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A GEOMETRIC INTERPRETATION OF THE SCHÜTZENBERGER GROUP OF A MINIMAL SUBSHIFT

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Abstract: The first author has associated in a natural way a profinite group to each irreducible subshift. The group in question was initially obtained as a maximal subgroup of a free profinite semigroup. In the case of minimal subshifts, the same group is shown in the present paper to also arise from geometric considerations involving the Rauzy graphs of the subshift. Indeed, the group is shown to be isomorphic to the inverse limit of the profinite completions of the fundamental groups of the Rauzy graphs of the subshift. A further result involving geometric arguments on Rauzy graphs is a criterion for freeness of the profinite group of a minimal subshift based on the Return Theorem of Berthé et. al.

Keywords: free profinite semigroup, profinite group, irreducible subshift, minimal subshift, Rauzy graph, return word, fundamental group, fundamental groupoid.
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1. Introduction

The importance of (relatively) free profinite semigroups in the study of pseudovarieties of finite semigroups is well established since the 1980's, which provides a strong motivation to understand their structure. The algebraic-topological structure of free profinite semigroups is far more complex than that of free semigroups. For instance, Rhodes and Steinberg showed that the (finitely generated) projective profinite groups are precisely the closed subgroups of (finitely generated) free profinite semigroups [31].

In the last decade, a connection introduced by the first author with the research field of symbolic dynamics provided new insight into the structure of free profinite semigroups, notably in what concerns their maximal subgroups [5, 3, 6]. This connection is made via the languages of finite blocks of symbolic dynamical systems, also known as subshifts [25]. In symbolic dynamics, irreducible subshifts deserve special attention: they are the ones

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which have a dense forward orbit. For each irreducible subshift $\mathcal{X}$ over a finite alphabet A, one may consider the topological closure in the A-generated free profinite semigroup $\overline{\Omega}_A\mathsf{S}$ of the language of finite blocks of $\mathcal{X}$. This closure is a union of $\mathcal{J}$-classes, among which there is a minimum one, $J(\mathcal{X})$, in the $\mathcal{J}$-ordering [7]. The $\mathcal{J}$-class $J(\mathcal{X})$ contains (isomorphic) maximal subgroups, which, as an abstract profinite group, the authors called in [9] the *Schützenberger group of* $\mathcal{X}$, denoted $G(\mathcal{X})$.

The approach used in [5, 9] consists in obtaining information about $G(\mathcal{X})$ using ideas, results and techniques borrowed from the theory of symbolic dynamical systems. The minimal subshifts, considered in those papers, are precisely the subshifts $\mathcal{X}$ for which the $\mathcal{J}$-class $J(\mathcal{X})$ consists of $\mathcal{J}$-maximal regular elements of $\overline{\Omega}_A\mathsf{S}$ [5].

The subshifts considered in [5, 9] are mostly substitutive systems [29, 22], that is, subshifts defined by (weakly) primitive substitutions. Substitutive subshifts are minimal subshifts which are described by a finite computable amount of data, which leads to various decision problems. The authors showed in [9] how to compute from a primitive substitution a finite profinite presentation of the Schützenberger group of the subshift defined by the substitution, and used this to show that it is decidable whether or not a finite group is a (continuous) homomorphic image of the subshift's Schützenberger group. The first examples of maximal subgroups of free profinite semigroups that are not relatively free profinite groups were also found as Schützenberger groups of substitutive systems [5, 9].

The Schützenberger group of the full shift $A^{\mathbb{Z}}$ is isomorphic to the maximal subgroups of the minimum ideal of $\overline{\Omega}_A\mathsf{S}$ and was first identified in [34], with techniques that were later extended to the general sofic case in [17] taking into account the invariance of $G(\mathcal{X})$ under conjugacy of symbolic dynamical systems [15]. This led to the main result of [17] that $G(\mathcal{X})$ is a free profinite group with rank $\aleph_0$ when $\mathcal{X}$ is a non-periodic irreducible sofic subshift.* From the viewpoint of the structure of the group $G(\mathcal{X})$, the class of irreducible sofic subshifts is thus quite different from that of substitutive (minimal) subshifts.

Substitutive systems are a small part of the realm of minimal subshifts, in the sense that substitutive systems have zero entropy [29], while there are minimal subshifts of entropy arbitrarily close to that of the full shift [19]. Therefore, it would be interesting to explore other techniques giving insight on the Schützenberger group of arbitrary minimal subshifts. That is one of

*Note that the minimal sofic subshifts are the periodic ones.

the main purposes of this paper. We do it by exploring the Rauzy graphs of subshifts, a tool that has been extensively used in the theory of minimal subshifts. For each subshift $\mathcal{X}$ and integer n, the Rauzy graph $\Sigma_n(\mathcal{X})$ is a De Bruijn graph where the vertices (words of length n) and edges (words of length $n+1$) not in the language of the subshift have been removed. This graph is connected if $\mathcal{X}$ is irreducible. In the irreducible case, we turn our attention to the profinite completion $\hat{\Pi}_n(\mathcal{X})$ of the fundamental group of $\Sigma_n(\mathcal{X})$. The subshift $\mathcal{X}$ can be seen in a natural way as an inverse limit of the graphs of the form $\Sigma_{2n}(\mathcal{X})$. The main result of this paper (Corollary 8.13) is that the induced inverse limit of the profinite groups $\hat{\Pi}_{2n}(\mathcal{X})$ is $G(\mathcal{X})$, provided $\mathcal{X}$ is minimal. We leave as an open problem whether this result extends to arbitrary irreducible subshifts.

The study of Rauzy graphs of a minimal subshift often appears associated with the study of sets of return words, as in the proof of the *Return Theorem* in [13]. We apply the Return Theorem, together with a technical result on return words giving a sufficient condition for freeness of the Schützenberger group of a minimal subshift, to show that if the minimal subshift involves n letters and satisfies the so-called tree condition [13], then its Schützenberger group is a free profinite group of rank n (Theorem 6.5). This result was obtained in [5] for the important special case of Arnoux-Rauzy subshifts, with a different approach: the result was there first proved for substitutive Arnoux-Rauzy subshifts, and then extended to arbitrary Arnoux-Rauzy subshifts using approximations by substitutive subshifts.

2. Profinite semigroups, semigroupoids, and groupoids

2.1. Free profinite semigroups. We refer to [6] as a useful introductory text about the theory of profinite semigroups. In [2] one finds an introduction to the subject via the more general concept of profinite algebra. We use the notation $\overline{\Omega}_A\mathsf{S}$ for the free profinite semigroup generated by the set A. Recall that $\overline{\Omega}_A\mathsf{S}$ is a profinite semigroup in which A embeds and which is characterized by the property that every continuous mapping $\varphi\colon A \to S$ into a profinite semigroup S extends in a unique way to a continuous semigroup homomorphism $\hat{\varphi}\colon \overline{\Omega}_A\mathsf{S} \to S$. Replacing the word "semigroup" by "group", we get the characterization of the free profinite group with basis A, which we denote by $\overline{\Omega}_A\mathsf{G}$. We shall use frequently the fact that the discrete subsemigroup of $\overline{\Omega}_A\mathsf{S}$ generated by A is the free semigroup A^+, and that its elements are the isolated elements of $\overline{\Omega}_A\mathsf{S}$ (for which reason the elements of A^+ are

said to be *finite*, while those in the subsemigroup $\overline{\Omega}_A\mathsf{S} \setminus A^+$ are *infinite*). The free group generated by A, denoted $FG(A)$, also embeds naturally into $\overline{\Omega}_A\mathsf{G}$, but its elements are not isolated.

2.2. Free profinite semigroupoids. Except stated otherwise, by a *graph* we mean a directed graph with possibly multiple edges. Formally: for us a graph is a pair of disjoint sets V, of *vertices*, and E, of *edges*, together with two incidence maps α and ω from E to V, the *source* and the *target*. An edge s with source x and target y will sometimes be denoted $s\colon x \to y$. Recall that a *semigroupoid* is a graph endowed with a partial associative operation, defined on consecutive edges (cf. [35, 24, 11]): for $s\colon x \to y$ and $t\colon y \to z$, their composite is an edge st such that $st\colon x \to z$. Alternatively, a semigroupoid may be seen a small category where some local identities are possibly missing.

Semigroups can be seen as being the one-vertex semigroupoids. If the set of loops of the semigroupoid S rooted at a vertex c is nonempty, then, for the composition law, it is a semigroup (for us an empty set is not a semigroup), the *local semigroup of S at c*, denoted $S(c)$.

The theory of topological/profinite semigroups inspires a theory of topological/profinite semigroupoids, but as seen in [7], there are some differences which have to be taken into account, namely in the case of semigroupoids with an infinite number of vertices. To begin with, the very definition of profinite semigroupoid is delicate. We use the following definition: a compact semigroupoid S is *profinite* if, for every pair u, v of distinct elements of S, there is a continuous semigroupoid homomorphism $\varphi\colon S \to F$ into a finite semigroupoid such that $\varphi(u) \neq \varphi(v)$. There is an unpublished example due to G. Bergman (mentioned in [30]) of an infinite-vertex semigroupoid that is profinite according to this definition, but that is not an inverse limit of finite semigroupoids. On the other hand, it is known that a topological graph Γ is an inverse limit of finite graphs if and only if for every $u, v \in \Gamma$ there is a continuous homomorphism of graphs $\varphi\colon \Gamma \to F$ into a finite graph F such that $\varphi(u) \neq \varphi(v)$ (see [32] for a proof), in which case Γ is said to be profinite.

For another delicate feature of infinite-vertex profinite semigroupoids, let Γ be a subgraph of a topological semigroupoid S, and let $\lceil\Gamma\rceil$ be the *closed subsemigroupoid of S generated by* Γ, that is, $\lceil\Gamma\rceil$ is the intersection of all closed subsemigroupoids of S that contain Γ. If S has a finite number of vertices, then $\lceil\Gamma\rceil$ is the topological closure $\overline{\langle\Gamma\rangle}$ of the discrete subsemigroupoid $\langle\Gamma\rangle$

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$\hat{\mu}(ts^{-1})$. Since $\hat{\mu}$ is faithful, we conclude that $ss^{-1} = ts^{-1}$, whence $s = t$. This establishes that $\hat{h}$ is injective. ∎

In view of Theorem 5.7, we may now obtain our main result as an immediate consequence of Theorem 8.12.

Corollary 8.13. *If $\mathcal{X}$ is a minimal subshift then $G(\mathcal{X})$ is isomorphic with $\varprojlim \hat{\Pi}_{2n}(\mathcal{X}, x)$ as a profinite group, for every $x \in \mathcal{X}$.* ∎

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of S generated by Γ. But if S has an infinite number of vertices, then $\overline{\langle\Gamma\rangle}$ may not be a semigroupoid and thus it is strictly contained in $\lceil\Gamma\rceil$ [7]. If Γ is a profinite graph, then the *free profinite semigroupoid generated by* Γ, denoted $\overline{\Omega}_\Gamma\mathsf{Sd}$, is a profinite semigroupoid, in which Γ embeds as a closed subgraph, characterized by the property that every continuous graph homomorphism $\varphi\colon \Gamma \to F$ into a finite semigroupoid F extends in a unique way to a continuous semigroupoid homomorphism $\hat{\varphi}\colon \overline{\Omega}_\Gamma\mathsf{Sd} \to F$. It turns out that $\lceil\Gamma\rceil = \overline{\Omega}_\Gamma\mathsf{Sd}$. The construction of $\overline{\Omega}_\Gamma\mathsf{Sd}$ is given in [7] (where some problems with the construction given in [11] are discussed), and consists in a reduction to the case where Γ is finite, previously treated in [24].

The free semigroupoid generated by Γ, denoted Γ^+, is the graph whose vertices are those of Γ, and whose edges are the paths of Γ with the obvious composition and incidence laws. The semigroupoid Γ^+ embeds naturally in $\overline{\Omega}_\Gamma\mathsf{Sd}$, with its elements being topologically isolated in $\overline{\Omega}_\Gamma\mathsf{Sd}$. Moreover, if Γ is an inverse limit $\varprojlim \Gamma_i$ of finite graphs, then $\Gamma^+ = \varprojlim \Gamma_i^+$ [7]. Also, one has a natural embedding of $\overline{\Omega}_\Gamma\mathsf{Sd}$ in $\varprojlim \overline{\Omega}_{\Gamma_i}\mathsf{Sd}$ [7]. A problem that we believe remains open and is studied in [7], is whether there exists some example where $\overline{\Omega}_\Gamma\mathsf{Sd} \neq \varprojlim \overline{\Omega}_{\Gamma_i}\mathsf{Sd}$.

Everything we said about semigroupoids has an analog for categories. We shall occasionally invoke the free category Γ^*, obtained from Γ^+ by adding an empty path 1_v at each vertex v.

2.3. Profinite completions of finite-vertex semigroupoids. A congruence on a semigroupoid S is an equivalence relation θ on the set of edges of S such that $u\,\theta\,v$ implies that u and v are *coterminal* (that is, they have the same source and the same target), and also that $xu\ \theta\ xv$ and $uy\ \theta\ vy$ whenever the products xu, xv, uy, vy are defined. The quotient S/θ is the semigroupoid with the same set of vertices of S and edges the classes u/θ with the natural incidence and composition laws. The relation that identifies coterminal edges is a congruence. Therefore, if S has a finite number of vertices, the set Λ of congruences on S such that S/θ is finite is nonempty. Note that if the congruences θ and ρ are such that $\theta \subseteq \rho$, then one has a natural semigroupoid homomorphism $S/\theta \to S/\rho$. Hence, when S has a finite number of vertices, we may consider the inverse limit $\hat{S} = \varprojlim_{\theta\in\Lambda} S/\theta$, which is a profinite semigroupoid, called the *profinite completion* of S. Let ι be the natural mapping $S \to \hat{S}$. Then $\iota(S)$ is a dense subsemigroupoid of $\hat{S}$ and $\hat{S}$ has the property that for every continuous semigroupoid homomorphism φ from

S into a profinite semigroupoid T there is a unique continuous semigroupoid homomorphism $\hat{\varphi}\colon \hat{S} \to T$ such that $\hat{\varphi} \circ \iota = \varphi$ [24]. If Γ is a finite-vertex graph, then $\overline{\Omega}_\Gamma \mathsf{Sd}$ is the profinite completion of the free semigroupoid Γ^+ [24].

2.4. Profinite groupoids. A groupoid is a (small) category in which every morphism has an inverse. The parallelism between the definitions of profinite semigroups and profinite groups carries on to an obvious parallelism between the definitions of topological/profinite semigroupoids and topological/profinite groupoids. As groupoids are special cases of semigroupoids some care is sometimes needed when relating corresponding concepts. The next lemma addresses one of such situations. For its proof, recall the well known fact that if t is an element of a compact semigroup T, then the closed subsemigroup $\overline{\langle t\rangle}$ has a unique idempotent, denoted t^ω; in case T is profinite, one has $t^\omega = \lim t^{n!}$ [2]. The inverse of $t \cdot t^\omega$ in the maximal subgroup of $\overline{\langle t\rangle}$ is denoted $t^{\omega-1}$.

Lemma 2.1. *Let G be a compact groupoid and suppose that A is a strongly connected subgraph that generates G as a topological groupoid. Then A also generates G as a topological semigroupoid.*

Proof: Denote by V_A and V_G the vertex sets of A and G, respectively. Let H be the subgraph of G with vertex set $\overline{V_A}$ and whose edges are the edges of G with source and target in $\overline{V_A}$. Clearly, H is closed and a subgroupoid. Since H contains A and A generates G as a topological groupoid, we conclude that $H = G$ and thus $\overline{V_A} = V_G$.

Consider an arbitrary closed subsemigroupoid S of G containing A. Let s be an edge of S. Since $\overline{V_A} = V_G$, there are nets $(a_i)_{i\in I}$ and $(b_j)_{j\in J}$ of elements of V_A respectively converging to $\alpha(s)$ and $\omega(s)$. Because A is strongly connected, for each $(i,j) \in I \times J$ there is some path $u_{i,j}$ in A from $(b_j)_{j\in J}$ to $(a_i)_{i\in I}$. Take an accumulation point u of the net $(u_{i,j})_{(i,j)\in I\times J}$. Then u is an element of S such that $\alpha(u) = \omega(s)$ and $\omega(u) = \alpha(s)$. In particular, we may consider the element $(su)^{\omega-1}$ of the local semigroup of S at $\alpha(s)$. We claim that $u(su)^{\omega-1} = s^{-1}$. Indeed, $s \cdot u(su)^{\omega-1} = (su)^\omega$ is the local identity of G at $\alpha(s)$, while $u(su)^{\omega-1} \cdot s = (us)^\omega$ is the local identity at $\omega(s)$. Hence $s^{-1} \in S$. Since S is an arbitrary closed subsemigroupoid of G containing A, we conclude that s^{-1} belongs to the closed subsemigroupoid K of G generated by A. Therefore, K is a closed subgroupoid of G containing A. Since G is generated by A as a topological groupoid, it follows that $K = G$. ∎

common prefix of $\gamma^{-1}_{\omega(s_i)}$ and $\delta_{\alpha(s_{i+1})}$ and let z_i and t_i be such that the equalities $\gamma_{\omega(s_i)} = z_i w_i^{-1}$ and $\delta_{\alpha(s_{i+1})} = w_i t_i$ hold in $\tilde{\Sigma}_{2n}(\mathcal{X})$. Note that

$$u_0 = \delta_{\alpha(s_1)},\ u_k = \gamma_{\omega(s_k)},\ \text{ and }\ u_i = z_i t_i \quad \text{for } i \in \{1,\dots,k-1\}. \tag{8.8}$$

It follows that

$$\hat{h}_n(u) = g_{s_1} g_{s_2} \cdots g_{s_k}$$

and so

$$\zeta_n(\hat{h}_n(u)) = \hat{\psi}_n\Big(\delta^+_{\alpha(s_1)} \cdot s_1 \cdot \gamma^+_{\omega(s_1)} \cdot \delta^+_{\alpha(s_2)} \cdot s_2 \cdot \gamma^+_{\omega(s_2)} \cdots \delta^+_{\alpha(s_k)} \cdot s_k \cdot \gamma^+_{\omega(s_k)}\Big). \tag{8.9}$$

On the other hand, by (8.8), we have $\delta^+_{\alpha(s_1)} = u_0$, $\gamma^+_{\omega(s_k)} = u_k$ and, in view of Lemmas 8.6 and 8.10, for $i \in \{1,\dots,k-1\}$, the following chain of equalities holds:

$$\begin{aligned}
\hat{\psi}_n(\gamma^+_{\omega(s_i)} \cdot \delta^+_{\alpha(s_{i+1})}) &= \hat{\psi}_n\big(z_i^+ (w_i^{-1})^+ \cdot w_i^+ t_i^+\big) \\
&= \hat{\psi}_n(z_i^+) \cdot \hat{\psi}_n\big((w_i^{-1})^+\big) \cdot \hat{\psi}_n(w_i^+) \cdot \hat{\psi}_n(t_i^+) \\
&= \hat{\psi}_n(z_i^+) \cdot \hat{\psi}_n(w_i^+)^{-1} \cdot \hat{\psi}_n(w_i^+) \cdot \hat{\psi}_n(t_i^+) \\
&= \hat{\psi}_n(z_i^+) \cdot \hat{\psi}_n(t_i^+) = \hat{\psi}_n(u_i).
\end{aligned}$$

Therefore, (8.9) simplifies to $\zeta_n(\hat{h}_n(u)) = \hat{\psi}_n(u)$, as we wished to show. ∎

Theorem 8.12. *Let $\mathcal{X}$ be a minimal subshift. Then, the restriction of the mapping $\hat{h}$ to $\hat{\Sigma}_\infty(\mathcal{X})$ is an isomorphism of topological groupoids onto $\varprojlim \hat{\Pi}_{2n}(\mathcal{X})$.*

Proof: By Lemma 2.3, $\hat{h}_n$ is onto for every $n \geq 1$, which shows that $\hat{h}$ is onto (cf. [36, Theorem 29.13]). Therefore, by Theorem 4.3, the equality $\hat{h}(\hat{\Sigma}(\mathcal{X})) = \varprojlim \hat{\Pi}_{2n}(\mathcal{X})$ holds. If s is a finite edge in $\hat{\Sigma}(\mathcal{X})$, then $\ell_{\alpha(s)}s$ is an edge in $\hat{\Sigma}_\infty(\mathcal{X})$ such that $\hat{h}(\ell_{\alpha(s)}s) = \hat{h}(s)$, whence $\hat{h}(\hat{\Sigma}_\infty(\mathcal{X})) = \varprojlim \hat{\Pi}_{2n}(\mathcal{X})$.

Let s, t be elements of $\hat{\Sigma}_\infty(\mathcal{X})$ such that $\hat{h}(s) = \hat{h}(t)$. Since $\hat{h}$ is the identity mapping on vertices, we may assume that s and t are edges and, therefore, they are coterminal edges. Then, for every $n \geq 1$, we have $\hat{h}_n(\hat{p}_n(ss^{-1})) = \hat{h}_n(\hat{p}_n(ts^{-1}))$, and so from Lemma 8.11 we deduce the equality

$$\hat{\psi}_n(\hat{p}_n(ss^{-1})) = \hat{\psi}_n(\hat{p}_n(ts^{-1})).$$

This shows that $\Psi_n(ss^{-1}) = \Psi_n(ts^{-1})$ every $n \geq 1$. From Proposition 8.7 we then obtain $\Psi(ss^{-1}) = \Psi(ts^{-1})$. By Lemma 8.3, this means that $\hat{\mu}(ss^{-1}) =$

which establishes the inductive step and concludes the proof. ∎

Recall that a *spanning tree* T of a graph Γ is a subgraph of Γ which, with respect to inclusion, is maximal for the property that the undirected graph underlying T is both connected and without cycles. In what follows, we say that a path $t_1 \dots t_k$ of $\widetilde{\Gamma}$ lies in T if for each $i \in \{1, \dots, k\}$ we have $t_i \in T$ or $t_i^{-1} \in T$.

Fix an element $x \in \mathcal{X}$ and, for each n, fix a spanning tree T of the graph $\Sigma_{2n}(\mathcal{X})$. For each pair of vertices v_1, v_2 of $\Sigma_{2n}(\mathcal{X})$, let T_{v_1,v_2} be the unique path of $\widetilde{\Sigma}_{2n}(\mathcal{X})$ from v_1 to v_2 that does not repeat vertices and lies in T. Note that $T_{v_1,v_2} = T_{v_2,v_1}^{-1}$, and that $T_{v,v}$ is the empty path at v. For each vertex v let $\gamma_v = T_{v,x_{[-n,n-1]}}$ and $\delta_v = T_{x_{[-n,n-1]},v}$. In particular, we have $\gamma_v = \delta_v^{-1}$.

For each edge s of $\Sigma_{2n}(\mathcal{X})$, consider the element g_s of the local group $\Pi_{2n}(\mathcal{X}, x)$ given by $g_s = (\delta_{\alpha(s)} \cdot s \cdot \gamma_{\omega(s)})/\!\sim$. Note that g_s is the identity of $\Pi_{2n}(\mathcal{X}, x)$ if s belongs to T. Denote by Y the set of edges of $\Sigma_{2n}(\mathcal{X})$ not in T. It is a well known fact that the set $B = \{g_s \mid s \in Y\}$ is a free basis of the fundamental group $\Pi_{2n}(\mathcal{X}, x)$ [27]. Hence, B is a basis of the free profinite group $\hat{\Pi}_{2n}(\mathcal{X}, x)$. In view of Lemma 8.6, we may therefore consider the unique continuous group homomorphism ζ_n from $\hat{\Pi}_{2n}(\mathcal{X}, x)$ into the local group of the profinite groupoid $K_{2n+1}(\mathcal{X})_E$ at $e_n = e_{x_{[-n,n-1]},n}$ such that

$$\zeta_n(g_s) = \hat{\psi}_n\Big(\delta^+_{\alpha(s)} \cdot s \cdot \gamma^+_{\omega(s)}\Big)$$

for every $s \in Y$.

Lemma 8.11. *Consider an irreducible subshift $\mathcal{X}$. Let u be a loop of $\hat{\Sigma}_{2n}(\mathcal{X})$ rooted at vertex $x_{[-n,n-1]}$. Then we have $\zeta_n(\hat{h}_n(u)) = \hat{\psi}_n(u)$.*

Proof: Since we are dealing with finite-vertex graphs, we have $\overline{\Sigma_{2n}(\mathcal{X})^+} = \hat{\Sigma}_{2n}(\mathcal{X})$. And since the vertex space of $\hat{\Sigma}_{2n}(\mathcal{X})$ is discrete, it follows that any loop of $\hat{\Sigma}_{2n}(\mathcal{X})$ rooted at $x_{[-n,n-1]}$ is the limit of a net of finite loops of $\hat{\Sigma}_{2n}(\mathcal{X})$ rooted at $x_{[-n,n-1]}$. Hence, since $\zeta_n \circ \hat{h}_n$ and $\hat{\psi}_n$ are continuous, the lemma is proved once we show that the equality $\zeta_n(\hat{h}_n(u)) = \hat{\psi}_n(u)$ holds whenever u is a finite loop rooted at $x_{[-n,n-1]}$. For such a finite loop u, let

$$u = u_0 s_1 u_1 s_2 u_2 \cdots u_{k-1} s_k u_k$$

be a factorization in $\Sigma_{2n}(\mathcal{X})^+$ such that $u_0, \dots, u_k$ are (possibly empty) paths that lie in T and $s_1, \dots, s_k$ are edges belonging to Y. Let w_i be the longest

A *groupoid congruence* is a semigroupoid congruence θ on a groupoid such that $u\ \theta\ v$ implies $u^{-1}\ \theta\ v^{-1}$. If S is a compact groupoid, then all closed semigroupoid congruences on S are groupoid congruences. Indeed, if $u, v \in S$ are coterminal edges then $v^{-1} = u^{-1}(vu^{-1})^{\omega-1}$, and if moreover $u\ \theta\ v$, then $u^{-1}(vu^{-1})^k\ \theta\ u^{-1}$ for every integer $k \geq 1$, whence $v^{-1}\ \theta\ u^{-1}$.

Replacing semigroupoid congruences by groupoid congruences, one gets the notion of *profinite completion of a finite-vertex groupoid* analogous to the corresponding one for semigroupoids. These notions generalize the more familiar ones of profinite completion of a group and of a semigroup, since (semi)groups are the one-vertex (semi)groupoids. The following lemma relates these concepts.

Lemma 2.2. *Let G be a connected groupoid with finitely many vertices. Then the profinite completion of a local group of G is a local group of the profinite completion of G.*

Proof: Denote by $\hat{G}$ the profinite completion of G and let x be a vertex of G. We must show that the local group $\hat{G}(x)$ is the profinite completion $\widehat{G(x)}$ of the local group $G(x)$.

Consider the natural homomorphism $\lambda\colon G \to \hat{G}$. Note that it maps $G(x)$ into the profinite group $\hat{G}(x)$, which is generated, as a topological group, by $\lambda(G(x))$. Thus, the restriction $\kappa = \lambda|_{G(x)}$ induces a unique continuous homomorphism $\psi : \widehat{G(x)} \to \hat{G}(x)$, which is onto.

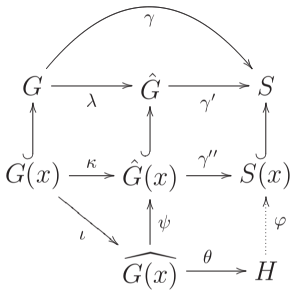
Suppose that $g \in \widehat{G(x)} \setminus \{1\}$. Since $\widehat{G(x)}$ is a profinite group, there exists a continuous homomorphism $\theta\colon \widehat{G(x)} \to H$ onto a finite group H such that $\theta(g) \neq 1$. For each vertex y in G, let $p_y\colon x \to y$ be an edge from G. It is easy to check that the following relation is a congruence on G: given two edges $u, v : y \to z$ in G, $u \sim v$ if $\theta \circ \iota(p_y u p_z^{-1}) = \theta \circ \iota(p_y v p_z^{-1})$. Moreover, note that, in case $u, v \in G(x)$, $u \sim v$ if and only if $\theta \circ \iota(u) = \theta \circ \iota(v)$. Therefore, if $S = G/\!\sim$, then $S(x)$ is finite, whence, since S is a connected groupoid, S is finite. As $\hat{G}$ is the profinite completion of G, it follows that the natural quotient mapping $\gamma\colon G \to S$ factors through λ as a continuous homomorphism $\gamma'\colon \hat{G} \to S$. The restriction $\hat{G}(x) \to S(x)$ of γ' is denoted by γ''.

Noting that $\theta \circ \iota$ is onto because the image of ι is dense, and since

$$\theta \circ \iota(u) = \theta \circ \iota(v) \iff u \sim v \iff \gamma(u) = \gamma(v),$$

there is an isomorphism $\varphi\colon H \to S(x)$ such that $\varphi\circ\theta\circ\iota = \gamma|_{G(x)} = \gamma''\circ\psi\circ\iota$. Again because the image of ι is dense, we deduce that $\varphi\circ\theta = \gamma''\circ\psi$.

All these morphisms are represented in Diagram (2.1).



(2.1)

As $\theta(g) \neq 1$, we get $\gamma''\circ\psi(g) = \varphi\circ\theta(g) \neq 1$,whence $\psi(g) \neq 1$. Therefore, ψ is an isomorphism of topological groups. ∎

2.5. The fundamental groupoid. For the reader's convenience, we write down a definition of the fundamental groupoid of a graph. Let Γ be a graph. Extend Γ to a graph $\widetilde{\Gamma}$ by injectively associating to each edge u a new formal inverse edge u^{-1} with $\alpha(u^{-1}) = \omega(u)$ and $\omega(u^{-1}) = \alpha(u)$. One makes $(u^{-1})^{-1} = u$. Graphs of the form $\widetilde{\Gamma}$ endowed with the mapping $u \mapsto u^{-1}$ on the edge set are precisely the graphs in the sense of J.-P. Serre. These are the graphs upon which the definition of fundamental groupoid of a graph is built in [27], the supporting reference we give for the next lines. Consider in the free category $\widetilde{\Gamma}^*$ the congruence $\sim$ generated by the identification of uu^{-1} with $1_{\alpha(u)}$ and $u^{-1}u$ with $1_{\omega(u)}$, where u runs over the set of edges of Γ. The quotient $\Pi(\Gamma) = \widetilde{\Gamma}^*/\!\sim$ is a groupoid, called the *fundamental groupoid* of Γ. Note that if $\varphi\colon \Gamma_1 \to \Gamma_2$ is a homomorphism of graphs, then the correspondence $\Pi(\varphi)\colon \Pi(\Gamma_1) \to \Pi(\Gamma_2)$ such that $\Pi(\varphi)(x/\!\sim) = \varphi(x)/\!\sim$ is a well defined homomorphism of groupoids, and the correspondence $\varphi \mapsto \Pi(\varphi)$ defines a functor from the category of graphs to the category of groupoids.

It is well known that the natural graph homomorphism from Γ^* to $\Pi(\Gamma)$ (that is, the restriction to Γ^* of the quotient mapping $\widetilde{\Gamma}^* \to \Pi(\Gamma)$) is injective. If Γ is connected (as an undirected graph), then the local groups of $\Pi(\Gamma)$ are isomorphic; their isomorphism class is the *fundamental group* of Γ. It is also well known that if Γ is a connected (finite) graph, then its fundamental group is a (finitely generated) free group.

Suppose $\mathcal{X}$ is an irreducible subshift. As the graph $\Sigma_{2n}(\mathcal{X})$ is strongly connected, for each edge $s\colon v_1 \to v_2$ of $\Sigma_{2n}(\mathcal{X})$ one can choose a path $s'\colon v_2 \to v_1$ in $\Sigma_{2n}(\mathcal{X})$. Denote by s^* the edge $(s's)^{\omega-1}s'$ of $\hat{\Sigma}_{2n}(\mathcal{X})$ from v_2 to v_1.

Remark 8.9. For every edge s of $\Sigma_{2n}(\mathcal{X})$, the loops $s^*\cdot s$ and $s\cdot s^*$ are idempotents. Therefore, if φ is a semigroupoid homomorphism from $\hat{\Sigma}_{2n}(\mathcal{X})$ into a groupoid, then $\varphi(s^*) = \varphi(s)^{-1}$ for every edge s.

Recall how in Section 2.5 we defined the graph $\widetilde{\Gamma}$ from a graph Γ, and denote $\widetilde{\Sigma_{2n}(\mathcal{X})}$ by $\widetilde{\Sigma}_{2n}(\mathcal{X})$. Let t^ε be an edge of $\widetilde{\Sigma}_{2n}(\mathcal{X})$, where t is an edge of $\Sigma_{2n}(\mathcal{X})$, $\varepsilon \in \{-1, 1\}$ and $t^1 = t$. We define

$$(t^\varepsilon)^+ = \begin{cases} t & \text{if } \varepsilon = 1,\\ t^* & \text{if } \varepsilon = -1.\end{cases}$$

If $s = s_1 s_2 \cdots s_k$ is a path, where each s_i is an edge of $\widetilde{\Sigma}_{2n}(\mathcal{X})$, then we define $s^+ = s_1^+ s_2^+ \cdots s_k^+$. Note that s^+ is an edge of $\hat{\Sigma}_{2n}(\mathcal{X})$ such that $\alpha(s^+) = \alpha(s)$ and $\omega(s^+) = \omega(s)$. We also follow the usual definition $s^{-1} = s_k^{-1}\cdots s_2^{-1}s_1^{-1}$.

If 1_v is the empty path at some vertex v of $\widetilde{\Sigma}_{2n}(\mathcal{X})$, then one takes $1_v = 1_v^{-1} = 1_v^* = 1_v^+$, and if φ is a semigroupoid homomorphism from $\hat{\Sigma}_{2n}(\mathcal{X})$ into a groupoid, then one defines $\varphi(1_v)$ as being the local unit at $\varphi(v)$.

Lemma 8.10. *Consider an irreducible subshift $\mathcal{X}$. Let φ be a semigroupoid homomorphism from $\hat{\Sigma}_{2n}(\mathcal{X})$ into a groupoid, and let t be a (possible empty) path of $\widetilde{\Sigma}_{2n}(\mathcal{X})$. Then we have*

$$\varphi(t^+)^{-1} = \varphi((t^{-1})^+). \tag{8.7}$$

Proof: The case where t is an empty path is immediate. We show (8.7) by induction on the length of t. Suppose that t has length 1. Either $t \in \Sigma_{2n}(\mathcal{X})$ or $t^{-1} \in \Sigma_{2n}(\mathcal{X})$. In the first case we have $t^+ = t$ and $(t^{-1})^+ = t^*$, while in the second case we have $t^+ = (t^{-1})^*$ and $(t^{-1})^+ = t^{-1}$. In either case, (8.7) follows from Remark 8.9.

Suppose that (8.7) holds for paths of length less than k, where $k > 1$. Let t be a path of $\widetilde{\Sigma}_{2n}(\mathcal{X})$ of length k, and consider a factorization in $\widetilde{\Sigma}_{2n}(\mathcal{X})^+$ of the form $t = rs$ with s an edge of $\widetilde{\Sigma}_{2n}(\mathcal{X})$.

Then $t^+ = r^+s^+$. Therefore, applying the inductive hypothesis, we get

$$\begin{aligned}\varphi(t^+)^{-1} &= \varphi(s^+)^{-1}\cdot\varphi(r^+)^{-1}\\ &= \varphi((s^{-1})^+)\cdot\varphi((r^{-1})^+) = \varphi\big((s^{-1}r^{-1})^+\big) = \varphi\big((t^{-1})^+\big),\end{aligned}$$

Let $\Psi_n = \gamma \circ \hat{\psi}_n \circ \hat{p}_n$ be the resulting composite.

For the next proposition, we take into account the metric d of $\overline{\Omega}_A\mathsf{S}$ such that if u and v are distinct elements of $\overline{\Omega}_A\mathsf{S}$, then $d(u,v) = 2^{-r(u,v)}$, where $r(u,v)$ is the minimum possible cardinality of a finite semigroup F for which there is a continuous homomorphism $\varphi\colon \overline{\Omega}_A\mathsf{S} \to F$ satisfying $\varphi(u) \neq \varphi(v)$. The hypothesis which we have been using that A is finite guarantees that the metric d generates the topology of $\overline{\Omega}_A\mathsf{S}$ [2, 3].

Proposition 8.7. *Suppose $\mathcal{X}$ is a minimal subshift. Endow the function space $(\overline{\Omega}_A\mathsf{S})^{\hat{\Sigma}(\mathcal{X})}$ with the pointwise topology. Then the sequence $(\Psi_n)_n$ converges uniformly to Ψ.*

Proof: Fix a positive integer k. We want to show that there is a positive integer N_k such that if $n \geq N_k$ then $d(\Psi_n(s), \Psi(s)) < \frac{1}{2^k}$ for every $s \in \hat{\Sigma}(\mathcal{X})$. For that purpose we use the following auxiliary lemma, whose proof is a standard exercise. It appears implicitly in the first part of the proof of Proposition 7.4 from [2].

Lemma 8.8. *Fix a positive integer k. There is a continuous semigroup homomorphism φ from $\overline{\Omega}_A\mathsf{S}$ onto a finite semigroup F such that*

$$d(u,v) < \frac{1}{2^k} \iff \varphi(u) = \varphi(v). \tag{8.3}$$

Proceeding with the proof of Proposition 8.7, let $\varphi : \overline{\Omega}_A\mathsf{S} \to F$ be a continuous homomorphism onto a finite semigroup F such that the equivalence (8.3) holds. Let N_φ be an integer as in Lemmas 8.4 and 8.5. Consider an integer n with $n \geq N_\varphi$. In view of equivalence (8.3), the proposition is proved once we show that, for every edge s of $\hat{\Sigma}(\mathcal{X})$, we have

$$\varphi(\Psi_n(s)) = \varphi(\Psi(s)). \tag{8.4}$$

If s has length 1, that is, if s is an edge $(x, \sigma(x))$ of $\mathcal{X}$, for some $x \in \mathcal{X}$, then we have

$$\Psi_n(s) = e_{\alpha(\hat{p}_n(s)),n} \cdot \hat{\mu}_n(\hat{p}_n(s)) \cdot e_{\omega(\hat{p}_n(s)),n}. \tag{8.5}$$

Because $n \geq N_\varphi$, it follows from Lemma 8.5 that

$$\varphi(e_{\alpha(\hat{p}_n(s)),n}) = \varphi(e_{\alpha(s)}) \quad \text{and} \quad \varphi(e_{\omega(\hat{p}_n(s)),n}) = \varphi(e_{\omega(s)}). \tag{8.6}$$

Since $\hat{\mu}_n \circ \hat{p}_n = \hat{\mu}$, from (8.5) and (8.6) we obtain (8.4) in the case where s has length 1. Hence, $\varphi \circ \Psi_n$ and $\varphi \circ \Psi$ are continuous semigroupoid homomorphisms coinciding in $\Sigma(\mathcal{X})$. Since $\Sigma(\mathcal{X})^+$ is dense in $\hat{\Sigma}(\mathcal{X})$ by Theorem 4.3, it follows that we actually have $\varphi \circ \Psi_n = \varphi \circ \Psi$, thereby establishing (8.4). ∎

Lemma 2.3. *Let Γ be a strongly connected finite-vertex profinite graph. Then the natural continuous homomorphism from the free profinite semigroupoid $\overline{\Omega}_\Gamma\mathsf{Sd}$ to the profinite completion of $\Pi(\Gamma)$, extending the natural graph homomorphism from Γ to $\Pi(\Gamma)$, is onto.* ∎

To prove Lemma 2.3 one uses the following fact [7, Corollary 3.20].

Lemma 2.4. *Let $\psi\colon S \to T$ be a countinuous homomorphism of compact semigroupoids. Let X be a subgraph of S. Then $\psi(\lceil X \rceil) \subseteq \lceil \psi(X) \rceil$. Moreover, $\psi(\lceil X \rceil) = \lceil \psi(X) \rceil$ if ψ is injective on the set of vertices of S.*

Proof of Lemma 2.3: Denote by $\hat{\Pi}(\Gamma)$ the profinite completion of $\Pi(\Gamma)$ and by h the natural continuous semigroupoid homomorphism $\overline{\Omega}_\Gamma\mathsf{Sd} \to \hat{\Pi}(\Gamma)$. By Lemma 2.4, the image of h is the closed subsemigroupoid of $\hat{\Pi}(\Gamma)$ generated by $h(\Gamma)$. Since $\hat{\Pi}(\Gamma)$ is generated by $h(\Gamma)$ as a profinite groupoid, it follows from Lemma 2.1, that h is onto. ∎

3. Subshifts and their connection with free profinite semigroups

A subset X of a semigroup S is *factorial* if the factors of elements of X also belong to X. The subset X is *prolongable* if for every $s \in S$ there are $x, y \in X$ such that $xs, sy \in S$. It is *irreducible* if for every $u, v \in X$ there is $w \in S$ such that $uwv \in X$. Using standard compactness arguments, one can show (see [17] for a proof) that if S is a compact semigroup and X is a nonempty, closed, factorial and irreducible subset of S, then X contains a regular $\mathcal{J}$-class, called the *apex of* X and denoted $J(X)$, such that every element of X is a factor of every element of $J(X)$.

Let A be a finite set. Endow $A^{\mathbb{Z}}$ with the product topology, where A is viewed as discrete space. Let σ be the homeomorphism $A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ defined by $\sigma((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}$, the *shift* mapping on $A^{\mathbb{Z}}$. A *subshift* of $A^{\mathbb{Z}}$ is a nonempty closed subset $\mathcal{X}$ of $A^{\mathbb{Z}}$ such that $\sigma(\mathcal{X}) = \mathcal{X}$. A *finite block* of an element $x = (x_i)_{i\in\mathbb{Z}}$ of $A^{\mathbb{Z}}$ is a word of the form $x_i x_{i+1} \dots x_{i+n}$ (which is also denoted by $x_{[i,i+n]}$) for some $n \geq 0$. For a subset $\mathcal{X}$ of $A^{\mathbb{Z}}$, denote by $L(\mathcal{X})$ the set of finite blocks of elements of $\mathcal{X}$. Then the correspondence $\mathcal{X} \mapsto L(\mathcal{X})$ is an isomorphism between the poset of subshifts of $A^{\mathbb{Z}}$ and the poset of factorial, prolongable languages of A^+ [25, Proposition 1.3.4]. A subshift $\mathcal{X}$ is *irreducible* if $L(\mathcal{X})$ is irreducible. We are interested in studying the topological closure of $L(\mathcal{X})$ in $\overline{\Omega}_A\mathsf{S}$, when $\mathcal{X}$ is a subshift of $A^{\mathbb{Z}}$. It was

noticed in [7] that $\overline{L(\mathcal{X})}$ is a factorial and prolongable subset of $\overline{\Omega}_A\mathsf{S}$, and that if $\mathcal{X}$ is irreducible then $\overline{L(\mathcal{X})}$ is an irreducible subset of $\overline{\Omega}_A\mathsf{S}$. Therefore, supposing $\mathcal{X}$ is irreducible, we can consider the apex $J(\mathcal{X})$ of $\overline{L(\mathcal{X})}$. Since $J(\mathcal{X})$ is regular, it has maximal subgroups, which are isomorphic as profinite groups; we denote by $G(\mathcal{X})$ the corresponding abstract profinite group.

In this paper we concentrate our attention on an important class of irreducible subshifts, the *minimal* subshifts, that is, those that do not contain proper subshifts. This class includes the *periodic* subshifts, finite subshifts $\mathcal{X}$ for which there is a positive integer n (called a *period*) and $x \in A^{\mathbb{Z}}$ such that $\sigma^n(x) = x$ and $\mathcal{X} = \{\sigma^k(x) \mid 0 \leq k < n\}$. It is well known that a subshift $\mathcal{X}$ is minimal if and only if $L(\mathcal{X})$ is *uniformly recurrent*, that is, if and only if for every $u \in L(\mathcal{X})$ there is an integer n such that every word of $L(\mathcal{X})$ with length at least n has u as a factor (cf. [26, Theorem 1.5.9]).

For a subshift $\mathcal{X}$ of $A^{\mathbb{Z}}$, denote by $\mathcal{M}(\mathcal{X})$ the set of elements u of $\overline{\Omega}_A\mathsf{S}$ such that all finite factors of u belong to $L(\mathcal{X})$. One has $\overline{L(\mathcal{X})} \subseteq \mathcal{M}(\mathcal{X})$, and there are simple examples of irreducible subshifts where this inclusion is strict [15]. In what follows, a *maximal regular element* of $\overline{\Omega}_A\mathsf{S}$ is a regular element of $\overline{\Omega}_A\mathsf{S}$ that is $\mathcal{J}$-equivalent with its regular factors. The maximal regular elements of $\overline{\Omega}_A\mathsf{S}$ are precisely the elements of $\overline{\Omega}_A\mathsf{S} \setminus A^+$ all of whose proper factors belong to A^+.

Theorem 3.1. *Let $\mathcal{X}$ be a minimal subshift. Then $\overline{L(\mathcal{X})} = \mathcal{M}(\mathcal{X})$ and $\overline{L(\mathcal{X})} \setminus A^+ = J(\mathcal{X})$. The correspondence $\mathcal{X} \mapsto J(\mathcal{X})$ is a bijection between the set of minimal subshifts of $A^{\mathbb{Z}}$ and the set of $\mathcal{J}$-classes of maximal regular elements of $\overline{\Omega}_A\mathsf{S}$.*

Theorem 3.1 is from [5]. In [7], an approach whose tools are recalled in the next section, distinct from that of [5], was used to deduce the equalities $\overline{L(\mathcal{X})} = \mathcal{M}(\mathcal{X}) = J(\mathcal{X}) \cup L(\mathcal{X})$, when $\mathcal{X}$ is minimal.

A fact that we shall use quite often is that every element of $\overline{\Omega}_A\mathsf{S} \setminus A^+$ has a unique prefix in A^+ with length k, and a unique suffix in A^+ with length k, for every $k \geq 1$ (cf. [1, Section 5.2]). Let $\mathbb{Z}_0^+$ and $\mathbb{Z}^-$ be respectively the sets of nonnegative integers and of negative integers. For $u \in \overline{\Omega}_A\mathsf{S} \setminus A^+$, we denote by $\overrightarrow{u}$ the unique element $(x_i)_{i\in\mathbb{Z}_0^+}$ of $A^{\mathbb{Z}_0^+}$ such that $x_{[0,k]}$ is a prefix of u, for every $k \geq 0$, and by $\overleftarrow{u}$ the unique element $(x_i)_{i\in\mathbb{Z}^-}$ of $A^{\mathbb{Z}^-}$ such that $x_{[-k,-1]}$ is a suffix of u, for every $k \geq 1$. Finally, we denote by $\overleftarrow{u}\,.\,\overrightarrow{u}$ the element of $A^{\mathbb{Z}}$ that restricts in $A^{\mathbb{Z}^-}$ to $\overleftarrow{u}$ and in $A^{\mathbb{Z}_0^+}$ to $\overrightarrow{u}$.

Proof: By Lemmas 4.5 and 4.6, the word $x_{[0,n-1]}$ is a common prefix of $e_{x_{[-n,n-1]},n}$ and e_x. Note also that, for $n \geq N_\varphi$, $x_{[0,n-1]}$, $e_{x_{[-n,n-1]},n}$, and e_x belong to $\mathcal{M}_{N_\varphi}(\mathcal{X})$. In view of Lemma 8.4, we conclude that the elements of the set

$$\{\varphi(x_{[0,n-1]}), \varphi(e_{x_{[-n,n-1]},n}), \varphi(e_x)\}$$

belong to $\varphi(J(\mathcal{X}))$. By stability of F, we deduce that

$$\varphi(e_{x_{[-n,n-1]},n})\ \mathcal{R}\ \varphi(x_{[0,n-1]})\ \mathcal{R}\ \varphi(e_x).$$

Similarly, we have

$$\varphi(e_{x_{[-n,n-1]},n})\ \mathcal{L}\ \varphi(x_{[-n,-1]})\ \mathcal{L}\ \varphi(e_x).$$

Hence $\varphi(e_{x_{[-n,n-1]},n})\ \mathcal{H}\ \varphi(e_x)$, and since $e_{x_{[-n,n-1]},n}$ and e_x are idempotents, we actually have $\varphi(e_{x_{[-n,n-1]},n}) = \varphi(e_x)$. ∎

Let $\mathcal{X}$ be an irreducible subshift. Consider the graph homomorphism $\psi_n\colon \Sigma_{2n}(\mathcal{X}) \to (\overline{\Omega}_A\mathsf{S})_E$ defined by

$$\psi_n(s) = (e_{\alpha(s),n}, e_{\alpha(s),n} \cdot \hat{\mu}_n(s) \cdot e_{\omega(s),n}, e_{\omega(s),n})$$

for each edge s of $\Sigma_{2n}(\mathcal{X})$. By the freeness of the profinite semigroupoid $\hat{\Sigma}_{2n}(\mathcal{X})$, the graph homomorphism ψ_n extends in a unique way to a continuous semigroupoid homomorphism $\hat{\psi}_n\colon \hat{\Sigma}_{2n}(\mathcal{X}) \to (\overline{\Omega}_A\mathsf{S})_E$.

Lemma 8.6. *For every irreducible subshift $\mathcal{X}$, the image of $\hat{\psi}_n$ is contained in the groupoid $K_{2n+1}(\mathcal{X})_E$.*

Proof: Let s be an edge of $\Sigma_{2n}(\mathcal{X})$. By their definition, the idempotents $e_{\alpha(s),n}$ and $e_{\omega(s),n}$ belong to $K_{2n+1}(\mathcal{X})$. Take $u = e_{\alpha(s),n} \cdot \hat{\mu}_n(s) \cdot e_{\omega(s),n}$. We have $u = \hat{\mu}_n(\ell_{\alpha(s),n} \cdot s \cdot \ell_{\omega(s),n})$. Since $\ell_{\alpha(s),n} \cdot s \cdot \ell_{\omega(s),n}$ belongs to $\hat{\Sigma}_{2n}(\mathcal{X})$, we must have $u \in \mathcal{M}_{2n+1}(\mathcal{X})$ by Lemma 4.4. But $u = e_{\alpha(s),n} \cdot u \cdot e_{\omega(s),n}$, and so $u \in K_{2n+1}(\mathcal{X})$ by the $\leq_{\mathcal{J}}$-minimality of $K_{2n+1}(\mathcal{X})$, establishing that $\hat{\psi}_n(s)$ belongs to $K_{2n+1}(\mathcal{X})_E$. Since $K_{2n+1}(\mathcal{X})_E$ is a closed subcategory of $(\overline{\Omega}_A\mathsf{S})_E$, applying Lemma 2.4 we conclude that the image of $\hat{\psi}_n$ is contained in $K_{2n+1}(\mathcal{X})_E$. ∎

Denote by γ the continuous semigroupoid homomorphism $(\overline{\Omega}_A\mathsf{S})_E \to \overline{\Omega}_A\mathsf{S}$ defined by $\gamma(e, u, f) = u$. Consider the following sequence of continuous semigroupoid homomorphisms:

$$\hat{\Sigma}(\mathcal{X}) \xrightarrow{\hat{p}_n} \hat{\Sigma}_{2n}(\mathcal{X}) \xrightarrow{\hat{\psi}_n} (\overline{\Omega}_A\mathsf{S})_E \xrightarrow{\gamma} \overline{\Omega}_A\mathsf{S}.$$

$e_{\alpha(s)} \cdot \hat{\mu}(t) \cdot e_{\omega(t)} = e_x \cdot x_0 \cdot x_0^{-1} e_x x_0 = e_{\alpha(s)} \cdot \hat{\mu}(t)$. Hence, (8.2) simplifies to

$$\Psi(s) = e_{\alpha(s)} \cdot \hat{\mu}(t) \cdot \hat{\mu}(r) \cdot e_{\omega(s)} = e_{\alpha(s)} \cdot \hat{\mu}(s) \cdot e_{\omega(s)},$$

which establishes the inductive step, and concludes the proof by induction that (8.1) holds for finite paths.

Denote by Φ the mapping $\hat{\Sigma}(\mathcal{X}) \to \overline{\Omega}_A \mathsf{S}$ such that $\Phi(s) = e_{\alpha(s)} \cdot \hat{\mu}(s) \cdot e_{\omega(s)}$ for every edge s of $\hat{\Sigma}(\mathcal{X})$. We proved that Ψ and Φ coincide in $\Sigma(\mathcal{X})^+$. By continuity of $\hat{\mu}$ and by Remark 7.6, we know that Φ is continuous. Hence, as $\Sigma(\mathcal{X})^+$ is dense in $\hat{\Sigma}(\mathcal{X})$ by Theorem 4.3, we conclude that $\Psi = \Phi$.

Suppose s is an infinite edge. Since s and $\ell_{\alpha(s)}$ have the same source, $\hat{\mu}(s)$ and $e_{\alpha(s)}$ have the same set of finite prefixes by Lemma 4.6. This means that $\hat{\mu}(s)$ and $e_{\alpha(s)}$ are $\mathcal{R}$-equivalent elements of $J(\mathcal{X})$, by Lemma 3.2. Similarly, $\hat{\mu}(s)$ and $e_{\omega(s)}$ are $\mathcal{L}$-equivalent. This establishes $\Psi(s) = \hat{\mu}(s)$. ∎

We begin a series of technical lemmas preparing a result (Proposition 8.7) about the approximation of Ψ by a special sequence of functions in the function space $(\overline{\Omega}_A \mathsf{S})^{\hat{\Sigma}(\mathcal{X})}$, endowed with the pointwise topology.

Lemma 8.4. *Suppose $\mathcal{X}$ is a minimal subshift. Let φ be a continuous semigroup homomorphism from $\overline{\Omega}_A \mathsf{S}$ into a finite semigroup F. Then there is an integer N_φ such that if u is an element of $\mathcal{M}_{N_\varphi}(\mathcal{X})$ with length at least N_φ, then $\varphi(u) \in \varphi(J(\mathcal{X}))$.*

Proof: Since $J(\mathcal{X}) \subseteq \overline{L(\mathcal{X})}$, there is $z \in L(\mathcal{X})$ such that $\varphi(z) \in \varphi(J(\mathcal{X}))$. By the uniform recurrence of $L(\mathcal{X})$, there is an integer M such that every word of $L(\mathcal{X})$ of length at least M contains z as a factor.

Let e be an idempotent of $J(\mathcal{X})$. Since $\mathcal{X}$ is a minimal subshift, by Theorem 3.1 we know that $K_\infty(\mathcal{X}) = J(\mathcal{X})$. Applying Lemma 7.8, we conclude that there is a sequence $(e_n)_n$ of idempotents converging to e such that $e_n \in K_n(\mathcal{X})$ for every $n \geq 1$. Hence, there is an integer N_φ with $N_\varphi \geq M$ for which we have $\varphi(e_n) = \varphi(e)$ whenever $n \geq N_\varphi$.

Let $u \in \mathcal{M}_{N_\varphi}(\mathcal{X})$ be such that the length of u is at least N_φ. Then z is a factor of u. We also have $e_{N_\varphi} \leq_{\mathcal{J}} u$ by the definition of $K_n(\mathcal{X})$. Hence, we obtain $\varphi(e_{N_\varphi}) \leq_{\mathcal{J}} \varphi(u) \leq_{\mathcal{J}} \varphi(z)$. But both $\varphi(z)$ and $\varphi(e_{N_\varphi}) = \varphi(e)$ belong to $\varphi(J(\mathcal{X}))$, thus $\varphi(u) \in \varphi(J(\mathcal{X}))$. ∎

Lemma 8.5. *Let $\mathcal{X}$, φ and N_φ be as in Lemma 8.4. For all $x \in \mathcal{X}$ and $n \geq N_\varphi$, the equality $\varphi(e_x) = \varphi(e_{x_{[-n,n-1]},n})$ holds.*

The part of the next lemma about Green's relations $\mathcal{R}$ and $\mathcal{L}$ was observed in [4] and in [7, Lemma 6.6]. The second part, about the $\mathcal{H}$ relation, is an easy consequence of the first part, and it is proved in a more general context in [8, Lemma 5.3].

Lemma 3.2. *Let $\mathcal{X}$ be a minimal subshift. Two elements u and v of $J(\mathcal{X})$ are $\mathcal{R}$-equivalent (respectively, $\mathcal{L}$-equivalent) if and only if $\overrightarrow{u} = \overrightarrow{v}$ (respectively, $\overleftarrow{u} = \overleftarrow{v}$). Moreover, if $x \in \mathcal{X}$, then the $\mathcal{H}$-class G_x formed by the elements u of $J(\mathcal{X})$ such that $\overleftarrow{u}\,.\,\overrightarrow{u} = x$ is a maximal subgroup of $J(\mathcal{X})$.*

We retain for the rest of the paper the notation G_x given in Lemma 3.2.

4. Free profinite semigroupoids generated by Rauzy graphs

Let $\mathcal{X}$ be a subshift of $A^{\mathbb{Z}}$. The *graph of* $\mathcal{X}$ is the graph $\Sigma(\mathcal{X})$ having $\mathcal{X}$ as the set of vertices and where the edges are precisely the pairs $(x, \sigma(x))$, with source and target being respectively equal to x and $\sigma(x)$. The graph $\Sigma(\mathcal{X})$ is a compact graph, with the topology on the edge set being naturally induced by that of $\mathcal{X}$.

Denote by $L_n(\mathcal{X})$ the set of elements of A^+ with length n. The *Rauzy graph of order n* of $\mathcal{X}$, denoted $\Sigma_n(\mathcal{X})$, is the graph defined by the following data: the set of vertices is $L_n(\mathcal{X})$, the set of edges is $L_{n+1}(\mathcal{X})$, and incidence of edges in vertices is given by

$$a_1 a_2 \cdots a_n \xrightarrow{a_1 a_2 \cdots a_n a_{n+1}} a_2 \cdots a_n a_{n+1},$$

where $a_i \in A$.

Remark 4.1. If $\mathcal{X}$ is irreducible, then $\Sigma_n(\mathcal{X})$ is strongly connected.

In the case of a Rauzy graph of even order $2n$, we consider a function μ_n, called *central labeling*, assigning to each edge $a_1 a_2 \cdots a_{2n} a_{2n+1}$ ($a_i \in A$) its middle letter a_{n+1}.

Remark 4.2. Extending the labeling μ_n as a semigroupoid homomorphism $\Sigma_{2n}(\mathcal{X})^+ \to A^+$, one sees that the set of images of paths of $\Sigma_{2n}(\mathcal{X})$ by that homomorphism is the set of elements of A^+ whose factors of length at most $2n+1$ belong to $L(\mathcal{X})$.

For $m \geq n$, we define a graph homomorphism $p_{m,n} \colon \Sigma_{2m}(\mathcal{X}) \to \Sigma_{2n}(\mathcal{X})$ as follows: if $w \in L_{2m}(\mathcal{X}) \cup L_{2m+1}(\mathcal{X})$ and if $w = vuv'$ with $v, v' \in A^{m-n}$, then $p_{m,n}(w) = u$. Note that p_n preserves the central labeling, that is,

$\mu_n \circ p_{m,n}(w) = \mu_m(w)$ for every edge w of $\Sigma_{2m}(\mathcal{X})$. The family of onto graph homomorphisms $\{p_{m,n} \,|\, n \leq m\}$ defines an inverse system of compact graphs. The corresponding inverse limit $\varprojlim \Sigma_{2n}(\mathcal{X})$ will be identified with $\Sigma(\mathcal{X})$ since the mapping from $\Sigma(\mathcal{X})$ to $\varprojlim \Sigma_{2n}(\mathcal{X})$ sending $x \in \mathcal{X}$ to $(x_{[-n,n-1]})_n$ and $(x, \sigma(x))$ to $(x_{[-n,n]})_n$ is a continuous graph isomorphism. The projection $\Sigma(\mathcal{X}) \to \Sigma_{2n}(\mathcal{X})$ is denoted by p_n. Let μ be the mapping defined on the set of edges of $\Sigma(\mathcal{X})$ by assigning x_0 to $(x, \sigma(x))$. Then $\mu = \mu_n \circ p_n$, for every $n \geq 1$.

We proceed with the setting of [7]. Like in that paper, denote by $\hat{\Sigma}_{2n}(\mathcal{X})$ and by $\hat{\Sigma}(\mathcal{X})$ the free profinite semigroupoids generated respectively by $\Sigma_{2n}(\mathcal{X})$ and by $\Sigma(\mathcal{X})$. The graph homomorphism $p_{m,n} \colon \Sigma_{2m}(\mathcal{X}) \to \Sigma_{2n}(\mathcal{X})$ extends uniquely to a continuous homomorphism $\hat{p}_{m,n} \colon \hat{\Sigma}_{2m}(\mathcal{X}) \to \hat{\Sigma}_{2n}(\mathcal{X})$ of compact semigroupoids. This establishes an inverse limit $\varprojlim \hat{\Sigma}_{2n}(\mathcal{X})$ in the category of compact semigroupoids, in which the graph $\overline{\Sigma(\mathcal{X})} = \varprojlim \Sigma_{2n}(\mathcal{X})$ naturally embeds. The canonical projection $\varprojlim \hat{\Sigma}_{2n}(\mathcal{X}) \to \hat{\Sigma}_{2k}(\mathcal{X})$ is denoted $\hat{p}_k$. Recall that the free profinite semigroupoid $\hat{\Sigma}(\mathcal{X})$ also embeds in $\varprojlim \hat{\Sigma}_{2n}(\mathcal{X})$, and that we do not know of any example where the inclusion is strict.

Theorem 4.3 ([7])**.** *If $\mathcal{X}$ is a minimal subshift then $\hat{\Sigma}(\mathcal{X}) = \varprojlim \hat{\Sigma}_{2n}(\mathcal{X}) = \overline{\Sigma(\mathcal{X})^+}$.*

In [7] one finds examples of irreducible subshifts $\mathcal{X}$ for which one has $\overline{\Sigma(\mathcal{X})^+} \neq \hat{\Sigma}(\mathcal{X})$.

Viewing A as a virtual one-vertex graph, whose edges are the elements of A, the graph homomorphism $\mu_n \colon \Sigma_{2n}(\mathcal{X}) \to A$ extends in a unique way to a continuous semigroupoid homomorphism $\hat{\mu}_n \colon \hat{\Sigma}_{2n}(\mathcal{X}) \to \overline{\Omega}_A\mathsf{S}$. The equality $\mu_m = \mu_n \circ p_{m,n}$ yields $\hat{\mu}_n \circ \hat{p}_{m,n} = \hat{\mu}_m$, when $m \geq n \geq 1$, and so we may consider the continuous semigroupoid homomorphism $\hat{\mu} \colon \varprojlim \hat{\Sigma}_{2n}(\mathcal{X}) \to \overline{\Omega}_A\mathsf{S}$ such that $\hat{\mu} = \hat{\mu}_n \circ \hat{p}_n$ for every $n \geq 1$. Recall that a graph homomorphism is *faithful* if distinct coterminal edges have distinct images. It turns out that $\hat{\mu}_n$ is faithful (cf. [7, Proposition 4.6]) and therefore so is $\hat{\mu}$.

Let us now turn our attention to the images of $\hat{\mu}_n$ and $\hat{\mu}$. For a positive integer n, let $\mathcal{M}_n(\mathcal{X})$ be the set of all elements u of $\overline{\Omega}_A\mathsf{S}$ such that all factors of u with length at most n belong to $L(\mathcal{X})$.

Lemma 4.4. *Let $\mathcal{X}$ be a subshift. For every positive integer n, the equality $\hat{\mu}_n(\hat{\Sigma}_{2n}(\mathcal{X})) = \mathcal{M}_{2n+1}(\mathcal{X})$ holds.*

an idempotent belonging to $\mathcal{M}(\mathcal{X})$. We also know that, for a fixed a positive integer k, the word $x_{[0,k]}$ is a prefix of $e_{x_{[-n,n-1],n}}$ whenever $n > k$, by Lemma 4.6. By continuity, we deduce that $x_{[0,k]}$ is a prefix of e. Similarly, $x_{[-k,-1]}$ is a suffix of e. Since k is arbitrary, we conclude from Lemma 3.2 that $e = e_x$. Hence, by compactness, the sequence $(e_{x_{[-n,n-1],n}})_n$ converges to e_x, as e_x is its sole accumulation point. ∎

Let $u \in \overline{\Omega}_A\mathsf{S}$. Suppose $z \in A^+$ is such that $u \in z \cdot \overline{\Omega}_A\mathsf{S}$. Then there is a unique w in $\overline{\Omega}_A\mathsf{S}$ such that $u = zw$ [1, Exercise 10.2.10]. We denote w by $z^{-1}u$. The product $(z^{-1}u)z$ is denoted simply by $z^{-1}uz$, as there is no risk of ambiguity. Observe that if u is idempotent then $z^{-1}uz$ is also idempotent. In terms of the element $x = (x_i)_{i\in\mathbb{Z}}$ of the minimal subshift $\mathcal{X}$, one sees that $e_x \in x_0 \cdot \overline{\Omega}_A\mathsf{S}$, and so we may consider the idempotent $x_0^{-1}e_x x_0$.

Lemma 8.2. *If $\mathcal{X}$ is a minimal subshift, then for every $x \in \mathcal{X}$ we have $e_{\sigma(x)} = x_0^{-1}e_x x_0$.*

Proof: Let $w = x_0^{-1}e_x x_0$. Then we have $\overleftarrow{w}.\overrightarrow{w} = \sigma(x)$. Hence, w is an idempotent in $G_{\sigma(x)}$, that is, $w = e_{\sigma(x)}$. ∎

By the freeness of the profinite semigroupoid $\hat{\Sigma}(\mathcal{X})$, we may consider the unique continuous semigroupoid homomorphism $\Psi \colon \hat{\Sigma}(\mathcal{X}) \to \overline{\Omega}_A\mathsf{S}$ such that $\Psi(s) = e_{\alpha(s)} \cdot \hat{\mu}(s) \cdot e_{\omega(s)}$ for every edge s of $\Sigma(\mathcal{X})$.

Lemma 8.3. *Suppose $\mathcal{X}$ is a minimal subshift. For every edge s of $\hat{\Sigma}(\mathcal{X})$, we have*

$$\Psi(s) = e_{\alpha(s)} \cdot \hat{\mu}(s) \cdot e_{\omega(s)}. \tag{8.1}$$

Moreover, if s is an infinite edge then $\Psi(s) = \hat{\mu}(s)$.

Proof: We first establish equality (8.1) for finite paths s belonging to $\Sigma(\mathcal{X})^+$, by induction on the length of s. The base case holds by the definition of Ψ.

Suppose that (8.1) holds for paths in $\Sigma(\mathcal{X})$ of length k, where $k \geq 1$, and let s be a path in $\Sigma(\mathcal{X})$ of length $k+1$. Factorize s as $s = tr$ with t being a path of length 1 and r a path of length k. Then, by the induction hypothesis, and since Ψ is a semigroupoid homomorphism, we have,

$$\Psi(s) = \Psi(t)\Psi(r) = e_{\alpha(s)} \cdot \hat{\mu}(t) \cdot e_{\omega(t)} \cdot \hat{\mu}(r) \cdot e_{\omega(s)}. \tag{8.2}$$

Since t has length 1, there is $x \in \mathcal{X}$ such that $t = (x, \sigma(x))$. As $\alpha(s) = x$, $\omega(t) = \sigma(x)$ and $\hat{\mu}(t) = x_0$, and taking into account Lemma 8.2, we obtain

Finally, it is an easy exercise to show that if H is a functor between two connected groupoids S and T that restricts to a bijection between the corresponding sets of vertices and to a bijection between some local group of S and some local group of T, then H is an isomorphism of groupoids. ∎

The following lemma is useful in the sequel.

Lemma 7.8. *Let $\mathcal{X}$ be an irreducible subshift. If e is an idempotent in $K_\infty(\mathcal{X})$, then there is a sequence $(e_n)_n$ of idempotents $e_n \in K_n(\mathcal{X})$ such that $\lim e_n = e$.*

Proof: For each positive integer n, choose $v_n \in K_n(\mathcal{X})$. Since $e \in \mathcal{M}_n(\mathcal{X})$ and $\mathcal{M}_n(\mathcal{X})$ is irreducible, there are $z_n, t_n \in \overline{\Omega}_A\mathsf{S}$ such that $ez_nv_nt_ne$ belongs to $\mathcal{M}_n(\mathcal{X})$, whence $(e, ez_nv_nt_ne, e)$ is a loop of $K_n(\mathcal{X})_E$, and so is $(e, ez_nv_nt_ne, e)^\omega$ in view of Proposition 7.5. Therefore, the idempotent $e_n = (ez_nv_nt_ne)^\omega$ belongs to $K_n(\mathcal{X})$.

Let f be an accumulation point of the sequence $(e_n)_n$. Note that f is an idempotent such that $f \leq_\mathcal{R} e$ and $f \leq_\mathcal{L} e$. As $m \geq n$ implies $e_m \in \mathcal{M}_n(\mathcal{X})$ and because $\mathcal{M}_n(\mathcal{X})$ is closed, we have $f \in \mathcal{M}_n(\mathcal{X})$ for every $n \geq 1$, whence $f \in \mathcal{M}_\infty(\mathcal{X})$. Therefore, since $e \in K_\infty(\mathcal{X})$ is a factor of f, we must have $f \in K_\infty(\mathcal{X})$. As $\overline{\Omega}_A\mathsf{S}$ is stable, we conclude that $f = e$. We have shown that e is the unique accumulation point of $(e_n)_n$, and so by compactness we conclude that $(e_n)_n$ converges to e. ∎

8. A geometric interpretation of $G(\mathcal{X})$ when $\mathcal{X}$ is minimal

In this section we present a series of technical results that culminate, for the case where $\mathcal{X}$ is a minimal subshift, in the geometric interpretation of $G(\mathcal{X})$ as an inverse limit of the profinite completions of the fundamental groups of the Rauzy graphs $\Sigma_{2n}(\mathcal{X})$ (Corollary 8.13). While some preliminary results are valid for all irreducible subshifts, we leave open whether our main result generalizes to that case.

By Corollary 7.4, if $\mathcal{X}$ is an irreducible subshift then, for each vertex w of $\Sigma_{2n}(\mathcal{X})$, we may choose an idempotent loop $\ell_{w,n}$ of $\hat{\Sigma}_{2n}(\mathcal{X})$ rooted at w such that the idempotent $e_{w,n} = \hat{\mu}_n(\ell_{w,n})$ belongs to $K_{2n+1}(\mathcal{X})$.

Lemma 8.1. *Suppose $\mathcal{X}$ is a minimal subshift. For every $x \in \mathcal{X}$, the sequence $(e_{x_{[-n,n-1]},n})_n$ converges to e_x.*

Proof: Since $\mathcal{M}(\mathcal{X})$ is the intersection of the descending chain of closed sets $(\mathcal{M}_{2n+1}(\mathcal{X}))_n$, we know that every accumulation point e of $(e_{x_{[-n,n-1]},n})_n$ is

Proof: We clearly have $\hat{\mu}_n(\Sigma_{2n}(\mathcal{X})^+) = \mathcal{M}_{2n+1}(\mathcal{X}) \cap A^+$ (cf. Remark 4.2). Noting that $\mathcal{M}_{2n+1}(\mathcal{X})$ is closed and open, that A^+ is dense in $\overline{\Omega}_A\mathsf{S}$, and that $\Sigma_{2n}(\mathcal{X})^+$ is dense in $\hat{\Sigma}_{2n}(\mathcal{X})$, the lemma follows immediately. ∎

Note that $\mathcal{M}_1(\mathcal{X}) \supseteq \mathcal{M}_2(\mathcal{X}) \supseteq \mathcal{M}_3(\mathcal{X}) \supseteq \cdots$ and $\mathcal{M}(\mathcal{X}) = \bigcap_{n\geq 1} \mathcal{M}_n(\mathcal{X})$. Therefore, the image of $\hat{\mu}$ is contained in $\mathcal{M}(\mathcal{X})$, by Lemma 4.4. One actually has $\hat{\mu}(\varprojlim \hat{\Sigma}_{2n}(\mathcal{X})) = \mathcal{M}(\mathcal{X})$ (cf. [7, Proposition 4.5]), but we shall not need this fact.

The next two lemmas where observed in [7, Lemmas 4.2 and 4.3]. We introduce some notation. We denote by $|u|$ the length of a word in A^+, and let $|u| = +\infty$ for $u \in \overline{\Omega}_A\mathsf{S} \setminus A^+$.

Lemma 4.5. *Consider a subshift $\mathcal{X}$. Let $q\colon x_{[-n,n-1]} \to y_{[-n,n-1]}$ be an edge of $\hat{\Sigma}_{2n}(\mathcal{X})$, where $x, y \in \mathcal{X}$. Let $u = \hat{\mu}_n(q)$. If $k = \min\{|u|, n\}$ then $x_{[0,k-1]}$ is a prefix of u and $y_{[-k,-1]}$ is a suffix of u.*

Lemma 4.6. *Consider a subshift $\mathcal{X}$. Let $q\colon x \to y$ be an edge of $\varprojlim \hat{\Sigma}_{2n}(\mathcal{X})$. Let $u = \hat{\mu}(q)$. If $u \in \overline{\Omega}_A\mathsf{S} \setminus A^+$ then $\overrightarrow{u} = (x_i)_{i\in\mathbb{Z}_0^+}$ and $\overleftarrow{u} = (y_i)_{i\in\mathbb{Z}^-}$. If $u \in A^+$ then q is the unique edge of $\Sigma(\mathcal{X})^+$ from x to $\sigma^{|u|}(x)$.*

We denote by $\Pi_{2n}(\mathcal{X})$ the fundamental groupoid of $\Sigma_{2n}(\mathcal{X})$, and by h_n the natural homomorphism $\Sigma_{2n}(\mathcal{X}) \to \Pi_{2n}(\mathcal{X})$. The graph homomorphism $p_{m,n}\colon \Sigma_{2m}(\mathcal{X}) \to \Sigma_{2n}(\mathcal{X})$ induces the groupoid homomorphism $q_{m,n} = \Pi(p_{m,n})\colon \Pi_{2m}(\mathcal{X}) \to \Pi_{2n}(\mathcal{X})$, characterized by the equality $q_{m,n} \circ h_m = h_n \circ p_{m,n}$. Let $\hat{\Pi}_{2n}(\mathcal{X})$ be the profinite completion of $\Pi_{2n}(\mathcal{X})$, and let $\hat{h}_n\colon \hat{\Sigma}_{2n}(\mathcal{X}) \to \hat{\Pi}_{2n}(\mathcal{X})$ and $\hat{q}_{m,n}\colon \hat{\Pi}_{2m}(\mathcal{X}) \to \hat{\Pi}_{2n}(\mathcal{X})$ be the natural homomorphisms respectively induced by h_n and $q_{m,n}$. Then the following diagram commutes:

$$\begin{array}{ccc} \hat{\Sigma}_{2m}(\mathcal{X}) & \xrightarrow{\hat{p}_{m,n}} & \hat{\Sigma}_{2n}(\mathcal{X}) \\ \downarrow{\scriptstyle \hat{h}_m} & & \downarrow{\scriptstyle \hat{h}_n} \\ \hat{\Pi}_{2m}(\mathcal{X}) & \xrightarrow[\hat{q}_{m,n}]{} & \hat{\Pi}_{2n}(\mathcal{X}). \end{array} \tag{4.1}$$

The family $(\hat{q}_{m,n})_{m,n}$ defines an inverse system of profinite groupoids. We denote by $\hat{h}$ the continuous semigroupoid homomorphism from $\varprojlim \hat{\Sigma}_{2n}(\mathcal{X})$ to $\varprojlim \hat{\Pi}_{2n}(\mathcal{X})$ established by the commutativity of Diagram (4.1).

For the remainder of this paper, we need to deal with the local semigroups of the various semigroupoids defined in this section. Given n, we denote respectively by $\Sigma_{2n}(\mathcal{X},x)^+$, $\hat{\Sigma}_{2n}(\mathcal{X},x)$, $\Pi_{2n}(\mathcal{X},x)$, $\hat{\Pi}_{2n}(\mathcal{X},x)$ the local semigroups at vertex $p_{2n}(x) = x_{[-n,n-1]}$ of $\Sigma_{2n}(\mathcal{X})^+$, $\hat{\Sigma}_{2n}(\mathcal{X})$, $\Pi_{2n}(\mathcal{X})$ and $\hat{\Pi}_{2n}(\mathcal{X})$.

Remark 4.7. If $\mathcal{X}$ is irreducible, then $\hat{\Pi}_{2n}(\mathcal{X},x)$ is the profinite completion of the fundamental group of the strongly connected graph $\Sigma_{2n}(\mathcal{X})$ (cf. Lemma 2.2).

5. Return words in the study of $G(\mathcal{X})$ in the minimal case

Consider a subshift $\mathcal{X}$ of $A^{\mathbb{Z}}$. Let $u \in L(\mathcal{X})$. The *return words*[†] of u in $\mathcal{X}$ are the elements of the set $R(u)$ of words $v \in A^+$ such that $vu \in L(\mathcal{X}) \cap uA^+$ and such that u occurs in vu only as both prefix and suffix. The characterization of minimal subshifts via the notion of uniform recurrence yields that the subshift $\mathcal{X}$ is minimal if and only if, for every $u \in L(\mathcal{X})$, the set $R(u)$ is finite.

Let $n \geq 0$ be such that $|u| \geq n$. Consider words u_1 and u_2 with $u = u_1u_2$ and $|u_1| = n$. Let $R(u_1,u_2)$ be the set of words v such that $u_1vu_2 \in L(\mathcal{X})$ and $u_1v \in R(u)u_1$. In other words, we have $R(u_1,u_2) = u_1^{-1}(R(u)u_1)$. In particular, $R(u_1,u_2)$ and $R(u)$ have the same cardinality. The elements of $R(u_1,u_2)$ are callled in [9] *n-delayed return words of u* in $\mathcal{X}$, and *return words of $u_1.u_2$* in [21]. Note that $R(u_1,u_2)$ is a code (actually, a circular code [21, Lemma 17]).

Fix $x \in \mathcal{X}$. Denote by $R_n(x)$ the set $R(x_{[-n,-1]}, x_{[0,n-1]})$. Clearly, if $\mathcal{X}$ is a periodic subshift with period N, then the elements of $R_n(x)$ have length at most N. On the other hand, we have the following result.

Lemma 5.1 (cf. [20, Lemma 3.2])**.** *If $\mathcal{X}$ is a minimal non-periodic subshift then $\lim_{n\to\infty} \min\{|r| : r \in R_n(x)\} = \infty$ for every $x \in \mathcal{X}$.*[‡]

Let $u \in R_n(x)$. The word $w = x_{[-n,-1]}ux_{[0,n-1]}$ belongs to $L(\mathcal{X})$. Its prefix and its suffix of length $2n$ is the word $x_{[-n,n-1]}$. Hence, the graph $\Sigma_{2n}(\mathcal{X})$

[†]What we call *return words* is sometimes in the literature designated *first return words*, as is the case of the article [13], which is cited later in this paper. The terminology that we adopt appears for instance in [20, 21, 12].

[‡]Lemma 5.1 is taken from [20, Lemma 3.2], but the limit which appears explicitly in [20, Lemma 3.2] is $\lim_{n\to\infty} \min\{|r| : r \in R(x_{[0,n-1]})\} = \infty$. However, $R(z,t)$ is clearly contained in the subsemigroup of A^+ generated by $R(t)$. In particular, $\min\{|r| : u \in R_n\} \geq \min\{|r| : u \in R(x_{[0,n-1]})\}$, and so our formulation of Lemma 5.1 follows immediately from the one in [20, Lemma 3.2].

To conclude the proof, it remains to show that $K_n(\mathcal{X})_E$ is residually finite as a topological groupoid. Since it is a subgroupoid of the category $(\overline{\Omega}_A\mathsf{S})_E$, which is residually finite as a topological category, the topological groupoid $K_n(\mathcal{X})_E$ is residually finite as the subcategory generated by the image of a homomorphism of a topological groupoid into a finite category is easily seen to be a groupoid. ∎

In the minimal case, we may combine Proposition 7.5 and Theorem 5.7 to obtain an alternative characterization of the profinite groupoid $\hat{\Sigma}_\infty(\mathcal{X})$ in terms of the local structure of the free profinite semigroup $\overline{\Omega}_A\mathsf{S}$. For this purpose, we introduce some notation that is also useful in the next section.

Suppose $\mathcal{X}$ is a minimal subshift. For each $x \in \mathcal{X}$, let ℓ_x be the identity at x in the groupoid $\hat{\Sigma}_\infty(\mathcal{X})$ (cf. Theorem 5.6). Let e_x be the idempotent $\hat{\mu}(\ell_x)$. Recall that e_x is the identity element of G_x (cf. Theorem 5.7).

Remark 7.6. For every minimal subshift, the mapping $x \in \mathcal{X} \mapsto \ell_x \in \hat{\Sigma}_\infty(\mathcal{X})$ is continuous, and therefore so is the mapping $x \in \mathcal{X} \mapsto e_x \in J(\mathcal{X})$.

By Theorem 3.1, we know that $K_\infty(\mathcal{X}) = J(\mathcal{X})$. By Proposition 7.5, we know that $J(\mathcal{X})_E$ is a profinite groupoid. Note that for each $x \in \mathcal{X}$, the profinite groups G_x and the local group of $J(\mathcal{X})_E$ are isomorphic, the mapping $u \in G_x \mapsto (e_x, u, e_x)$ being a continuous isomorphism between them. The following gives a sort of first geometric characterization of the groupoid $J(\mathcal{X})_E$.

Theorem 7.7. *For every minimal subshift $\mathcal{X}$, we have a continuous groupoid isomorphism $F\colon \hat{\Sigma}_\infty(\mathcal{X}) \to J(\mathcal{X})_E$ defined on vertices by $F(x) = e_x$ and on edges by $F(s) = (e_{\alpha(s)}, \hat{\mu}(s), e_{\omega(s)})$.*

Proof: Note first that F is clearly a functor between categories, as $\hat{\mu}$ is itself a semigroupoid homomorphism. The continuity of F follows from the continuity of $\hat{\mu}$ and Remark 7.6. Let e be an idempotent of $J(\mathcal{X})$, and take $x = \overleftarrow{e}.\overrightarrow{e}$. Since $e_x \in G_x$, we have $e_x = e$ in view of Lemma 3.2, whence $F(x) = e$. On the other hand, if $F(x) = F(y)$, then $x = y$, also in view of Lemma 3.2. This establishes that F is bijective on vertices.

Fix an element $x \in \mathcal{X}$. Consider the isomorphism $u \in G_x \mapsto (e_x, u, e_x)$, from G_x onto the local group of $J(\mathcal{X})_E$ at e. Composing it with the restriction of $\hat{\mu}$ to the local group $\hat{\Sigma}_\infty(\mathcal{X},x)$ we get, thanks to Theorem 5.7, a continuous isomorphism, which is precisely the restriction of F mapping $\hat{\Sigma}_\infty(\mathcal{X},x)$ onto the local group of $J(\mathcal{X})_E$ at e.

of $\overline{\Omega}_A\mathsf{S}$. Hence, we have $w = \hat{\mu}_n(\ell)w = \hat{\mu}_n(\ell t)$. Since $\ell t \in \operatorname{Ker}\hat{\Sigma}_{2n}(\mathcal{X})$, this shows the reverse inclusion $K_{2n+1}(\mathcal{X}) \subseteq \hat{\mu}_n(\operatorname{Ker}\hat{\Sigma}_{2n}(\mathcal{X}))$. ∎

Corollary 7.4. *Let $\mathcal{X}$ be an irreducible subshift. Fix a positive integer n. For every vertex v of $\hat{\Sigma}_{2n}(\mathcal{X})$, there is an idempotent loop ℓ of $\hat{\Sigma}_{2n}(\mathcal{X})$ rooted at v such that $\hat{\mu}(\ell) \in K_{2n+1}(\mathcal{X})$.*

Proof: The graph $\hat{\Sigma}_{2n}(\mathcal{X})$ is strongly connected, and so every element of $\operatorname{Ker}\hat{\Sigma}_{2n}(\mathcal{X})$ is a factor of a loop q rooted at v. The loop $\ell = q^\omega$ then satisfies the desired conditions, by Lemma 7.3. ∎

Let S be a semigroup. The category S_E is defined by the following data:

(1) the vertex set is the set of idempotents of S;
(2) the edges from e to f are the triples (e, u, f) with $u \in eSf$;
(3) the composition is defined by $(e, u, f)(f, v, g) = (e, uv, g)$.

Note that (e, e, e) is a local identity at each idempotent e of S. The category S_E was introduced in semigroup theory by Tilson in his fundamental paper [35]. Since the construction $S \mapsto S_E$ is functorial, if S is profinite, then S_E becomes a profinite category by considering the product topology in $S \times S \times S$. In this paper we are interested in dealing with the profinite category $(\overline{\Omega}_A\mathsf{S})_E$. For an irreducible subshift $\mathcal{X}$ and $n \in \mathbb{Z}^+ \cup \{\infty\}$, denote by $K_n(\mathcal{X})_E$ the subgraph of $(\overline{\Omega}_A\mathsf{S})_E$ whose vertices are the idempotents of $K_n(\mathcal{X})$ and whose edges are the edges (e, u, f) of $(\overline{\Omega}_A\mathsf{S})_E$ with $u \in K_n(\mathcal{X})$.

Proposition 7.5. *Let $\mathcal{X}$ be an irreducible subshift. For every $n \in \mathbb{Z}^+ \cup \{\infty\}$, the graph $K_n(\mathcal{X})_E$ is a closed subcategory of $(\overline{\Omega}_A\mathsf{S})_E$. Moreover, $K_n(\mathcal{X})_E$ is a profinite groupoid.*

Proof: We know that $K_n(\mathcal{X})_E$ is topologically closed in $(\overline{\Omega}_A\mathsf{S})_E$ because the set of idempotents of $\overline{\Omega}_A\mathsf{S}$ and every $\mathcal{J}$-class of $\overline{\Omega}_A\mathsf{S}$ are closed.

As shown in [10, Lemma 8.2], if w is a finite factor of a product pqr with $p, q, r \in \overline{\Omega}_A\mathsf{S}$ and $q \notin A^+$, then w is a factor of pq or of qr. Therefore, the composition in $(\overline{\Omega}_A\mathsf{S})_E$ of two edges of $K_n(\mathcal{X})_E$ belongs to $K_n(\mathcal{X})_E$, and so $K_n(\mathcal{X})_E$ is a subcategory of $(\overline{\Omega}_A\mathsf{S})_E$.

If (e, u, f) is an edge of $K_n(\mathcal{X})_E$, then $e \mathcal{R} u \mathcal{L} f$ by stability of $\overline{\Omega}_A\mathsf{S}$. It follows from the basic properties of Green's relations that there is some v in $K_n(\mathcal{X})$ such that $f \mathcal{R} v \mathcal{L} e$, $uv = e$ and $vu = f$. Hence (f, v, e) is an edge of $K_n(\mathcal{X})_E$ that is an inverse of (e, u, f), thereby establishing that $K_n(\mathcal{X})_E$ is a groupoid.

has a cycle s rooted at the vertex $x_{[-n,n-1]}$ such that $\mu_n(s) = u$. Since μ_n is faithful, we may therefore define a function $\lambda_n \colon R_n(x) \to \Sigma_{2n}(\mathcal{X}, x)^+$ such that $\mu_n \circ \lambda_n$ is the identity $1_{R_n(x)}$ on $R_n(x)$.

To extract consequences from these facts at the level of the free profinite semigroup $\overline{\Omega}_A\mathsf{S}$, we use the following theorem from [28].

Theorem 5.2. *If X is a finite code of A^+, then the closed subsemigroup of $\overline{\Omega}_A\mathsf{S}$ generated by X is a profinite semigroup freely generated by X.*

Assuming that $\mathcal{X}$ is a minimal subshift, as we do throughout along this section, the code $R_n(x)$ is finite. Therefore it follows from Theorem 5.2 that the profinite subsemigroup $\overline{\langle R_n(x)\rangle}$ of $\overline{\Omega}_A\mathsf{S}$ is freely generated by $R_n(x)$, and so the mapping λ_n extends in a unique way to a continuous homomorphism $\hat{\lambda}_n \colon \overline{\langle R_n(x)\rangle} \to \hat{\Sigma}_{2n}(\mathcal{X}, x)$ of profinite semigroups. Note that the following equality holds by definition of λ_n:

$$\hat{\mu}_n \circ \hat{\lambda}_n = 1_{\overline{\langle R_n(x)\rangle}}. \tag{5.1}$$

If $m \geq n$, then the inclusion $R_m(x) \subseteq \langle R_n(x)\rangle$ clearly holds.

Lemma 5.3. *Let $\mathcal{X}$ be a minimal non-periodic subshift and let $x \in \mathcal{X}$. Then we have $\bigcap_{n\geq 1} \overline{\langle R_n(x)\rangle} = G_x$.*

Proof: Denote by I the intersection $\bigcap_{n\geq 1} \overline{\langle R_n(x)\rangle}$. The inclusion $G_x \subseteq I$ appears in [9, Lemma 5.1]. Let us show the reverse inclusion. If w is an element of $\langle R_n(x)\rangle$, then it labels a closed path of $\Sigma_{2n}(\mathcal{X})$ at $x_{[-n,n-1]}$. Therefore, every factor of w of length at most $2n + 1$ belongs to $L(\mathcal{X})$. Since w is an arbitrary element of $\langle R_n(x)\rangle$, this implies that every factor of length at most $2n + 1$ of an element of $\overline{\langle R_n(x)\rangle}$ belongs to $L(\mathcal{X})$. Therefore, if $u \in I$, then every finite factor of u belongs to $L(\mathcal{X})$. On the other hand, by Lemma 5.1 the elements of I do not belong to A^+. We conclude from Theorem 3.1 that $I \subseteq J(\mathcal{X})$. Let $n > 0$. By Lemma 5.1, there is $m > n$ such that the length of every element of $R_m(x)$ is greater than n. Since the elements of $R_m(x)$ label closed paths at $x_{[-m,m-1]}$, we know that $R_m(x) \subseteq x_{[0,n-1]}A^+ \cap A^+x_{[-n,-1]}$. Hence, we have $I \subseteq \overline{\langle R_m(x)\rangle} \subseteq x_{[0,n-1]}\overline{\Omega}_A\mathsf{S} \cap \overline{\Omega}_A\mathsf{S}x_{[-n,-1]}$. Since n is arbitrary, we deduce from the definition of G_x that $I \subseteq G_x$. ∎

If $\mathcal{X} = \{x\}$ is the singleton periodic subshift given by $x = \cdots aaa.aaa \cdots$, then $R_n(x) = \{a\}$ for all n, and Lemma 5.3 does not hold in this case. However, denoting by $\overline{\langle R_n(x)\rangle}_\infty$ the profinite semigroup $\overline{\langle R_n(x)\rangle} \setminus A^+$, we get the following result, which can be easily seen to apply to periodic subshifts.

Lemma 5.4. *Let $\mathcal{X}$ be a minimal subshift and let $x \in \mathcal{X}$. Then we have $\bigcap_{n\geq 1} \overline{\langle R_n(x)\rangle}_\infty = G_x$.* ∎

We shall consider the inverse systems with connecting morphisms the inclusions $i_{m,n}\colon \overline{\langle R_m(x)\rangle} \to \overline{\langle R_n(x)\rangle}$ and $i_{m,n}|\colon \overline{\langle R_m(x)\rangle}_\infty \to \overline{\langle R_n(x)\rangle}_\infty$. Note that we can identify G_x with $\varprojlim \overline{\langle R_n(x)\rangle}_\infty$ via Lemma 5.4 (each $g \in G_x$ is identified with the sequence $(g)_{n\geq 1}$). Also, one has $G_x \subseteq \varprojlim \overline{\langle R_n(x)\rangle}$, with equality in the non-periodic case, as seen in Lemma 5.3.

Let $m \geq n$, and let $r \in \overline{\langle R_m(x)\rangle}$. Then, the equalities

$$\hat{\mu}_n(\hat{p}_{m,n} \circ \hat{\lambda}_m(r)) = \hat{\mu}_m(\hat{\lambda}_m(r)) = r = \hat{\mu}_n(\hat{\lambda}_n(r))$$

hold by (5.1). Since $\hat{\mu}_n$ is faithful, this shows that the following diagram commutes:

$$\begin{array}{ccc} \overline{\langle R_m(x)\rangle} & \xrightarrow{\ i_{m,n}\ } & \overline{\langle R_n(x)\rangle} \\ \downarrow{\scriptstyle \hat{\lambda}_m} & & \downarrow{\scriptstyle \hat{\lambda}_n} \\ \hat{\Sigma}_{2m}(\mathcal{X}, x) & \xrightarrow[\ \hat{p}_{m,n}\ ]{} & \hat{\Sigma}_{2n}(\mathcal{X}, x). \end{array} \tag{5.2}$$

The commutativity of Diagram (5.2) yields the existence of the homomorphism $\hat{\lambda} = \varprojlim \hat{\lambda}_n$ from $\varprojlim \overline{\langle R_n(x)\rangle}$ to $\varprojlim \hat{\Sigma}_{2n}(\mathcal{X}, x)$. Note that $\varprojlim \hat{\Sigma}_{2n}(\mathcal{X}, x)$ is the local semigroup $\hat{\Sigma}(\mathcal{X}, x)$ of $\hat{\Sigma}(\mathcal{X})$ at vertex x (cf. Theorem 4.3).

Let $\hat{\Sigma}_\infty(\mathcal{X})$ be the subgraph of $\hat{\Sigma}(\mathcal{X}) \setminus \Sigma(\mathcal{X})^+$ obtained by deleting the edges in $\Sigma(\mathcal{X})^+$.

Remark 5.5. When $\mathcal{X}$ is a minimal non-periodic subshift, the local semigroup $\hat{\Sigma}_\infty(\mathcal{X}, x)$ of $\hat{\Sigma}_\infty(\mathcal{X})$ at x coincides with $\hat{\Sigma}(\mathcal{X}, x)$.

It turns out that $\hat{\Sigma}_\infty(\mathcal{X}, x)$ is a profinite group whenever $\mathcal{X}$ is minimal. Indeed, the following theorem was announced in [3] and shown in [7, Theorem 6.7].

Theorem 5.6. *Let $\mathcal{X}$ be a minimal subshift. Then $\hat{\Sigma}_\infty(\mathcal{X})$ is a profinite connected groupoid.*

It should be noted that the notion of profiniteness for semigroupoids is being taken as compactness plus residual finiteness in the category of semigroupoids. If the semigroupoid turns out to be a groupoid, one may ask

In view of Lemma 7.1, and since clearly $\mathcal{M}_n(\mathcal{X})$ is closed and factorial (irrespectively of $\mathcal{X}$ being irreducible or not), we may consider the apex $K_n(\mathcal{X})$ of $\mathcal{M}_n(\mathcal{X})$ when $\mathcal{X}$ is irreducible.

The irreducibility of $\mathcal{X}$ also implies that, for every positive integer n, the semigroupoid $\hat{\Sigma}_{2n}(\mathcal{X})$ is strongly connected, since $\Sigma_{2n}(\mathcal{X})$ is then itself strongly connected.

A subsemigroupoid T of a semigroupoid S is an *ideal* if for every $t \in T$ and every $s \in S$, $\omega(s) = \alpha(t)$ implies $st \in T$, and $\omega(t) = \alpha(s)$ implies $ts \in T$. In a strongly connected compact semigroupoid S, there is an underlying *minimum ideal* $\operatorname{Ker} S$. This ideal $\operatorname{Ker} S$ may be defined as follows. Consider any vertex v of S and the local semigroup $S(v)$ of S at v. Then $S(v)$ is a compact semigroup, and therefore it has a minimum ideal K_v. Let $\operatorname{Ker} S$ be the subsemigroupoid of S with the same set of vertices as S and whose edges are those edges of S that admit some (and therefore every) element of K_v as a factor. Note that $K_v = (\operatorname{Ker} S)(v)$.

The next lemma is folklore. The relations $\leq_\mathcal{J}$ and $\mathcal{J}$ in semigroupoids extend naturally the corresponding notions for semigroups, namely, in a semigroupoid $s \leq_\mathcal{J} t$ means the edge t is a factor of the edge s.

Lemma 7.2. *If S is a strongly connected compact semigroupoid, then* $\operatorname{Ker} S$ *is a closed ideal of S that does not depend on the choice of v. Moreover, the edges in* $\operatorname{Ker} S$ *are $\mathcal{J}$-equivalent in S; more precisely, they are $\leq_\mathcal{J}$-below every edge of S.*

We next relate $\operatorname{Ker} \hat{\Sigma}_{2n}(\mathcal{X})$ with $K_{2n+1}(\mathcal{X})$.

Lemma 7.3. *Consider an irreducible subshift $\mathcal{X}$ and a positive integer n. Then we have the equality $\hat{\mu}_n(\operatorname{Ker} \hat{\Sigma}_{2n}(\mathcal{X})) = K_{2n+1}(\mathcal{X})$.*

Proof: Let $s \in \operatorname{Ker} \hat{\Sigma}_{2n}(\mathcal{X})$ and let $w \in K_{2n+1}(\mathcal{X})$.

By Lemma 4.4, there is $t \in \hat{\Sigma}_{2n}(\mathcal{X})$ such that $\hat{\mu}_n(t) = w$. But t is a factor of s by Lemma 7.2, and so w is a factor of $\hat{\mu}_n(s)$. Again by Lemma 4.4, we have $\hat{\mu}_n(s) \in \mathcal{M}_{2n+1}(\mathcal{X})$. The $\leq_\mathcal{J}$-minimality of $K_{2n+1}(\mathcal{X})$ then yields $\hat{\mu}_n(s) \in K_{2n+1}(\mathcal{X})$, establishing the inclusion $\hat{\mu}_n(\operatorname{Ker} \hat{\Sigma}_{2n}(\mathcal{X})) \subseteq K_{2n+1}(\mathcal{X})$.

On the other hand, since $\hat{\Sigma}_{2n}(\mathcal{X})$ is strongly connected, there is an edge r in $\hat{\Sigma}_{2n}(\mathcal{X})$ having s has a factor and such that tr is a loop. Let $\ell = (tr)^\omega$. Since $\operatorname{Ker} \hat{\Sigma}_{2n}(\mathcal{X})$ is an ideal, we have $\ell \in \operatorname{Ker} \hat{\Sigma}_{2n}(\mathcal{X})$, and so the idempotent $\hat{\mu}_n(\ell)$ belongs to $K_{2n+1}(\mathcal{X})$ by the already proved inclusion. But $w = \hat{\mu}_n(t) \in K_{2n+1}(\mathcal{X})$ is a prefix of the idempotent $\hat{\mu}_n(\ell)$, and so $w \mathcal{R} \hat{\mu}_n(\ell)$ by stability

result is called the Return Theorem in [13]. Combining the Return Theorem with Theorem 6.1, noting that, for every $x \in \mathcal{X}$, the set $R_n(x)$ is conjugate to $R_n(x_{[-n,n-1]})$, we immediately deduce the following theorem.

Theorem 6.5. *If $\mathcal{X}$ is a minimal subshift satisfying the tree condition, then $G(\mathcal{X})$ is a free profinite group with rank M, where M is the number of letters occurring in $\mathcal{X}$.* ∎

There are other cases of minimal subshifts $\mathcal{X}$, not satisfying the tree condition, for which $G(\mathcal{X})$ is known to be a free profinite group. Indeed, it is shown in [5] that if $\mathcal{X}$ is the subshift defined by a weakly primitive substitution φ which is group invertible, then $G(\mathcal{X})$ is a free profinite group. The weakly primitive substitution

$$\varphi(a) = ab, \quad \varphi(b) = cda, \quad \varphi(c) = cd, \quad \varphi(d) = abc$$

is group invertible, but the minimal subshift defined by $\mathcal{X}$ is a subshift that fails the tree condition [13, Example 3.4].

The special case of Theorem 6.5 in which the subshift is an Arnoux-Rauzy subshift was previously established in [5] by the first author by extending the case of substitution Arnoux-Rauzy subshifts, for which the substitutions are group invertible.

7. The groupoids $K_n(\mathcal{X})_E$

Let $\mathcal{X}$ be a subshift of $A^{\mathbb{Z}}$. For every positive integer n, if $\mathcal{X}_n$ is the subshift of $A^{\mathbb{Z}}$ consisting on those elements x of $A^{\mathbb{Z}}$ such that $x_{[k,k+n-1]} \in L(\mathcal{X})$ for every $k \in \mathbb{Z}$, then one clearly has $L(\mathcal{X}_n) = \mathcal{M}_n(\mathcal{X}) \cap A^+$. Since $\mathcal{M}_n(\mathcal{X})$ is a clopen subset of $\overline{\Omega}_A\mathsf{S}$, it follows that $\overline{L(\mathcal{X}_n)} = \mathcal{M}_n(\mathcal{X})$. From this fact one deduces the following lemma. For the sake of uniformity, we denote $\mathcal{M}(\mathcal{X}) = \bigcap_{n\geq 1} \mathcal{M}_n(\mathcal{X})$ by $\mathcal{M}_\infty(\mathcal{X})$.

Lemma 7.1. *For every $n \in \mathbb{Z}^+ \cup \{\infty\}$, the set $\mathcal{M}_n(\mathcal{X})$ is irreducible.*

Proof: Clearly, for every $n \geq 1$, if $\mathcal{X}$ is irreducible then so is $\mathcal{X}_n$, whence $\mathcal{M}_n(\mathcal{X}) = \overline{L(\mathcal{X}_n)}$ is irreducible. Let $u, v \in \mathcal{M}_\infty(\mathcal{X})$. For each $n \geq 1$, there is $w_n \in \mathcal{M}_n(\mathcal{X})$ such that $uw_nv \in \mathcal{M}_n(\mathcal{X})$. If w is an accumulation point of $(w_n)_{n\in\mathbb{Z}^+}$ then $w \in \mathcal{M}_n(\mathcal{X})$ for every n, since $\mathcal{M}_n(\mathcal{X})$ is closed and $w_m \in \mathcal{M}_n(\mathcal{X})$ for every $m \geq n$. This shows $\mathcal{M}_\infty(\mathcal{X})$ is irreducible. ∎

whether profiniteness in the category of groupoids is an equivalent property. The answer is affirmative since it is easy to verify that, if $\varphi : G \to S$ is a semigroupoid homomorphism and G is a groupoid, then the subsemigroupoid of S generated by $\varphi(G)$ is a groupoid.

In the statement of [7, Theorem 6.7], it is only indicated that $\hat{\Sigma}_\infty(\mathcal{X})$ is a connected groupoid but we note that, if a compact semigroupoid is a groupoid, then edge inversion and the mapping associating to each vertex the identity at that vertex are continuous operations. Thus, $\hat{\Sigma}_\infty(\mathcal{X})$ is in fact a topological groupoid.

A preliminary version of the next theorem was also announced in [3], and a proof appears in the doctoral thesis [16]. We present here a different proof, based on Lemma 5.3.

Theorem 5.7. *For every minimal subshift $\mathcal{X}$ and every $x \in \mathcal{X}$, the restriction $\hat{\lambda}| : G_x \to \hat{\Sigma}_\infty(\mathcal{X}, x)$ is an isomorphism. Its inverse is the restriction $\hat{\mu}| : \hat{\Sigma}_\infty(\mathcal{X}, x) \to G_x$.*

Proof: By Lemma 5.4, we know that $G_x = \bigcap_{n\geq 1} \overline{\langle R_n(x)\rangle}_\infty$, and so from (5.1) we deduce that $\hat{\mu} \circ \hat{\lambda}(g) = g$ for every $g \in G_x$. This shows in particular that $\hat{\lambda}(g)$ must be an infinite path whenever $g \in G_x$, whence $\hat{\lambda}(G_x)$ is indeed contained in $\hat{\Sigma}_\infty(\mathcal{X}, x)$. It also shows that the restriction $\hat{\mu}| : \hat{\lambda}(G_x) \to G_x$ is onto. Such a restriction is also injective, as $\hat{\lambda}(G_x) \subseteq \hat{\Sigma}(\mathcal{X}, x)$ and $\hat{\mu}$ is faithful. Therefore, all it remains to show is the equality $\hat{\lambda}(G_x) = \hat{\Sigma}_\infty(\mathcal{X}, x)$.

Let $s \in \hat{\Sigma}_\infty(\mathcal{X}, x)$ and let $g = \hat{\mu}(s)$. By Theorem 4.3, s is the limit of a net of finite paths of the graph $\Sigma(\mathcal{X})$. Since the labeling $\hat{\mu}$ of finite paths clearly belongs to $L(\mathcal{X})$, we have $g = \hat{\mu}(s) \in \overline{L(\mathcal{X})} \setminus A^+$ by continuity of $\hat{\mu}$. It follows that $\hat{\mu}(s) \in J(\mathcal{X})$ by Theorem 3.1. Since s is a loop rooted at x, applying Lemma 4.6, we conclude that $\hat{\mu}(s) \in G_x$. Hence, we have $\hat{\mu}(s) = g = \hat{\mu}(\hat{\lambda}(g))$. As $\hat{\mu}$ is faithful, we get $s = \hat{\lambda}(g)$, concluding the proof. ∎

The notion of isomorphism between subshifts is called *conjugacy*. If $\mathcal{X}$ and $\mathcal{Y}$ are conjugate subshifts, then $\Sigma(\mathcal{X})$ and $\Sigma(\mathcal{Y})$ are isomorphic, which combined Theorem 5.7 leads to the following result.

Corollary 5.8. *If $\mathcal{X}$ and $\mathcal{Y}$ are conjugate minimal subshifts, then the profinite groups $G(\mathcal{X})$ and $G(\mathcal{Y})$ are isomorphic.* ∎

Actually, a more general result was proved in [15] using different techniques: if $\mathcal{X}$ and $\mathcal{Y}$ are conjugate irreducible subshifts, then the profinite groups $G(\mathcal{X})$ and $G(\mathcal{Y})$ are isomorphic.

6. An application: a sufficient condition for freeness

In this section, we establish the next theorem, where $FG(A)$ denotes the free group generated by A.

Theorem 6.1. *Let $\mathcal{X}$ be a minimal non-periodic subshift, and take $x \in \mathcal{X}$. Let A be the set of letters occurring in $\mathcal{X}$. Suppose there is a subgroup K of $FG(A)$ and an infinite set P of positive integers such that, for every $n \in P$, the set $R_n(x)$ is a free basis of K. Let $\overline{K}$ be the topological closure of K in $\overline{\Omega}_A\mathsf{G}$. Then the restriction to G_x of the canonical projection $p_\mathsf{G}\colon \overline{\Omega}_A\mathsf{S} \to \overline{\Omega}_A\mathsf{G}$ is a continuous isomorphism from G_x onto $\overline{K}$.*

The following proposition, taken from [9, Proposition 5.2], plays a key role in the proof of Theorem 6.1.

Proposition 6.2. *Let $\mathcal{X}$ be a minimal non-periodic subshift of $A^{\mathbb{Z}}$ and let $x \in \mathcal{X}$. Suppose there are $M \geq 1$ and strictly increasing sequences $(p_n)_n$ and $(q_n)_n$ of positive integers such that $R(x_{[-p_n,-1]}, x_{[0,q_n]})$ has exactly M elements $r_{n,1}, \ldots, r_{n,M}$, for every n. Let $(r_1, \ldots, r_M)$ be an arbitrary accumulation point of the sequence $(r_{n,1}, \ldots, r_{n,M})_n$ in $(\overline{\Omega}_A\mathsf{S})^M$. Then $\overline{\langle r_1, \ldots, r_M\rangle}$ is the maximal subgroup G_x of $J(\mathcal{X})$.*

In the proof of Theorem 6.5 we shall apply the following lemma, whose proof is an easy and elementary exercise that we omit.

Lemma 6.3. *Let $S_1 \supseteq S_2 \supseteq S_3 \supseteq \cdots$ be a descending sequence of compact subspaces of a compact space S_1. Suppose that $\varphi\colon S_1 \to T$ is a continuous mapping such that $\varphi(S_n) = T$ for every $n \geq 1$. If $I = \bigcap_{n\geq 1} S_n$, then we have $\varphi(I) = T$.*

We shall also use the following tool.

Proposition 6.4 ([18, Corollary 2.2])**.** *Suppose that B is the basis of a finitely generated subgroup K of $FG(A)$. Let $\overline{K}$ be the topological closure of K in $\overline{\Omega}_A\mathsf{G}$. Then $\overline{K}$ is a free profinite group with basis B.*

We are ready to prove Theorem 6.1.

Proof of Theorem 6.1: By Lemma 5.3, we have $G_x = \bigcap_{n\in P} \overline{\langle R_n(x)\rangle}$. On the other hand, for every $n \in P$, since by hypothesis the set $p_\mathsf{G}(R_n(x)) = R_n(x)$ is a basis of K, we have $p_\mathsf{G}\big(\overline{\langle R_n(x)\rangle}\big) = \overline{K}$. It then follows from Lemma 6.3 that $p_\mathsf{G}(G_x) = \overline{K}$.

By assumption, for every $n \in P$, the set $R_n(x)$ has M elements, where M is the rank of K. Therefore, by Proposition 6.2, we know that G_x is generated by M elements. On the other hand, $\overline{K}$ is a free profinite group of rank M, by Proposition 6.4. Hence, there is a continuous onto homomorphism $\psi\colon \overline{K} \to G_x$. We may then consider the continuous onto endomorphism φ of $\overline{K}$ such that $\varphi(g) = p_\mathsf{G}(\psi(g))$ for every $g \in \overline{K}$. Every onto continuous endomorphism of a finitely generated profinite group is an isomorphism [33, Proposition 2.5.2], whence φ is an isomorphism. Since ψ is onto, we conclude that ψ is an isomorphism. This shows that the restriction $p_G|\colon G_x \to \overline{K}$ is the continuous isomorphism $\varphi \circ \psi^{-1}\colon G_x \to \overline{K}$. ∎

We proceed to apply Theorem 6.1 and two of the main results of [13] to deduce the freeness of the Schützenberger group of the minimal subshifts satisfying the *tree condition*, which we next describe.

Let $\mathcal{X}$ be a subshift of $A^{\mathbb{Z}}$. Given $w \in L(\mathcal{X}) \cup \{1\} \subseteq A^*$, let

$$\begin{aligned} L_w &= \{a \in A \mid aw \in L(\mathcal{X})\},\\ R_w &= \{a \in A \mid wa \in L(\mathcal{X})\},\\ E_w &= \{(a,b) \in A \times A \mid awb \in L(\mathcal{X})\}. \end{aligned}$$

The *extension graph* G_w is the bipartite undirected graph whose vertex set is the union of disjoint copies of L_w and R_w, and whose edges are the pairs $(a,b) \in E_w$, with incidence in $a \in L_w$ and $b \in R_w$. The subshift $\mathcal{X}$ satisfies the tree condition if G_w is a tree for every $w \in L(\mathcal{X}) \cup \{1\}$.

The class of subshifts satisfying the tree condition contains two classes that have deserved strong attention in the literature: the class of *Arnoux-Rauzy subshifts*[§] (see the survey [23]), and the class of subshifts defined by *regular interval exchange transformations* (see [13, 14]).

It is shown in [13, Theorem 4.5] that if the minimal subshift $\mathcal{X}$ satisfies the tree condition, then, for every $w \in L(\mathcal{X})$, the set of return words $R(w)$ is a basis of the free group generated by the set of letters occurring in $\mathcal{X}$. This

[§]The Arnoux-Rauzy subshifts over two-letter alphabets are the extensively studied *Sturmian subshifts*, but we warn that in [13] the Arnoux-Rauzy subshifts are called Sturmian.