dot{\mathcal{S}}(x,\mathrm{h}-), \mathcal{D} \right\}_{\mathrm{bi}}$$

Proposition 7.3 (Pointwise pseudo-Kan extension)**.** *Let $\mathcal{S}, \dot{S}$ be small 2-categories and $\mathfrak{A}$ be a bicategorically complete 2-category. If $\mathrm{h} : \mathcal{S} \to \dot{S}$ is a pseudofunctor, then*

$$\text{Ps-}\mathcal{R}an_{\mathrm{h}}\mathcal{D}x = \left\{ \dot{\mathcal{S}}(x,\mathrm{h}-), \mathcal{D} \right\}_{\mathrm{bi}}$$

They also showed that the 2-category of pseudocoalgebras of the induced 2-comonad is $[\mathcal{S},\mathfrak{A}]_{PS}$. It actually works more generally: precisely,

$$[\mathcal{S},\mathfrak{A}]_{PS} \to [\mathcal{S}_0,\mathfrak{A}]_{PS}$$

is pseudocomonadic whenever there is a right biadjoint

$$\text{Ps-}\mathcal{R}an_{\mathrm{h}}\mathcal{D} : [\mathcal{S}_0,\mathfrak{A}]_{PS} \to [\mathcal{S},\mathfrak{A}]_{PS},$$

because existing bilimits of $\mathfrak{A}$ are constructed objectwisely (and, therefore, the hypotheses of Lemma 5.4 is satisfied). Thus, we get the following commutative square:

$$\begin{array}{ccc} [\dot{\mathcal{S}},\mathfrak{A}]_{PS} & \xrightarrow{[\mathrm{h},\mathfrak{A}]_{PS}} & [\mathcal{S},\mathfrak{A}]_{PS} \\ \downarrow & & \downarrow \\ [\dot{\mathcal{S}}_0,\mathfrak{A}] & \xrightarrow{[\mathrm{h},\mathfrak{A}]_{PS}} & [\mathcal{S}_0,\mathfrak{A}] \end{array}$$

Thereby, Corollary 7.1 gives a way to study pseudo-Kan extensions, even in the absence of strict 2-limits. That is to say, on the one hand, if the 2-category $\mathfrak{A}$ is complete, our results give pseudo-Kan extensions as descent objects of strict 2-limits. On the other hand, in the absence of strict 2-limits and, in particular, assuming that $\mathfrak{A}$ is bicategorically complete, we can construct the following pseudo-Kan extensions:

$$\begin{aligned} \text{Ps-}\mathcal{R}an_{\mathcal{S}_0\to\dot{\mathcal{S}}_0} : [\mathcal{S}_0,\mathfrak{A}] &\to [\dot{\mathcal{S}}_0,\mathfrak{A}]_{PS} \\ \mathcal{D} &\mapsto \text{Ps-}\mathcal{R}an_{\mathcal{S}_0\to\dot{\mathcal{S}}_0}\mathcal{D} : \left(x \mapsto \prod_{\mathrm{h}(a)=x} \mathcal{D}a\right) \\ \text{Ps-}\mathcal{R}an_{\dot{\mathcal{S}}_0\to\dot{\mathcal{S}}} : [\dot{\mathcal{S}}_0,\mathfrak{A}] &\to [\dot{\mathcal{S}},\mathfrak{A}]_{PS} \\ \mathcal{D} &\mapsto \text{Ps-}\mathcal{R}an_{\dot{\mathcal{S}}_0\to\dot{\mathcal{S}}}\mathcal{D} : \left(x \mapsto \prod_{y\in\dot{\mathcal{S}}_0} \dot{\mathcal{S}}(x,y)\pitchfork\mathcal{D}y\right) \\ \text{Ps-}\mathcal{R}an_{\mathcal{S}_0\to\mathcal{S}} : [\mathcal{S}_0,\mathfrak{A}] &\to [\mathcal{S},\mathfrak{A}]_{PS} \\ \mathcal{D} &\mapsto \text{Ps-}\mathcal{R}an_{\mathcal{S}_0\to\mathcal{S}}\mathcal{D} : \left(a \mapsto \prod_{b\in\mathcal{S}_0} \mathcal{S}(a,b)\pitchfork\mathcal{D}b\right) \end{aligned}$$

Reciprocally, if G is right V-adjoint to J, then $\mathcal{A}(-,GY) \cong \mathcal{B}(J-,Y)$ is an equalizer of

$$\mathcal{A}(-,q_Y),\mathcal{A}(-,r_Y) : \mathcal{A}(-,RLY) \to \mathcal{A}(-,RLULY).$$

And, hence, GY is the V-equalizer of

$$RLY \underset{q_Y}{\overset{r_Y}{\rightrightarrows}} RLULY$$

And this completes the proof that, indeed, the V-equalizers of q_Y, r_Y are also necessary. ∎

In the proof above, observe that, indeed, $r_Y = RL(U(\mu_{LY})\eta_{JRLY})\rho_{RLY}$ and $q_Y = RL(\eta_Y)$. That is to say, for each object Y of the category $\mathcal{B}$, the equalizer of the parallel arrows

$$RLY \underset{RL(\eta_Y)}{\overset{RL(U(\mu_{LY})\eta_{JRLY})\rho_{RLY}}{\rightrightarrows}} RLULY$$

are precisely the needed equalizers.

Proposition 1.1 applies to the case of **CAT**-enriched category theory. But it does not give results about pseudomonad theory. For instance, the (dual of the) construction above does not give the left biadjoint constructed in [1, 10]

$$\textbf{Ps-Alg} \to \textbf{Alg}_s$$

2. Bilimits

We denote by 2-**CAT** the 3-category of 2-categories, pseudofunctors (homomorphisms), pseudonatural transformations (strong transformations) and modifications. The precise definitions can be found in [14]. And, since this is our main setting, we recall some results and concepts related to 2-**CAT**. Most of them can be found in [14], and a few of them are direct consequences of results given there.

Firstly, we have the bicategorical Yoneda lemma. Denoting by $[\mathcal{S},\mathbf{CAT}]_{PS}$ the 2-category of pseudofunctors $\mathcal{S} \to \mathbf{CAT}$, pseudonatural transformations and modifications, the bicategorical Yoneda lemma says that there is a pseudonatural equivalence

$$[\mathcal{S},\mathbf{CAT}]_{PS}(\mathcal{S}(a,-),\mathcal{D}) \simeq \mathcal{D}(a)$$

given by the evaluation at the identity. And this Yoneda lemma gives the Yoneda embedding.

Lemma 2.1 (Yoneda Embedding [14])**.** *The Yoneda* 2*-functor*

$$Y : \mathfrak{A} \to [\mathfrak{A}^{\mathrm{op}}, \mathbf{CAT}]_{PS}$$

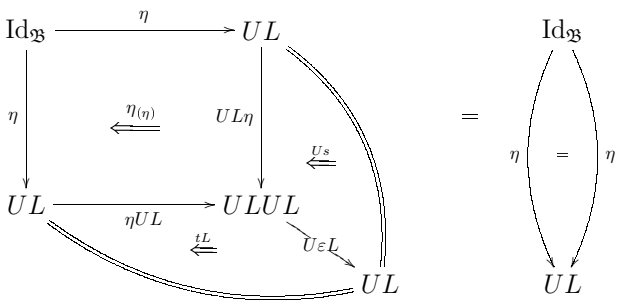
is locally an equivalence (that is to say, it induces equivalences between the hom-categories).

Considering pseudofunctors $L : \mathfrak{B} \to \mathfrak{C}$ and $U : \mathfrak{C} \to \mathfrak{B}$, we say that U is right biadjoint to L, denoted by $L \dashv U$, if we have a pseudonatural equivalence $\mathfrak{C}(L-,-) \simeq \mathfrak{B}(-,U-)$. This concept can be also defined in terms of unit and counit as it is done at Definition 2.2. The equivalence of the definitions is a consequence of the (bicategorical) Yoneda Lemma.

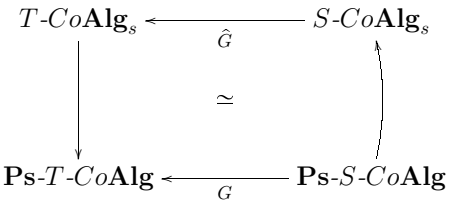
Definition 2.2. [[5, 11]] Let $\mathfrak{B}$ and $\mathfrak{C}$ be 2-categories and let $U : \mathfrak{C} \to \mathfrak{B}$ and $L : \mathfrak{B} \to \mathfrak{C}$ be pseudofunctors. L is left biadjoint to U if there exist

(1) pseudonatural transformations $\eta : \mathrm{Id}_{\mathfrak{B}} \to UL$ and $\varepsilon : LU \to \mathrm{Id}_{\mathfrak{C}}$
(2) invertible modifications $s : \mathrm{Id}_L \Rightarrow (\varepsilon L)\circ(L\eta)$ and $t : (U\varepsilon)\circ(\eta U) \Rightarrow \mathrm{Id}_U$

such that the following equations hold [5, 11]:



in which E has a right 2*-adjoint, then $\widehat{J}$ has a right* 2*-adjoint $\widehat{G}$ and J has a right biadjoint G. Moreover, recall that T-$Co\mathbf{Alg}_s \to \mathbf{Ps}$-$T$-$Co\mathbf{Alg}$ and S-$Co\mathbf{Alg}_s \to \mathbf{Ps}$-$S$-$Co\mathbf{Alg}$ have right* 2*-adjoints such that their counits are pseudonatural equivalences. Therefore there is a pseudonatural equivalence*



in which the curved arrow denotes the right 2*-adjoint to the inclusion*

$$S\text{-}Co\mathbf{Alg}_s \to \mathbf{Ps}\text{-}S\text{-}Co\mathbf{Alg}.$$

7.1. On pseudo-Kan extensions. One simple application of Corollary 7.1 is about pseudo-Kan extensions. In the 3-category 2-**CAT**, the natural notion of Kan extension is that of pseudo-Kan extension. More precisely, a *right pseudo-Kan extension of a pseudofunctor*

$$\mathcal{D} : \mathcal{S} \to \mathfrak{A}$$

along a pseudofunctor $\mathrm{h} : \mathcal{S} \to \dot{\mathcal{S}}$*, denoted by* Ps-$\mathcal{R}an_{\mathrm{h}}\mathcal{D}$*, is (if it exists) a birepresentation of the pseudofunctor* $W \to [\mathcal{S}, \mathfrak{A}]_{PS}(W \circ \mathrm{h}, \mathcal{D})$. Recall that birepresentations are unique up to equivalence and, therefore, right pseudo-Kan extensions are unique up to pseudonatural equivalence.

Assuming that $\mathcal{S}$ and $\dot{\mathcal{S}}$ are small 2-categories, in the setting described above, the following are natural problems on pseudo-Kan extensions: (1) investigating the left biadjointness of the pseudofunctor $W \to W \circ \mathrm{h}$, namely, investigating whether all right pseudo-Kan extensions along h exist; (2) understanding pointwise pseudo-Kan extensions (that is to say, proving the existence of right pseudo-Kan extensions provided that $\mathfrak{A}$ has all bilimits).

It is shown in [1] that, if $\mathcal{S}_0$ denotes the discrete 2-category of the objects of $\mathcal{S}$, the restriction

$$[\mathcal{S}, \mathfrak{A}] \to [\mathcal{S}_0, \mathfrak{A}]$$

is 2-comonadic, provided that $[\mathcal{S}, \mathfrak{A}] \to [\mathcal{S}_0, \mathfrak{A}]$ has a right 2-adjoint

$$\mathcal{R}an_{\mathrm{h}} : [\mathcal{S}_0, \mathfrak{A}] \to [\mathcal{S}, \mathfrak{A}].$$

is a commutative diagram, such that E has a right 2-adjoint R. Then Proposition 1.1 gives necessary and sufficient conditions to construct a right 2-adjoint to J. And, of course, as a consequence of Theorem 5.2, we have the analogous version for pseudocomonads.

Corollary 7.1. *Let $T : \mathfrak{A} \to \mathfrak{A}$ and $S : \mathfrak{C} \to \mathfrak{C}$ be pseudocomonads. If the diagram*

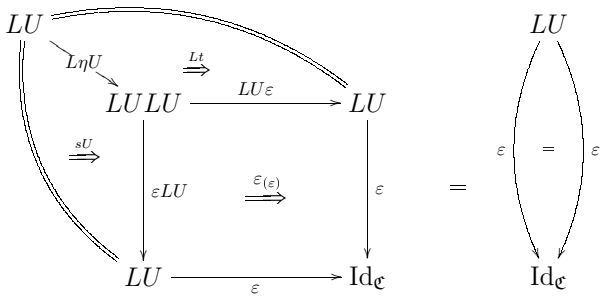
$$\begin{array}{ccc} \mathbf{Ps}\text{-}T\text{-}Co\mathbf{Alg} & \xrightarrow{J} & \mathbf{Ps}\text{-}S\text{-}Co\mathbf{Alg} \\ \downarrow{\scriptstyle \hat{L}} & & \downarrow{\scriptstyle L} \\ \mathfrak{A} & \xrightarrow{E} & \mathfrak{C} \end{array}$$

commutes and E has a right biadjoint, then J has a right biadjoint provided that $\mathbf{Ps}$*-T-Co**Alg** *has descent objects.*

Actually, if, furthermore, $T : \mathfrak{A} \to \mathfrak{A}$ and $S : \mathfrak{C} \to \mathfrak{C}$ are 2-comonads such that $\mathfrak{A}, \mathfrak{C}$ have (and T, S preserve) strict descent objects and $\mathfrak{A}$ has (and T preserves) **Cat**-equalizers, as a consequence of the enriched adjoint triangle theorem and the strict version of the biadjoint triangle theorem/coherence result [10], we prove Corollary 7.2.

Corollary 7.2. *Let $\mathfrak{A}, \mathfrak{C}$ be 2-categories such that $\mathfrak{A}$ and $\mathfrak{C}$ have strict descent objects. Assume that $T : \mathfrak{A} \to \mathfrak{A}$ and $S : \mathfrak{C} \to \mathfrak{C}$ are 2-comonads which preserve strict descent objects. If, furthermore, T preserves (and $\mathfrak{A}$ has)* **Cat**-*equalizers, we have that, given a commutative diagram*

$$\begin{array}{ccc} T\text{-}Co\mathbf{Alg}_s & \xrightarrow{\hat{J}} & S\text{-}Co\mathbf{Alg}_s \\ \downarrow & & \downarrow \\ \mathbf{Ps}\text{-}T\text{-}Co\mathbf{Alg} & \xrightarrow{J} & \mathbf{Ps}\text{-}S\text{-}Co\mathbf{Alg} \\ \downarrow & & \downarrow \\ \mathfrak{A} & \xrightarrow{E} & \mathfrak{C} \end{array}$$



Remark 2.3. A biadjunction $L \dashv U$ has at least one associated data of counit and unit $(L, U, \eta, \varepsilon, s, t)$ as described above. Unlike the strict case, this data is not uniquely determined up to isomorphism, although it is determined up to equivalence. Still, to construct the pseudofunctors of our biadjoint triangle theorems, we often need a chosen data of units and counits. Thus, in these cases, herein, we say "the biadjunction $(L, U, \eta, \varepsilon, s, t)$".

If it exists, a birepresentation of a pseudofunctor $\mathcal{U} : \mathfrak{C} \to \mathbf{CAT}$ is an object X of $\mathfrak{C}$ endowed with a pseudonatural equivalence $\mathfrak{C}(X, -) \simeq \mathcal{U}$. When $\mathcal{U}$ has a birepresentation, we say that $\mathcal{U}$ is birepresentable. And, in this case, by Lemma 2.1, its birepresentation is unique up to equivalence.

Lemma 2.4 ([14])**.** *Assume that $\mathcal{U} : \mathfrak{C} \to [\mathfrak{B}^{\mathrm{op}}, \mathbf{CAT}]_{PS}$ is a pseudofunctor such that, for each object X of $\mathfrak{C}$, $\mathcal{U}X$ has a birepresentation*

$$e_X : \mathcal{U}X \simeq \mathfrak{B}(-, UX).$$

Then there is a pseudofunctor $U : \mathfrak{C} \to \mathfrak{B}$ such that the pseudonatural equivalences e_X are the components of a pseudonatural equivalence $\mathcal{U} \simeq \mathfrak{B}(-, U-)$, in which $\mathfrak{B}(-, U-)$ denotes the pseudofunctor

$$\begin{array}{rcl} \mathfrak{C} & \to & [\mathfrak{B}^{\mathrm{op}}, \mathbf{CAT}]_{PS} \\ X & \mapsto & \mathfrak{B}(-, UX) \end{array}$$

As a consequence, a pseudofunctor $L : \mathfrak{B} \to \mathfrak{C}$ has a right biadjoint if and only if, for each object X of $\mathfrak{C}$, the pseudofunctor $\mathfrak{C}(L-, X)$ is birepresentable. *Id est*, each object X is endowed with an object UX of $\mathfrak{B}$ and a pseudonatural equivalence

$$\mathfrak{C}(L-, X) \simeq \mathfrak{B}(-, UX).$$

The natural notion of limit in our context is that of (weighted) bilimit, instead of the rather restrictive notion of (weighted) strict 2-limit. Namely, assume that $\mathcal{S}$ is a small 2-category, if $W : \mathcal{S} \to \mathbf{Cat}, \mathcal{D} : \mathcal{S} \to \mathfrak{A}$ are pseudofunctors, the (weighted) bilimit, denoted herein by $\{W, \mathcal{D}\}_{\text{bi}}$, if it exists, is a birepresentation of the 2-functor

$$\begin{array}{rcl} \mathfrak{A}^{\text{op}} & \to & \mathbf{CAT} \\ X & \mapsto & [\mathcal{S}, \mathbf{CAT}]_{PS}(W, \mathfrak{A}(X, \mathcal{D}-)) \end{array}$$

That is to say, if it exists, a bilimit is an object $\{W, \mathcal{D}\}_{\text{bi}}$ endowed with a pseudonatural equivalence (in X) $\mathfrak{A}(X, \{W, \mathcal{D}\}_{\text{bi}}) \simeq [\mathcal{S}, V]_{PS}(W, \mathfrak{A}(X, \mathcal{D}-))$. Since, by the Yoneda lemma, $\{W, \mathcal{D}\}_{\text{bi}}$ is unique up to equivalence, we say *the* (weighted) bilimit.

Still, if W and $\mathcal{D}$ are 2-functors, recall that the (strict) weighted limit $\{W, \mathcal{D}\}$ is, if it exists, a 2-representation of the 2-functor

$$X \mapsto [\mathcal{S}, \mathbf{CAT}](W, \mathfrak{A}(X, \mathcal{D}-)),$$

in which $[\mathcal{S}, \mathbf{CAT}]$ is the 2-category of 2-functors $\mathcal{S} \to \mathbf{CAT}$, 2-natural transformations and modifications.

It is easy to see that $\mathbf{CAT}$ is bicategorically complete. More precisely, if $W : \mathcal{S} \to \mathbf{Cat}$ and $\mathcal{D} : \mathcal{S} \to \mathbf{Cat}$ are pseudofunctors, then

$$\{W, \mathcal{D}\}_{\text{bi}} \simeq [\mathcal{S}, \mathbf{CAT}]_{PS}(W, \mathcal{D}).$$

Moreover, from the bicategorical Yoneda lemma of [14], we get the (strong) Yoneda lemma.

Lemma 2.5 ((Strong) Yoneda Lemma)**.** *Let $\mathcal{D} : \mathcal{S} \to \mathfrak{A}$ be a pseudofunctor between 2-categories. There is a pseudonatural equivalence $\{\mathcal{S}(a, -), \mathcal{D}\}_{\text{bi}} \simeq \mathcal{D}(a)$.*

Proof: By the bicategorical Yoneda lemma, we have a pseudonatural equivalence (in X and a) $[\mathcal{S}, \mathbf{CAT}]_{PS}(\mathcal{S}(a, -), \mathfrak{A}(X, \mathcal{D}-)) \simeq \mathfrak{A}(X, \mathcal{D}(a))$. Therefore $\mathcal{D}(a)$ is *the* bilimit $\{\mathcal{S}(a, -), \mathcal{D}\}_{\text{bi}}$. ∎

Let $\mathcal{S}$ be a small 2-category and $\mathcal{D} : \mathcal{S} \to \mathfrak{A}$ be a pseudofunctor. Consider the pseudofunctor

$$\begin{array}{rcl} [\mathcal{S}, \mathfrak{C}]_{PS} & \to & [\mathfrak{A}^{\text{op}}, \mathbf{CAT}]_{PS} \\ W & \mapsto & \mathbb{D}_W \end{array}$$

Also, it is known that, if $\mathfrak{A}$ has strict descent objects and T is a 2-comonad on $\mathfrak{A}$ which preserves strict descent objects, then the forgetful 2-functor $T\text{-}Co\mathbf{Alg}_s \to \mathfrak{A}$ creates strict descent objects. This fact gives what we need to formulate the result of [10] as it is done there: namely, there is a right 2-adjoint to the inclusion $T\text{-}Co\mathbf{Alg}_s \to \mathfrak{A}$, and its counit is a pseudonatural equivalence, if $\mathfrak{A}$ has strict descent objects and T preserves them.

7. On lifting biadjunctions

One of the most elementary corollaries of the adjoint triangle theorem [2] is about lifting adjunctions to adjunctions between the Eilenberg Moore categories. That is to say, let $T : \mathbb{A} \to \mathbb{A}$ and $S : \mathbb{C} \to \mathbb{C}$ be comonads. Assume that the diagram

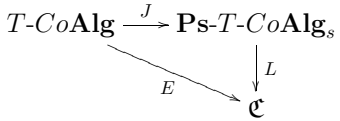
$$\begin{array}{ccc} T\text{-}Co\mathbf{Alg} & \xrightarrow{J} & S\text{-}Co\mathbf{Alg} \\ \downarrow{\scriptstyle \hat{L}} & & \downarrow{\scriptstyle L} \\ \mathbb{A} & \xrightarrow{E} & \mathbb{C} \end{array}$$

commutes and E has a right adjoint R. As a particular case of Eduardo Dubuc's result, we know sufficient (and necessary) conditions under which the adjunction $E \dashv R$ can be lifted to an adjunction $J \dashv G$ between the coalgebras, since this setting gives the commutative diagram

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{J} & S\text{-}Co\mathbf{Alg} \\ & {\scriptstyle E\circ\hat{L}} \searrow \quad \swarrow {\scriptstyle L} & \\ & \mathbb{C} & \end{array}$$

in which L is comonadic and $E \circ \hat{L}$ has a right adjoint. Therefore, by the enriched version 1.1, we have the analogous result for 2-comonads and 2-categories. Namely, let $T : \mathbb{A} \to \mathbb{A}$ and $S : \mathbb{C} \to \mathbb{C}$ be 2-comonads. Assume that

$$\begin{array}{ccc} T\text{-}Co\mathbf{Alg}_s & \xrightarrow{J} & S\text{-}Co\mathbf{Alg}_s \\ \downarrow{\scriptstyle \hat{L}} & & \downarrow{\scriptstyle L} \\ \mathfrak{A} & \xrightarrow{E} & \mathfrak{C} \end{array}$$



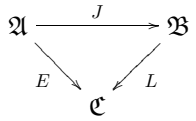
The firs part of Stephen Lack's result on this triangle says that J has a right 2-adjoint G if the diagrams $\mathcal{D}_Y : \Delta \to T\text{-}Co\mathbf{Alg}$ have descent objects (for every object Y) . And, hence, Theorem 5.2 can be seen as a generalization of his construction. And, actually, his result is a consequence of the strict version, that is to say, Theorem 6.1.

The second part says that the counit of the 2-adjunction $J \dashv G$ is a pseudonatural equivalence if $E \circ \mathrm{Ps}\mathcal{R}an_{\mathrm{j}}\mathcal{D}_Y$ is of effective descent (for every object Y). Therefore, Corollary 5.3 and Corollary 5.5 are generalizations of the second part. Also, when using Theorem 6.1 (which is the Theorem 4.4) and Corollary 5.5, we get a proof of the result of [10] on pseudoalgebras and possible (strict) generalizations (because $E \dashv R$ and $L \dashv U$ induce the same pseudocomonad).

Theorem 6.1 (Strict Biadjoint Triangle Theorem)**.** *Let*

$$E : \mathfrak{A} \to \mathfrak{C}, J : \mathfrak{A} \to \mathfrak{B}, L : \mathfrak{B} \to \mathfrak{C}$$

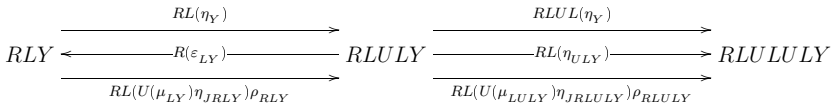
be 2*-functors such that*



commutes. Assume that (E, R, ρ, μ) *is a* 2*-adjunction and* $(L, U, \eta, \varepsilon, s, t)$ *is a biadjunction such that* L *is strictly pseudoprecomonadic and* $J(\mathfrak{A}(A, B))$ *is contained in the subcategory of strict morphisms of*

$$\mathfrak{B}(JA, JB) \cong \mathbf{Ps}\text{-}Co\mathbf{Alg}(KJA, KJB).$$

Then J *has a right* 2*-adjoint if and only if, for every object* Y *of* $\mathfrak{B}$*, the strict descent object of the diagram* $\mathcal{D}_Y : \Delta \to \mathfrak{A}$



(with the obvious 2*-cells) exists in* $\mathfrak{A}$*.*

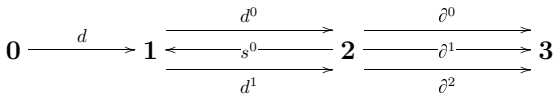
in which the 2-functor $\mathbb{D}_W$ is given by $X \mapsto [\mathcal{S}, \mathbf{CAT}]_{PS}(W, \mathfrak{A}(X, \mathcal{D}-))$. By Lemma 2.4, we conclude that it is possible to get a pseudofunctor $\{-, \mathcal{D}\}_{\mathrm{bi}}$ defined in a full sub-2-category of $[\mathcal{S}, \mathbf{CAT}]_{PS}$ of weights $W : \mathcal{S} \to \mathbf{CAT}$ such that $\mathfrak{A}$ has the bilimit $\{W, \mathcal{D}\}_{\mathrm{bi}}$.

3. Descent Object

In this section, we describe the 2-categorical limits called *descent objects*. We deal with these 2-limits via pointwise right Kan extensions, since it seems more natural to give the "exact conditions" when using this approach.

We need both constructions, strict descent objects and descent objects [15]. Our domain 2-category, denoted by Δ, is the dual of that defined at Definition 2.1 in [11].

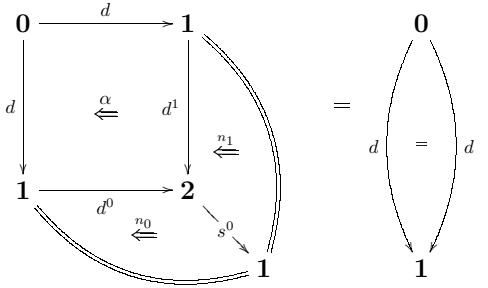
Definition 3.1. We denote by $\dot{\Delta}$ the 2-category generated by the diagram



with the 2-cells:

$$\begin{aligned} \sigma_{ik} &: \partial^k d^i \cong \partial^i d^{k-1}, \text{ if } i < k \\ n_0 &: s^0 d^0 \cong \mathrm{Id}_{\mathbf{1}} \\ n_1 &: \mathrm{Id}_{\mathbf{1}} \cong s^0 d^1 \\ \alpha &: d^1 d \cong d^0 d \end{aligned}$$

satisfying the equation below



The 2-category Δ is, herein, by definition, the full sub-2-category of $\dot{\Delta}$ with objects $\mathbf{1}, \mathbf{2}, \mathbf{3}$. We denote the inclusion by $\mathrm{j} : \Delta \to \dot{\Delta}$.

Given a 2-functor $D : \Delta \to \mathfrak{A}$, we denote by $\mathcal{R}an_{\mathrm{j}}D : \dot{\Delta} \to \mathfrak{A}$ the pointwise right Kan extension of D along j (if it exists). Recall that

$$\mathcal{R}an_{\mathrm{j}}D(a) := \left\{\dot{\Delta}(a, \mathrm{j}-), D\right\}$$

and that $\mathcal{R}an_{\mathrm{j}}D$ is, actually, an extension of D, since j is **Cat**-fully faithful [7].

Assuming that $\mathcal{R}an_{\mathrm{j}}D$ exists, we call $\mathcal{R}an_{\mathrm{j}}D(\mathbf{0})$ herein the strict descent object of D. This terminology is coherent with [10, 14]. Moreover, assume that $\mathfrak{D} : \dot{\Delta} \to \mathfrak{A}$ is a 2-functor such that $\mathcal{R}an_{\mathrm{j}}(\mathfrak{D} \circ \mathrm{j})$ exists, then we get a comparison 1-cell $\mathfrak{D} \to \mathcal{R}an_{\mathrm{j}}(\mathfrak{D} \circ \mathrm{j})$. *We say that $\mathfrak{D}$ is of strict descent if this comparison 1-cell is an isomorphism.* That is to say, $\mathfrak{D}$ is of strict descent if $\mathfrak{D}$ is the pointwise right Kan extension of $\mathfrak{D} \circ \mathrm{j}$.

In analogy to the strict case, we define the pointwise pseudo-Kan extension, which is stronger then the notion of quasi-Kan extension already considered by John Gray [4, 6].

Definition 3.2. [Pointwise pseudo-Kan Extension] Assume that $D : \Delta \to \mathfrak{A}$ is a pseudofunctor. If $\left\{\dot{\Delta}(\mathbf{0}, \mathrm{j}-), D\right\}_{\mathrm{bi}}$ exists, we define the *pointwise right pseudo-Kan extension* as follows:

$$\begin{aligned} \mathrm{Ps}\mathcal{R}an_{\mathrm{j}}D : \ \dot{\Delta} &\to \mathfrak{A} \\ a &\mapsto \left\{\dot{\Delta}(a, \mathrm{j}-), D\right\}_{\mathrm{bi}} \end{aligned}$$

Note that a pointwise right pseudo-Kan extension is unique up to pseudonatural equivalence.

Since j is fully faithful, by the Yoneda Lemma 2.5, $\mathrm{Ps}\mathcal{R}an_{\mathrm{j}}D$ is an extension of $D : \Delta \to \mathfrak{A}$ up to pseudonatural equivalence. More precisely, $(\mathrm{Ps}\mathcal{R}an_{\mathrm{j}}D) \circ \mathrm{j}$ is pseudonaturally equivalent to D. Also, as in the strict case, if $\mathfrak{D} : \dot{\Delta} \to \mathfrak{A}$ is a pseudofunctor such that the pointwise pseudo-Kan extension $\mathrm{Ps}\mathcal{R}an_{\mathrm{j}}(\mathfrak{D} \circ \mathrm{j})$ exists, we get a comparison 1-cell

$$\mathfrak{D} \to \mathrm{Ps}\mathcal{R}an_{\mathrm{j}}(\mathfrak{D} \circ \mathrm{j}).$$

But, postponing its construction (to section 7), for our purposes herein, we say that $\mathfrak{D}$ *is of effective descent if* $\mathrm{Ps}\mathcal{R}an_{\mathrm{j}}(\mathfrak{D} \circ \mathrm{j})$ *is pseudonaturally equivalent to* $\mathfrak{D}$.

If $D : \Delta \to \mathfrak{A}$ is a pseudofunctor, $\mathrm{Ps}\mathcal{R}an_{\mathrm{j}}D(\mathbf{0})$ is called the descent object of D. Thus, if it exists, the descent object is unique up to equivalence. Moreover, the inclusion $\mathrm{j} : \Delta \to \dot{\Delta}$ has the following special property: if the

a biequivalence. And this proves that E is pseudocomonadic if and only if E has a right biadjoint and it creates absolute effective descent diagrams.

Lemma 5.4 (Pseudocomonadicity [11])**.** *A left biadjoint pseudofunctor*

$$L : \mathfrak{B} \to \mathfrak{C}$$

is pseudocomonadic if and only if it creates absolute effective descent diagrams.

As a consequence of Lemma 5.4, compositions of pseudocomonadic functors are pseudocomonadic. Furthermore, within the setting of Theorem 4.1, if J has a right biadjoint and E is pseudocomonadic, then J is pseudocomonadic as well.

Corollary 5.5. *Let $LJ = E$ be a biadjoint triangle satisfying the hypotheses of Theorem 5.2 such that L is pseudocomonadic. We denote by G the obtained right biadjoint of J. And we assume that the biadjunctions $E \dashv R$ and $L \dashv U$ induce the same pseudocomonad.*

Then the pseudofunctor $E : \mathfrak{A} \to \mathfrak{C}$ preserves the effective descent diagram

$$\mathcal{D}_Y : \Delta \to \mathfrak{A}$$

for every object Y of $\mathfrak{B}$ if and only if the counit $\overline{\varepsilon}$ of the biadjunction $J \dashv G$ is a pseudonatural equivalence. That is to say,

$$E \circ \mathrm{Ps}\mathcal{R}an_{\mathrm{j}}(\mathcal{D}_Y)$$

is of effective descent for every object Y of $\mathfrak{B}$ if and only if the counit

$$\overline{\varepsilon} : JG \to \mathrm{Id}_{\mathfrak{B}}$$

is a pseudonatural equivalence.

Proof: Since L creates absolute effective descent diagrams, it is easy to see that $J \circ \mathcal{D}_Y : \dot{\Delta} \to \mathfrak{A}$ is effective descent (for every object Y of $\mathfrak{B}$) if and only if $LJ \circ \mathcal{D}_Y \simeq E \circ \mathcal{D}_Y$ is effective descent (for every object Y of $\mathfrak{B}$). ∎

6. Coherence

In the setting of [10], we have a 2-comonad T on a 2-category $\mathfrak{C}$. If

$$T\text{-}Co\mathbf{Alg}_s$$

denotes the 2-category of strict algebras, morphisms and transformations, then the canonical 2-adjunction $E \dashv R : Co\mathbf{Alg} \to \mathfrak{C}$ induces an Eilenberg Moore factorization w.r.t. the pseudocomonadic functor $\mathbf{Ps}\text{-}T\text{-}Co\mathbf{Alg} \to \mathfrak{C}$.

Furthermore, the unit $\overline{\eta}$ of the biadjunction $J \dashv G$ is a pseudonatural equivalence if and only if the 2-functor $\overline{D_A} : \dot{\Delta} \to \mathfrak{A}$ (with omitted 2-cells)

$$A \xrightarrow{\rho_A} REA \underset{RE(\rho_A)}{\overset{\rho_{REA}}{\underset{R(\mu_{EA})}{\rightleftarrows}}} REREA \underset{RERE(\rho_A)}{\overset{\rho_{REREA}}{\underset{RE(\rho_{REA})}{\rightrightarrows}}} REREREA$$

is of effective descent for every object A of $\mathfrak{A}$.

Once we know that pseudocomonadic pseudofunctors create absolute effective descent diagrams, we can get the pseudocomonadicity theorem [11] as a consequence of the Biadjoint Theorem 5.2 and Corollary 5.3. To do so, assuming that $E : \mathfrak{A} \to \mathfrak{C}$ is a left biadjoint which creates absolute effective descent diagrams, we have to consider the commutative triangle of the Eilenberg Moore factorization

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{J} & \textbf{Ps-}Co\textbf{Alg} \\ & \searrow^{E} & \downarrow{\scriptstyle L} \\ & & \mathfrak{C} \end{array}$$

Note that, for each object Y of $\textbf{Ps-}Co\textbf{Alg}$, the diagram $\mathcal{D}_Y : \dot{\Delta} \to \mathfrak{A}$ is such that $E \circ \mathcal{D}_Y$ has a pointwise right pseudo-Kan extension

$$\mathrm{Ps}\mathcal{R}an_{\mathrm{j}}(E \circ \mathcal{D}_Y) \simeq L \circ D_Y$$

which is an absolute effective descent diagram. Therefore, since E creates absolute effective descent diagrams, we conclude that $\mathcal{D}_Y$ has a descent object. And, then, by Theorem 5.2, J is left biadjoint to a pseudofunctor G. Moreover, since

$$E \circ \mathrm{Ps}\mathcal{R}an_{\mathrm{j}}(\mathcal{D}_Y) = LJ \circ \mathrm{Ps}\mathcal{R}an_{\mathrm{j}}(\mathcal{D}_Y) \simeq L \circ D_Y$$

is an absolute effective descent diagram and L creates absolute effective descent diagrams, we conclude that $J \circ \mathrm{Ps}\mathcal{R}an_{\mathrm{j}}(\mathcal{D}_Y)$ is of effective descent. Thus, by Corollary 5.3, we conclude that the counit of the biadjunction $J \dashv G$ is a pseudonatural equivalence.

Furthermore, for each object A of $\mathfrak{A}$, $E \circ \overline{D}_A$ is an absolute effective descent diagram. And, since E creates absolute effective descent diagrams, we conclude that $\overline{D}_A$ is of effective descent. Thus, by Corollary 5.3, the unit of the biadjunction $J \dashv G$ is a pseudonatural equivalence. Therefore J is, indeed,

pointwise right Kan extension of a 2-functor D along j exists, it is a pointwise right pseudo-Kan extension. In particular, strict descent objects are descent objects.

Theorem 3.3. *Let $D : \Delta \to \mathfrak{B}$ be a pseudofunctor. The right pointwise pseudo-Kan extension $\mathrm{Ps}\mathcal{R}an_{\mathrm{j}}D$ exists and $\mathrm{Ps}\mathcal{R}an_{\mathrm{j}}D = \mathfrak{D}$ if and only if we have a pseudonatural equivalence*

$$\mathrm{Ps}\mathcal{R}an_{\mathrm{j}}\mathfrak{B}(a, D-) \simeq \mathfrak{B}(a, \mathfrak{D}-).$$

Proof: This follows from the definition of weighted bilimit. That is to say, by definition,

$$\left\{\dot{\Delta}(\mathbf{0}, \mathrm{j}-), \mathfrak{B}(a, D-)\right\}_{\mathrm{bi}} \simeq \mathfrak{B}\left(a, \mathfrak{D}(\mathbf{0})\right)$$

is a pseudonatural equivalence in a iff $\left\{\dot{\Delta}(\mathbf{0}, \mathrm{j}-), D\right\}_{\mathrm{bi}}$ exists and

$$\left\{\dot{\Delta}(\mathbf{0}, \mathrm{j}-), D\right\}_{\mathrm{bi}} = \mathfrak{D}(\mathbf{0}).$$

■

Herein, we say that an effective descent diagram $D : \dot{\Delta} \to \mathfrak{B}$ is preserved by a pseudofunctor $L : \mathfrak{B} \to \mathfrak{C}$ if $L \circ D$ is of effective descent. While, $D : \dot{\Delta} \to \mathfrak{B}$ is an absolute effective descent diagram if D is preserved by any pseudofunctor L.

In this setting, a pseudofunctor $L : \mathfrak{B} \to \mathfrak{C}$ is said to reflect absolute effective descent diagrams if, whenever a 2-functor $D : \dot{\Delta} \to \mathfrak{B}$ is such that $L \circ D$ is an absolute effective descent diagram, D is of effective descent. Moreover, we say herein that a pseudofunctor $L : \mathfrak{B} \to \mathfrak{C}$ creates absolute effective descent diagrams if L reflects absolute effective descent diagrams and, whenever a diagram $D : \Delta \to \mathfrak{B}$ is such that $L \circ D$ has a pointwise right pseudo-Kan extension

$$\mathrm{Ps}\mathcal{R}an_j(L \circ D) : \dot{\Delta} \to \mathfrak{C}$$

which is an absolute effective descent diagram, D has a pointwise right pseudo-Kan extension such that $L \circ \mathrm{Ps}\mathcal{R}an_{\mathrm{j}}(D) \simeq \mathrm{Ps}\mathcal{R}an_j(L \circ D)$.

Remark 3.4. The dual notion of descent object is that of codescent object, described by Stephen Lack [10] and Ivan Le Creurer, Francisco Marmolejo, Enrico Vitale [11]. Of course, to approach such weighted colimit, we can consider the dual of the concepts given above. That is to say, considering the inclusion $v : \Delta^{\mathrm{op}} \to \dot{\Delta}^{\mathrm{op}}$, if $D : \Delta^{\mathrm{op}} \to \mathfrak{A}$ is a 2-functor, we call $\mathcal{L}an_v D(\mathbf{0})$

the strict codescent object of D, when $\mathcal{L}an_v D$ denotes the pointwise left Kan extension of D along v (provided that $\mathcal{L}an_v D$ exists). And we say that $\mathfrak{D} : \dot{\Delta}^{\mathrm{op}} \to \mathfrak{A}$ is strictly effective codescent if the comparison 1-cell $\mathcal{L}an_v(\mathfrak{D} \circ v) \to \mathfrak{D}$ is an isomorphism.

And, also, we can define the pointwise left pseudo-Kan extension $\mathrm{Ps}\mathcal{L}an_v D$, via weighted bicolimits. Again, we say that $\mathfrak{D} : \dot{\Delta} \to \mathfrak{A}$ is of effective codescent if $\mathrm{Ps}\mathcal{L}an_v(\mathfrak{D} \circ v)$ is pseudonaturally equivalent to $\mathfrak{D}$.

4. Biadjoint Triangles

In this section, we give a 2-dimensional version of the adjoint triangle theorem [2]. As it is shown in [11], for each object X of an 2-category $\mathfrak{B}$, the unit $\eta : \mathrm{Id}_{\mathfrak{A}} \to UL$ of a biadjunction $L \dashv U : \mathfrak{B} \to \mathfrak{C}$ gives rise to a 2-functor $D_X : \dot{\Delta} \to \mathfrak{B}$. That is to say, the diagram below with the obvious 2-cells induced by Definition 2.2 of unit and counit.

$$X \xrightarrow{\eta_X} ULX \underset{UL(\eta_X)}{\overset{\eta_{ULX}}{\underset{U(\varepsilon_{LX})}{\rightleftarrows}}} ULULX \underset{ULUL(\eta_X)}{\overset{\eta_{ULULX}}{\underset{UL(\eta_{ULX})}{\rightrightarrows}}} ULULULX$$

More precisely, if $(L, U, \eta, \varepsilon, s, t) : \mathfrak{B} \to \mathfrak{C}$ is the biadjunction above, for each object X of $\mathfrak{B}$, we define the 2-cells of the diagram $D_X : \dot{\Delta} \to \mathfrak{B}$ as follows: $D_X(\alpha) = \left(\eta_{\eta_X} : UL(\eta_X)\eta_X \cong \eta_{ULX}\eta_X\right)$ is the isomorphism 2-cell component of the pseudonatural transformation η at the morphism η_X. And, analogously, the images of the 2-cells σ_{ik} are defined to be the corresponding components of η.

$$\begin{aligned}
D_X(\sigma_{01}) &= \left(\eta_{\eta_{ULX}}\right) : UL(\eta_{ULX})\eta_{ULX} \cong \eta_{(UL)^2X}\eta_{ULX} \\
D_X(\sigma_{02}) &= \left(\eta_{UL(\eta_X)}\right) : (UL)^2(\eta_X)\eta_{ULX} \cong \eta_{(UL)^2X}UL(\eta_X) \\
D_X(\sigma_{12}) &= \left(\left(\phi^{UL}_{\eta_{ULX},\eta_X}\right)^{-1} UL(\eta_X)\phi^{UL}_{UL(\eta_X),\eta_X}\right) : (UL)^2(\eta_X)UL(\eta_X) \cong UL(\eta_{ULX})UL(\eta_X)
\end{aligned}$$

In which, if v, w are composable morphisms of $\mathfrak{A}$,

$$\phi^{UL}_{v,w} : UL(v)UL(w) \cong UL(vw)$$

originally in [11]), that is to say, a left biadjoint pseudofunctor $L : \mathfrak{B} \to \mathfrak{C}$ is pseudoprecomonadic if and only if the diagram $D_X : \dot{\Delta} \to \mathfrak{B}$ (with omitted 2-cells)

$$X \xrightarrow{\eta_X} ULX \underset{UL(\eta_X)}{\overset{\eta_{ULX}}{\underset{U(\varepsilon_{LX})}{\rightleftarrows}}} ULULX \underset{ULUL(\eta_X)}{\overset{\eta_{ULULX}}{\underset{UL(\eta_{ULX})}{\rightrightarrows}}} ULULULX$$

is of effective descent for every object X of $\mathfrak{B}$. And, therefore, we can reformulate our Biadjoint Triangle Theorem 4.1.

Theorem 5.2 (Biadjoint Triangle Theorem)**.** *Let $E : \mathfrak{A} \to \mathfrak{C}$, $J : \mathfrak{A} \to \mathfrak{B}$, $L : \mathfrak{B} \to \mathfrak{C}$ be pseudofunctors such that there is a pseudonatural equivalence*

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{J} & \mathfrak{B} \\ & {\scriptstyle E}\searrow \quad \swarrow{\scriptstyle L} & \\ & \mathfrak{C} & \end{array}$$

Assume that (E, R, ρ, μ, v, w) and $(L, U, \eta, \varepsilon, s, t)$ are biadjunctions and L is pseudoprecomonadic. Then J has a right biadjoint if and only if, for every object Y of $\mathfrak{B}$, $\mathfrak{A}$ has the descent object of the diagram $\mathcal{D}_Y : \dot{\Delta} \to \mathfrak{A}$ (with the obvious 2-cells)

$$RLY \underset{RL(\eta_Y)}{\overset{RL(U(\mu_{LY})\eta_{JRLY})\rho_{RLY}}{\underset{R(\varepsilon_{LY})}{\rightleftarrows}}} RLULY \underset{RLUL(\eta_Y)}{\overset{RL(U(\mu_{LULY})\eta_{JRLULY})\rho_{RLULY}}{\underset{RL(\eta_{ULY})}{\rightrightarrows}}} RLULULY$$

Corollary 5.3. *Let $LJ = E$ be a biadjoint triangle satisfying the hypotheses of Theorem 5.2. We denote by G the obtained right biadjoint of J. And we assume that the biadjunction $E \dashv R$ induces the same pseudocomonad as $L \dashv U$.*

Then the 2-functor $J : \mathfrak{A} \to \mathfrak{B}$ preserves the effective descent diagram

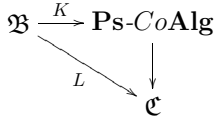
$$\mathcal{D}_Y : \dot{\Delta} \to \mathfrak{A}$$

for every object Y of $\mathfrak{B}$ if and only if the counit $\overline{\varepsilon}$ of the biadjunction $J \dashv G$ is a pseudonatural equivalence. That is to say, $J \circ \mathrm{Ps}\mathcal{R}an_{\mathrm{j}}(\mathcal{D}_Y)$ is of effective descent for every object Y of $\mathfrak{B}$ if and only if the counit

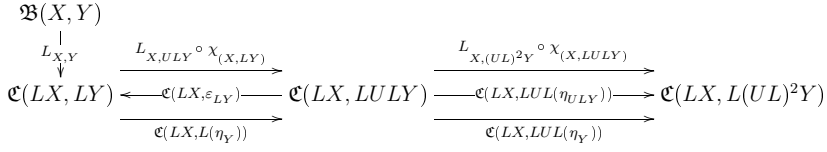
$$\overline{\varepsilon} : JG \to \mathrm{Id}_{\mathfrak{B}}$$

is a pseudonatural equivalence.

Proposition 5.1. *Let $L : \mathfrak{B} \to \mathfrak{C}$ be a pseudofunctor such that L is left biadjoint to U. Then, as described above, we get the corresponding Eilenberg Moore factorization*

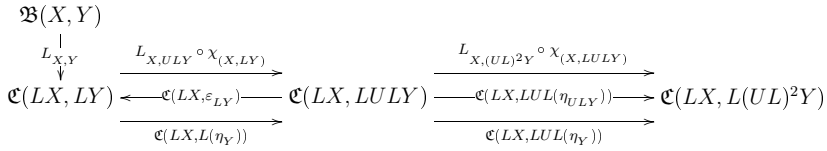


The comparison pseudofunctor K is locally an isomorphism (locally an equivalence) if and only if the diagram $\mathbb{D}_Y^X : \dot{\Delta} \to \mathbf{CAT}$ (with omitted 2-cells)



is of strict descent (effective descent) for every pair of objects X, Y of $\mathfrak{B}$.

Proof: This proposition is easy to check (only using definitions). And, actually, it follows from an observation already given in [13]: if L is strictly pseudocomonadic (that is to say, the comparison 2-functor is an isomorphism), then



is of strict descent for every pair X, Y of objects in $\mathfrak{B}$. ∎

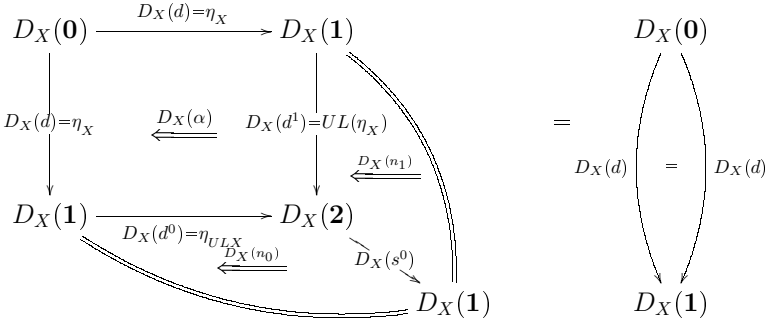
We say that a left biadjoint functor $L : \mathfrak{B} \to \mathfrak{C}$ is strictly pseudoprecomonadic if the comparison pseudofunctor K is fully faithful (which means, K is locally an isomorphism). Thereby, Propostion 5.1 gives a characterization os pseudoprecomonadic and strictly pseudoprecomonadic pseudofunctors.

Also, in the setting of Proposition 5.1, by the biadjunction $L \dashv U$, we conclude that, for each pair X, Y of objects in $\mathfrak{B}$, the diagram $\mathbb{D}_Y^X$ is pseudonaturally equivalent to the diagram $\mathfrak{B}(X, D_Y)$. Thus, by Theorem 3.3, D_Y is of effective descent for every object Y of $\mathfrak{B}$ if and only if $\mathbb{D}_Y^X$ is of effective descent for every pair of objects X, Y of $\mathfrak{B}$. We conclude, therefore, the characterization of pseudoprecomonadic pseudofunctors (which was proven

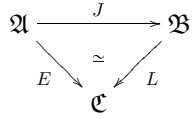
denotes the 2-cell isomorphism of the pseudofunctor UL. At last, we define the images of n_0 and n_1.

$$\begin{aligned} D_X(n_0) &= (t_{LX}) : U(\varepsilon_{LX})\eta_{ULX} \cong \mathrm{Id}_{ULX} \\ D_X(n_1) &= (Us)_X \\ &= \left((\phi^U_{\varepsilon_{LX}, L(\eta_X)})^{-1} U(s_X)\phi^U_{ULX}\right) : \mathrm{Id}_{ULX} \cong U(\varepsilon_{LX})UL(\eta_X) \end{aligned}$$

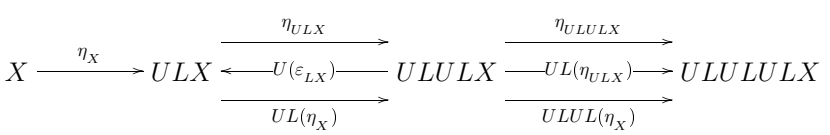
Recall that, as a consequence of Definition 2.2, indeed, D_X is well defined. More precisely, in fact, the following equation holds.



Theorem 4.1 (Biadjoint Triangle)**.** *Let $E : \mathfrak{A} \to \mathfrak{C}$, $J : \mathfrak{A} \to \mathfrak{B}$, $L : \mathfrak{B} \to \mathfrak{C}$ be pseudofunctors such that there is a pseudonatural equivalence*



Assume that (E, R, ρ, μ, v, w) and $(L, U, \eta, \varepsilon, s, t)$ are biadjunctions. For each object X of the 2-category $\mathfrak{B}$, the unit $\eta : \mathrm{Id}_{\mathfrak{B}} \to UL$ induces a 2-functor $D_X : \dot{\Delta} \to \mathfrak{B}$,



(with the 2-cells described above). If D_X is of effective descent for every object X of $\mathfrak{B}$, then J has a right biadjoint if and only if, for every object

Y of $\mathfrak{B}$, the descent object of the diagram $\mathcal{D}_Y : \dot{\Delta} \to \mathfrak{A}$ (with the obvious 2-cells)

$$RLY \underset{RL(\eta_Y)}{\overset{RL(U(\mu_{LY})\eta_{JRLY})\rho_{RLY}}{\underset{\longleftarrow R(\varepsilon_{LY}) \longleftarrow}{\longrightarrow}}} RLULY \underset{RLUL(\eta_Y)}{\overset{RL(U(\mu_{LULY})\eta_{JRLULY})\rho_{RLULY}}{\underset{\longrightarrow RL(\eta_{ULY}) \longrightarrow}{\longrightarrow}}} RLULULY$$

exists in $\mathfrak{A}$.

Proof: Our main argument follows from Yoneda Lemma 2.1 and from the fact that, by Theorem 3.3, pointwise right pseudo-Kan extensions are determined and preserved by 2-representable functors.

Firstly, observe that we can replace the hypothesis of a pseudonatural equivalence $LJ \simeq E$ by an equality $LJ = E$, since everything works up to equivalence. More precisely, since, by hypothesis, LJ is pseudonaturally equivalent to a left biadjoint, LJ is a left biadjoint as well. Define, then, $E := LJ$.

By the biadjunctions $L \dashv U$ and $E \dashv R$, for each object Y of $\mathfrak{B}$, we get the following pseudonatural equivalences:

$$\begin{aligned}
\mathfrak{B}(J-, ULY) &\simeq \mathfrak{C}(E-, LY) \\
&\simeq \mathfrak{A}(-, RLY) \\
\mathfrak{B}(J-, ULULY) &\simeq \mathfrak{C}(E-, LULY) \\
&\simeq \mathfrak{A}(-, RLULY) \\
\mathfrak{B}(J-, ULULULY) &\simeq \mathfrak{C}(E-, LULULY) \\
&\simeq \mathfrak{A}(-, RLULULY)
\end{aligned}$$

Recall that, by hypothesis, for each object Y of $\mathfrak{B}$, the diagram $D_Y : \dot{\Delta} \to \mathfrak{B}$

$$Y \xrightarrow{\eta_Y} ULY \underset{UL(\eta_Y)}{\overset{\eta_{ULY}}{\underset{\longleftarrow U(\varepsilon_{LY}) \longleftarrow}{\longrightarrow}}} ULULY \underset{ULUL(\eta_Y)}{\overset{\eta_{ULULY}}{\underset{\longrightarrow UL(\eta_{ULY}) \longrightarrow}{\longrightarrow}}} ULULULY$$

is of effective descent. And, therefore, for each object Y of $\mathfrak{B}$, taking the image by the 2-representable functor $\mathfrak{B}(J-, -)$, we conclude that

$$\mathfrak{B}(J-, D_Y) : \dot{\Delta} \to \mathbf{CAT},$$

Reciprocally, assume that we have a 2-adjunction $J \dashv G$. By hypothesis, the diagram below is of strict descent for every object Y of $\mathfrak{B}$. And this implies that GY is, actually, the strict descent object of $\mathcal{D}_Y$, which proves that the existence of such strict descent object is also needed.

$$\mathfrak{A}(-, GY) \cong \mathfrak{B}(J-, Y) \longrightarrow \mathfrak{A}(-, RLY) \rightrightarrows \mathfrak{A}(-, RLULY) \rightrightarrows \mathfrak{A}(-, RLULULY)$$

■

Actually, in Theorem 4.4, unlike the 2-adjoint version, to get the right biadjoint we do not need the hypothesis of $\eta * \mathrm{Id}_J$ being a 2-natural transformation. This hypothesis was used only to conclude that the pseudonatural equivalence $\mathbb{D}_Y^{JA} \circ \mathrm{j} \simeq \mathfrak{A}(A, \mathcal{D}_Y)$ is actually a 2-natural isomorphism.

5. Pseudoprecomonadicity

Herein, a pseudomonad is the same as a doctrine, whose definition can be found in page 123 of [14], while a pseudocomonad is the dual notion. As in the 1-dimensional case, for each pseudocomonad T on a 2-category $\mathfrak{A}$, there is an associated biadjunction $\mathbf{Ps}\text{-}T\text{-}Co\mathbf{Alg} \to \mathfrak{A}$, in which $\mathbf{Ps}\text{-}T\text{-}Co\mathbf{Alg}$ is the 2-category of pseudocoalgebras, pseudomorphisms and transformations [10]. Also, every biadjunction $L \dashv U$ induces a comparison pseudofunctor and an Eilenberg Moore factorization [11] below, in which $T = LU$ denotes the induced pseudocomonad. See, for instance, the formal theory of pseudomonads developed by Stephen Lack [9].

$$\begin{array}{ccc} \mathfrak{B} & \xrightarrow{K} & \mathbf{Ps}\text{-}T\text{-}Co\mathbf{Alg} \\ & \searrow^{L} & \downarrow \\ & & \mathfrak{C} \end{array}$$

If K is locally an equivalence, we say that L is pseudoprecomonadic. While, if K is a biequivalence (*i.e.* K is locally an equivalence and surjective on objects up to a pseudonatural equivalence), we say that L is pseudocomonadic. Ivan Le Creurer, Francisco Marmolejo and Enrico Vitale [11] characterized pseudoprecomonadic and pseudocomonadic functors.

In this section, we give some remarks about the results of [11]. Firstly, the pseudo(co)monadicity theorem [11] can be seen as a corollary of the biadjoint triangle theorem. Secondly, the pseudoprecomonadicity characterization [11] shows that one of the hypotheses of Theorem 4.1 is equivalent to saying that L is pseudoprecomonadic. More precisely, we have the following results.

we know that the unit $\eta : \mathrm{Id} \to UL$ *induces the diagram*

$$\mathbb{D}_X^{JA} : \dot{\Delta} \to \mathbf{CAT}$$

$$\begin{array}{ccccc} \mathfrak{B}(JA,X) & & & & \\ {\scriptstyle L_{JA,X}}\downarrow & & & & \\ \mathfrak{C}(EA,LX) & \underset{\mathfrak{C}(EA,L(\eta_X))}{\overset{L_{JA,ULX}\circ\chi_{(JA,LX)}}{\rightleftarrows}} & \mathfrak{C}(EA,LULX) & \underset{\mathfrak{C}(EA,LUL(\eta_X))}{\overset{L_{JA,(UL)^2X}\circ\chi_{(JA,LULX)}}{\rightleftarrows}} & \mathfrak{C}(EA,L(UL)^2X) \end{array}$$

(with omitted 2*-cells). If* $\mathbb{D}_X^{JA}$ *is of strict descent (effective descent) for every object* A *of* $\mathfrak{A}$ *and every object* X *of* $\mathfrak{B}$*, then* J *has a right* 2*-adjoint (biadjoint) if and only if, for every object* Y *of* $\mathfrak{B}$*, the strict descent object (descent object) of* $\mathcal{D}_Y : \dot{\Delta} \to \mathfrak{A}$ *(with the obvious* 2*-cells)*

$$\begin{array}{ccccc} & & RL(\eta_Y) & & RLUL(\eta_Y) & & \\ & \longrightarrow & & \longrightarrow & & & \\ RLY & \longleftarrow & R(\varepsilon_{LY}) & RLULY & RL(\eta_{ULY}) & RLULULY \\ & \longrightarrow & & \longrightarrow & & & \\ & & RL(U(\mu_{LY})\eta_{JRLY})\rho_{RLY} & & RL(U(\mu_{LULY})\eta_{JRLULY})\rho_{RLULY} & & \end{array}$$

exists in $\mathfrak{A}$*.*

Proof: Indeed, by the 2-adjunction (E, R, ρ, μ) and, since $\eta * \mathrm{Id}_{_J}$ is a 2-natural transformation, for every object A of $\mathfrak{A}$ and every object Y of $\mathfrak{B}$, the diagram $\mathbb{D}_Y^{JA} \circ \mathrm{j}$ is 2-naturally isomophic to

$$\mathfrak{A}(A, \mathcal{D}_Y) : \Delta \to \mathbf{CAT}$$

Therefore, for every object Y of $\mathfrak{B}$, $\mathcal{D}_Y$ has a pointwise right Kan extension $\mathcal{R}an_{\mathrm{j}}\mathcal{D}_Y$ if and only if, for every object A of $\mathfrak{A}$,

$$\begin{array}{rcl} \mathbb{D}_Y^{JA} & \cong & \mathcal{R}an_{\mathrm{j}}(\mathbb{D}_Y^{JA} \circ \mathrm{j}) \\ & \cong & \mathcal{R}an_{\mathrm{j}}\mathfrak{A}(A, \mathcal{D}_Y) \\ & \cong & \mathfrak{A}(A, \mathcal{R}an_{\mathrm{j}}\mathcal{D}_Y) \end{array}$$

In particular, if we assume that there is a strict descent object GY of $\mathcal{D}_Y$ (for every object Y of $\mathfrak{B}$), then

$$\begin{array}{rcl} \mathfrak{B}(JA, Y) & = & \mathbb{D}_Y^{JA}(\mathbf{0}) \\ & \cong & \mathcal{R}an_{\mathrm{j}}\mathfrak{A}(A, \mathcal{D}_Y(\mathbf{0})) \\ & \cong & \mathfrak{A}(A, \mathcal{R}an_{\mathrm{j}}\mathcal{D}_Y(\mathbf{0})) \\ & \cong & \mathfrak{A}(A, GY) \end{array}$$

(for every object A of $\mathfrak{A}$). And this completes the proof that $J \dashv G$, in which G can be defined pointwisely by $GY = \mathcal{R}an_{\mathrm{j}}\mathcal{D}_Y(\mathbf{0})$.

given by

$$\begin{array}{ccccc} \mathfrak{B}(J-,Y) & & & & \\ {\scriptstyle \mathfrak{B}(J-,\eta_Y)}\downarrow & & & & \\ \mathfrak{B}(J-,ULY) & \underset{\mathfrak{B}(J-,UL(\eta_Y))}{\overset{\mathfrak{B}(J-,\eta_{ULY})}{\rightleftarrows}} & \mathfrak{B}(J-,(UL)^2Y) & \underset{\mathfrak{B}(J-,(UL)^2(\eta_Y))}{\overset{\mathfrak{B}(J-,\eta_{(UL)^2Y})}{\rightleftarrows}} & \mathfrak{B}(J-,(UL)^3Y) \end{array}$$

is of effective descent. And, by our previous observations, the diagram above is pseudonaturally equivalent to $\mathfrak{D}_Y : \dot{\Delta} \to \mathbf{CAT}$ below.

$$\mathfrak{B}(J-,Y) \longrightarrow \mathfrak{A}(-,RLY) \rightrightarrows \mathfrak{A}(-,RLULY) \rightrightarrows \mathfrak{A}(-,RLULULY)$$

Thus $\mathfrak{D}_Y$ is of effective descent. When considering the restriction $\mathfrak{D}_Y \circ \mathrm{j}$, by the Yoneda Lemma, we get $\mathcal{D}_Y$ in $\mathfrak{A}$,

$$RLY \rightrightarrows RLULY \rightrightarrows RLULULY$$

such that $\mathfrak{A}(-, \mathcal{D}_Y)$ is pseudonaturally equivalent to $\mathfrak{D}_Y \circ \mathrm{j}$. Assume that, for each object Y of $\mathfrak{B}$, $\mathcal{D}_Y : \Delta \to \mathfrak{A}$ has a descent object, *i.e.* there is a pointwise right pseudo-Kan extension $\mathrm{Ps}\mathcal{R}an_{\mathrm{j}}\mathcal{D}_Y : \dot{\Delta} \to \mathfrak{A}$. We define $GY := \mathrm{Ps}\mathcal{R}an_{\mathrm{j}}\mathcal{D}_Y(\mathbf{0})$.

Assuming the existence of $\mathrm{Ps}\mathcal{R}an_{\mathrm{j}}\mathcal{D}_Y$, since $\mathfrak{D}_Y$ is of effective descent, we have the following pseudonatural equivalences:

$$\begin{array}{rcl} \mathfrak{D}_Y & \simeq & \mathrm{Ps}\mathcal{R}an_{\mathrm{j}}(\mathfrak{D}_Y \circ \mathrm{j}) \\ & \simeq & \mathrm{Ps}\mathcal{R}an_{\mathrm{j}}\mathfrak{A}(-, \mathcal{D}_Y) \\ & \simeq & \mathfrak{A}(-, \mathrm{Ps}\mathcal{R}an_{\mathrm{j}}\mathcal{D}_Y) \end{array}$$

And, in particular, $\mathfrak{B}(J-,Y) \simeq \mathfrak{D}_Y(\mathbf{0}) \simeq \mathfrak{A}(-, \mathrm{Ps}\mathcal{R}an_{\mathrm{j}}\mathcal{D}_Y(\mathbf{0})) \simeq \mathfrak{A}(-,GY)$. And this completes the proof that $J \dashv G$, in which G can be defined pointwisely by $GY = \mathrm{Ps}\mathcal{R}an_{\mathrm{j}}\mathcal{D}_Y(\mathbf{0})$.

Reciprocally, assume that we have a biadjunction $J \dashv G$. By hypothesis, the diagram below is of effective descent for every object Y of $\mathfrak{B}$. And this implies that GY is, actually, the descent object of $\mathcal{D}_Y$, which proves that the existence of such descent object is also needed.

$$\mathfrak{A}(-,GY) \simeq \mathfrak{B}(J-,Y) \longrightarrow \mathfrak{A}(-,RLY) \rightrightarrows \mathfrak{A}(-,RLULY) \rightrightarrows \mathfrak{A}(-,RLULULY)$$

■

As in the 1-dimensional case [2], there is a special type of biadjoint triangles: namely, if the induced pseudocomonads from $E \dashv R$ and $L \dashv U$ are equivalent. More precisely, in the setting of Theorem 4.1, we define $\theta := (\mathrm{Id}_{_U} * \mu)(\eta * \mathrm{Id}_{JR})$ and get the following corollary.

Corollary 4.2. *Let $LJ = E$ be a biadjoint triangle satisfying the hypotheses of Theorem 4.1. We denote by G the obtained right biadjoint of J. And we assume that $\theta := (\mathrm{Id}_U * \mu)(\eta * \mathrm{Id}_{JR})$ is isomorphic to the identity Id_U (and, of course, we also assume that $U = JR$).*

Then the pseudofunctor $J : \mathfrak{A} \to \mathfrak{B}$ preserves the effective descent diagram

$$\mathcal{D}_Y : \Delta \to \mathfrak{A}$$

for every object Y of $\mathfrak{B}$ if and only if the counit $\overline{\varepsilon}$ of the biadjunction $J \dashv G$ is a pseudonatural equivalence. That is to say, $J \circ \mathrm{Ps}\mathfrak{Ran}_{\mathrm{j}}(\mathcal{D}_Y)$ is of effective descent for every object Y of $\mathfrak{B}$ if and only if the counit

$$\overline{\varepsilon} : JG \to \mathrm{Id}_{\mathfrak{B}}$$

is a pseudonatural equivalence.

Also, the unit $\overline{\eta}$ of the biadjunction $J \dashv G$ is a pseudonatural equivalence if and only if the 2-functor $\overline{D_A} : \dot{\Delta} \to \mathfrak{A}$ (with omitted 2-cells)

$$A \xrightarrow{\rho_A} REA \underset{R(\mu_{EA})\circ \rho_{REA}}{\overset{\rho_{REA}}{\rightrightarrows}} REREA \underset{RE(\rho_{REA})\circ \rho_{REREA}}{\overset{\rho_{REREA}}{\rightrightarrows}} REREREA$$

is of effective descent for every object A of $\mathfrak{A}$.

There is a trivial result on biadjoint triangles: if a biadjunction

$$J \dashv G : \mathfrak{A} \to \mathfrak{B}$$

has a pseudonatural equivalence as counit, then a pseudofunctor $L : \mathfrak{B} \to \mathfrak{C}$ has a right biadjoint if and only if $L \circ J$ also has a right biadjoint R. In this case, the right biadjoint of L is given by $J \circ R$. Thereby, if the first part of the Corollary 4.2 holds, we may conclude that $U \simeq J \circ R$.

We establish below the obvious dual result of Theorem 4.1, which is the relevant theorem to the usual context of pseudopremonadicity. For being able to give this dual version, we have to employ the observations given in Remark 3.4 on codescent objects, and the terminology established there.

Theorem 4.3 (Biadjoint Triangle)**.** *Let $R : \mathfrak{A} \to \mathfrak{C}$, $J : \mathfrak{A} \to \mathfrak{B}$, $U : \mathfrak{B} \to \mathfrak{C}$ be pseudofunctors such that we have a pseudonatural equivalence $UJ \simeq R$. Assume that E, L are left biadjoints, with $E \dashv R$ and $L \dashv U$. We know*

that, each object X of the 2-category $\mathfrak{B}$, the counit $\varepsilon : LU \to \mathrm{Id}_{\mathfrak{B}}$ induces the 2-functor D_X below.

$$X \xleftarrow{\varepsilon_X} LUX \underset{LU(\varepsilon_X)}{\overset{\varepsilon_{LUX}}{\rightrightarrows}} LULUX \underset{LULU(\varepsilon_X)}{\overset{\varepsilon_{LULUX}}{\rightrightarrows}} LULULUX$$

If D_X is of effective codescent for every object X of $\mathfrak{B}$, then J has a left biadjoint if and only if, for every object Y of $\mathfrak{B}$, $\mathfrak{A}$ has the codescent object of the diagram (with the obvious 2-cells):

$$EUY \underset{EU(L(\rho_{UY})\varepsilon_{JEUY})\mu_{EUY}}{\overset{EU(\varepsilon_Y)}{\rightrightarrows}} EULUY \underset{EU(L(\rho_{ULUY})\varepsilon_{JEULUY})\mu_{EULUY}}{\overset{EULU(\varepsilon_Y)}{\rightrightarrows}} EULULUY$$

4.1. Strict Version. The employed techniques to prove strict versions of Theorem 4.1 are virtually the same. That is to say, we just need to repeat the same constructions, but, now, by means of strict descent objects and 2-adjoints.

Let $U : \mathfrak{C} \to \mathfrak{B}$, $L : \mathfrak{B} \to \mathfrak{C}$ be 2-functors, such that $(L, U, \eta, \varepsilon, s, t)$ is a biadjunction. We denote by $\chi : \mathfrak{C}(L-,-) \simeq \mathfrak{B}(-,U-)$ the associated pseudonatural equivalence, that is to say, for every object X of $\mathfrak{B}$ and every object Z of $\mathfrak{C}$,

$$\begin{aligned} \chi_{(X,Z)} : \ \mathfrak{C}(LX, Z) &\to \mathfrak{B}(X, UZ) \\ f &\mapsto U(f)\eta_X \end{aligned}$$

Observe that, for every pair of objects (X, Y) of $\mathfrak{B}$, the diagram $D_Y : \dot{\Delta} \to \mathfrak{B}$ induces the 2-functor $\mathbb{D}_Y^X : \dot{\Delta} \to \mathbf{CAT}$ (with omitted 2-cells)

$$\begin{array}{l} \mathfrak{B}(X,Y) \\ \quad\downarrow{\scriptstyle L_{X,Y}} \\ \mathfrak{C}(LX,LY) \end{array} \quad \underset{\mathfrak{C}(LX,L(\eta_Y))}{\overset{L_{X,ULY}\circ \chi_{(X,LY)}}{\rightrightarrows}} \quad \mathfrak{C}(LX, LULY) \quad \underset{\mathfrak{C}(LX,LUL(\eta_Y))}{\overset{L_{X,(UL)^2Y}\circ \chi_{(X,LULY)}}{\rightrightarrows}} \quad \mathfrak{C}(LX, L(UL)^2Y)$$

Theorem 4.4 (Strict Biadjoint Triangle)**.** *Let $E : \mathfrak{A} \to \mathfrak{C}$, $J : \mathfrak{A} \to \mathfrak{B}$, $L : \mathfrak{B} \to \mathfrak{C}$ be 2-functors such that $LJ = E$. Assume that (E, R, ρ, μ) is a 2-adjunction and $(L, U, \eta, \varepsilon, s, t)$ is a biadjunction such that $\eta * \mathrm{Id}_J$ is a (strict) 2-natural transformation. For every object A of $\mathfrak{A}$ and every object X of $\mathfrak{B}$,*