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A NON-SASAKIAN LEFSCHETZ K-CONTACT MANIFOLD OF TIEVSKY TYPE

BENIAMINO CAPPELLETTI-MONTANO, ANTONIO DE NICOLA, JUAN CARLOS MARRERO AND IVAN YUDIN

Abstract: We find a family of five dimensional completely solvable compact manifolds that constitute the first examples of K-contact manifolds which satisfy the Hard Lefschetz Theorem and have a model of Tievsky type just as Sasakian manifolds but do not admit any Sasakian structure.

1. Introduction

It is well known that Sasakian geometry is closely related to Kähler geometry. Indeed, on the one hand, an odd dimensional Riemannian manifold (M, g) is Sasakian if and only if its metric cone $(M \times \mathbb{R}^+, r^2 g + dr^2)$ is a Kähler manifold. On the other hand, the Reeb vector field of a Sasakian manifold generates a transversely Kähler foliation so that, in particular, if the foliation is regular the leaf space is a Kähler manifold.

A natural problem in Kähler geometry is the study of obstructions to the existence of a Kähler structure on a given compact symplectic manifold. Similarly, in Sasakian geometry a problem of interest is the study of obstructions to the existence of a Sasakian structure on a given compact contact manifold. On any Sasakian manifold the Reeb vector field is Killing. Hence the only interesting case to consider is when a contact manifold has Killing Reeb vector field with respect to some metric. The problem of constructing explicit examples of K-contact manifolds with no Sasakian structures arises.

The study of various obstructions to the existence of a Sasakian metric on a given compact contact manifold intensified after the problem was presented in a systematic form in [4, Chapter 7]. Several obstructions are now known.

For a contact manifold (M, η), we write $H^*_B(M)$ for the basic cohomology with respect to the foliation induced by the Reeb vector field. In 2008 Tievsky proved in his Ph.D. thesis [18] that in order to admit a compatible Sasakian structure the de Rham cohomology algebra $(\Omega^*(M), d)$ of a compact contact manifold (M, η) has to be quasi-isomorphic as CDGA to a elementary Hirsch

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extension of $H_B(M)$. Namely, one has the model $(\mathcal{T}^*(M), d)$ given by

$$\mathcal{T}^*(M) := H^*_B(M) \otimes \mathbb{R}[y]/(y^2), \qquad d([\alpha]_B + [\beta]_B y) := [\beta \wedge d\eta]_B, \quad (1.1)$$

where we put $\deg(y) := 1$, α is a basic p-form and β a basic $(p-1)$-form. We will refer to a compact contact manifold (M, η) admitting the model $(\mathcal{T}^*(M), d)$ as to a manifold of *Tievsky type*.

Using Tievsky's result, we proved in [5] that a compact nilmanifold is of Tievsky type if and only if it is a compact quotient of the Heisenberg group $H(1, n)$ of dimension $2n+1$ by a co-compact discrete subgroup. In particular, any compact nilmanifold carries a Sasakian structure if and only if it is of this type. To the knowledge of the authors, so far, there are no examples of non-Sasakian compact contact manifolds of Tievsky type.

Another obstruction to the existence of Sasakian structures, lately discovered in [7], is provided by the so-called *Hard Lefschetz Theorem* for Sasakian manifolds. Let (M^{2n+1}, η) be a compact contact manifold with Reeb vector field ξ. Consider the following relation between $H^p(M)$ and $H^{2n+1-p}(M)$:

$$\begin{aligned}\mathcal{R}_{\mathrm{Lef}_p} = \Big\{ \big([\beta] , [\eta \wedge (d\eta)^{n-p} \wedge \beta] \big) \;\Big|\; & \beta \in \Omega^p(M), \\ & d\beta = 0,\ i_\xi \beta = 0,\ (d\eta)^{n-p+1} \wedge \beta = 0 \Big\} ,\end{aligned}$$

for $p \leq n$. In [7] it is proved that if (M^{2n+1}, η) admits a compatible Sasakian metric, then $\mathcal{R}_{\mathrm{Lef}_p}$ is the graph of an isomorphism

$$\mathrm{Lef}_p : H^p(M) \longrightarrow H^{2n+1-p}(M).$$

In such a case we say that (M, η) is a *Lefschetz contact manifold*. In any Lefschetz contact manifold the p-th Betti number is even for p odd with $1 \leq p \leq n$. Note that for Sasakian manifolds this fact was known since [9]. We point out that the property given by the Hard Lefschetz Theorem is stronger than the above mentioned obstruction in terms of Betti numbers. Namely, in [6] we constructed examples of K-contact manifolds satisfying these restrictions on the Betti numbers but for which the Hard Lefschetz Theorem does not hold.

Several examples of K-contact manifolds with no Sasakian structure have recently appeared in [4, 6, 12, 14, 15, 16]. Each of them uses some of the known obstructions on Betti numbers, on Hard Lefschetz property or on higher Massey products to prove that the given manifold does not admit any Sasakian structure. Examples of non-formal Sasakian manifolds as well as examples of formal K-contact manifold with no Sasakian structure are presented in [3].

The aim of the paper is to give the first example of a compact 5-dimensional K-contact manifold which has all the above listed features of a compact Sasakian manifold, even the same type of real homotopy model, and yet it does not admit any Sasakian structure. More precisely, we present a family of 5-dimensional compact completely solvable K-contact formal manifolds of Tievsky type with no Sasakian structure. We give two descriptions of our family of examples. Namely, in Section 2 for each real number p we construct a 5-dimensional compact K-contact completely solvable manifold $\widehat{G(p)}$ starting from a nilpotent Lie group of dimension 3 and then we compute its de Rham cohomology. In Section 3 we show that each $\widehat{G(p)}$ can be also be constructed from a completely solvable Lie group of dimension 3 and has the structure of a principal S^1-bundle over a symplectic compact solvmanifold of dimension 4 which was considered in [8]. This second description allows us to prove that the manifold $\widehat{G(p)}$ possesses the claimed properties.

2. A first description of the compact solvmanifold

2.1. The completely solvable Lie group $G(p)$. Let $H(1,1)$ be the Heisenberg group of dimension 3 which may be described as the group of real matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

The group $H(1,1)$ is a connected simply connected nilpotent Lie group. We will denote by (x,y,z) the standard global coordinates on $H(1,1)$.

Then, our connected simply connected Lie group of dimension 5 is the semi-direct product

$$G(p) = (H(1,1) \rtimes_\psi \mathbb{R}) \rtimes_\phi \mathbb{R},$$

where $\psi : \mathbb{R} \to Aut(H(1,1))$ and $\phi : \mathbb{R} \to Aut(H(1,1) \rtimes_\psi \mathbb{R}u)$ are the representations defined by

$$\psi(u)(x,y,z) = (e^{pu}x, e^{-pu}y, z), \quad \phi(t)(x,y,z,u) = (x,y,z+tu,u). \tag{2.1}$$

Here $p \in \mathbb{R}\setminus\{0\}$ is a parameter and (x,y,z,u,t) are the standard coordinates on $G(p) = \mathbb{R}^5$.

The Lie group $H(1,1) \rtimes_\psi \mathbb{R}$ was considered previously in [10] (see page 756 in [10]).

From (2.1) we deduce that the multiplication in $G(p)$ is given by

$$\begin{aligned}(x,y,z,u,t)(x',y',z',u',t') \;=\; & (x + e^{pu}x', y + e^{-pu}y', \\ & z + z' + xe^{-pu}y' + tu', u + u', t + t').\end{aligned} \tag{2.2}$$

The 1-forms

$$\alpha = e^{-pu}dx, \quad \beta = e^{pu}dy, \quad \theta = du, \quad \gamma = dt, \quad \eta = dz - xdy - tdu \tag{2.3}$$

give a basis of left-invariant 1-forms on $G(p)$. Note that

$$d\alpha = p\alpha\wedge\theta, \quad d\beta = -p\beta\wedge\theta, \quad d\theta = 0, \quad d\gamma = 0, \quad d\eta = -\alpha\wedge\beta - \gamma\wedge\theta. \tag{2.4}$$

Next, we consider the dual basis $\{A, B, U, R, \xi\}$ of left-invariant vector fields. It follows that

$$A = e^{pu}\frac{\partial}{\partial x}, \quad B = e^{-pu}(\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}), \quad U = \frac{\partial}{\partial u} + t\frac{\partial}{\partial z}, \quad R = \frac{\partial}{\partial t}, \quad \xi = \frac{\partial}{\partial z}, \tag{2.5}$$

and, therefore,

$$[A,B] = \xi, \quad [A,U] = -pA, \quad [B,U] = pB, \quad [R,U] = \xi, \tag{2.6}$$

the rest of the basic Lie brackets being zero.

This implies that $G(p)$ is a non-nilpotent, but completely solvable Lie group.

2.2. A co-compact discrete subgroup of $G(p)$. In this section, we will describe a co-compact discrete subgroup of $G(p)$.

In fact, we will prove the following result

Proposition 2.1. *Let p be a real number, $p \neq 0$, and N a unimodular 2×2 matrix with integer entries and eigenvalues e^p and e^{-p}. Suppose that (x_0, x_1) (resp. (y_0, y_1)) is an eigenvector for the eigenvalue e^p (resp. e^{-p}). Then, one may find a co-compact discrete subgroup $\Gamma(p)$ of $G(p)$ of the form*

$$\Gamma(p) = (\Gamma_N \rtimes_\psi \mathbb{Z}) \rtimes_\phi z_2\mathbb{Z},$$

where $z_2 = x_1y_0 - x_0y_1$ and Γ_N is a co-compact discrete subgroup of $H(1,1)$ which satisfies the following conditions:

(1) Γ_N is invariant under the restriction to $\mathbb{Z}$ of the representation ψ and
(2) $\Gamma_S = \Gamma_N \rtimes_\psi \mathbb{Z}$ is a co-compact discrete subgroup of the Lie group $S = H(1,1) \rtimes_\psi \mathbb{R}$ which is invariant under the restriction to $z_2\mathbb{Z}$ of the representation ϕ.

Proof: In order to obtain co-compact discrete subgroups of the Lie group S, a general construction was developed in [10] (see pages 756 and 757 in [10]).

We will see that it is possible to choose a co-compact discrete subgroup Γ_S of S of the form $\Gamma_N \rtimes_\psi \mathbb{Z}$, with Γ_N a co-compact discrete of $H(1,1)$ such that Γ_S is invariant under the restriction to $z_2\mathbb{Z}$ of the representation ϕ. In order to obtain Γ_S, we will follow the construction in [10].

Suppose that

$$N = \begin{pmatrix} n_{00} & n_{01} \\ n_{10} & n_{11} \end{pmatrix} \in Sl(2,\mathbb{Z}).$$

Then, we take the matrices in $H(1,1)$

$$h_0 = \begin{pmatrix} 1 & x_0 & z_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{pmatrix}, \quad h_1 = \begin{pmatrix} 1 & x_1 & z_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.7}$$

with $z_0, z_1 \in \mathbb{R}$ which will be fixed later.

We have that

$$h_2(h_0h_1) = h_1h_0,$$

with

$$h_2 = \begin{pmatrix} 1 & 0 & z_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and } z_2 = x_1y_0 - x_0y_1 \neq 0, \tag{2.8}$$

(note that $\{(x_0,y_0),(x_1,y_1)\}$ is a basis of $\mathbb{R}^2$).

Thus, we deduce that

$$h_2h_0 = h_0h_2, \quad h_1h_2 = h_2h_1, \quad h_1^{m_1}h_0^{m_0} = h_2^{m_0m_1}h_0^{m_0}h_1^{m_1}, \qquad \text{for } m_0, m_1 \in \mathbb{Z}. \tag{2.9}$$

Denote by Γ_N the subgroup of $H(1,1)$ which is generated by h_0 and h_1.

From (2.9), it follows that

$$\Gamma_N = \{h_0^{m_0}h_1^{m_1}h_2^{m_2} \ / \ m_0, m_1, m_2 \in \mathbb{Z}\} \subseteq H(1,1). \tag{2.10}$$

In fact, the map

$$\mathbb{Z}^3 \to \Gamma_N, \quad (m_0,m_1,m_2) \to h_0^{m_0}h_1^{m_1}h_2^{m_2}$$

is a group isomorphism, where we consider the multiplication

$$(m_0,m_1,m_2)(m_0',m_1',m_2') = (m_0+m_0', m_1+m_1', m_2+m_2'+m_1m_0')$$

on $\mathbb{Z}^3$.

Next, we need to ensure that the subgroup Γ_N is invariant under the restriction to $\mathbb{Z}$ of the representation ψ. This will produce some restrictions on the real numbers z_0 and z_1.

In fact, using that the eigenvalues of N are different from 1, one may choose $z_0, z_1 \in \mathbb{R}$ such that

$$\psi(1)(h_0) = h_0^{n_{00}} h_1^{n_{01}} h_2^r, \quad \psi(1)(h_1) = h_0^{n_{10}} h_1^{n_{11}} h_2^s$$

with $r, s \in \mathbb{Z}$.

This implies that Γ_N is invariant under the restriction to $\mathbb{Z}$ of ψ. Thus, we have that $\Gamma_S = \Gamma_N \rtimes_\psi \mathbb{Z}$ is a co-compact discrete subgroup of the Lie group $S = H(1,1) \rtimes_\psi \mathbb{R}$.

On the other hand, from (2.1) we deduce that the multiplication in the Lie group S is given by

$$(x, y, z, u)(x', y', z', u') = (x + e^{pu}x', y + e^{-pu}y', z + z' + xe^{-pu}y', u + u'). \quad (2.11)$$

Therefore, using again (2.1), it follows that

$$\phi(t)(x, y, z, u) = (x, y, z, u)(0, 0, tu, 0).$$

Consequently, if $m \in \mathbb{Z}$ and $(x, y, z, u) \in \Gamma_S$ then we deduce that

$$\phi(z_2 m)(x, y, z, u) = (x, y, z, u)(h_2^{mu}, 0) \in \Gamma_S.$$

In other words, Γ_S is invariant under the restriction to $z_2\mathbb{Z}$ of the representation ϕ. This implies that

$$\Gamma(p) = \Gamma_S \rtimes_\phi z_2\mathbb{Z} = (\Gamma_N \rtimes_\psi \mathbb{Z}) \rtimes_\phi z_2\mathbb{Z} \quad (2.12)$$

is a co-compact discrete subgroup of $G(p)$. ■

Remark 2.2. The Lie group S is completely solvable and the quotient manifold S/Γ_S is a compact solvmanifold. In fact, S/Γ_S is diffeomorphic to an Inoue surface of type S^+ (see page 757 in [10]).

2.3. De Rham cohomology of the compact solvmanifold $\widehat{G(p)} = G(p)/\Gamma(p)$. The Lie group $G(p)$ is completely solvable. So, in order to compute the de Rham cohomology of $\widehat{G(p)}$, we may use Hattori's theorem [11] and we have that

$$H^*(\widehat{G(p)}) \simeq H^*(\mathfrak{g}^*),$$

where $\mathfrak{g}$ is the Lie algebra of $G(p)$.

Thus, if we denote by $\hat{\alpha}$, $\hat{\beta}$, $\hat{\theta}$, $\hat{\gamma}$ and $\hat{\eta}$ the 1-forms on $\widehat{G(p)}$ which are induced by the left-invariant 1-forms α, β, θ, γ and η, respectively, we deduce the following result.

Corollary 2.3. *The de Rham cohomology groups of $\widehat{G(p)}$ are the vector spaces*

$$\begin{array}{rcl} H^0(\widehat{G(p)}) & = & \langle 1 \rangle, \\ H^1(\widehat{G(p)}) & = & \langle [\hat{\theta}], [\hat{\gamma}] \rangle, \\ H^2(\widehat{G(p)}) & = & \langle [\hat{\theta} \wedge \hat{\gamma}] \rangle = \langle [\hat{\theta} \wedge \hat{\gamma} - \hat{\alpha} \wedge \hat{\beta}] \rangle, \\ H^3(\widehat{G(p)}) & = & \langle [\hat{\eta} \wedge (\hat{\theta} \wedge \hat{\gamma} - \hat{\alpha} \wedge \hat{\beta})] \rangle, \\ H^4(\widehat{G(p)}) & = & \langle [\hat{\eta} \wedge \hat{\gamma} \wedge \hat{\alpha} \wedge \hat{\beta}], [\hat{\alpha} \wedge \hat{\beta} \wedge \hat{\theta} \wedge \hat{\eta}] \rangle, \\ H^5(\widehat{G(p)}) & = & \langle [\hat{\alpha} \wedge \hat{\beta} \wedge \hat{\theta} \wedge \hat{\gamma} \wedge \hat{\eta}] \rangle. \end{array} \tag{2.13}$$

3. A second description of the contact compact solvmanifold $\widehat{G(p)}$

In this section, we will present a different description of the compact solvmanifold $\widehat{G(p)}$ as a principal S^1-bundle over a symplectic compact solvmanifold $(K/\Gamma_K) \times S^1$ of dimension 4. In fact, from (2.4), we deduce that $\hat{\eta}$ defines a contact structure on $\widehat{G(p)}$ and, in addition, we will see that $(K/\Gamma_K) \times S^1$ is just the space of orbits of the Reeb vector field associated with $\hat{\eta}$ and the symplectic structure on this space is induced by the contact structure of $\widehat{G(p)}$.

Define K to be the connected simply connected Lie group of dimension 3 given by

$$K = \mathbb{R}^2 \rtimes_\zeta \mathbb{R}$$

where $\zeta : \mathbb{R} \to Aut(\mathbb{R}^2)$ is the representation given by

$$\zeta(u)(x, y) = (e^{pu}x, e^{-pu}y),$$

for $u \in \mathbb{R}$ and $(x, y) \in \mathbb{R}^2$ (see [1]).

Thus, if we consider the standard coordinates (u, x, y) on K then the multiplication is given by

$$(x, y, u)(x', y', u') = (x + e^{pu}x', y + e^{-pu}y', u + u'). \tag{3.1}$$

We have that a basis of left-invariant 1-forms is $\{\alpha_K, \beta_K, \theta_K\}$, with

$$\alpha_K = e^{-pu}dx, \quad \beta_K = e^{pu}dy, \quad \theta_K = du. \tag{3.2}$$

Note that

$$d\alpha_K = p\alpha_K \wedge \theta_K, \;\; d\beta_K = -p\beta_K \wedge \theta_K, \;\; d\theta_K = 0. \tag{3.3}$$

Now, we consider the dual basis $\{A_K, B_K, U_K\}$ of left-invariant vector fields on K. It follows that

$$A_K = e^{pu}\frac{\partial}{\partial x}, \;\; B_K = e^{-pu}\frac{\partial}{\partial y}, \;\; U_K = \frac{\partial}{\partial u}$$

and, thus,

$$[A_K, U_K] = -pA_K, \;\; [B_K, U_K] = pB_K,$$

the rest of the basic Lie brackets being zero.

Therefore, K is a non-nilpotent completely solvable Lie group.

The group K admits co-compact discrete subgroups (see [2]). In fact, suppose that $N \in Sl(2, \mathbb{Z})$, with eigenvalues e^p and e^{-p}, and that (x_0, x_1), (y_0, y_1) are eigenvectors of N as in Section 2.2. Then, we have that the integer lattice $\Gamma_{\mathbb{R}^2}$ on $\mathbb{R}^2$, which is generated by the basis $\{(x_0, y_0), (x_1, y_1)\}$, is invariant under the restriction to $\mathbb{Z}$ of ζ and

$$\Gamma_K = \Gamma_{\mathbb{R}^2} \rtimes_\zeta \mathbb{Z} \tag{3.4}$$

is a co-compact discrete subgroup of K (see [2]).

This implies that $\hat{K} = K/\Gamma_K$ is a compact solvmanifold.

Next, we consider the completely solvable Lie group $K\times\mathbb{R}$ and the discrete co-compact subgroup

$$\Gamma_{K\times\mathbb{R}} = \Gamma_K \times z_2\mathbb{Z}, \tag{3.5}$$

with $z_2 = x_1y_0 - x_0y_1 \in \mathbb{R}$. The corresponding compact solvmanifold is

$$\frac{K\times\mathbb{R}}{\Gamma_{K\times\mathbb{R}}} = \hat{K} \times (\mathbb{R}/z_2\mathbb{Z}) \simeq \hat{K} \times S^1. \tag{3.6}$$

On the other hand, using (3.3), we have that

$$\Omega = -(\alpha_K \wedge \beta_K + dt \wedge \theta_K) \tag{3.7}$$

is a left-invariant symplectic 2-form on $K\times\mathbb{R}$. Thus, Ω induces a symplectic form $\hat{\Omega}$ on $\hat{K}\times S^1$. In fact, if $\hat{\omega}$ is the length element of S^1 and $\{\widehat{\alpha_K}, \widehat{\beta_K}, \widehat{\theta_K}\}$ is the global basis of 1-forms on $\hat{K}$ induced by the left-invariant 1-forms α_K, β_K and θ_K, respectively, then

$$\hat{\Omega} = -(\widehat{\alpha_K} \wedge \widehat{\beta_K} + \hat{\omega} \wedge \widehat{\theta_K}). \tag{3.8}$$

Moreover, we may prove the following result

Theorem 3.1. *The contact compact solvmanifold* $(\widehat{G(p)}, \hat{\eta})$ *is regular, the orbit space of the Reeb vector field* $\hat{\xi}$ *is the compact solvmanifold* $\hat{K} \times S^1$ *and the symplectic structure on* $\hat{K} \times S^1$ *is just the* 2*-form* $\hat{\Omega}$ *given by (3.8).*

Proof: Using (2.2) and (3.1), we deduce that the canonical projection

$$\pi : G(p) \to K \times \mathbb{R}, \quad (x, y, z, u, t) \to (x, y, u, t) \tag{3.9}$$

is a Lie group epimorphism. In fact, from (2.5), it follows that $K \times \mathbb{R}$ is just the space of orbits of the left-invariant vector field ξ on $G(p)$.

In addition, if $\Gamma(p)$ is the discrete co-compact subgroup of $G(p)$ given by (2.12) then, using (2.7), (2.8), (2.10) and (2.12), we obtain that

$$\pi(\Gamma(p)) = (\Gamma_{\mathbb{R}^2} \rtimes_\zeta \mathbb{Z}) \times z_2\mathbb{Z}. \tag{3.10}$$

Thus, from (3.4), (3.5), (3.6) and (3.10), we have that the Lie group epimorphism $\pi : G(p) \to K \times \mathbb{R}$ induces a principal S^1-bundle

$$\hat{\pi} : \widehat{G(p)} \to \hat{K} \times S^1$$

where the action of S^1 on $\widehat{G(p)}$ is just the flow of the Reeb vector field $\hat{\xi}$ and, therefore, the base space is just the orbit space of $\hat{\xi}$.

On the other hand, using (2.3), (2.4), (3.2), (3.7) and (3.9), it follows that $\pi^*(\Omega) = d\eta$ and, consequently,

$$\hat{\pi}^*(\hat{\Omega}) = d\hat{\eta}.$$

This implies that our contact structure $\hat{\eta}$ on $\widehat{G(p)}$ is regular. ■

Remark 3.2. The left-invariant 2-form Ω on $K \times \mathbb{R}$ was used, in [8], in order to obtain examples of Lefschetz symplectic structures on compact solvmanifolds of the form $(K \times \mathbb{R})/D$, with D a co-compact discrete subgroup. We remark that these manifolds are formal (see [8] for more details).

4. The K-contact structure on $\widehat{G(p)}$ and its properties

In this section, we will introduce a K-contact structure on $\widehat{G(p)}$ and we will discuss its properties.

Denote by $\hat{A}$, $\hat{B}$, $\hat{U}$, $\hat{R}$ and $\hat{\xi}$ the vector fields on $\widehat{G(p)}$ which are induced by the left-invariant vector fields A, B, U, R and ξ, respectively. Then, it is clear that $\{\hat{A}, \hat{B}, \hat{U}, \hat{R}, \hat{\xi}\}$ is a global basis of vector fields on $\widehat{G(p)}$. Moreover, the dual basis of 1-forms is just $\{\hat{\alpha}, \hat{\beta}, \hat{\theta}, \hat{\gamma}, \hat{\eta}\}$.

In the following theorem we will use the following notions. Given a commutative differential graded algebra (CDGA) (A, d) we define an *elementary extension* of (A, d) with respect to the closed element $b \in A_2$, to be $A \otimes \Lambda \langle y \rangle$, with $\deg(y) = 1$ and $dy = b$. We will say that a contact manifold (M, η) is of *Tievsky type*, if the de Rham cohomology algebra $\Omega^*(M)$ is quasi-isomorphic (as CDGA) to the elementary extension of $H_B(M)$ with respect to $[d\eta]_B$. Note, that according to [18], every Sasakian manifold is of Tievsky type.

Theorem 4.1. *Let $\hat{\phi}$ be the $(1,1)$-tensor field on $\widehat{G(p)}$ which is characterized by the following conditions:*

$$\hat{\phi}(\hat{A}) = -\hat{B}, \;\; \hat{\phi}(\hat{B}) = \hat{A}, \;\; \hat{\phi}(\hat{U}) = \hat{R}, \;\; \hat{\phi}(\hat{R}) = -\hat{U}, \;\; \hat{\phi}(\hat{\xi}) = 0, \tag{4.1}$$

and $\hat{g}$ the Riemannian metric on $\widehat{G(p)}$ whose matrix associated with respect to the basis $\{\hat{A}, \hat{B}, \hat{U}, \hat{R}, \hat{\xi}\}$ is the identity, that is, the basis $\{\hat{A}, \hat{B}, \hat{U}, \hat{R}, \hat{\xi}\}$ is orthonormal. Then:

(1) The almost contact metric structure $(\hat{\phi}, \hat{\xi}, \hat{\eta}, \hat{g})$ on $\widehat{G(p)}$ is K-contact.

(2) The contact structure $\hat{\eta}$ on $\widehat{G(p)}$ has the Hard Lefschetz property.

(3) The minimal model (over $\mathbb{R}$) of $\widehat{G(p)}$ is isomorphic to the exterior algebra $\Lambda\langle a_1, a_2, b\rangle$ with zero differential, where the grading is determined by the requirement that a_1, a_2 have degree 1, and b has degree 3. In particular, $\widehat{G(p)}$ is formal.

(4) The contact manifold $(\widehat{G(p)}, \eta)$ is of Tievsky type.

(5) The manifold $\widehat{G(p)}$ does not admit Sasakian structures.

Proof: *(1)* From (2.4) and(4.1), we deduce that $(\hat{\phi}, \hat{\xi}, \hat{\eta}, \hat{g})$ is a contact metric structure on $\widehat{G(p)}$. In addition, using (2.6), we have that the element in $\mathfrak{g}$, which induces the left-invariant vector field ξ on $G(p)$, belongs to the center of $\mathfrak{g}$. So, the Reeb vector field $\hat{\xi}$ is Killing with respect to the Riemannian metric $\hat{g}$.

(2) A direct computation, using (2.13) and the fact that $d\hat{\eta} = -\hat{\alpha}\wedge\hat{\beta}-\hat{\gamma}\wedge\hat{\theta}$, proves that the Lefschetz relation in degree 1 (resp. degree 2) for the compact contact manifold $(\widehat{G(p)}, \hat{\eta})$ is the graph of an isomorphism between $H^1(\widehat{G(p)})$ and $H^4(\widehat{G(p)})$ (resp. $H^2(\widehat{G(p)})$ and $H^3(\widehat{G(p)})$). This proves *(2)*.

(3) Define the map $\tau\colon \Lambda\langle a_1, a_2, b\rangle \to \Omega^*(M)$ by

$$\tau(a_1) = \hat{\gamma}, \qquad \tau(a_2) = \hat{\theta}, \qquad \tau(b) = \hat{\eta} \wedge (\hat{\theta} \wedge \hat{\gamma} - \hat{\alpha} \wedge \hat{\beta}),$$

where the 1-forms $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$, $\hat{\theta}$ are defined in Subsection 2.3. Since, the forms $\hat{\gamma}$, $\hat{\theta}$, and $\hat{\eta}\wedge(\hat{\theta}\wedge\hat{\gamma}-\hat{\alpha}\wedge\hat{\beta})$ are closed, the map τ is a well-defined homomorphism of CDGAs. It follows from Corollary 2.3 that τ is a quasi-isomorphism. Since $\Lambda\langle a_1, a_2, b\rangle$ is a free CDGA with zero differential, it is a minimal Sullivan algebra. Moreover, it is quasi-isomorphic to its cohomology, and thus is formal.

(4) We will show that there is a quasi-isomorphism from $\Lambda\langle a_1, a_2, b\rangle$ into $H_B(M)\otimes\Lambda\langle y\rangle$ with $dy := [d\eta]_B$. Since ξ is regular, we get that $H_B(\widehat{G(p)})$ is isomorphic as graded algebra to $H((K/\Gamma_K)\times S^1)$. The cohomology ring of $H((K/\Gamma_K)\times S^1)$ was computed in [8] and it is isomorphic to

$$\Lambda\langle u, v\rangle\otimes\mathbb{R}[w]/(w^2)$$

with elements u, v of degree 1 and w of degree 2. The element $[d\eta]_B = [\hat{\Omega}] = -uv - w$ in this description. We define the map ρ from $\Lambda\langle a_1, a_2, b\rangle$ to

$$A := H((K/\Gamma_K)\times S^1)\otimes\Lambda\langle y\rangle \quad (\text{with } dy = -uv - w),$$

by

$$\rho(a_1) = u, \qquad \rho(a_2) = v, \qquad \rho(b) = (uv - w)y.$$

The elements u and v are closed in A. Moreover,

$$d((uv - w)y) = -(uv - w)(uv + w) = -uvw + wuv = 0.$$

Thus the map ρ is well-defined. It is easy to check that ρ induces an isomorphism in the cohomology. Thus A is quasi-isomorphic to the de Rham algebra $\Omega^*(\widehat{G(p)})$, which shows that $\widehat{G(p)}$ is a manifold of Tievsky type.

(5) The completely solvable Lie algebra $\mathfrak{g}$ is not the Heisenberg Lie algebra of dimension 5. Therefore, the result follows using Corollary 1.5 in [13]. ∎

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Beniamino Cappelletti-Montano

Dipartimento di Matematica e Informatica, Universtà degli Studi di Cagliari, Via Ospedale 72, 09124 Cagliari

E-mail address: b.cappellettimontano@gmail.com

Antonio De Nicola

CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal

E-mail address: antondenicola@gmail.com

Juan Carlos Marrero
Unidad Asociada ULL-CSIC "Geometría Diferencial y Mecánica Geométrica" Departamento de Matemáticas, Estadística e Investigación Operativa, Facultad de Ciencias, Universidad de La Laguna, La Laguna, Tenerife, Spain

E-mail address: jcmarrer@ull.edu.es

Ivan Yudin
CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal

E-mail address: yudin@mat.uc.pt