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GENERALIZED GOLDBERG FORMULA

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Abstract: In this paper we prove a useful formula for the graded commutator of the Hodge codifferential with the left wedge multiplication by a fixed p-form acting on the de Rham algebra of a Riemannian manifold. Our formula generalizes a formula stated by Samuel I. Goldberg for the case of 1-forms. As first examples of application we obtain new identities on locally conformal Kähler manifolds and quasi-Sasakian manifolds.

1. Introduction

Since the early days of Differential Geometry it became apparent the importance of formulae that relate various differential objects on a manifold. Let us mention among others Bianchi identities, Weitzenböck formula, Frölicher-Nijenhuis calculus. It should be noted that all the above results can be obtained by elementary, although tedious and long, computations. Their importance lies in the psychological/practical plane, as they permit to work with the quantities in question without undergoing into error-prone calculations, thus forming a swiss-knife kit of a differential geometer. In this article we prove a formula that we hope will deserve its place in the kit.

Let (M, g) be a Riemannian manifold. As usual, $\Omega^*(M)$ denotes the de Rham algebra of differential forms on M and $\delta : \Omega^*(M) \to \Omega^{*-1}(M)$ the Hodge codifferential. Given a k-form ω, we denote by ϵ_ω the operator on $\Omega^*(M)$ defined by $\epsilon_\omega \theta = \omega \wedge \theta$, for every $\theta \in \Omega^l(M)$. In Theorem 4, we prove the following expression for the graded commutator of δ with ϵ_ω in terms of Frölicher-Nijenhuis operators

$$[\delta, \epsilon_\omega] = \epsilon_{\delta\omega} - \mathcal{L}_{\omega^\#} - (-1)^p \, i_{\omega^\diamondsuit}. \tag{1}$$

Here, $\omega^\# \in \Omega^{k-1}(M, TM)$ denotes the vector valued form obtained from $\omega \in \Omega^k(M)$ by metric contraction and $\omega^\diamondsuit \in \Omega^k(M, TM)$ is a vector valued k-form defined in Section 3.

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Let ξ be a vector field and η its metric dual 1-form. In Corollary 5 we show that in this case Formula (1) takes the form

$$\{\delta, \epsilon_\eta\} + \mathcal{L}_\xi = \epsilon_{\delta\eta} + i_{(\mathcal{L}_\xi g)^\#}, \tag{2}$$

where the curly bracket denotes the anticommutator. Equation (2) was stated by Goldberg in [9] and on page 109 of [10]. In both cases, Goldberg refrained from explicitly proving this result. Nevertheless, he proved a partial case of (2) on pages 110-111 of [10] under the condition that ξ generates a flow of conformal transformations. The absence of a published proof can be one of the causes that Equation (2) is not widely known.

Let us give a simple example of use of (1). Let (M, g, J) be a Kähler manifold and let $\Omega(X, Y) = g(X, JY)$ be its fundamental 2-form. Then $\Omega^\# = J$ is parallel and Ω is closed and coclosed. One gets easily that the associated vector valued 2-form $\Omega^\diamondsuit$ vanishes (see equation (21)). Thus (1) becomes

$$[\delta, \epsilon_\Omega] + \mathcal{L}_J = 0. \tag{3}$$

This is of course a well-known formula in Kähler geometry, but usually it takes several pages of local computations to prove it.

In Theorem 6 we show the importance of the condition

$$[\delta, \epsilon_\omega] + \mathcal{L}_{\omega^\#} = 0 \tag{4}$$

for a p-form ω. Namely, we prove that if (4) holds for all $\omega \in S$, where S is subset of $\Omega^*(M)$, then the subalgebra

$$\Omega^*_{\mathcal{L}_{S^\#}}(M) := \{\, \beta \,|\, \mathcal{L}_{\omega^\#}\beta = 0,\ \omega \in S\}$$

of $\Omega^*(M)$ is quasi-isomorphic to $\Omega^*(M)$ as CDGA, with the quasi-isomorphism given by the embedding. Note that in the case M is Kähler manifold, this quasi-isomorphism is the first step in the proof of formality of Kähler manifolds given in [4]. Employing Formula (1), in Theorem 7 we give a complete characterization of all forms ω that satisfy (4).

In Section 4 we consider the case of locally conformal Kähler manifolds. By applying Formula (1), we get the following result which in a sense generalizes Equation (3). Let (M, J, g) be a locally conformal Kähler manifolds with fundamental 2-form Ω, Lee 1-form θ, and anti-Lee 1-form η. Then, for any p-form β we have

$$[\delta, \epsilon_\Omega]\beta = (p - n)\eta \wedge \beta - \mathcal{L}_J\beta + \Omega \wedge i_{\theta^\#}\beta. \tag{5}$$

Finally, in Section 5 we show how our formula works in the context of quasi-Sasakian manifolds. In Theorem 9 we prove the following result. Let (M, ϕ, ξ, η, g) be a quasi-Sasakian manifold and let $A := -\phi \circ \nabla\xi$. Then

$$[\delta, \epsilon_\Phi] = -\operatorname{tr}(A)\epsilon_\eta - \mathcal{L}_\phi + 2\epsilon_\eta i_A. \tag{6}$$

The special case of Formula (6) for Sasakian manifolds was first proved by Fujitani in [8] by complicated computation in local coordinates. This formula was crucial for the proof of the main result in our recent article [3] on Hard Lefschetz Theorem for Sasakian manifolds. We hope that (6) will allow us to obtain a suitable generalization of Hard Lefschetz Theorem for quasi-Sasakian manifold.

2. Preliminaries

In this section we remind some notions and results of Frölicher-Nijenhuis calculus [6, 7] which will be used later. Let M be a smooth manifold of dimension n. The direct sum $\Omega^*(M) := \bigoplus_{k=1}^n \Omega^k(M)$ has a structure of a commutative differential graded algebra (CDGA) with respect to the wedge product $\wedge$ and the exterior derivative $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$. We write $\Omega^k(M, TM)$ for the space of skew-symmetric TM-valued k-forms on M.

Denote by Σ_m the permutation group on $\{1, \dots, m\}$. For k and s such that $k+s = m$, let $\mathrm{Sh}_{k,s}$ be the subset of (k, s)-shuffles in Σ_m. Thus for $\sigma \in \mathrm{Sh}_{k,s}$, we have

$$\sigma(1) < \sigma(2) < \cdots < \sigma(k), \qquad \sigma(k+1) < \cdots < \sigma(k+s).$$

Let $\phi \in \Omega^p(M, TM)$. We define the operator i_ϕ of degree $p-1$ on $\Omega^*(M)$ by

$$(i_\phi\omega)(Y_1, \dots, Y_{p+k-1}) = \sum_{\sigma \in \mathrm{Sh}_{p,k-1}} (-1)^\sigma \omega\left(\phi(Y_{\sigma(1)}, \dots, Y_{\sigma(p)}), Y_{\sigma(p+1)}, \dots, Y_{\sigma(p+k-1)}\right)$$

where $\omega \in \Omega^k(M)$. The Lie derivative $\mathcal{L}_\phi$ is an operator of degree p on $\Omega^*(M)$ defined as the graded commutator $[i_\phi, d]$.

We recall now the fundamental theorem of Frölicher-Nijenhuis calculus.

Theorem 1 ([7]). *Let $\partial\colon \Omega^*(M) \to \Omega^*(M)$ be a derivation of degree p. Then there are unique $\phi \in \Omega^p(M, TM)$ and $\psi \in \Omega^{p+1}(M, TM)$, such that $\partial = \mathcal{L}_\phi + i_\psi$.*

As a consequence of the above theorem, we get:

(i) If a TM-valued p-form ϕ is different from 0, then $i_\phi \neq 0$.

(ii) If $\partial\colon \Omega^*(M) \to \Omega^*(M)$ is a derivation such that $[\partial, d] = 0$, then there is a unique $\phi \in \Omega^p(M, TM)$, such that $\partial = \mathcal{L}_\phi$.

For a k-form $\omega \in \Omega^k(M)$ and TM-valued p-form ϕ , we define the TM-valued $(p+k)$-form $\omega \wedge \phi$ by

$$(\omega \wedge \phi)(Y_1, \dots, Y_{p+k}) = \sum_{\sigma \in \mathrm{Sh}_{k,p}} (-1)^\sigma \omega\left(Y_{\sigma(1)}, \dots, Y_{\sigma(k)}\right) \phi(Y_{\sigma(k+1)}, \dots, Y_{\sigma(k+p)}).$$

Following [6], we will define the *contraction* (sometimes called *trace*) operator $\mathrm{C}\colon \Omega^p(M, TM) \to \Omega^{p-1}(M)$ as follows. Every $\phi \in \Omega^p(M, TM)$ can be written locally as a finite sum $\sum_{i\in I} \omega_i \wedge X_i$, where X_i are vector fields and $\omega_i \in \Omega^p(M)$. Then

$$\mathrm{C}(\phi) := \sum_{i \in I} i_{X_i} \omega_i.$$

One can check that $\mathrm{C}(\phi)$ does not depend on the choice of the local presentation for ϕ. We will use the following property [6, eq. (2.12)]

$$\mathrm{C}(\omega \wedge \phi) = (-1)^k \omega \wedge \mathrm{C}(\phi) + (-1)^{(k+1)p} i_\phi \omega, \tag{7}$$

for any $\omega \in \Omega^k(M)$ and $\phi \in \Omega^p(M, TM)$. Given $\omega \in \Omega^k(M)$, we define

$$\begin{aligned} \epsilon_\omega\colon \Omega^p(M, TM) &\to \Omega^{p+k}(M, TM) \\ \phi &\mapsto \omega \wedge \phi. \end{aligned}$$

For an operator $A\colon \Omega^*(M) \to \Omega^*(M)$ and $\omega \in \Omega^*(M)$ we abbreviate the composition $\epsilon_\omega \circ A$ by $\omega \wedge A$. It is easy to check that

$$\omega \wedge i_\phi = i_{\omega \wedge \phi}. \tag{8}$$

We will need the following fact.

Proposition 2. *Let M be a smooth manifold, $\omega \in \Omega^k(M)$, and $\phi \in \Omega^p(M, TM)$. Then,*

$$\omega \wedge \mathcal{L}_\phi = \mathcal{L}_{\omega \wedge \phi} - (-1)^{p+k} i_{(d\omega)\wedge\phi}. \tag{9}$$

Proof: The computation

$$\mathcal{L}_{\omega\wedge\phi} = [i_{\omega\wedge\phi}, d] = [\omega \wedge i_\phi, d] = (-1)^{k+p} (d\omega) \wedge i_\phi + \omega \wedge \mathcal{L}_\phi.$$

proves the claim. ∎

3. Generalized Goldberg Formula

In this section we prove the main result of the article. Let M be a smooth manifold equipped with a Riemannian metric g and let ∇ denote the corresponding Levi-Civita connection. Using ∇, we can define the map $d^\nabla\colon \Omega^p(M,TM) \to \Omega^{p+1}(M,TM)$ similarly to the standard exterior derivative, as follows

$$\begin{aligned} d^\nabla \phi\,(Y_1,\ldots,Y_{p+1}) = &\sum_{s=1}^{p+1}(-1)^{s-1}\nabla_{Y_s}\left(\phi(Y_1,\ldots,\widehat{Y}_s,\ldots,Y_{p+1})\right) \\ &+\sum_{s<t}(-1)^{s+t}\phi\left([Y_s,Y_t]\,,Y_1,\ldots,\widehat{Y}_s,\ldots,\widehat{Y}_t,\ldots,Y_{p+1}\right). \end{aligned}$$

Since for the Levi-Civita connection we have $[Y,Z]=\nabla_Y Z-\nabla_Z Y$, one can easily check that

$$(d^\nabla\phi)(Y_1,\ldots,Y_{p+1}) = \sum_{s=1}^{p+1}(-1)^{s+1}(\nabla_{Y_s}\phi)(Y_1,\ldots,\widehat{Y}_s,\ldots,Y_{p+1}). \tag{10}$$

Moreover, note that d^∇ is related to the Riemann curvature by the formula

$$(d^\nabla)^2\phi(Y_1,\ldots,Y_{p+2}) = \sum_{\sigma\in \mathrm{Sh}_{2,p}}(-1)^\sigma R(Y_{\sigma(1)},Y_{\sigma(2)})\left(\phi(Y_{\sigma(3)},\ldots,Y_{\sigma(p+2)})\right).$$

For $\omega\in\Omega^k(M)$ and $\phi\in\Omega^p(M,TM)$, we have

$$d^\nabla\left(\omega\wedge\phi\right) = (d\omega)\wedge\phi + (-1)^k\omega\wedge\left(d^\nabla\phi\right).$$

Note that for any vector field $X\in\Omega^0(M,TM)$, we get

$$d^\nabla X\,(Y) = \nabla_Y X.$$

Hence, $d^\nabla X=\nabla X$. Thus we can think about ∇-parallel vector fields as a generalization of harmonic functions. For any k-form ω and any vector field X, we get

$$\mathcal{L}_X\omega = \nabla_X\omega + i_{\nabla X}\omega.$$

In other words

$$\nabla_X = \mathcal{L}_X - i_{d^\nabla X}. \tag{11}$$

This equation suggests the following generalization of the covariant derivative. Namely, for $\phi\in\Omega^p(M,TM)$ we define

$$\nabla_\phi := \mathcal{L}_\phi - (-1)^p i_{d^\nabla\phi}. \tag{12}$$

We get

$$\omega \wedge \nabla_\phi = \omega \wedge \mathcal{L}_\phi - \omega \wedge i_{d^\nabla \phi} = \mathcal{L}_{\omega\wedge\phi} - (-1)^{p+k} i_{(d\omega)\wedge\phi} - (-1)^p i_{\omega\wedge d^\nabla\phi}$$

$$= \mathcal{L}_{\omega\wedge\phi} - (-1)^{p+k} i_{d\omega\wedge\phi + (-1)^k \omega \wedge d^\nabla \phi} = \mathcal{L}_{\omega\wedge\phi} - (-1)^{p+k} i_{d^\nabla(\omega\wedge\phi)}$$

that is,

$$\omega \wedge \nabla_\phi = \nabla_{\omega\wedge\phi}. \tag{13}$$

This equation is a generalization of the property

$$f\nabla_X = \nabla_{fX}$$

for the usual covariant derivative, where $f \in C^\infty(M)$ and $X \in \Omega^0(M, TM)$.

The Hodge codifferential is abstractly defined as the Hodge dual of the operator d on Ω. It is well known that given a local orthonormal frame X_1, …, X_n on $U \subset M$, the following local expression for the codifferential holds

$$\delta = -\sum_{t=1}^n i_{X_t} \circ \nabla_{X_t}.$$

Since both i_{X_t} and ∇_{X_t} are derivations of $\Omega^*(U)$, we see that δ is a differential operator of order 2 on $\Omega^*(U)$, and thus also on $\Omega^*(M)$.

Let $\omega \in \Omega^p(M)$. Then $[\delta, \epsilon_\omega]$ is a differential operator of order 1 and of degree $p-1$ on $\Omega^*(M)$. Thus it can be expressed in a unique way as a sum

$$\epsilon_\alpha + \nabla_\phi + i_\psi$$

for suitable $(p-1)$-form α, TM-valued $(p-1)$-form ϕ, and TM-valued $(p+1)$-form ψ. Our aim is to identify α, ϕ and ψ for a given ω.

For $\omega \in \Omega^p(M)$, we define $\omega^\# \in \Omega^{p-1}(M, TM)$ and $\omega^\nabla \in \Omega^p(M, TM)$ by

$$\omega^\# = \sum_{t=1}^n (i_{X_t}\omega) \wedge X_t \qquad \omega^\nabla = \sum_{t=1}^n (\nabla_{X_t}\omega) \wedge X_t. \tag{14}$$

It is easy to see that $\omega^\#$ and ω^∇ do not depend on the choice of the orthonormal frame X_1, …, X_n. Therefore $\omega^\#$ and ω^∇ are well-defined. By applying the contraction operator C to (14), we get

$$\mathrm{C}(\omega^{\#}) = \sum_{t=1}^{n} i_{X_t}^2 \omega = 0 \tag{15}$$

$$\mathrm{C}(\omega^{\nabla}) = \sum_{t=1}^{n} i_{X_t} \nabla_{X_t} \omega = -\delta\omega. \tag{16}$$

Proposition 3. *For any $\omega \in \Omega^p(M)$, we have $d^{\nabla}\left(\omega^{\#}\right) + (d\omega)^{\#} = \omega^{\nabla}$.*

Proof: Let $X_1, \ldots, X_n$ be an orthonormal frame on an open set U in M. By definition of ω^{∇} and the Leibniz rule for d^{∇}, we get

$$d^{\nabla}\left(\omega^{\#}\right) = \sum_{t=1}^{n} d\left(i_{X_t}\omega\right) \wedge X_t + (-1)^{p-1} \sum_{t=1}^{n} i_{X_t}\omega \wedge \nabla X_t. \tag{17}$$

Further,

$$(d\omega)^{\#} = \sum_{t=1}^{n} i_{X_t}\left(d\omega\right) \wedge X_t. \tag{18}$$

Note that for every $1 \leq t \leq n$, we have

$$d\left(i_{X_t}\omega\right) + i_{X_t}\left(d\omega\right) = \mathcal{L}_{X_t}\omega = \nabla_{X_t}\omega + i_{\nabla X_t}\omega.$$

Therefore, summing (17) with (18) we get

$$\begin{aligned} d^{\nabla}\left(\omega^{\#}\right) + (d\omega)^{\#} &= \sum_{t=1}^{n} \nabla_{X_t}\omega \wedge X_t + \sum_{t=1}^{n} i_{\nabla X_t}\omega \wedge X_t + (-1)^{p-1} \sum_{t=1}^{n} i_{X_t}\omega \wedge \nabla X_t \\ &= \omega^{\nabla} + \sum_{t=1}^{n} i_{\nabla X_t}\omega \wedge X_t + (-1)^{p-1} \sum_{t=1}^{n} i_{X_t}\omega \wedge \nabla X_t. \end{aligned}$$

Let us denote the expression

$$\sum_{t=1}^{n} i_{\nabla X_t}\omega \wedge X_t + (-1)^{p-1} \sum_{t=1}^{n} i_{X_t}\omega \wedge \nabla X_t$$

by T. Since $T = d^{\nabla}\left(\omega^{\#}\right) + (d\omega)^{\#} - \omega^{\nabla}$, we see that T does not depend on the choice of the orthonormal basis $X_1, \ldots, X_n$ and that T is a tensor on M. Let $x \in M$. Then there is an local orthonormal frame $X_1, \ldots, X_n$ on an open neighbourhood of x such that $\left(\nabla X_t\right)_x = 0$ for every $1 \leq t \leq n$. Computing T_x with respect to this basis, we see that $T_x = 0$. Since x is an arbitrary point of M, we see that $T \equiv 0$. ∎

Let us define for every $\omega \in \Omega^p(M)$ the TM-valued form

$$\omega^\diamondsuit = d^\nabla \left(\omega^\#\right) + \omega^\nabla. \tag{19}$$

Note that by Proposition 3 we can write it in two other ways

$$\omega^\diamondsuit = 2d^\nabla \left(\omega^\#\right) + (d\omega)^\#, \tag{20}$$

$$\omega^\diamondsuit = 2\omega^\nabla - (d\omega)^\#. \tag{21}$$

Now (15) and (16) give the following expression for $\delta\omega$ in terms of $\omega^\diamondsuit$

$$\delta\omega = -\frac{1}{2}\,\mathrm{C}(\omega^\diamondsuit). \tag{22}$$

We can now prove a formula for the commutator of the codifferential with the left wedge multiplication by a k-form.

Theorem 4. *Let $\omega \in \Omega^p(M)$. Then*

$$[\delta, \epsilon_\omega] = \epsilon_{\delta\omega} - \nabla_{\omega^\#} - (-1)^p i_{\omega^\nabla}. \tag{23}$$

Or, using the Lie derivative instead of the covariant derivative,

$$[\delta, \epsilon_\omega] = \epsilon_{\delta\omega} - \mathcal{L}_{\omega^\#} - (-1)^p\, i_{\omega^\diamondsuit}. \tag{24}$$

Proof: Let X be a vector field and $\omega \in \Omega^p(M)$. Then

$$\begin{aligned}
[i_X \circ \nabla_X, \epsilon_\omega] &= [i_X, \epsilon_\omega] \circ \nabla_X + i_X \circ [\nabla_X, \epsilon_\omega] \\
&= \epsilon_{i_X\omega}\nabla_X + i_X\epsilon_{\nabla_X\omega} \\
&= \epsilon_{i_X\omega}\nabla_X + [i_X, \epsilon_{\nabla_X\omega}] + (-1)^p\epsilon_{\nabla_X\omega} i_X \\
&= \nabla_{i_X\omega\wedge X} + \epsilon_{i_X\nabla_X\omega} + (-1)^p\epsilon_{\nabla_X\omega} i_X \\
&= \epsilon_{i_X\nabla_X\omega} + \nabla_{i_X\omega\wedge X} + (-1)^p i_{\nabla_X\omega\wedge X}.
\end{aligned}$$

Now (23) follows by substituting X_t instead of X and summing up over t.

Since $\omega^\# \in \Omega^{p-1}(M, TM)$, from (12) we get

$$\nabla_{\omega^\#} = \mathcal{L}_{\omega^\#} - (-1)^{p-1}\, i_{d^\nabla(\omega^\#)} = \mathcal{L}_{\omega^\#} + (-1)^p\, i_{d^\nabla(\omega^\#)}.$$

Therefore

$$[\delta, \epsilon_\omega] = \epsilon_{\delta\omega} - \mathcal{L}_{\omega^\#} - (-1)^p \left(i_{d^\nabla(\omega^\#)} + i_{\omega^\nabla}\right).$$

■

As a corollary we can get Formula (4) in Goldberg's article [9].

Corollary 5. *Let ξ be a vector field on a Riemannian manifold M, and η its metric dual 1-form. Then $\eta^\diamondsuit = (\mathcal{L}_\xi g)^\#$, that is*

$$\{\delta, \epsilon_\eta\} + \mathcal{L}_\xi = \epsilon_{\delta\eta} + i_{(\mathcal{L}_\xi g)^\#}, \tag{25}$$

where $\{-,-\}$ denotes the anti-commutator of operators and $(\mathcal{L}_\xi g)^\#$ is the metric contraction of the $(0,2)$-tensor $\mathcal{L}_\xi g$.

Proof: We have to check that $d^\nabla \eta^\# + \eta^\nabla = (\mathcal{L}_\xi g)^\#$. Since $\eta^\# = \xi$, we have for any vector field Y

$$(d^\nabla \eta^\#)(Y) = (d^\nabla \xi)(Y) = \nabla_Y \xi = \sum_{t=1}^n g(X_t, \nabla_Y \xi) X_t, \tag{26}$$

where $X_1, \ldots, X_n$ is a local orthonormal frame on M. Further,

$$\eta^\nabla(Y) = \sum_{t=1}^n (\nabla_{X_t}\eta)(Y) X_t = \sum_{t=1}^n g(\nabla_{X_t}\xi, Y) X_t. \tag{27}$$

It is well known that

$$(\mathcal{L}_\xi g)(Y, Z) = g(\nabla_Y \xi, Z) + g(\xi, \nabla_Z \xi), \tag{28}$$

for any vector fields ξ, Y and Z. Therefore, adding (26) and (27), we get

$$(d^\nabla \xi + \eta^\nabla)(Y) = \sum_{t=1}^n (\mathcal{L}_\xi g)(X_t, Y) X_t = (\mathcal{L}_\xi g)^\#(Y).$$

■

Let S be a set of differential forms on M. We will denote by $S^\#$ the set of vector valued forms $\omega^\#$, where $\omega \in S$. Further we write $\Omega^*_{\mathcal{L}_{S^\#}}(M)$ for the intersection of the kernels of operators $\mathcal{L}_{\omega^\#}$, $\omega \in S$. We have the following theorem that generalizes several known facts.

Theorem 6. *Let (M, g) be a compact Riemannian manifold. Suppose $S \subset \Omega^*(M)$ is such that $[\delta, \epsilon_\omega] + \mathcal{L}_{\omega^\#} = 0$ for all $\omega \in S$. Then the inclusion $j \colon \Omega^*_{\mathcal{L}_{S^\#}}(M) \hookrightarrow \Omega^*(M)$ is a quasi-isomorphism of CDGAs.*

Proof: Let $\omega \in S$. Since $[\delta, \epsilon_\omega] + \mathcal{L}_{\omega^\#} = 0$ and $\delta^2 = 0$, we get that

$$[\delta, \mathcal{L}_{\omega^\#}] = -[\delta, [\delta, \epsilon_\omega]] = 0.$$

Since the Hodge Laplacian Δ is the graded commutator of d and δ, we have also that $[\Delta, \mathcal{L}_{\omega^\#}] = 0$.

Let β be a harmonic p-form. We are going to show that $\beta \in \Omega^p_{\mathcal{L}_{S^\#}}(M)$. This will imply by Hodge theory that j induces a surjection in cohomology. Since $[\Delta, \mathcal{L}_{\omega^\#}] = 0$ for all $\omega \in S$, we get immediately, that $\Delta(\mathcal{L}_{\omega^\#}\beta) = 0$, i.e. $\mathcal{L}_{\omega^\#}\beta$ is harmonic. But, since β is closed, we have $\mathcal{L}_{\omega^\#}\beta = di_{\omega^\#}\beta$ is an exact form. Thus by Hodge theory, $\mathcal{L}_{\omega^\#}\beta = 0$.

It is left to show that j induces an injection in cohomology. Let $\beta \in \Omega^p_{\mathcal{L}_{S^\#}}(M)$ such that $[\beta] = 0$ in $H^p(M)$. Then $\beta = dG\delta\beta$, where G is the Green operator for Δ. We are going to show that $G\delta\beta \in \Omega^p_{\mathcal{L}_{S^\#}}(M)$. For this, it is enough to prove that $\mathcal{L}_{\omega^\#}G = G\mathcal{L}_{\omega^\#}$ for every $\omega \in S$. In fact, then

$$\mathcal{L}_{\omega^\#}G\delta\beta = G\delta\mathcal{L}_{\omega^\#}\beta = 0, \ \forall \omega \in S.$$

We have

$$I - G\Delta = \Pi_\Delta, \qquad\qquad I - \Delta G = \Pi_\Delta, \tag{29}$$

where Π_Δ is the orthogonal projection on the set of harmonic forms. Now we multiply the equation $\mathcal{L}_{\omega^\#}\Delta = \Delta\mathcal{L}_{\omega^\#}$ by G on the left and right hand sides. We get

$$G\mathcal{L}_{\omega^\#}\Delta G = G\Delta\mathcal{L}_{\omega^\#}G.$$

Applying (29) we obtain

$$G\mathcal{L}_{\omega^\#} - G\mathcal{L}_{\omega^\#}\Pi_\Delta = \mathcal{L}_{\omega^\#}G - \Pi_\Delta\mathcal{L}_{\omega^\#}G. \tag{30}$$

As we saw above, $\mathcal{L}_{\omega^\#}$ annihilates harmonic forms, hence $\mathcal{L}_{\omega^\#}\Pi_\Delta = 0$. To finish the proof it is enough to check that $\Pi_\Delta\mathcal{L}_{\omega^\#} = 0$. Let $\alpha \in \Omega^k(M)$. By Hodge theory, we can write α as $\alpha_\delta + \alpha_\Delta + \alpha_d$, where α_δ is in the image of δ, α_d is in the image of d, and α_Δ is harmonic. Note that $\mathcal{L}_{\omega^\#}\alpha_\Delta = 0$. Further, $\mathcal{L}_{\omega^\#}\alpha_d = \pm di_{\omega^\#}\alpha_d$, where the sign depends on the degree of ω. In particular, $\mathcal{L}_{\omega^\#}\alpha_d$ is exact, and therefore $\Pi_\Delta\mathcal{L}_{\omega^\#}\alpha_d = 0$. Finally, since $[\delta, \epsilon_\omega] + \mathcal{L}_{\omega^\#} = 0$, we get

$$\mathcal{L}_{\omega^\#}\alpha_\delta = -[\delta, \epsilon_\omega]\alpha_\delta = -\delta(\omega \wedge \alpha_\delta).$$

Hence, $\mathcal{L}_{\omega^\#}\alpha_\delta$ is a coexact form and thus $\Pi_\Delta\mathcal{L}_{\omega^\#}\alpha_\delta = 0$. ■

The previous theorem shows the importance of the property $[\delta, \omega] + \mathcal{L}_{\omega^\#} = 0$ for a differential form ω. In the following theorem we characterize all the forms with this property.

Theorem 7. *Let (M, g) be a Riemannian manifold and ω a p-form on M, with $p \geq 1$. Then*

$$[\delta, \epsilon_\omega] + \mathcal{L}_{\omega^\#} = 0$$

if and only if one of the following conditions holds

(i) $p = 1$ *and* $\omega^{\#}$ *is a Killing vector field;*
(ii) $p \geq 2$ *and* ω *is parallel.*

Proof: Let us consider first the case $p = 1$. Suppose $\xi = \omega^{\#}$ is Killing. Then $\mathcal{L}_\xi g = 0$. By Corollary 5, we have

$$\omega^{\diamondsuit} = (\mathcal{L}_\xi g)^{\#} = 0.$$

Applying (22), we get $\delta\omega = -\frac{1}{2}\,\mathrm{C}(\omega^{\diamondsuit}) = 0$. By (25), we obtain that $\{\delta, \epsilon_\omega\} + \mathcal{L}_\xi = 0$.

Now, suppose that $\{\delta, \epsilon_\omega\} + \mathcal{L}_\xi = 0$. Then from (25)

$$\epsilon_{\delta\omega} + i_{(\mathcal{L}_\xi g)^{\#}} = 0. \tag{31}$$

Applying (31) to the constant function with the value 1, we get $\delta\omega = 0$. Thus $i_{(\mathcal{L}_\xi g)^{\#}} = 0$. By Theorem 1, we have $\mathcal{L}_\xi g = 0$, and thus ξ is a Killing vector field.

Now suppose $p \geq 2$ and $\nabla\omega = 0$. Then, by looking at defining formulae one readily sees that $\delta\omega = 0$, $d\omega = 0$, and $\omega^{\nabla} = 0$. Thus, by (24) we get that $[\delta, \epsilon_\omega] + \mathcal{L}_{\omega^{\#}} = 0$.

Finally, suppose that $[\delta, \epsilon_\omega] + \mathcal{L}_{\omega^{\#}} = 0$. Then, by (24) we have

$$\epsilon_{\delta\omega} - (-1)^p i_{\omega^{\diamondsuit}} = 0. \tag{32}$$

Applying (32) to the constant function 1, we get that $\delta\omega = 0$. Therefore $i_{\omega^{\diamondsuit}} = 0$ and, by Theorem 1, we have $\omega^{\diamondsuit} = 0$. Using (21) and (14), we obtain

$$0 = \omega^{\diamondsuit} = \sum_{t=1}^{n} 2\nabla_{X_t}\omega \wedge X_t - \sum_{t=1}^{n} i_{X_t}\omega \wedge X_t = \sum_{t=1}^{n} (2\nabla_{X_t}\omega - i_{X_t}d\omega) \wedge X_t,$$

where $X_1, \ldots, X_n$ is a local orthonormal frame on M. Since $X_1, \ldots, X_n$ are linearly independent at every point, we obtain that

$$2\nabla_{X_t}\omega = i_{X_t}d\omega$$

for all t. But this implies

$$2\nabla_Z\omega = i_Z d\omega \tag{33}$$

for every vector field Z.

Let $Y_0, \dots, Y_p$ be vector fields. Then, by using (33) we get

$$\begin{aligned}
2(d\omega)(Y_0,\dots,Y_p) &= \sum_{s=0}^{p}(-1)^s(2\nabla_{Y_s}\omega)(Y_0,\dots,\widehat{Y_s},\dots,Y_p)\\
&= \sum_{s=0}^{p}(-1)^s(i_{Y_s}d\omega)(Y_0,\dots,\widehat{Y_s},\dots,Y_p)\\
&= \sum_{s=0}^{p}(d\omega)(Y_0,\dots,Y_p) = (p+1)d\omega(Y_0,\dots,Y_p).
\end{aligned}$$

Since $p \neq 1$, we obtain $d\omega = 0$. Now (33) implies $\nabla\omega = 0$. ■

4. Locally conformal Kähler manifolds

In this section, we show how Theorem 4 works in the context of locally conformal Kähler manifolds.

Let (M^{2n+2}, g) be a Riemannian manifold and J a complex structure on M. Then (M, J, g) is called *Hermitian* if $g(JX, JY) = g(X, Y)$ for all vector fields X, Y on M. For an Hermitian manifold (M, J, g), we define its *fundamental 2-form* Ω by $\Omega(X, Y) = g(X, JY)$. Thus $\Omega^{\#} = J$. An Hermitian manifold (M, J, g) is called *locally conformal Kähler* (l.c.K.) if there exists a 1-form θ (called the *Lee form*) such that

$$d\Omega = \theta \wedge \Omega.$$

We are going to apply Theorem 4 to $\omega = \Omega$. For this we have to compute $\Omega^{\diamondsuit}$ and $\delta\Omega$. We define $\eta = i_J\theta$. It is proved in [5, Corollary 1.1] that

$$(\nabla_X J)Y = \frac{1}{2}\left(\eta(Y)X - \theta(Y)JX - g(X,Y)\eta^{\#} - \Omega(X,Y)\theta^{\#}\right).$$

Thus

$$\begin{aligned}
d^{\nabla}J(X,Y) &= (\nabla_X J)Y - (\nabla_Y J)X\\
&= \frac{1}{2}\left(\eta(Y)X - \theta(Y)JX - \eta(X)Y + \theta(X)JY - 2\Omega(X,Y)\theta^{\#}\right)\\
&= \frac{1}{2}(-(\eta\wedge\mathrm{Id})(X,Y) + (\theta\wedge J)(X,Y)) - (\Omega\wedge\theta^{\#})(X,Y).
\end{aligned}$$

Hence, we get

$$d^{\nabla}J = \frac{1}{2}(\theta\wedge J - \eta\wedge\mathrm{Id}) - \Omega\wedge\theta^{\#}.$$

Using the definition of $\#$, it is easy to check that

$$(d\Omega)^{\#} = (\theta \wedge \Omega)^{\#} = \Omega\wedge\theta^{\#} - \theta\wedge\Omega^{\#} = \Omega\wedge\theta^{\#} - \theta\wedge J. \tag{34}$$

Thus by (20)

$$\Omega^{\diamondsuit} = 2d^{\nabla}J + (d\Omega)^{\#} = -\eta\wedge\mathrm{Id} - \Omega\wedge\theta^{\#}. \tag{35}$$

Moreover, due to (15), by contracting (34) we get

$$\mathrm{C}(\Omega\wedge\theta^{\#}) = \mathrm{C}(\theta\wedge J)$$

Hence by (22), we obtain from (35)

$$\delta\Omega = -\frac{1}{2}\,\mathrm{C}(\Omega^{\diamondsuit}) = \frac{1}{2}(\mathrm{C}(\eta\wedge\mathrm{Id}) + \mathrm{C}(\Omega\wedge\theta^{\#})) = \frac{1}{2}(\mathrm{C}(\eta\wedge\mathrm{Id}) + \mathrm{C}(\theta\wedge J)).$$

Using (7), we have

$$\begin{aligned}\mathrm{C}(\eta\wedge\mathrm{Id}) &= -\,\mathrm{C}(\mathrm{Id})\eta + i_{\mathrm{Id}}\eta = -(2n+2)\eta + \eta = -(2n+1)\eta,\\ \mathrm{C}(\theta\wedge J) &= -\,\mathrm{C}(J)\theta + i_J\theta = \eta.\end{aligned}$$

Therefore

$$\delta\Omega = \frac{1}{2}(\eta - (2n+1)\eta) = -n\eta.$$

Applying Theorem 4, we get the following formula that in a sense generalizes Equation (3) which holds for Kähler manifolds.

Theorem 8. *Let (M, J, g) be a locally conformal Kähler manifold. Let Ω be the fundamental 2-form, θ the Lee 1-form, and $\eta = i_J\theta$. Then, for any p-form β we have*

$$[\delta, \epsilon_\Omega]\beta = (p-n)\eta\wedge\beta - \mathcal{L}_J\beta + \Omega\wedge i_{\theta^{\#}}\beta. \tag{36}$$

5. Quasi-Sasakian manifolds

In this section we will show how Theorem 4 can be used to get useful formulae for commutators on quasi-Sasakian manifolds.

An *almost contact metric* structure on a manifold M^{2n+1} is a quadruple (ϕ, ξ, η, g), where ϕ is an endomorphism of TM, ξ is a vector field, η is a 1-form, and g is a Riemannian metric such that

$$\begin{aligned}\phi^2 &= -\mathrm{Id} + \eta\otimes\xi, &\qquad \eta(\xi) &= 1,\\ g(\phi X, Y) &= -g(X, \phi Y), &\qquad \eta(X) &= g(X, \xi),\end{aligned}$$

for any vector fields X and Y. As a consequence, one easily gets that $\phi(\xi) = 0$ and $\eta \circ \phi = 0$. We define an almost complex structure J on $M \times \mathbb{R}$ by

$$J\Big(X, f\frac{d}{dt}\Big) = \Big(\phi X - f\xi, \eta(X)\frac{d}{dt}\Big),$$

where f is a smooth function on $M \times \mathbb{R}$. If J is integrable, the almost contact metric structure (ϕ, ξ, η, g) on M is called *normal*. We define a 2-form Φ by

$$\Phi(X, Y) = g(X, \phi Y), \text{ for any } X, Y \in \mathfrak{X}(M).$$

A normal almost contact metric structure (ϕ, ξ, η, g) on M is called *quasi-Sasakian* if Φ is closed.

Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a quasi-Sasakian manifold. We define

$$A := -\phi \circ \nabla\xi.$$

We are going to apply Theorem 4 to $\omega = \Phi$. For this we have to compute $\Phi^{\#}$, $\Phi^{\diamondsuit}$, and $\delta\Phi$. From the definition of Φ, we have that $\Phi^{\#} = \phi$. Since Φ is closed, from (20), we get

$$\Phi^{\diamondsuit} = 2d^{\nabla}\phi.$$

In [11] it was shown that

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \qquad g(AX, Y) = g(X, AY).$$

Thus by (10), we have

$$\begin{aligned}(d^{\nabla}\phi)(X, Y) &= (\nabla_X\phi)(Y) - (\nabla_Y\phi)(X)\\ &= \eta(Y)AX - g(AX, Y)\xi - \eta(X)AY + g(X, AY)\xi\\ &= -(\eta\wedge A)(X, Y).\end{aligned}$$

Therefore

$$\Phi^{\diamondsuit} = -2\eta\wedge A. \tag{37}$$

Further by (22)

$$\delta\Phi = -\frac{1}{2}\,\mathrm{C}(\Phi^{\diamondsuit}) = \mathrm{C}(\eta\wedge A). \tag{38}$$

By (7), we have

$$\mathrm{C}(\eta\wedge A) = -\eta\wedge \mathrm{C}(A) + i_A\eta = -\mathrm{C}(A)\eta + i_A\eta. \tag{39}$$

Since $A = -\phi \circ \nabla\xi$ and $\eta \circ \phi = 0$, combining (38) and (39), we finally get

$$\delta\Phi = -\mathrm{C}(A)\eta.$$

Thus by Theorem 4 and (37), we have

$$[\delta, \epsilon_\Phi] = -\epsilon_{\mathrm{C}(A)\eta} - \mathcal{L}_\phi + i_{2\eta\wedge A}.$$

Since A is an endomorphism of TM, we actually have $\mathrm{C}(A) = \mathrm{tr}(A)$. Hence we have proved the following result.

Theorem 9. *Let (M, ϕ, ξ, η, g) be a quasi-Sasakian manifold. Then*

$$[\delta, \epsilon_\Phi] = -\operatorname{tr}(A)\epsilon_\eta - \mathcal{L}_\phi + 2\epsilon_\eta i_A. \tag{40}$$

The most important examples of quasi-Sasakian manifolds are co-Kähler manifolds (see [2]) and Sasakian manifolds (see [1]). For every co-Kähler manifold, one has $\nabla\xi = 0$ and thus $A = 0$. Therefore in co-Kähler case, we get

$$[\delta, \epsilon_\Phi] = -\mathcal{L}_\phi,$$

which could also have been achieved by using the fact that ϕ is parallel on a co-Kähler manifold and Theorem 7.

For Sasakian manifold, one has $\nabla\xi = -\phi$, and thus $A = \phi^2 = -\mathrm{Id} + \eta\wedge\xi$. Therefore $\operatorname{tr} A = -2n$ in this case. Applying Theorem 9, we get

$$\begin{aligned} [\delta, \epsilon_\phi] &= 2n\epsilon_\eta - \mathcal{L}_\Phi + 2\epsilon_\eta(-i_{\mathrm{Id}} + \epsilon_\eta i_\xi) \\ &= 2n\epsilon_\eta - \mathcal{L}_\phi - 2\epsilon_\eta i_{\mathrm{Id}}. \end{aligned} \tag{41}$$

The formula (41) was first proved by Fujitani in [8] by complicated computation in local coordinates. This formula was crucial for some proofs in our recent article [3] on Hard Lefschetz Theorem for Sasakian manifolds. We hope that Theorem 9 will permit us to find a suitable generalization of Hard Lefschetz Theorem for quasi-Sasakian manifold.

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