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ON AN ASPECT OF SCATTEREDNESS IN THE POINTFREE SETTING

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ABSTRACT: It is well known that a locale is subfit iff each of its open sublocales is a join of closed ones, and fit iff each of its closed sublocales is a meet of open ones. This formulation, however, exaggerates the parallelism between the behavior of fitness and subfitness. For it can be shown that a locale is fit iff each of its sublocales is a meet of closed ones, but it is not the case that a locale is subfit iff each of its sublocales is a join of closed ones.

Thus we are led to take up the very natural question of which locales have the feature that every sublocale is a join of closed sublocales. In this note we show that these are precisely the subfit locales which are scattered in the pointfree sense of [13], and we add a variation for spatial frames.

KEYWORDS: frame, locale, sublocale, open and closed sublocales, fit, subfit, scattered.

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Introduction

The problem we solve in this paper is one of point-free topology, but it can be explained in terms familiar to a reader unacquainted with this area. Think of the objects of point-free topology, the *locales*, as generalized (topological) spaces and of their *sublocales* as generalized subspaces. It is important to understand that a locale that represents a classical space typically has more sublocales then just those that correspond to classical subspaces. In fact, only very special spaces, the so called weakly scattered ones, have only classical subspaces ([18, 9]; see also 4.2 below.) Nevertheless, the family of all sublocales of a given locale constitutes a nice complete lattice,

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although in general not a Boolean algebra, as subspaces of space do. However, there are well-defined *open* and *closed* sublocales which, in the case of spaces, correspond precisely to classical open and closed subspaces.

Two separation properties, subfitness and fitness – see 1.3 below – play an important role in pointfree topology. Subfitness, when applied to a space, is a condition slightly weaker than T_1, and fitness is akin to, but weaker than, regularity. These properties have the following elegant characterizations in terms of the lattice of sublocales:

- A locale is subfit iff each of its open sublocales is a join of closed ones.
- A locale is fit iff each of its closed sublocales is a meet of open ones.

In fact, these were the original definitions given by Isbell when he introduced these concepts in [7]. One also has a formally stronger condition characterizing fitness.

- A locale is fit iff each of its sublocales is a meet of open ones.

This immediately begs the question of which locales have the feature that every sublocale is a join of closed ones. One sees at once that this is too strong to be equivalent to subfitness; in fact, it is not even equivalent for spaces and subspaces (see, e.g., [12] – of course such a strong parallelism could have been hardly expected: we do not have the necessary De Morgan law in the coframe of sublocales).

In this paper we show that the property that every sublocale is a join of closed sublocales is equivalent to the point-free variant of scatteredness in the sense of [13, 14]), together with subfitness. Furthermore, we show that in the case of locales representing classical spaces, this condition is equivalent to classical scatteredness together with T_1.

1. Preliminaries

Basic notation. For a subset A of a poset $(X, \leqslant)$ we write

$$\uparrow A \equiv \{x \colon \exists\ a \in A\ (x \geqslant a)\},$$

and we abbreviate $\uparrow\{a\}$ to $\uparrow a$. The subsets A for which $A = \uparrow A$ are referred to as *up-sets*. Down-sets are defined and designated dually. If we wish to emphasize the context, we write $\uparrow_X A$ for $\{x \in X \colon \exists a \in A\ (x \geqslant a)\}$.

Monotone maps $f\colon (X, \leqslant_1) \to (Y, \leqslant_2)$ and $g\colon (Y, \leqslant_2) \to (X, \leqslant_1)$ are in a *Galois adjunction*, f to the left and g to the right, if

$$f(x) \leqslant y \iff x \leqslant g(y).$$

Recall that left adjoints preserve suprema and right adjoints preserve infima, and conversely, if the posets are complete lattices then any monotone map preserving suprema (infima) has a right (left) adjoint. The dual of a poset $P = (X, \leqslant)$ will be designated by P^{op}, and the dual of a category $\mathbf{C}$ will be designated by $\mathbf{C}^{op}$.

A *frame* is a complete lattice L satisfying the distributive law

$$(\bigvee_{i\in J} a_i) \wedge b = \bigvee_{i\in J}(a_i \wedge b)$$

for all subsets $\{a_i\colon i \in J\} \subseteq L$ and elements $b \in L$. A *frame homomorphism* $h\colon L \to M$ preserves all joins and all finite meets.

A typical frame is the lattice $\Omega(X)$ of all open sets of a topological space X, and if $f\colon X \to Y$ is a continuous map then we have a frame homomorphism $\Omega(f) = (U \mapsto f^{-1}[U])\colon \Omega(Y) \to \Omega(X)$. This constitutes a functor

$$\Omega\colon \mathbf{Top} \to \mathbf{Frm}^{op};$$

the category $\mathbf{Frm}^{op}$ is often referred to as the category of *locales*, designated $\mathbf{Loc}$. If L is isomorphic to an $\Omega(X)$ we speak of a *spatial frame* L.

The Heyting structure. The frame distributivity law given above can be interpreted as saying that each map $(-) \wedge b$ preserves all suprema. Thus there are right adjoints $b \to (-)$ providing the frame with a Heyting structure such that

$$a \wedge b \leqslant c \iff a \leqslant b \to c.$$

In particular, the symbol $x \to y$ will receive heavy use in the sequel.

Note that the Heyting structure also provides self-adjunctions $(-) \to c\colon L \to L^{op}$ and $(-) \to c\colon L^{op} \to L$, which yield the formula

$$\bigvee_i a_i \to b = \bigwedge_i (a_i \to b).$$

In particular, each frame is pseudocomplemented with the pseudocomplements given by

$$a^* = a \to 0.$$

In $\Omega(X)$ this works out to $U^* = X \smallsetminus \overline{U}$. If a happens to have a complement then it coincides, of course, with the pseudocomplement, and we use the symbol a^* in that case as well.

We write $a \prec b$ to mean that $a^* \vee b = \top$. Note that in $\Omega(X)$,

$$U \prec V \iff \overline{U} \subseteq V.$$

A frame L is *regular* if

$$a = \bigvee_{b \prec a} b, \qquad a \in L.$$

We immediately see that $\Omega(X)$ is regular iff the space X is regular in the standard sense.

A frame L is *zero-dimensional* if it is join-generated by its complemented elements. Obviously $\Omega(X)$ is zero-dimensional iff X is zero-dimensional in the standard sense, namely that each open set is a union of its clopen subsets. Since for a complemented a we have $a \prec a$, each zero-dimensional frame is regular.

Both the Heyting arrow and the pseudocomplementation operation have useful reformulations in regular frames. The following theorem appears in the literature proved for the zero-dimensional case. It holds more generally, however, so we prove it here.

Theorem 1.1. *In a regular frame,*

$$a \to b = \bigwedge\{u \colon u \geqslant b, u \vee a = \top\}.$$

In particular,

$$a^* = \bigwedge\{u \colon u \vee a = \top\}.$$

Proof: We verify the first displayed equation. To do so fix a, put

$$U \equiv \{u \colon u \geqslant b \text{ and } u \vee a = \top\},$$

and suppose that $v \leqslant a \to b$ so that $v \wedge a \leqslant b$. If $u \in U$ then

$$v = v \wedge (u \vee a) = (v \wedge u) \vee (v \wedge a) \leqslant u \vee b = u,$$

hence $a \to b \leqslant \bigwedge U$. Now we will use the regularity. We have

$$a \to b = \bigvee_{x^* \vee a = \top} x \to b = \bigwedge_{x^* \vee a = \top} (x \to b) \geqslant \bigwedge U$$

because $x \to b \geqslant b$ and $x \to b \geqslant x \to 0 = x^*$. ∎

If L^{op} is a frame, i.e., if the frame law holds with joins and meets interchanged, we speak of a *coframe*. From Theorem 1.1 we immediately obtain the following.

Corollary 1.2. *A regular frame which is a coframe is a Boolean algebra.*

Proof: Indeed, $a \vee a^* = a \vee \bigwedge_{u \vee a = \top} u = \bigwedge_{u \vee a = \top}(a \vee u) = \top$. ■

Primes and the spectrum. An element $p \in L$ is *prime* if $a \wedge b \leqslant p$ implies that either $a \leqslant p$ or $b \leqslant p$. The functor $\Omega\colon \mathbf{Top} \to \mathbf{Frm}^{op}$ has a right adjoint Σ, the *spectrum*. One of its descriptions is

$$\Sigma L = \big(\{p \in L\colon p \text{ prime}\}, \{\Sigma_a\colon a \in L\}\big),$$

where $\Sigma_a = \{p\colon p \not\leqslant a\}$. We easily see that $\Sigma_0 = \emptyset$, $\Sigma_\top = \Sigma L$, $\Sigma_{a \wedge b} = \Sigma_a \cap \Sigma_b$, and $\Sigma_{\bigvee_i a_i} = \bigcup_i \Sigma_{a_i}$, so that $\{\Sigma_a\colon a \in L\}$ is a topology. For a frame homomorphism $h\colon L \to M$ we have Σh defined by $\Sigma h(p) = h_*(p)$, where h_* is the right Galois adjoint of h.

Definitions 1.3. Sobriety. A topological space X is T_D (a.k.a. $T_{\frac{1}{2}}$, see [1] and also [4]) if every point x has a neighborhood U such that $U \smallsetminus \{x\}$ is open.

Each $X \smallsetminus \overline{\{x\}}$ is prime in $\Omega(X)$. If there are no other primes in $\Omega(X)$, and if X is T_0, then X is said to be *sober* ([6]). Each spectrum ΣL is sober, and the subcategory of sober spaces is reflective in **Top**, with reflector

$$(x \mapsto X \smallsetminus \overline{\{x\}})\colon X \to \Sigma\Omega(X),$$

the unit of the adjunction $\Omega \dashv \Sigma$.

Subfitness and fitness. In the Introduction we defined, following [7], a frame to be *subfit* if each open sublocale is a join of closed ones. This property has a first order equivalent formulation:

$$a \not\leqslant b \implies \exists c,\ (a \vee c = \top \neq b \vee c).$$

This property was introduced in [18] under the name *conjunctivity*. Note that this condition makes good sense in a classical context, for if U and V are open sets such that $U \not\subseteq V$ then there is an open W with $U \cup W = X \neq V \cup W$. This is implied by T_1 because we can choose an $x \in U \smallsetminus V$ and take $W = X \smallsetminus \{x\}$. Another equivalent in classical topology is the following ([18, 7]).

- A space X is subfit iff for each open U and $x \in U$ there is $y \in \overline{\{x\}}$ such that $\overline{\{y\}} \subseteq U$.

The following fact is important, though very easy to prove.

Proposition 1.4. *A space is* T_1 *iff it is* T_D *and subfit.*

The property of fitness, characterized above by each closed sublocale being a meet of open ones, can also be characterized by a first order sentence (see, e.g., [10]). This is not so important here; rather, we will need the following well known fact (see, e.g.,[7, 8]).

Proposition 1.5. *A frame is fit iff each of its sublocales is subfit.*

For more about subfitness, fitness, and the relationship between them, see [11].

2. Sublocales and subspaces

Nuclei and congruences. A *nucleus* on a frame L is a monotone mapping $\nu\colon L \to L$ such that $a \leqslant \nu(a)$, $\nu(\nu(a)) = \nu(a)$, and $\nu(a \wedge b) = \nu(a) \wedge \nu(b)$. The image $\nu[L]$ is a frame in the order inherited from L, and though meets in $\nu[L]$ agree with those in L, joins do not. Nevertheless, $\nu\colon L \to \nu[L]$ is a frame homomorphism. Nuclei are one way of representing subobjects in **Frm**, or more precisely, in **Loc**; and we make use of this representation at one juncture, although we will, as a rule, use the representation given in Definition 2.1 below.

Another representation of subobjects in **Loc** uses frame congruences. The translation between nuclei and congruences is as follows:

$$\nu \mapsto C = \{(x, y) \colon \nu(x) = \nu(y)\} \quad \text{and} \quad C \mapsto \nu = (a \mapsto \bigvee Ca).$$

Sublocales and the coframe $\mathcal{S}(L)$. We will mostly use the following representation of subobjects.

Definition 2.1. A *sublocale* of L is a subset $S \subseteq L$ such that

- $M \subseteq S \implies \bigwedge M \in S$, and
- $\forall x \in L, \forall s \in S, x \to s \in S$.

A sublocale is a frame with the meets as in L while the joins typically differ. The range $\nu[L]$ of a nucleus ν is a sublocale, and each sublocale S can be obtained this way, namely as the range of the left Galois adjoint of the embedding $S \to L$.

The sublocales of L constitute a complete lattice (in fact, a coframe) which we denote by

$$\mathcal{S}(L).$$

The bottom element of $\mathcal{S}(L)$, which we denote by O, is $\{\top\}$; the first condition in Definition 2.1 applies when M is taken to be the empty set, and implies that every sublocale contains $\top$. Thus O represents the void subspace. The top element of $\mathcal{S}(L)$ is L, $\bigwedge_i S_i = \bigcap_i S_i$, and the join is given by the rule

$$\bigvee_i S_i = \left\{ \bigwedge M \colon M \subseteq \bigcup_i S_i \right\}.$$

The fact that the joins in a sublocale $S \subseteq L$ do not coincide with those in L should not be confused with the behavior of the joins in $\mathcal{S}(L)$.

Obviously a sublocale T of a sublocale S of L is also a sublocale of L, so that $\mathcal{S}(S) \subseteq \mathcal{S}(L)$. But we have more, for $\mathcal{S}(S)$ is a principal down-set in $\mathcal{S}(L)$.

Proposition 2.2. *If S is a sublocale of L then $\mathcal{S}(S) = {\downarrow_{\mathcal{S}(L)}} S$.*

Open and closed sublocales, zero-dimensionality of $\mathcal{S}(L)^{\mathrm{op}}$. The open and the closed sublocales are the following:

$$\mathfrak{o}(a) = \{x \colon a \to x = x\} = \{a \to x \colon x \in L\} \quad \text{and} \quad \mathfrak{c}(a) = {\uparrow} a\,, \quad a \in L.$$

We will use both symbols $\mathfrak{c}(a)$ and ${\uparrow}a$, the former when emphasizing its role as a subobject and the latter in calculations. Here are a few of the rules governing such calculations (see, e.g., [10]).

- $\mathfrak{o}(\perp) = O$, $\mathfrak{o}(\top) = L$, $\mathfrak{o}(a \wedge b) = \mathfrak{o}(a) \cap \mathfrak{o}(b)$, $\mathfrak{o}(\bigvee a_i) = \bigvee \mathfrak{o}(a_i)$.
- $\mathfrak{c}(\perp) = L$, $\mathfrak{c}(\top) = O$, $\mathfrak{c}(a \wedge b) = \mathfrak{c}(a) \vee \mathfrak{c}(b)$, $\mathfrak{c}(\bigvee a_i) = \bigcap \mathfrak{c}(a_i)$.

Finally, $\mathfrak{o}(a)$ and $\mathfrak{c}(a)$ are complements of one another in $\mathcal{S}(L)$.

The following well known fact (see, e.g., [8], [10]) will play an important role in our investigation.

Lemma 2.3. *Each sublocale can be represented in the form*

$$S = \bigwedge_i (\mathfrak{o}(a_i) \vee \mathfrak{c}(b_i)).$$

Consequently, in $\mathcal{S}(L)^{\mathrm{op}}$ each element is a join of complemented elements. That is, $\mathcal{S}(L)^{\mathrm{op}}$ is a zero-dimensional, and hence regular, frame.

Definition 2.4 (Scattered frame)**.** A frame is *scattered* ([13, 14], see also [5]) if $\mathcal{S}(L)$ is a frame.

In view of Corollary 1.2, L is scattered iff $\mathcal{S}(L)$ is a Boolean algebra.

One-point sublocales. The sublocale $\mathsf{O} = \{\top\}$ represents the empty subspace. Thus the smallest non-trivial sublocales are those of the form $\{p, \top\}$, termed *one-point sublocales*. Note the following.

- A two-element set $\{p, \top\}$ is a sublocale iff p is a prime. Thus such a sublocale really represents a point of L, that is, a point of the spectrum.
- L is spatial iff it is the join $\bigvee\{P\colon P \text{ is a one-point sublocale of } L\}$.

3. When every sublocale is the join of closed sublocales: the general case

We are interested in those frames L with the feature that every sublocale is the join of closed sublocales, an attribute we shall refer to by the acronym

$$\text{ESJC.}$$

We begin by pointing out in Corollary 3.2 that such frames are fit.

Proposition 3.1. *ESJC is a hereditary property, that is, ESJC holds in every sublocale of a frame satisfying ESJC.*

Proof: Obviously, if $\mathfrak{c}(a)$ is a closed sublocale of L and if $\mathfrak{c}(a) \subseteq S$ for a sublocale S then it is a closed sublocale of S. Thus, the statement immediately follows from 2.2. ∎

Recall Proposition 1.5. Since ESJC obviously implies subfitness, we have this.

Corollary 3.2. *A frame satisfying ESJC is fit.*

We turn our attention now to showing that the frames with ESJC are scattered, Proposition 3.5. This will require some machinery. We denote the frame of nonempty up-sets of L by

$$\mathfrak{U}(L) \equiv \{A \subseteq L\colon \emptyset \neq A = {\uparrow}A\}.$$

Note that the meets are the intersections and the joins are the unions, with one exception, namely $\bigvee \emptyset = \mathsf{O} = \{\top\}$, which does not interfere with the distribution law. Also note that every subset $S \subseteq L$ contains a largest member of $\mathfrak{U}(L)$; we denote this by

$$\mathrm{U}(S) = \{a\colon \uparrow a \subseteq S\} = \bigcup\{\uparrow a : \uparrow a \subseteq S\}.$$

On the other hand, if A is an up-set then every sublocale containing A contains all the $\mathfrak{c}(a) = \uparrow a \subseteq A$ and hence we have the smallest sublocale containing A,

$$\mathrm{J}(A) = \bigvee\{\uparrow a : \uparrow a \subseteq A\} = \bigvee\{\uparrow a : a \in A\} = \left\{\bigwedge B\colon B \subseteq A\right\}.$$

Lemma 3.3 captures two salient features of this setup.

Lemma 3.3. *Let* U *and* J *be as above. Then* $\mathrm{J}(A)$ *is a join of closed sublocales and for any up-set* A *and sublocale* S*,* $\mathrm{J}(A) \subseteq S$ *iff* $S \subseteq \mathrm{U}(S)$*. Thus we have the Galois adjunction*

$$\mathfrak{U}(L) \underset{\mathrm{J}}{\overset{\mathrm{U}}{\leftrightarrows}} \mathcal{S}(L).$$

Proof: It follows immediately from the definitions. ■

Lemma 3.4. *The map* $\sigma \equiv \mathrm{UJ}$ *is a nucleus on* $\mathfrak{U}(L)$*.*

Proof: σ is monotone, $A \subseteq \mathrm{UJ}(A) = \sigma(A)$, and $\sigma\sigma = \mathrm{UJUJ} = \mathrm{UJ} = \sigma$ by the adjunction. Furthermore,

$$\begin{aligned}\mathrm{J}(A) \cap \mathrm{J}(B) &= \bigvee_A \uparrow a \cap \bigvee_B \uparrow b = \bigvee_{A,B} (\uparrow a \cap \uparrow b) = \bigvee_{A,B} \uparrow (a \vee b) \subseteq \bigvee_{A \cap B} \uparrow c \\ &= \mathrm{J}(A \cap B) \subseteq \mathrm{J}(A) \cap \mathrm{J}(B),\end{aligned}$$

and

$$\begin{aligned}\mathrm{U}(S) \cap \mathrm{U}(T) &= \{a\colon \uparrow a \subseteq S\} \cap \{a\colon \uparrow a \subseteq T\} = \{a\colon \uparrow a \subseteq S \cap T\} \\ &= \mathrm{U}(S \cap T) \subseteq \mathrm{U}(S) \cap \mathrm{U}(T),\end{aligned}$$

so that $\sigma(A \cap B) = \mathrm{UJ}(A \cap B) = \sigma(A) \cap \sigma(B)$. ■

Set

$$\mathfrak{U}_\sigma(L) \equiv \sigma[\mathfrak{U}(L)].$$

Because U and J constitute an adjunction, the restrictions to their ranges are inverse isomorphisms. The range of J is $\mathrm{JU}[\mathcal{S}(L)]$ and the range of U is $\mathrm{UJ}[\mathfrak{U}(L)] = \sigma[\mathfrak{U}(L)] = \mathfrak{U}_\sigma(L)$.

Proposition 3.5. *If each sublocale of* L *is a join of closed sublocales then* $\mathfrak{U}$ *maps* $\mathcal{S}(L)$ *isomorphically onto* $\mathfrak{U}_\sigma(L)$*. Consequently,* $\mathcal{S}(L)$ *is a frame and* L *is scattered.*

Proof: To say that sublocale S is the join of the closed sublocales contained within it is precisely to say that $S = J\mathfrak{U}(S)$, i.e., that S lies in the range of J. If this is the case for every sublocale of L then $\mathfrak{U}$ provides an isomorphism from $\mathcal{S}(L)$ onto $\mathfrak{U}_\sigma(L)$. ∎

Theorem 3.6. *The following statements about a frame* L *are equivalent.*

(1) L *satisfies ESJC.*
(2) L *is scattered and subfit.*
(3) L *is scattered and fit.*

Proof: (1)⇒(3): L is fit by 3.2 and scattered by 3.5. (3)⇒(2) is trivial. (2)⇒(1): Take an arbitrary sublocale S and consider its complement $T = S^*$. We have the standard representation (recall 2.3)

$$T = \bigwedge_{i\in J}(\mathfrak{o}(a_i) \vee \mathfrak{c}(b_i))$$

so that

$$S = T^* = \bigvee_{i\in J}(\mathfrak{c}(a_i) \wedge \mathfrak{o}(b_i)).$$

Then $\mathfrak{o}(b_i) = \bigvee_{j\in J_i} \mathfrak{c}(b_{ij})$ by virtue of the subfitness of L, hence

$$S = \bigvee_{i\in J_i}(\mathfrak{c}(a_i) \wedge \bigvee_{j\in J_i}\mathfrak{c}(b_{ij}) = \bigvee_{i\in J_i}\bigvee_{j\in J_i}(\mathfrak{c}(a_i) \wedge \mathfrak{c}(b_{ij}) = \bigvee_{i\in J, j\in J_i}\mathfrak{c}(a_i \vee b_{ij}). \qquad ∎$$

The second equality in the last line of the foregoing proof holds because of the assumption that $\mathcal{S}(L)$ is a frame as well as a coframe. It is, however, worth noting that in the case of complemented elements, and more generally of the so called linear elements, one has frame distributivity in any $\mathcal{S}(L)$ (see, e.g., [7, 8, 10]).

ESJC does not follow from scatteredness alone.

Example 3.7 (A scattered frame that is not subfit). Let

$$L = \{\bot = a_0 < a_1 < \cdots < a_n = \top\}$$

be a finite chain. It is obviously not subfit for $n \geqslant 2$. We have the Heyting operation given by

$$x \to y = \begin{cases} \top & \text{if } x \leqslant y, \\ y & \text{if } x > y \end{cases}$$

Furthermore, $\bigwedge M = \min M \in M$ for any $\emptyset \neq M \subseteq L$. Consequently, every $S \subseteq L$ that contains $\top$ is a sublocale, and the family of all such subsets is a Boolean algebra.

Note 3.8. Recall that for a frame L we have a one-to-one frame homomorphism

$$\nabla \colon L \to \mathcal{S}(L)^{\mathrm{op}} = (a \mapsto \mathfrak{c}(a)).$$

The frame $\mathcal{S}(L)^{\mathrm{op}} \equiv \mathfrak{C}(L)$ is an extension of L in which each element of L becomes complemented. Repeating this extension, we obtain a transfinite sequence

$$L \to \mathfrak{C}(L) \to \mathfrak{C}^2(L) \to \cdots \to \mathfrak{C}^\alpha(L) \to \cdots,$$

the so called *assembly tower of* L. This tower may grow through all ordinal indices, or it may stabilize at some index. It is not known at present whether the first step at which it stops can be greater than 4. The fact above can be interpreted as saying that the tower of a frame with ESJC stops at the second step at the latest.

4. The spatial case

Subspaces and sublocales. A classical subspace (subset) of a space X can be represented as the congruence

$$C_A = \{(U, V) \colon A \cap U = A \cap V\}$$

on $\Omega(X)$. Sublocales of $\Omega(X)$ which arise in this manner are referred to as *induced* sublocales. The representation of subspaces as sublocales is faithful iff the space is T_D. We have the following.

Proposition 4.1. *The mapping $A \mapsto C_A$ is one-to-one if and only if X is T_D.*

Proof: See [15] or [10, pp. 99–100]. ■

In fact, T_D is already implied by distinguishing two subspaces, one of which is open [2]. On the other hand, aside from exceptional spaces, there are always sublocales that are not induced. The following was proved in [19]; see also [9] and [16].

Theorem 4.2. *All the sublocales of $\Omega(X)$ are induced iff X is weakly scattered.*

A space X is *weakly scattered*, or *corrupted*, if every non-empty closed set A contains a weakly isolated point, that is, a point x such that $x \in A \cap U \subseteq \overline{\{x\}}$ for some open set U. Under T_D, each weakly isolated point is isolated. Scattered spaces are characterized by the property that every non-empty closed set contains an isolated point. Thus a weakly scattered T_D-space is scattered. On the other hand, a scattered space is T_D ([19]), so that the scattered spaces are precisely the weakly scattered T_D ones. Thus, taking into account Proposition 4.1, we can conclude the following.

Proposition 4.3. *The correspondence $A \mapsto C_A$ between subspaces and sublocales of a space X is invertible iff X is scattered.*

It is worth noting that Simmons has a finer analysis of these conditions in [19], showing that scattered is the same as T_0 and dispersed, where dispersed is the combination of corrupted and T_B. The last says that the topology generated by the open sets together with the closed sets is a Boolean algebra.

We will need the following fact (see, e.g., [10]).

Proposition 4.4. *Each complemented sublocale of a spatial frame is induced.*

Thus, for a spatial frame $L = \Omega(X)$, $\mathcal{S}(L)$ is a Boolean algebra only if there are no non-induced sublocales, that is, if the space is scattered. Compare this fact with the pointfree Definition 2.4 of scatteredness.

The following fact can be found in [8] (see also [3]), but to help the reader, and because it is important in the sequel, we will present a short proof here.

Proposition 4.5. *If the spectrum $\Sigma\Omega(X)$ is T_D then X is homeomorphic to $\Sigma\Omega(X)$. In particular, X is sober.*

Proof: Start with the standard observation that, for $U \in \Omega(X)$, $U \nsubseteq X \smallsetminus \overline{\{x\}}$ iff $x \in U$. Now consider a point $p \in \Sigma\Omega(X)$ and use the T_D property to find open subsets $U, V \subseteq X$ be such that $p \in \Sigma_U$ and $\Sigma_U \smallsetminus \{p\} = \Sigma_V$. Hence $U \nsubseteq p$ and $V \subseteq p$ so that $U \smallsetminus V \neq \emptyset$. Choose $x \in U \smallsetminus V$. Then $U \nsubseteq X \smallsetminus \overline{\{x\}}$. If $p \neq X \smallsetminus \overline{\{x\}}$ then $X \smallsetminus \overline{\{x\}} \in \Sigma_U \smallsetminus \{p\} = \Sigma_V$, hence $x \in V$, a contradiction. ∎

In [17], a frame is said to be a T_1-*frame* if each prime in L is maximal. We have this.

Proposition 4.6. *If every sublocale of a frame is a join of closed ones then it is a* T_1*-frame.*

Proof: Let p be prime in L. Then $\{p, \top\}$ is a sublocale, and since it can be written as a join of closed sublocales, there is an $\uparrow a \neq 0$ such that $\uparrow a \subseteq \{p, \top\}$. It follows that $a = p$ and $\uparrow p = \{p, \top\}$, that is, p is maximal. ■

Corollary 4.7. *A space* X *such that every sublocale of its topology* $\Omega(X)$ *is a join of closed ones is both* T_1 *and sober.*

Proof: By Proposition 4.6 $\Omega(X)$ is a T_1-frame, making $\Sigma\Omega(X)$ a T_1 space and hence also a T_D space. Thus X is homeomorphic to $\Sigma\Omega(X)$ by Proposition 4.5, and is therefore a sober T_1-space. ■

Theorem 4.8. *For a spatial frame* $L \cong \Sigma\Omega(X)$*, all sublocales are joins of closed ones iff* X *is a sober scattered* T_1*-space, i.e., a sober* T_1*-space such that each of its sublocales is induced by a uniquely determined subspace. The space* X *is uniquely determined by* L.

Proof: $\Leftarrow$: If X is T_1 it is T_D and hence the induction of sublocales by subspaces is one-to-one. If each of the sublocales of $\Omega(X)$ is induced then $\mathcal{S}(L)$ is isomorphic to the Boolean algebra $\mathfrak{P}(X)$ of all subsets of X. Because X is T_1, $\Omega(X)$ is subfit (and Boolean), and hence all the sublocales are joins of closed ones by Theorem 3.6.

$\Rightarrow$: Suppose every sublocale of $L \equiv \Omega(X)$ is a join of closed ones. Then X is a sober T_1 space by Corollary 4.7, and each sublocale of $\Omega(X)$ is complemented by Theorem 4.8. By Proposition 4.3, all such complemented sublocales are induced. The unicity follows from 4.5. ■

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