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HARD LEFSCHETZ THEOREM FOR VAISMAN MANIFOLDS

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ABSTRACT: We establish a Hard Lefschetz theorem for the de Rham cohomology of compact Vaisman manifolds. A similar result is proved for the the basic cohomology with respect to the Lee vector field. Motivated by these results, we introduce the notions of a Lefschetz and of a basic Lefschetz locally conformal symplectic (l.c.s.) manifold of the first kind. We prove that the two notions are equivalent if there exists a Riemannian metric such that the Lee vector field is unitary and parallel and its metric dual 1-form coincides with the Lee 1-form. Finally, we discuss several examples of compact l.c.s. manifolds of the first kind which do not admit compatible Vaisman metrics.

1. Introduction

1.1. Antecedents and motivation. It is well known that the global scalar product on the space of k-forms in an oriented compact Riemannian manifold M of dimension m induces an isomorphism between the kth de Rham cohomology group $H^k(M)$ and the dual space of the $(m-k)$th de Rham cohomology group $H^{m-k}(M)$. So, using that the dimension of the de Rham cohomology groups is finite, we deduce the Poincaré-duality: the dimension of $H^k(M)$ is equal to the dimension of $H^{m-k}(M)$.

In addition, in some special cases, one may define a canonical isomorphism between the vector spaces $H^k(M)$ and $H^{m-k}(M)$. For instance, if M is a compact Kähler manifold of dimension $2n$ then, using the $(n-k)$th exterior power of the symplectic 2-form, one obtains an explicit isomorphism between $H^k(M)$ and $H^{2n-k}(M)$. This is known as the Hard Lefschetz isomorphism for compact Kähler manifolds (see [11]).

On the other hand, it is well known that the odd dimensional counterparts of Kähler manifolds are Sasakian and co-Kähler manifolds (see [1, 2]). In these cases, one may also obtain a Hard Lefschetz isomorphism as shown in [5] for Sasakian and in [6] for co-Kähler manifolds. For a compact co-Kähler manifold the Hard Lefschetz isomorphism depends only on the underlying

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cosymplectic structure and for a Sasakian manifold it depends only on the corresponding contact structure.

A particular class of Hermitian manifolds which are related to Kähler, co-Kähler and Sasakian manifolds are Vaisman manifolds introduced in [21, 22]. A Vaisman manifold is a locally conformal Kähler manifold M which has non-zero parallel Lee 1-form ω. In this paper, we will assume, without the loss of generality, that ω is unitary. If J is the complex structure of M then the 1-form $\eta := -\omega \circ J$ is called the anti-Lee 1-form of M. The *Lee vector field* U is defined as the metric dual of ω, while the metric dual of η is called the *anti-Lee vector field* and is denoted by V. The following properties of Vaisman manifolds can be found in [7] and [15]:

- the couple (U, V) defines a flat foliation of rank 2 on M which is transversely Kähler;
- the foliation on M defined by V is transversely co-Kähler;
- the orthogonal bundle to the foliation on M defined by U is integrable and the leaves of the corresponding foliation are c-Sasakian manifolds.

The above results show that there is a close relationship between Vaisman manifolds on the one side and Kähler, co-Kähler, and Sasakian manifolds on the other side. In fact, in a recent paper [17], Ornea and Verbitsky proved that a compact Vaisman manifold of dimension $2n+2$ is holomorphically isometric to the mapping torus of a compact $\frac{1}{2}$-Sasakian manifold N of dimension $2n+1$. Conversely, the mapping torus of a compact $\frac{1}{2}$-Sasakian manifold is a compact Vaisman manifold.

So, a natural question arise: is there a Hard Lefschetz theorem for a compact Vaisman manifold? The aim of this paper is to give a positive answer to this question.

1.2. The results in the paper. The main result of this paper is the Hard Lefschetz theorem for a compact Vaisman manifold which may be formulated as follows.

Theorem 1.1. *Let M be a compact Vaisman manifold of dimension $2n+2$ with Lee 1-form ω, anti-Lee 1-form η, Lee vector field U, and anti-Lee vector field V. Then for each integer k between 0 and n, there exists an isomorphism*

$$\mathrm{Lef}_k : H^k(M) \longrightarrow H^{2n+2-k}(M)$$

which can be computed by using the following properties:

(*i*) *for every* $[\gamma] \in H^k(M)$, *there is* $\gamma' \in [\gamma]$ *such that*

$$\mathcal{L}_U\gamma' = 0, \;\; i_V\gamma' = 0, \;\; L^{n-k+2}\gamma' = 0, \;\; L^{n-k+1}\epsilon_\omega\gamma' = 0;$$

(*ii*) *if* $\gamma' \in [\gamma]$ *satisfies the conditions in* (*i*) *then*

$$\mathrm{Lef}_k[\gamma] = [\epsilon_\eta L^{n-k}(Li_U\gamma' - \epsilon_\omega\gamma')].$$

In this theorem, we write ϵ_β for the operator of the exterior multiplication by a differential form β, and L is defined to be $\frac{1}{2}\epsilon_{d\eta}$.

The map $\mathrm{Lef}_k : H^k(M) \longrightarrow H^{2n+2-k}(M)$, $0 \leq k \leq n$, in Theorem 1.1 will be called the *Lefschetz isomorphism in degree* k for the compact Vaisman manifold M.

In order to prove Theorem 1.1, we will first use a result which relates the de Rham cohomology of an oriented compact Riemannian manifold with the basic cohomology with respect to a unitary and parallel vector field.

Theorem 1.2. *Let* W *be a unitary and parallel vector field on an oriented compact Riemannian manifold* (P, g) *of dimension* p *and let the* 1*-form* w *be the metric dual of* W. *Denote by* $H_B^*(P)$ *the basic de Rham cohomology of* P *with respect to* W. *Then for each integer* k *between* 0 *and* p, *the map*

$$[(Id, \epsilon_w)] : H_B^k(P) \oplus H_B^{k-1}(P) \longrightarrow H^k(P)$$

defined by

$$[(Id, \epsilon_w)]([\beta]_B, [\beta']_B) = [\beta + w \wedge \beta'] \tag{1.1}$$

is an isomorphism.

Another result that we will use for proving of Theorem 1.1 is the basic Hard Lefschetz theorem below.

Theorem 1.3. *Let* M *be a compact Vaisman manifold of dimension* $2n + 2$ *with Lee* 1*-form* ω, *anti-Lee* 1*-form* η, *Lee vector field* U, *and anti-Lee vector field* V. *Denote by* $H_B^*(M)$ *the basic cohomology of* M *with respect to* U. *Then for each integer* k *between* 0 *and* n, *there exists an isomorphism*

$$\mathrm{Lef}_k^B : H_B^k(M) \longrightarrow H_B^{2n+1-k}(M)$$

which can be computed by using the following properties:

(*i*) *for every* $[\beta]_B \in H_B^k(M)$, *there is* $\beta' \in [\beta]_B$ *such that*

$$i_V\beta' = 0, \;\; L^{n-k+1}\beta' = 0; \tag{1.2}$$

(ii) if $\beta' \in [\beta]_B$ satisfies the conditions in (i) then

$$\mathrm{Lef}_k^B[\beta]_B = [\epsilon_\eta L^{n-k}\beta']_B.$$

The map $\mathrm{Lef}_k^B : H_B^k(M) \longrightarrow H_B^{2n+1-k}(M)$, for $0 \leq k \leq n$, will be called the *basic Lefschetz isomorphism in degree k* associated with the compact Vaisman manifold M.

For a Vaisman manifold M of dimension $2n+2$, the couple (ω, η) of the Lee and anti-Lee 1-forms defines a locally conformal symplectic (l.c.s.) structure of the first kind with anti-Lee vector field V and infinitesimal automorphism U (see [23] and Section 2.2 for the definition of an l.c.s. structure of the first kind). Note that the definition of the Hard Lefschetz and the basic Hard Lefschetz isomorphism associated with M only depends on the l.c.s. structure of the first kind. Hence, both isomorphisms provide obstructions for an l.c.s. manifold to admit Vaisman structures.

Now, let M be a compact manifold of dimension $2n+2$ endowed with an l.c.s. structure of the first kind (ω, η). Suppose that U and V are the anti-Lee and Lee vector field, respectively, on M.

Then, the previous results suggest us to introduce the following *Lefschetz relation* between the cohomology groups $H^k(M)$ and $H^{2n+2-k}(M)$, for $0 \leq k \leq n$,

$$R_{\mathrm{Lef}_k} = \left\{ ([\gamma], [\epsilon_\eta L^{n-k}(Li_U\gamma - \epsilon_\omega\gamma)]) \,\middle|\, \gamma \in \Omega^k(M),\ d\gamma = 0,\ \mathcal{L}_U\gamma = 0,\ i_V\gamma = 0, \right.$$
$$\left. L^{n-k+2}\gamma = 0,\ L^{n-k+1}\epsilon_\omega\gamma = 0 \right\}.$$

Similarly, we define the *basic Lefschetz relation* between the basic cohomology groups $H_B^k(M)$ and $H_B^{2n+1-k}(M)$, for $0 \leq k \leq n$, by

$$R^B_{\mathrm{Lef}_k} = \left\{ ([\beta]_B, [\epsilon_\eta L^{n-k}\beta]_B) \,\middle|\, \beta \in \Omega_B^k(M),\ d\beta = 0,\ i_V\beta = 0,\ L^{n-k+1}\beta = 0 \right\}.$$

An l.c.s. structure on M of the first kind is said to be:

- *Lefschetz* if, for every $0 \leq k \leq n$, the relation R_{Lef_k} is the graph of an isomorphism $\mathrm{Lef}_k : H^k(M) \longrightarrow H^{2n+2-k}(M)$;
- *Basic Lefschetz* if, for every $0 \leq k \leq n$, the relation $R^B_{\mathrm{Lef}_k}$ is the graph of an isomorphism $\mathrm{Lef}_k^B : H_B^k(M) \longrightarrow H_B^{2n+1-k}(M)$.

It is not clear what is the relation between the Lefschetz property and the basic Lefschetz property in general. However, we may prove the following result.

Theorem 1.4. *Let M be a compact manifold of dimension $2n+2$ endowed with an l.c.s. structure of the first kind (ω, η) such that the Lee vector field U is unitary and parallel with respect to a Riemannian metric g on M and*

$$\omega(X) = g(X, U), \quad \textit{for } X \in \mathfrak{X}(M).$$

Then:

(1) *The structure (ω, η) is Lefschetz if and only if it is basic Lefschetz.*
(2) *If the structure (ω, η) is Lefschetz (or, equivalently, basic Lefschetz), then for each $1 \leq k \leq n$ there exists a non-degenerate bilinear form*

$$\psi : H_B^k(M) \times H_B^k(M) \longrightarrow \mathbb{R}$$
$$\psi([\beta]_B, [\beta']_B) = \int_M \mathrm{Lef}_k[\beta] \cup [\beta']$$

which is skew-symmetric for odd k and symmetric for even k. As a consequence,

$$b_k(M) - b_{k-1}(M) \text{ is even if } k \text{ is odd and } 1 \leq k \leq n, \tag{1.3}$$

where $b_r(M)$ is the rth *Betti number of M.*

We remark that relations in (1.3) are well-known properties of a compact Vaisman manifold of dimension $2n+2$ (see [7, 22]).

1.3. Organization of the paper. In Section 2 we review the construction of a mapping torus and how it can be used to construct a Vaisman manifold. In Sections 3, 4, 5, we will prove Theorems 1.2, 1.3, and 1.4, respectively. As a consequence Theorem 1.1 is also proved in Section 5. Finally, in Section 6, we give several examples of compact l.c.s. manifolds of the fist kind which do not admit compatible Vaisman metrics. Some of these examples satisfy the Lefschetz property and the basic Lefschetz property and others not.

All manifolds considered in this paper will be assumed to be smooth and connected. For wedge product, exterior derivative and interior product we use the conventions as in Goldberg's book [8].

2. Mapping torus and compact Vaisman manifolds

2.1. Mapping torus by an isometry. In this section, we will review the notion of a mapping torus by an isometry. More details can be found in [10].

Let N be a compact smooth manifold, $f : N \longrightarrow N$ a diffeomorphism of N and α a positive real number.

We will denote by

$$(f, T_\alpha) : N \times \mathbb{R} \longrightarrow N \times \mathbb{R}$$

the transformation of the product manifold $N \times \mathbb{R}$ given by

$$(f, T_\alpha)(x, t) = (f(x), t + \alpha).$$

The map (f, T_α) induces a free and properly discontinuous action of the discrete subgroup $\mathbb{Z}$ on $N \times \mathbb{R}$ defined by

$$\mathbb{Z} \times (N \times \mathbb{R}) \longrightarrow N \times \mathbb{R}, \quad (k, (x, t)) \mapsto (f, T_\alpha)^k(x, t) = (f^k(x), t + k\alpha),$$

for $(x, t) \in N \times \mathbb{R}$.

The mapping torus of N by the couple (f, α) is the space of orbits of this action

$$N_{f,\alpha} = \frac{N \times \mathbb{R}}{\mathbb{Z}}.$$

It is a compact smooth manifold and we have a canonical projection

$$\pi : N_{f,\alpha} \longrightarrow S^1 = \frac{\mathbb{R}}{\alpha\mathbb{Z}}$$

from $N_{f,\alpha}$ onto the circle $S^1 = \dfrac{\mathbb{R}}{\alpha\mathbb{Z}}$.

We will denote by θ the closed 1-form on $N_{f,\alpha}$ given by

$$\theta = \pi^*(\theta_{S^1}),$$

where θ_{S^1} is the length element of the circle S^1.

Note that if $\tau_{f,\alpha} : N \times \mathbb{R} \longrightarrow N_{f,\alpha}$ is the canonical projection then

$$\tau^*_{f,\alpha}(\theta) = dt. \tag{2.1}$$

On the other hand, it is clear that the vector field $\dfrac{\partial}{\partial t}$ on $N \times \mathbb{R}$ is (f, T_α)-invariant. Thus, it induces a vector field U on the mapping torus $N_{f,\alpha}$ in such a way that

$$\theta(U) = 1. \tag{2.2}$$

Now, suppose that h is a Riemannian metric on N and that $f : N \longrightarrow N$ is an isometry. Then, we can consider the metric

$$h + dt^2$$

on the product manifold $N \times \mathbb{R}$. It follows that

$$(f, T_\alpha)^*(h + dt^2) = h + dt^2$$

and, thus, the Riemannian metric $h + dt^2$ is $\mathbb{Z}$-invariant.

This implies that $h + dt^2$ induces a Riemannian metric g on $N_{f,\alpha}$ which is characterized by the following condition

$$\tau^*_{f,\alpha} g = h + dt^2. \tag{2.3}$$

Moreover, one may prove the following result.

Proposition 2.1. *The 1-form θ on $N_{f,\alpha}$ is unitary and parallel with respect to the Riemannian metric g and*

$$\theta(X) = g(X, U), \quad \text{for } X \in \mathfrak{X}(N_{f,\alpha}). \tag{2.4}$$

2.2. Compact Sasakian and Vaisman manifolds. In this section, we will review the relation between compact Sasakian manifolds and compact Vaisman manifolds.

First of all, we will recall the notions of a contact and of a c-Sasakian manifold. The reader is referred to monographs [1] and [2] for detailed exposition on this subject.

A 1-form η on a smooth manifold N of dimension $2n+1$ is said to be a *contact* 1*-form* if $\eta \wedge (d\eta)^n$ is a volume form on N. In this case, the pair (N, η) is called a *contact manifold*. One can show that in a contact manifold (N, η) there exists a global vector field ξ, called *Reeb vector field*, characterized by the properties

$$\eta(\xi) = 1, \quad i_\xi d\eta = 0.$$

We will denote by $I : TN \longrightarrow TN$ the identity map acting on the tangent bundle.

Let c be a positive real number. A *c-Sasakian manifold* is a contact manifold (N, η) which carries a Riemannian metric h such that ξ is the metric dual of η and a $(1,1)$-tensor field ϕ satisfying the following conditions

(*i*) $\phi^2 = -I + \eta \otimes \xi$;
(*ii*) $d\eta(X, Y) = 2c\, h(X, \phi Y)$ for any vector fields X, Y on N;
(*iii*) the tensor field J on $N \times \mathbb{R}$ defined by

$$J\left(X, f\frac{\partial}{\partial t}\right) := \left(\phi X + f\xi, -\eta(X)\frac{\partial}{\partial t}\right), \tag{2.5}$$

where $X, Y \in \mathfrak{X}(N)$ and $f \in C^\infty(N \times \mathbb{R})$, is a complex structure.

If $c = 1$, then N is called a *Sasakian manifold*. Note that given a Sasakian structure (ϕ, ξ, η, h) on a manifold N, we can obtain a c-Sasakian structure

(ϕ, ξ, η, h') on N with the same underlying contact structure by the following transformation

$$\phi' = \phi, \quad \xi' = \xi, \quad \eta' = \eta, \quad h' = c^{-1}\, h + \left(1 - c^{-1}\right) \eta \otimes \eta. \tag{2.6}$$

Next, we will recall the definition of a locally conformal symplectic (l.c.s.) structure of the first kind and of a Vaisman manifold. More details can be found in [7, 16, 23].

An *l.c.s. structure of the first kind* on a manifold M of dimension $2n+2$ is a couple (ω, η) of 1-forms such that:

(i) ω is closed;
(ii) the rank of $d\eta$ is $2n$ and $\omega \wedge \eta \wedge (d\eta)^n$ is a volume form.

The form ω is called the *Lee* 1*-form* while η is said to be the *anti-Lee* 1*-form*.

If (ω, η) is an l.c.s. structure of the first kind on M then there exists a unique vector field V, *the anti-Lee vector field* of M, which is characterized by the following conditions

$$\omega(V) = 0, \;\; \eta(V) = 1, \;\; i_V d\eta = 0.$$

Moreover, there exists a unique vector field U, *the Lee vector field* of M, which is characterized by the following conditions

$$\omega(U) = 1, \;\; \eta(U) = 0, \;\; i_U d\eta = 0.$$

Remark 2.2. If (ω, η) is an l.c.s. structure of the first kind then the 2-form

$$\Omega := d\eta + \eta \wedge \omega$$

is non-degenerate and

$$d\Omega = \omega \wedge \Omega.$$

Moreover, the Lee vector field U satisfies the condition $\mathcal{L}_U \Omega = 0$. In other words, Ω is an l.c.s. structure of the first kind in the sense of Vaisman [23] with Lee 1-form ω and infinitesimal automorphism U. Conversely, if Ω is an l.c.s. structure of the first kind with Lee 1-form ω and infinitesimal automorphism U then the rank of $d\eta$ is $2n$ and $\omega \wedge \eta \wedge (d\eta)^n$ is a volume form, with η the 1-form on M given by

$$\eta = -i_U \Omega.$$

A *Vaisman manifold* is an l.c.s. manifold of the first kind (M, ω, η) which carries a Riemannian metric g such that:

(1) the tensor field J of type $(1,1)$, given by

$$g(X, JY) = \Omega(X, Y), \text{ for } X, Y \in \mathfrak{X}(M),$$

is a complex structure which is compatible with g, that is,

$$g(JX, JY) = g(X, Y);$$

(2) the Lee 1-form ω is parallel with respect to g.

Remark 2.3. If ω is the Lee 1-form of a Vaisman manifold M then $\|\omega\|$ is a constant $\lambda > 0$. We will assume, without loss of generality, that ω is unitary which implies that the anti-Lee and Lee vector fields U and V also are unitary.

If (M, J, g) is a Vaisman manifold one may prove that U is parallel (and, thus, Killing), V is Killing and

$$[U, V] = 0, \ \ \mathcal{L}_U J = 0, \ \ \mathcal{L}_V J = 0.$$

Moreover, under the assumption in Remark 2.3, the leaves of the foliation $\omega = 0$ have an induced $\frac{1}{2}$-Sasakian structure (for more details, see [7, Chapter 5]).

Now, suppose that N is a compact Sasakian manifold with Sasakian structure (ϕ, ξ, η, h) and that $f : N \longrightarrow N$ is a contact isometry. This means that

$$f^*\eta = \eta, \quad f^*h = h, \tag{2.7}$$

which implies that $f^*(d\eta) = d\eta$ and, thus,

$$\phi_{f(x)} \circ T_x f = T_{f(x)} f \circ \phi_x, \ \ T_x f(\xi_x) = \xi_{f(x)}, \tag{2.8}$$

for every $x \in M$, where $T_x f$ is the tangent map to f at x.

By considering a homothetic transformation as in (2.6) we can obtain a c-Sasakian structure (ϕ, ξ, η, h') on N with $c = \frac{1}{2}$. Note that f is a contact isometry also with respect to the new structure.

In addition, let α be a positive real number and $N_{f,\alpha} = \dfrac{N \times \mathbb{R}}{\mathbb{Z}}$ the mapping torus of (N, ϕ, ξ, η, h') defined by f and α.

In Section 2.1, we constructed a Riemannian metric g on $N_{f,\alpha}$ from the metric h' on N. Moreover, using (2.7) and (2.8), we deduce that the complex structure on $N \times \mathbb{R}$ given by (2.5) is $\mathbb{Z}$-invariant. Therefore, it induces a complex structure J on $N_{f,\alpha}$ in such a way that $(N_{f,\alpha}, J, g)$ is a compact Vaisman manifold.

Note that the exact 1-form dt on $N \times \mathbb{R}$ is $\mathbb{Z}$-invariant and it induces a closed 1-form ω on $N_{f,\alpha}$ which is in fact the Lee 1-form of $N_{f,\alpha}$.

The anti-Lee vector field V is just the vector field on $N_{f,\alpha}$ which is induced by the Reeb vector field ξ, while the Lee vector field U is induced by $\frac{\partial}{\partial t}$.

In particular, the leaves of the foliation $\omega = 0$ are diffeomorphic to N.

In fact, in [17], the authors prove the following result.

Theorem 2.4 ([17])**.** *Let M be a compact Vaisman manifold of dimension $2n+2$. Then, there exists a compact Sasakian manifold N of dimension $2n+1$, a contact isometry $f : N \longrightarrow N$ and a positive real number α such that M is holomorphically isometric to $N_{f,\alpha} = \dfrac{N \times \mathbb{R}}{\mathbb{Z}}$ with the structure defined above.*

3. Proof of Theorem 1.2

Proof: Denote by $\star$ the Hodge star isomorphism on P and by

$$\delta = (-1)^{pk+p+1} \star d \star$$

the codifferential on the space of k-forms on P. Note that (see page 97 in [8])

$$\star\, \star = (-1)^{k(p-k)} Id, \qquad \star\, \delta = (-1)^p d \star . \tag{3.1}$$

Let $\{W_1, \ldots, W_p\}$ be a local orthonormal basis of vector fields on P and $\{W^1, \ldots, W^p\}$ the corresponding dual basis of 1-forms. For a k-form θ on P, we have that

$$d\theta = \sum_{j=1}^{p} \epsilon_{W^j} \nabla_{W_j} \theta, \qquad \delta\theta = -\sum_{j=1}^{p} i_{W_j} \nabla_{W_j} \theta, \tag{3.2}$$

where ∇ is the Levi-Civita connection on P and ϵ_{W^j} is the operator of the exterior multiplication by the 1-form W^j.

Using (3.2), the following result follows.

Lemma 3.1. *If θ is a parallel k-form on P, then θ is harmonic.*

Now, we will proceed in two steps.

First step. We will prove that if γ is a closed k-form on P which is basic with respect to W (that is, $i_W \gamma = 0$ and $\mathcal{L}_W \gamma = 0$) and $\mathcal{H}\gamma$ is the harmonic representative of $[\gamma]$, then $\mathcal{H}\gamma$ also is basic with respect to W and $w \wedge \mathcal{H}\gamma$ is a harmonic $(k+1)$-form.

Lemma 3.2. *Under the assumptions of Theorem 1.2, if α is a harmonic form, then $i_W\alpha$ and $w \wedge \alpha$ are also harmonic.*

Proof: By [8, Theorem 3.7.1] we have $\mathcal{L}_W\alpha = 0$, since W is Killing and α is harmonic. As $d\alpha = 0$, we get $di_W\alpha = 0$, that is $i_W\alpha$ is closed.

Now, we compute the codifferential of $i_W\alpha$. Using (3.2), we get

$$\delta i_W\alpha = -\sum_{j=1}^{p} i_{W_j}\nabla_{W_j} i_W\alpha.$$

Since W is parallel, we have $\left[\nabla_{W_j}, i_W\right] = i_{\nabla_{W_j}W} = 0$. Thus

$$\delta i_W\alpha = \sum_{j=1}^{p} i_W i_{W_j}\nabla_{W_j}\alpha = -i_W\delta\alpha = 0.$$

Therefore $i_W\alpha$ is also coclosed and hence harmonic. The second part of the claim follows from the first by using Hodge duality. ■

Now, let $\beta := \gamma - \mathcal{H}\gamma$. By [8, Theorem 3.7.1], we get $\mathcal{L}_W\mathcal{H}\gamma = 0$ as W is Killing. Since γ is basic, we obtain that $\mathcal{L}_W\beta = 0$. Since β is exact, by Hodge theory we have $dG\delta\beta = \beta$, where G denotes the Green operator on P. As $\delta\mathcal{H}\gamma = 0$, we get

$$dG\delta\gamma = \gamma - \mathcal{H}\gamma. \tag{3.3}$$

Let us apply i_W to both sides of (3.3). We get

$$i_W\mathcal{H}\gamma = -i_W dG\delta\gamma = -\mathcal{L}_W G\delta\gamma + di_W G\delta\gamma = -G\delta\mathcal{L}_W\gamma + di_W G\delta\gamma = di_W G\delta\gamma,$$

where we use that $\mathcal{L}_W$ commutes with G and δ, because W is Killing. Therefore, $i_W\mathcal{H}\gamma$ is exact. But it is also harmonic by Lemma 3.2. Hence $i_W\mathcal{H}\gamma = 0$. This shows that $\mathcal{H}\gamma$ is basic. Moreover, the form $w \wedge \mathcal{H}\gamma$ is harmonic by Lemma 3.2.

Second step. We will prove that the map

$$[(Id, \epsilon_w)] : H^k_B(P) \oplus H^{k-1}_B(P) \longrightarrow H^k(P)$$

defined by (1.1) is an isomorphism.

Since w is a parallel 1-form, we have that w is closed by Lemma 3.1. This implies that the map $[(Id, \epsilon_w)]$ is a well-defined linear morphism.

Next, denote by $\mathcal{H}\gamma$ the harmonic representative of a closed k-form γ on P. Note that

$$[\mathcal{H}\gamma] = [\gamma]. \tag{3.4}$$

Then, we can consider the map

$$([\mathcal{H} - \epsilon_w i_W \mathcal{H}]_B, [i_W \mathcal{H}]_B) : H^k(P) \longrightarrow H^k_B(P) \oplus H^{k-1}_B(P) \tag{3.5}$$

defined by

$$([\mathcal{H} - \epsilon_w i_W \mathcal{H}]_B, [i_W \mathcal{H}]_B)[\gamma] = ([\mathcal{H}\gamma - w \wedge i_W \mathcal{H}\gamma]_B, [i_W \mathcal{H}\gamma]_B). \tag{3.6}$$

First of all, we show that this map is well defined. In fact, we have that

$$[\gamma] = [\gamma'] \Rightarrow \mathcal{H}\gamma = \mathcal{H}\gamma'. \tag{3.7}$$

This implies that the right hand side of (3.6) does not depend on the choice of the representative in the cohomology class $[\gamma]$. It is left to check that the forms $\mathcal{H}\gamma - w \wedge i_W \mathcal{H}\gamma$ and $i_W \mathcal{H}\gamma$ are basic and closed. It follows from Lemma 3.2 that they are harmonic and, hence, closed. It is obvious that $\mathcal{H}\gamma - w \wedge i_W \mathcal{H}\gamma$ and $i_W \mathcal{H}\gamma$ are in the kernel of i_W. Thus both of them are closed and basic, and the map (3.5) is well defined.

Now, from (3.4), it follows that

$$[(Id, \epsilon_w)] \circ ([\mathcal{H} - \epsilon_w i_W \mathcal{H}]_B, [i_W \mathcal{H}]_B) = Id.$$

Therefore the map $[(Id, \epsilon_w)]$ is surjective. Next, let $\beta \in \Omega^k_B(M)$ and $\beta' \in \Omega^{k-1}_B(M)$ be closed basic forms. Using Lemma 3.2, we deduce that $\mathcal{H}\beta + w \wedge \mathcal{H}\beta' \in [\beta + w \wedge \beta']$ is a harmonic form, and thus

$$\mathcal{H}(\beta + w \wedge \beta') = \mathcal{H}\beta + w \wedge \mathcal{H}\beta'. \tag{3.8}$$

Now, from (3.8) and since $\mathcal{H}\beta$, $\mathcal{H}\beta'$ are basic (see Step 1), we conclude that

$$([\mathcal{H} - \epsilon_w i_W \mathcal{H}], [i_W \mathcal{H}]) \circ [(Id, \epsilon_w)] = Id.$$

This proves that the map $[(Id, \epsilon_w)]$ is a injection and, thus, an isomorphism. ∎

4. Proof of Theorem 1.3

Thanks to Theorem 2.4, it is sufficient to prove the result for the case when M is a mapping torus of a compact Sasakian manifold N. We will denote by L_N the exterior multiplication by the 2-form $\frac{1}{2}d\eta$, acting on the space of forms on N.

We will proceed in three steps.

First step. We will show that the claim of Theorem 1.3 follows from an analogous statement for the cohomology of invariant forms on the Sasakian manifold N.

Let $\Omega^k(N)^f$ be the subspace of k-forms on N which are f-invariant.

Denote by $\mathrm{pr}_1 : N \times \mathbb{R} \longrightarrow N$ the canonical projection on the first factor. We have that if $\beta \in \Omega^k(N)^f$ then the k-form $\mathrm{pr}_1^* \beta$ on $N \times \mathbb{R}$ is $\mathbb{Z}$-invariant. Thus, there exists a unique basic k-form $\widehat{\beta}$ on $N_{f,\alpha}$ such that

$$\tau_{f,\alpha}^* \widehat{\beta} = \mathrm{pr}_1^* \beta, \tag{4.1}$$

where we recall that $\tau_{f,\alpha} : N \times \mathbb{R} \longrightarrow N_{f,\alpha} = \frac{N\times\mathbb{R}}{\mathbb{Z}}$ is the canonical projection.

On the other hand, it is clear that

$$d(\Omega^k(N)^f) \subseteq \Omega^{k+1}(N)^f.$$

Therefore, we can consider the subcomplex

$$\ldots \longrightarrow \Omega^{k-1}(N)^f \xrightarrow{d} \Omega^k(N)^f \xrightarrow{d} \Omega^{k+1}(N)^f \longrightarrow \ldots$$

of the de Rham complex of M and the corresponding cohomology $H^*(N)^f$. For a closed $\beta \in \Omega^k(N)^f$, we write $[\beta]_f$ for the corresponding class in $H^*(N)^f$.

Denote by $\Omega^l_B(N_{f,\alpha})$ the subspace of l-forms which are basic with respect to the Lee vector field U of $N_{f,\alpha}$, that is

$$\Omega^l_B(N_{f,\alpha}) = \{\gamma \in \Omega^l(N_{f,\alpha}) \mid i_U\gamma = 0, \;\; \mathcal{L}_U\gamma = 0\}.$$

Then, the basic de Rham cohomology complex is

$$\ldots \longrightarrow \Omega^{l-1}_B(N_{f,\alpha}) \xrightarrow{d} \Omega^l_B(N_{f,\alpha}) \xrightarrow{d} \Omega^{l+1}_B(N_{f,\alpha}) \longrightarrow \ldots$$

We recall that U is the vector field on $N_{f,\alpha}$ which is induced by the $\mathbb{Z}$-invariant vector field $\frac{\partial}{\partial t}$ on $N \times \mathbb{R}$. Using this fact, it is easy to prove that the map

$$\widehat{\ } : \Omega^*(N)^f \longrightarrow \Omega^*_B(N_{f,\alpha}) \tag{4.2}$$

defined by (4.1) is an isomorphism of chain complexes. Moreover, the map (4.2) has the following compatibility properties with respect to the action of the operators ϵ_η, i_ξ, L_N on $\Omega^k(N)^f$ and ϵ_η, i_V, L on $\Omega^k_B(N_{f,\alpha})$:

$$\widehat{L_N\beta} = L\widehat{\beta}, \qquad \widehat{\epsilon_\eta\beta} = \epsilon_\eta\widehat{\beta}, \qquad \widehat{i_\xi\beta} = i_V\widehat{\beta}. \tag{4.3}$$

In addition, using this isomorphism, we see that for proving the claim of the theorem it is enough to prove that there exists an isomorphism

$$\mathrm{Lef}_k^{N,f} : H^k(N)^f \longrightarrow H^{2n+1-k}(N)^f \tag{4.4}$$

and that the following properties hold:

Condition 4.1. For every $[\beta]_f \in H^k(N)^f$ there is $\beta' \in [\beta]_f$ such that

$$i_\xi \beta' = 0, \qquad L_N^{n-k+1}\beta' = 0.$$

Condition 4.2. For every closed f-invariant form β that satisfies Condition 4.1, one has

$$\mathrm{Lef}_k^{N,f}[\beta]_f = [\epsilon_\eta L_N^{n-k}\beta]_f.$$

In the next step we will prove the existence of the isomorphism (4.4). The fact that such an isomorphism satisfies Conditions 4.1 and 4.2 will be proved in the third step.

Second step. Let

$$[i] : H^l(N)^f \longrightarrow H^l(N)$$

be the linear morphism induced by the canonical inclusion $i : \Omega^l(N)^f \longrightarrow \Omega^l(N)$, $0 \leq l \leq 2n+1$, and let

$$\mathrm{Lef}_k^N : H^k(N) \longrightarrow H^{2n+1-k}(N)$$

be the Lefschetz isomorphism associated with the Sasakian structure on N (see [5, Theorem 4.5]), which is defined as follows:

$$\mathrm{Lef}_k^N[\beta] = [\eta \wedge L_N^{n-k}\beta'], \tag{4.5}$$

where β' is any representative of $[\beta] \in H^k(N)$ such that

$$i_\xi \beta' = 0,\ \ L_N^{n-k+1}\beta' = 0. \tag{4.6}$$

Note that such a representative β' exists, due to [5].

Now, we prove that $[i]$ is a monomorphism, for all $0 \leq l \leq 2n+1$. In fact, assume that $[\beta]_f \in H^l(N)^f$ and

$$[i][\beta]_f = 0.$$

It follows that β is exact and hence

$$\beta = d(G\delta\beta),$$

where G is the Green operator on N. Therefore, using that f is an isometry and the fact that $\beta \in \Omega^k(N)^f$, we obtain

$$f^*(G\delta\beta) = G\delta(f^*\beta) = G\delta\beta.$$

Thus β is exact in the complex $\Omega^*(N)^f$ and hence $[\beta]_f = 0$. Therefore $[i]$ is a monomorphism.

Let $[f^*] : H^l(N) \longrightarrow H^l(N)$ be the isomorphism induced by the contact isometry $f : N \longrightarrow N$. Now, we check that

$$\mathrm{Lef}_k^N \circ [i](H^k(N)^f) \subset [i](H^{2n+1-k}(N)^f). \tag{4.7}$$

Let $[\beta]_f \in H^k(N)^f$. Choose $\beta' \in [\beta]$ that satisfies (4.6). Then also $f^*\beta'$ satisfies (4.6), as f is an automorphism of the Sasakian manifold N. Moreover, $f^*\beta'$ belongs to $[\beta]$, since $f^*\beta = \beta$. Thus we can compute $\mathrm{Lef}_k^N([\beta])$ using either β' or $f^*\beta'$ in (4.5). Hence

$$[f^*] \circ \mathrm{Lef}_k^N([\beta]) = [f^*]([\epsilon_\eta L_N^{n-k}\beta']) = [\epsilon_\eta L_N^{n-k}(f^*\beta')] = \mathrm{Lef}_k^N([\beta]).$$

This shows that the cohomology class $\mathrm{Lef}_k^N([\beta])$ is $[f^*]$-invariant.

Let γ be the harmonic representative in $\mathrm{Lef}_k^N \circ [i]([\beta]_f)$. Then $f^*\gamma$ is also a harmonic representative of the same class. Hence $f^*\gamma = \gamma$, so that $[\gamma]_f$ is a well-defined class in $H^k(N)^f$ and $\mathrm{Lef}_k^N \circ [i]([\beta]_f) = [i]([\gamma]_f)$. This proves (4.7).

Since $[i]$ is a monomorphism and (4.7) holds, we can define

$$\mathrm{Lef}_k^{N,f} \colon H^k(N)^f \longrightarrow H^{2n+1-k}(N)^f$$

as the unique map such that

$$[i] \circ \mathrm{Lef}_k^{N,f} = \mathrm{Lef}_k^N \circ [i], \quad \text{for } 0 \leq k \leq n. \tag{4.8}$$

Since the right hand side of (4.8) is a monomorphism, we deduce that $\mathrm{Lef}_k^{N,f}$ is also a monomorphism.

Next, we will prove that $\mathrm{Lef}_k^{N,f}$ is an epimorphism.

If $\beta' \in [\beta]$ satisfies the conditions (4.6), then also $f^*\beta' \in [f^*][\beta]$ satisfies (4.6). Thus

$$[f^*]\,\mathrm{Lef}_k^N[\beta] = [f^*][\eta \wedge L_N^{n-k}\beta'] = [\eta \wedge L_N^{n-k} f^*\beta'] = \mathrm{Lef}_k^N[f^*][\beta].$$

Hence,

$$[f^*] \circ \mathrm{Lef}_k^N = \mathrm{Lef}_k^N \circ [f^*], \qquad \text{for } 0 \leq k \leq n. \tag{4.9}$$

Now, let $[\gamma]_f$ be an element in $H^{2n+1-k}(N)^f$.

As we know, Lef_k^N is an isomorphism (see [5]). This implies that there exists a unique $[\beta] \in H^k(N)$ such that

$$\mathrm{Lef}_k^N[\beta] = [i][\gamma]_f. \tag{4.10}$$

Thus, from (4.9) and since $[\gamma]_f \in H^{2n+1-k}(N)^f$, we have that

$$\begin{aligned}\mathrm{Lef}_k^N[\beta] &= [\gamma] = [f^*][\gamma] = [f^*][i][\gamma]_f \\ &= ([f^*] \circ \mathrm{Lef}_k^N)[\beta] = (\mathrm{Lef}_k^N \circ [f^*])[\beta].\end{aligned}$$

Therefore, using that Lef_k^N is an isomorphism, it follows that

$$[f^*][\beta] = [\beta].$$

Note that both $\mathcal{H}\beta$ and $f^*\mathcal{H}\beta$ are harmonic representatives of the cohomology class $[\beta] = [f^*][\beta]$. Therefore $f^*\mathcal{H}\beta = \mathcal{H}\beta$. So, we can consider $[\mathcal{H}\beta]_f \in H^k(N)^f$ and, using (4.8) and (4.10), we have that

$$[i](\mathrm{Lef}_k^{N,f}[\mathcal{H}\beta]_f) = [i][\gamma]_f.$$

Finally, as $[i]$ is a monomorphism, we conclude that

$$\mathrm{Lef}_k^{N,f}[\mathcal{H}\beta]_f = [\gamma]_f.$$

This proves that $\mathrm{Lef}_k^{N,f}$ is also surjective and hence it is an isomorphism.

Third step. We will prove that $\mathrm{Lef}_k^{N,f}$ satisfies Conditions 4.1 and 4.2.

To prove that every class in $H^k(N)^f$ has a representative satisfying Condition 4.1, we start with $[\beta]_f \in H^k(N)^f$ and then, proceeding as above, we deduce that $f^*(\mathcal{H}\beta) = \mathcal{H}\beta$. Since $[i]$ is a monomorphism, we get that $[i][\mathcal{H}\beta]_f = [i][\beta]_f$ implies that $[\mathcal{H}\beta]_f = [\beta]_f$. Moreover, the form $\mathcal{H}\beta$ satisfies Condition 4.1 (see [5]).

Now, let $[\beta]_f \in H^k(N)^f$ and $\beta' \in [\beta]_f$ satisfying (4.6). Then $\beta' \in [\beta]$. Thus by [5, Theorem 4.5] we have $\mathrm{Lef}_k^N[\beta] = [\epsilon_\eta L_N^{n-k}\beta']$ in $H^{2n+1-k}(N)$. Since $f^*\beta' = \beta'$, we get also that $f^*(\epsilon_\eta L_N^{n-k}\beta') = \epsilon_\eta L_N^{n-k}\beta'$. Thus by (4.8) we get

$$[i]\,\mathrm{Lef}_k^{N,f}[\beta]_f = [i][\epsilon_\eta L_N^{n-k}\beta']_f.$$

Since $[i]$ is a monomorphism, we get $\mathrm{Lef}_k^{N,f}[\beta]_f = [\epsilon_\eta L_N^{n-k}\beta']$ which ends the proof.

5. Proof of Theorem 1.4

Proof: **Proof of (1) in Theorem 1.4**

(Lefschetz $\Rightarrow$ basic Lefschetz)

Assume that the l.c.s. structure (ω, η) is Lefschetz and denote by

$$[i] : H_B^k(M) \longrightarrow H^k(M)$$

the canonical monomorphism, that is, $[i] = [(Id, \epsilon_\omega)]|_{H^k_B(M)\oplus\{0\}}$ (cf. Theorem 1.2).

Let β be a closed basic k-form on M, with $k \leq n$. Then, there exists a closed k-form β' on M such that $\beta' \in [\beta]$ and

$$\mathcal{L}_U\beta' = 0, \;\; i_V\beta' = 0, \;\; L^{n-k+2}\beta' = 0, \;\; L^{n-k+1}\epsilon_\omega\beta' = 0. \tag{5.1}$$

Let $\bar{\beta} = G\delta(\beta' - \beta)$, where G is the Green operator on M. Then $\mathcal{L}_U\bar{\beta} = 0$ since $\mathcal{L}_U\beta = 0$ and $\mathcal{L}_U$ commutes with G and δ because U is a Killing vector field.

From the properties of the Green operator, it follows that

$$\beta' = \beta + d\bar{\beta}.$$

Therefore, if we take

$$\beta'' = \beta' - \omega \wedge i_U\beta' \tag{5.2}$$

then $d\beta'' = 0$, $\beta'' \in \Omega^k_B(M)$ and

$$\beta'' = \beta + d(\bar{\beta} - \omega \wedge i_U\bar{\beta}).$$

This implies that

$$\beta'' \in [\beta]_B.$$

Moreover, from (5.1) and (5.2), it follows that

$$i_V\beta'' = 0, \quad L^{n-k+1}\beta'' = 0.$$

This shows that every class $[\beta]_B$ contains a representative β'' that satisfies (1.2).

We define $\mathrm{Lef}^B_k : H^k_B(M) \longrightarrow H^{2n+1-k}_B(M)$ by

$$\mathrm{Lef}^B_k[\beta]_B = [\epsilon_\eta L^{n-k}\beta']_B, \tag{5.3}$$

where β' is any representative of $[\beta]_B$ that satisfies (1.2). Note that $\epsilon_\eta L^{n-k}\beta'$ is closed as $d(\epsilon_\eta L^{n-k}\beta') = 2L^{n-k+1}\beta' = 0$ and basic because $i_U\epsilon_\eta L^{n-k}\beta' = \epsilon_\eta L^{n-k} i_U\beta' = 0$. We must prove that Lef^B_k is well defined and is an isomorphism. To show that Lef^B_k is well defined, we have to check that for any β', $\beta'' \in [\beta]_B$ that satisfy (1.2), one has

$$[\epsilon_\eta L^{n-k}\beta']_B = [\epsilon_\eta L^{n-k}\beta'']_B.$$

Since β' and β'' are basic and

$$i_V\beta' = 0, \qquad L^{n-k+1}\beta' = 0, \qquad i_V\beta'' = 0, \qquad L^{n-k+1}\beta'' = 0,$$

it is easy to check that β' and β'' satisfy (5.1). Moreover, they are representatives of $[\beta] \in H^k(M)$. Thus by the defining properties of Lef_k, we have

$$[\epsilon_\eta L^{n-k}\beta'] = \mathrm{Lef}_k[\beta] = [\epsilon_\eta L^{n-k}\beta'']. \tag{5.4}$$

Thus $[i][\epsilon_\eta L^{n-k}\beta']_B = [i][\epsilon_\eta L^{n-k}\beta'']_B$. Since $[i]$ is a monomorphism, we get $[\epsilon_\eta L^{n-k}\beta']_B = [\epsilon_\eta L^{n-k}\beta'']_B$. Hence Lef^B_k is well defined and satisfies the required property for its computation.

From (5.4), we get that

$$\mathrm{Lef}_k \circ [i][\beta]_B = [i] \circ \mathrm{Lef}^B_k[\beta]_B$$

for any $[\beta]_B \in H^k_B(M)$. Thus if $\mathrm{Lef}^B_k[\beta]_B = 0$ then $\mathrm{Lef}_k[i][\beta]_B = 0$. As Lef_k is an isomorphism, we get that $[i][\beta]_B = 0$. Since $[i]$ is monomorphism this implies that $[\beta]_B = 0$, that is Lef^B_k is injective.

By Poincaré duality for basic cohomology [20, Theorem 7.54] (see also calculations on page 69 of [20]), it follows that

$$\dim H^k_B(M) = \dim H^{2n+1-k}_B(M).$$

Thus Lef^k_B is an isomorphism, being an injective map between two vector spaces of the same dimension.

(*Basic Lefschetz* $\Rightarrow$ *Lefschetz*)

Assume that the l.c.s. structure on M is basic Lefschetz.

Let γ be a closed k-form on M, with $k \leq n$. We have to show that there is $\gamma' \in [\gamma]$ satisfying (5.1). Note that

$$[(Id, \epsilon_\omega)]^{-1}[\gamma] = ([\gamma_1]_B, [\gamma_2]_B) \in \Omega^k_B(M) \oplus \Omega^{k-1}_B(M)$$

for some basic forms γ_1 and γ_2 on M. Since the basic Lefschetz property holds, we can choose $\gamma_1' \in [\gamma_1]_B$ and $\gamma_2' \in [\gamma_2]_B$ such that

$$i_V\gamma_1' = 0, \quad L^{n-k+1}\gamma_1' = 0,$$

and

$$i_V\gamma_2' = 0, \quad L^{n-k+2}\gamma_2' = 0.$$

Thus, we can consider the closed k-form

$$\gamma' = \gamma_1' + \omega \wedge \gamma_2'$$

and it is easy to see that $\gamma' \in [\gamma]$ and

$$\mathcal{L}_U\gamma' = 0, \quad i_V\gamma' = 0, \quad L^{n-k+2}\gamma' = 0, \quad L^{n-k+1}\epsilon_\omega\gamma' = 0.$$

Now, consider the following isomorphisms

$$[(Id, \epsilon_\omega)]^{-1} : H^k(M) \longrightarrow H^k_B(M) \oplus H^{k-1}_B(M),$$

$$\mathrm{Lef}^B_k \oplus \mathrm{Lef}^B_{k-1} : H^k_B(M) \oplus H^{k-1}_B(M) \longrightarrow H^{2n+1-k}_B(M) \oplus H^{2n+2-k}_B(M),$$

$$\sigma : H^{2n+1-k}_B(M) \oplus H^{2n+2-k}_B(M) \longrightarrow H^{2n+2-k}_B(M) \oplus H^{2n+1-k}_B(M),$$

$$[(Id, \epsilon_\omega)] : H^{2n+2-k}_B(M) \oplus H^{2n+1-k}_B(M) \longrightarrow H^{2n+2-k}(M),$$

where σ is the canonical involution and $\mathrm{Lef}^B_r : H^r_B(M) \longrightarrow H^{2n+1-r}_B(M)$ is the isomorphism whose graph is the basic Lefschetz relation $R^B_{\mathrm{Lef}_r}$. We define the map $\mathrm{Lef}_k\colon H^k(M) \longrightarrow H^{2n+2-k}(M)$ by

$$\mathrm{Lef}_k = [(Id, \epsilon_\omega)] \circ \sigma \circ (\mathrm{Lef}^B_k \oplus \mathrm{Lef}^B_{k-1}) \circ [(Id, \epsilon_\omega)]^{-1}.$$

It is straightforward that Lef_k is an isomorphism. It is left to show that for any closed form $\gamma \in \Omega^k(M)$ that satisfies (5.1), we have

$$\mathrm{Lef}_k[\gamma] = [\epsilon_\eta L^{n-k}(Li_U\gamma - \epsilon_\omega\gamma)]. \tag{5.5}$$

Let $\gamma_1 = i_U(\omega\wedge\gamma)$ and $\gamma_2 = i_U\gamma$. Note that the forms γ_1 and γ_2 are closed and obviously $i_U\gamma_1 = i_U\gamma_2 = 0$. Thus they represent certain classes in $H^*_B(M)$. We have

$$[(Id, \epsilon_\omega)]^{-1}[\gamma] = ([\gamma_1]_B, [\gamma_2]_B).$$

Moreover, γ_1 satisfies the conditions

$$i_V\gamma_1 = 0, \quad L^{n-k+1}\gamma_1 = 0,$$

and γ_2 satisfies

$$i_V\gamma_2 = 0, \quad L^{n-k+2}\gamma_2 = 0.$$

Thus

$$\mathrm{Lef}^B_k[\gamma_1]_B = [\epsilon_\eta L^{n-k}\gamma_1]_B = [\epsilon_\eta L^{n-k} i_U(\omega\wedge\gamma)]_B \tag{5.6}$$

and

$$\mathrm{Lef}^B_{k-1}[\gamma_2]_B = [\epsilon_\eta L^{n-k+1}\gamma_2]_B = [\epsilon_\eta L^{n-k+1} i_U\gamma]_B. \tag{5.7}$$

Now, we get by definition of the map Lef_k that

$$\mathrm{Lef}_k[\gamma] = [\epsilon_\eta L^{n-k+1} i_U\gamma + \epsilon_\omega\epsilon_\eta L^{n-k} i_U(\omega\wedge\gamma)] = [\epsilon_\eta L^{n-k}(Li_U\gamma - \epsilon_\omega\gamma)].$$

This proves the first part of Theorem 1.4.

Proof of (2) **in Theorem 1.4**

Assume that the l.c.s. structure (ω, η) is Lefschetz (or, equivalently, basic Lefschetz) and denote by

$$\mathrm{Lef}_k^B : H_B^k(M) \longrightarrow H_B^{2n+1-k}(M), \;\; 1 \leq k \leq n,$$

the isomorphism whose graph is the basic Lefschetz relation $R^B_{\mathrm{Lef}_k}$. Since U is a parallel vector field, we get that the mean curvature κ of the foliation $\langle U \rangle$ is zero. Moreover, it is shown on page 69 of [20] that the characteristic form of $\langle U \rangle$ is ω. Therefore, by Theorem 7.54 in [20], we have a non-degenerate pairing

$$\langle \cdot, \cdot \rangle : H_B^{2n+1-k}(M) \otimes H_B^k(M) \longrightarrow \mathbb{R}$$

given by

$$\langle [\mu]_B, [\beta]_B \rangle = \int_M \omega \wedge \mu \wedge \beta. \tag{5.8}$$

Since Lef_k^B is an isomorphism, we get that the bilinear form

$$\psi := \langle \cdot, \cdot \rangle \circ (\mathrm{Lef}_k^B, Id) \tag{5.9}$$

on $H_B^k(M)$ is non-degenerate. Let α and β be closed basic k-forms on M. Then, for any $\alpha' \in [\alpha]_B$ and $\beta' \in [\beta]_B$ such that

$$i_V \alpha' = 0, \qquad L^{n-k+1}\alpha' = 0, \qquad i_V \beta' = 0, \qquad L^{n-k+1}\beta' = 0,$$

we get that

$$\begin{aligned} \psi([\alpha]_B, [\beta]_B) &= \int_M \omega \wedge \epsilon_\eta L^{n-k} \alpha' \wedge \beta' \\ &= (-1)^{k^2} \int_M \omega \wedge \epsilon_\eta L^{n-k} \beta' \wedge \alpha' = (-1)^k \psi([\beta]_B, [\alpha]_B). \end{aligned}$$

This shows that ψ is symmetric if k is even, and skew-symmetric if k is odd.

Consequently, for odd k between 1 and n, $c_k(M) := \dim H_B^k(M)$ is even. On the other hand, from Theorem 1.2 we deduce that

$$b_k(M) = c_k(M) + c_{k-1}(M),$$

which implies that

$$b_k(M) - b_{k-1}(M) = c_k(M) - c_{k-2}(M).$$

Since $c_k(M)$ and $c_{k-2}(M)$ are both even, we get that $b_k(M) - b_{k-1}(M)$ is also even. ∎

Finally, we derive Theorem 1.1 from the previously proved results.

(Proof of Theorem 1.1). By Theorem 1.3, every compact Vaisman manifold satisfies the basic Lefschetz property. Thus from Theorem 1.4, we get that every Vaisman manifold has the Lefschetz property. ■

6. Examples of non Vaisman compact Lefschetz l.c.s. manifolds

In this section, we will construct examples of compact l.c.s. manifolds of the first kind which do not satisfy the Lefschetz property. We will also present an example of a compact Lefschetz l.c.s. manifold of the first kind which does not admit compatible Vaisman metrics.

In order to do this we will use the following proposition.

Proposition 6.1. *Let M be a $(2n+2)$-dimensional compact l.c.s. manifold of the first kind such that the space of orbits N of the Lee vector field U of M is a quotient manifold.*

(i) If $\pi\colon M \longrightarrow N$ is the canonical projection then there exists a contact 1-form η_N on N such that $\pi^*\eta_N = \eta$, η being the anti-Lee 1-form of M. Moreover, the anti-Lee vector field V of M is π-projectable and its projection $\xi \in \mathfrak{X}(N)$ is the Reeb vector field of the contact manifold (N, η_N).

(ii) There exists a Riemannian metric g on M such that U is parallel and unitary with respect to g, and, in addition,

$$\omega(X) = g(X, U), \quad \textit{for all } X \in \mathfrak{X}(M),$$

where ω is the Lee 1-form of M.

Proof: The first claim is well known and can be found in [23].

We will prove (ii). Denote by $\mathcal{F}$ the vector subbundle induced by the foliation $\omega = 0$. Then $TM = \mathcal{F} \oplus \langle U\rangle$. From an arbitrary Riemannian metric g_N on N, we can define a Riemannian metric g on M such that

$$\pi\colon (M, g) \longrightarrow (N, g_N)$$

is a Riemannian submersion with the horizontal subbundle $\mathcal{F}$ and the vector field U is unitary with respect to g. This implies that $\omega(X) = g(X, U)$. Moreover, it is easy to prove that U is a Killing vector field. Thus, since the dual 1-form to U with respect to g is ω and it is closed, we conclude that U is parallel. ■

Under the same conditions as in Proposition 6.1, it is clear that the basic Lefschetz property for the l.c.s. manifold M is equivalent to the Lefschetz property for the base contact manifold N. Therefore, using Theorem 1.4 and Proposition 6.1, we deduce the following result.

Corollary 6.2. *Let M be a $(2n+2)$-dimensional compact l.c.s. manifold of the first kind such that the space of orbits of the Lee vector field is the contact manifold N. Then, the following conditions are equivalent:*

(1) The l.c.s. structure on M satisfies the Lefschetz property.
(2) The l.c.s. structure on M satisfies the basic Lefschetz property.
(3) The contact structure on N satisfies the Lefschetz property.

Now, let N be a compact contact manifold and consider the product manifold $M = N \times S^1$ with its standard l.c.s. structure of the first kind, that is, the Lee 1-form is the length element of S^1 and the anti-Lee 1-form is the contact 1-form on N. Then, it is clear that the space of orbits of the Lee vector field of M is N. Thus, using Corollary 6.2 and taking as N the examples of non-Lefschetz compact contact manifolds considered in [3], we obtain examples of compact l.c.s. manifolds of the first kind which satisfy the following conditions:

(1) Their Betti numbers satisfy relations (1.3) in Theorem 1.4.
(2) They do not satisfy neither the Lefschetz property nor the basic Lefschetz property (and, therefore, they do not admit compatible Vaisman metrics).

Note that (1) follows using that $b_k(N)$ is even if k is odd and $k \leq n$, with $dim N = 2n+1$ and $b_k(N)$ the k-th Betti number of N.

On the other hand, in [4], we present an example of a compact Lefschetz contact manifold N which does not admit any Sasakian structure. So, the standard l.c.s. structure of the first kind on $M = N \times S^1$ is Lefschetz and basic Lefschetz. However, M does not admit compatible Vaisman metrics.

We conclude stating some open problems concerning these topics. An interesting problem is to find examples of compact l.c.s. manifolds of the first kind which satisfy the basic Lefschetz property but they do not satisfy the Lefschetz property and vice versa. These topics will be the subject of forthcoming papers.

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