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CHARACTERIZATIONS OF Δ-VOLTERRA LATTICE: A SYMMETRIC ORTHOGONAL POLYNOMIALS INTERPRETATION

I. AREA, A. BRANQUINHO, A. FOULQUIÉ MORENO AND E. GODOY

ABSTRACT: In this paper we introduce the Δ-Volterra lattice which is interpreted in terms of symmetric orthogonal polynomials. It is shown that the measure of orthogonality associated with these systems of orthogonal polynomials evolve in t like $(1+x^2)^{1-t}\mu(x)$ where μ is a given positive Borel measure. Moreover, the Δ-Volterra lattice is related to the Δ-Toda lattice from Miura or Bäcklund transformations. The main ingredients are orthogonal polynomials which satisfy an Appell condition with respect to the forward difference operator Δ and the characterization of the point spectrum of a Jacobi operator that satisfies a Δ-Volterra equation (Lax type theorem). We also provide an explicit example of solutions of Δ-Volterra and Δ-Toda lattices, and connect this example with the results presented in the paper.

KEYWORDS: Orthogonal polynomials, Difference operators, Operator theory, Toda lattices.
AMS SUBJECT CLASSIFICATION (2010): 33C45, 37K10, 65Q10.

1. Introduction

Nonlinear evolution equations have been used as models to describe various physical phenomena as shallow water waves and ion-acoustic waves in plasmas. In 1967, M. Toda [28] introduced a model, that he named as exponential lattice, for a one-dimensional crystal in solid state physics with a nearest neighbor interaction, with potential

$$\phi(r) = \frac{a}{b}\exp(-r) + ar + \text{const.}\,, \qquad a, b > 0\,,$$

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such that the particles are subject to

$$\frac{dp_k(t)}{dt} = \exp(q_{k-1}(t) - q_k(t)) - \exp(q_k(t) - q_{k+1}(t)), \qquad \frac{dq_k(t)}{dt} = p_k(t)\,,$$

where $q_k(t)$ and $p_k(t)$ are the displacement of the k-th particle from its equilibrium position, and its momentum, respectively and the mass is assumed to be equal to the unity [5]. The latter Toda lattice describe the oscillations of an infinite system of points joined by spring masses, where the interaction is exponential in the distance between two spring masses [29]. Later on, Henón [13] and Flaschka [10] proved that the non–periodic Toda lattice is a completely Hamiltonian integrable system, with Hamiltonian function

$$H(q_1,\dots,q_m,p_1,\dots,p_m) = \frac{1}{2}\sum_{n=1}^{m} p_n^2 + \sum_{n=1}^{m-1} \exp\big(q_n - q_{n+1}\big)\,.$$

By using the Flaschka transformation

$$a_n(t) = \exp\big(q_{n-1}(t) - q_n(t)\big)\,, \qquad b_n(t) = \frac{dq_n(t)}{dt}\,,$$

the semi-infinite Toda lattice in one time variable is the system of ordinary differential equations

$$a_{-1}(t) \equiv 0\,,\ a_0(t) \equiv 1\,, \begin{cases} \dfrac{da_n(t)}{dt} = a_n(t)\big(b_{n-1}(t) - b_n(t)\big)\,, \\ \dfrac{db_n(t)}{dt} = a_n(t) - a_{n+1}(t)\,, \end{cases} \qquad n \in \mathbb{N}\,. \tag{1}$$

The Toda lattice is integrable in the sense of Liouville and it is mainly a theoretical mathematical model due to the rich mathematical structure encoded in it.

There exists a closed relation between the Toda system (1) and orthogonal polynomials shown by Moser [22, 23] and Kac and Moerbecke [15], that we briefly describe. Let $t_0 \in \mathbb{R}$ and $\mu(x;t_0)$ be a measure such that all the moments

$$u_n = \int_{\mathbb{R}} x^n \, d\mu(x;t_0)\,, \qquad n \in \mathbb{N}\,, \tag{2}$$

exist and are finite, and $P_n(x)$ be the sequence of monic orthogonal polynomials with respect to $\mu(x;t_0)$,

$$\int_{\mathbb{R}} P_n(x)P_m(x)\, d\mu(x;t_0) = h_n^2\, \delta_{n,m}\,,$$

where $\delta_{i,j}$ denotes the Kronecker delta. As it is very well-known [7, 14, 27], the monic polynomials $P_n(x;t_0) \equiv P_n(x)$ satisfy a three-term recurrence relation

$$P_{n+1}(x) = (x - b_n)P_n(x) - a_n P_{n-1}(x)\,,$$

with initial conditions $P_0(x) = 1$ and $P_1(x) = x - b_0$.

The dynamic of the solutions of the Toda lattice (1) corresponds to the evolution of the spectral measure [23, 24],

$$d\mu(x;t) = \frac{\exp(-xt)d\mu(x,t_0)}{\int \exp(-xt)d\mu(x,t_0)}\,,$$

of a operator $J(t)$, defined in the standard basis of $\ell_2(0,\infty)$,

$$e_k = (0,\dots,0,1,0,\dots)^{\mathsf{T}}\,, \quad k \in \mathbb{N}\,,$$

by a Jacobi matrix

$$J(t) = \big(J_{i,j}(t)\big) = \begin{pmatrix} b_0(t) & 1 & 0 & & & \\ a_1(t) & b_1(t) & 1 & 0 & & \\ 0 & a_2(t) & b_2(t) & 1 & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & \end{pmatrix}, \tag{3}$$

where the monic polynomials $P_n(x;t)$ orthogonal with respect to the modified weight $\mu(x;t)$ satisfy

$$P_{n+1}(x;t) = (x - b_n(t))P_n(x;t) - a_n(t)P_{n-1}(x;t)\,, \quad n = 1,\dots\,, \tag{4}$$

with initial conditions $P_0(x;t) = 1$ and $P_1(x;t) = x - b_0(t)$.

Let $\mathcal{P}$ be the column vector of monic orthogonal polynomials, *i.e.* $\mathcal{P} = (P_0, P_1, \dots)^{\mathsf{T}}$, with respect to a linear functional $u(t)$, defined in terms of its moments (2) by (*cf.* [19])

$$u(t) : \mathcal{P} \to \mathbb{R}\,, \quad \text{with} \quad \big\langle u(t), x^n \big\rangle = u_n(t)\,,\ n \in \mathbb{N}\,,$$

and $J(t)$ the corresponding Jacobi matrix (3). Then, the recurrence relation for the monic orthogonal polynomials can be written as

$$J(t)\,\mathcal{P} = x\,\mathcal{P}\,.$$

Next, we define the Stieltjes function [24], $S(z;t) = e_0^{\mathsf{T}} R_z(t)\, e_0$, for the resolvent operator, $R_z(t) = \big[J(t) - z\,\mathcal{I}\big]^{-1}$, associated with the operator $J(t)$ (*cf.* [1]). We shall assume that linear functional, $u(t)$, is normalized, *i.e.*

$$u_0(t) = 1\,. \tag{5}$$

By using (*cf.* [3])

$$\langle u(t), x^n \rangle = J^n_{1,1}(t)\,, \qquad n \in \mathbb{N}\,,$$

the Stieltjes function reads as

$$\begin{aligned} S(z;t) &= e_0^{\mathsf{T}} R_z(t)\, e_0 = e_0^{\mathsf{T}} \big[J(t) - z\,\mathcal{I}\big]^{-1} e_0 = e_0^{\mathsf{T}} \sum_{n=0}^{\infty} \frac{J(t)^n}{z^{n+1}} e_0 = \sum_{n=0}^{\infty} \frac{J^n_{1,1}(t)}{z^{n+1}} \\ &= \sum_{n=0}^{\infty} \frac{u_n(t)}{z^{n+1}} = \Big\langle u(t), \frac{1}{z-x} \Big\rangle\,. \end{aligned}$$

A difference analogue of a Korteweg-de Vries equation,

$$a_1(t) \equiv 0\,, \quad \frac{da_n(t)}{dt} = a_n(t)\big(a_{n+1}(t) - a_{n-1}(t)\big)\,, \quad n = 2, 3, \dots\,,$$

is called Langmuir lattice, due to its applications in modeling Langmuir oscillations in plasmas [12] or finite difference KDV equation [25], whose dynamic is given by

$$d\mu(x;t) = \frac{\exp(-x^2 t)\, d\mu(x, t_0)}{\int \exp(-x^2 t)\, d\mu(x, t_0)}\,. \tag{6}$$

In [12] it was studied the construction of a solution of the Toda lattice

$$\begin{cases} \dfrac{da_n(t)}{dt} = a_n(t)\big(b_{n-1}(t) - b_n(t)\big)\,, \\[2ex] \dfrac{db_n(t)}{dt} = a_n(t) - a_{n+1}(t)\,, \end{cases} \qquad n \in \mathbb{Z}, \tag{7}$$

from another given solution, considering sequences $\{a_n(t)\}_{n\in\mathbb{Z}}$, $\{b_n(t)\}_{n\in\mathbb{Z}}$, of real functions. Both solutions of (7) were linked by *Bäcklund* or *Miura transformations*

$$\begin{aligned} a_n(t) &= \gamma_{2n}(t)\gamma_{2n-1}(t)\,, \quad b_n(t) = \gamma_{2n+1}(t) + \gamma_{2n}(t) + c\,, \quad n \in \mathbb{Z}, \\ \tilde{a}_n(t) &= \gamma_{2n+1}(t)\gamma_{2n}(t)\,, \quad \tilde{b}_n(t) = \gamma_{2n+2}(t) + \gamma_{2n+1}(t) + c\,, \quad n \in \mathbb{Z}\,, \end{aligned}$$

with c an arbitrary complex constant independent of t and where $\{\gamma_n(t)\}_{n\in\mathbb{Z}}$ is a solution of the Volterra lattice or Langmuir lattice (*cf.* [25, Theorem 1])

$$\dot{\gamma}_{n+1}(t) = \gamma_{n+1}(t)\big(\gamma_{n+2}(t) - \gamma_n(t)\big)\,, \quad n \in \mathbb{Z}\,. \tag{8}$$

This Volterra system, also known as the KM system, was solved in [15] using a discrete version of the inverse scattering method. The Lax pair for (8) can be found in [23]. There exists a relation, first discovered by Hénon, between the Volterra system and the non–periodic Toda lattice (*cf.* [8, 23] for more details).

In [6], this kind of analysis has been generalized to the full hierarchy of Toda and Volterra lattices studied in [1, 2] (*cf.* also [9]).

Recently in [4], the following system of nonlinear difference equations, named Δ-Toda lattice:

$$\begin{cases} \Delta_t a_n(t) = \alpha_1^n(t)\big(b_{n-1}(t) - b_n(t+1)\big)\,, \\ \Delta_t b_n(t) = \alpha_1^n(t) - \alpha_1^{n+1}(t)\,, \end{cases} \qquad n \in \mathbb{N}\,, \tag{9}$$

and his characterization has been presented, where

$$\alpha_1^n(t) = \frac{g_n(t)}{b_0(t+1)+1}\,, \quad \text{and} \quad g_n(t) = \prod_{k=1}^{n} \frac{a_k(t+1)}{a_{k-1}(t)}\,,$$

assuming that $b_0(t+1)+1 \neq 0$ and $a_0(t) = 1$, where the forward difference operator Δ_t is defined by

$$\Delta_t g(t) = g(t+1) - g(t)\,.$$

The Δ-Toda lattice (9) can be written in a Lax-type representation as a first–order linear difference system

$$\Delta_t J(t) = A(t)\,J(t) - J(t+1)\,A(t)\,,$$

where

$$A(t) = \begin{pmatrix} b_0(t+1) & 0 & & \\ g_1(t) & b_0(t+1) & 0 & \\ 0 & g_2(t) & b_0(t+1) & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix}\,,$$

and $J(t)$ was defined in (3). Let us now introduce the Δ-Volterra lattice (or Δ-Langmuir lattice) by means of a new Lax–type pair representation

$$\Delta_t \Gamma(t) = B(t)\,\Gamma(t) - \Gamma(t+1)\,B(t)\,, \tag{10}$$

where

$$\Gamma(t) = \big(\Gamma_{i,j}(t)\big) = \begin{pmatrix} 0 & 1 & 0 & & \\ \gamma_1(t) & 0 & 1 & 0 & \\ 0 & \gamma_2(t) & 0 & 1 & \ddots \\ & \ddots & \ddots & \ddots & \ddots \end{pmatrix}\,, \tag{11}$$

$$B(t) = \begin{pmatrix} \gamma_1(t+1) & 0 & & & \\ 0 & \gamma_1(t+1) & 0 & & \\ \eta_1(t) & 0 & \gamma_1(t+1) & \ddots & \\ 0 & \eta_2(t) & 0 & \ddots & \\ & \ddots & \ddots & \ddots & \end{pmatrix}, \tag{12}$$

and

$$\eta_1(t) = \gamma_1(t+1)\gamma_2(t+1)\,,\ \eta_n(t) = \frac{\gamma_1(t+1)\cdots\gamma_{n+1}(t+1)}{\gamma_1(t)\cdots\gamma_{n-1}(t)}\,,\ n = 2, \ldots\,, \tag{13}$$

with $1+\gamma_1(t+1) \neq 0$ and $\gamma_n(t) \neq 0$.

The main goal of this work is to obtain characterizations of the Δ-Volterra lattice (10). This will be done in terms of the moments for the associated linear functional, the Stieltjes function, and in terms of the Appell type equation that these families of symmetric orthogonal polynomials satisfy. Besides, it is shown that the solutions of Δ-Toda lattice (9) are connected to Δ-Volterra lattice (10) through Miura or Bäcklund transformations [12, 18].

The structure of the paper is the following: In section 2, we present the main theorem of the Δ-Volterra lattices. We give a representation of the symmetric orthogonality functional and a Lax-type theorem. In section 3, we present the connection between the Bäcklund or Miura transformations in terms of the theory of orthogonal polynomials. Finally, in section 4, an explicit example of solutions of Δ-Volterra and Δ-Toda lattices related to Jacobi polynomials is given, and connected with the results presented in this paper.

2. Δ-Volterra system

Let us consider the following Δ-Volterra lattice (or Δ-Langmuir lattice) equivalent to (10):

$$\begin{cases} \Delta_t\gamma_1(t) = -\dfrac{\gamma_1(t+1)\gamma_2(t+1)}{1+\gamma_1(t+1)}\,, \\[2ex] \Delta_t\gamma_n(t) = \dfrac{\gamma_n(t+1)\cdots\gamma_1(t+1)\big(\gamma_{n-1}(t)-\gamma_{n+1}(t+1)\big)}{(1+\gamma_1(t+1))\gamma_{n-1}(t)\cdots\gamma_1(t)}, n=2,\ldots \end{cases} \tag{14}$$

assuming that $1+\gamma_1(t+1) \neq 0$ and $\gamma_n(t) \neq 0$.

We shall also consider the backward difference operator, ∇_t, defined by

$$\nabla_t g(t) = g(t) - g(t-1)\,.$$

Theorem 1. *Let us assume that the sequence $\{\gamma_n(t)\}_{n\in\mathbb{N}}$ is uniformly bounded. The following conditions are equivalent:*

(1) *The Jacobi matrix $\Gamma(t)$ defined in (11) satisfies the matrix difference equation (10).*

(2) *The moments $u_n(t)$ associated to a symmetric functional $u(t)$, defined by (2), satisfy*

$$\Delta_t u_n(t) = -u_{n+2}(t+1) + u_2(t+1)u_n(t)\,, \qquad \text{when } n \text{ is even}\,, \tag{15}$$

since $u_{2n+1}(t) = 0$.

(3) *The Stieltjes function associated with $\Gamma(t)$ satisfies*

$$\Delta_t S(z;t) = -z^2\, S(z;t+1) + u_2(t+1)S(z,t) + z\,. \tag{16}$$

(4) *The linear functional $u(t)$ associated with $\Gamma(t)$ satisfies*

$$\Delta_t u(t) = -x^2\, u(t+1) + u_2(t+1)u(t)\,. \tag{17}$$

(5) *The sequence of monic symmetric polynomials, $\{R_n(x;t)\}_{n\in\mathbb{N}}$, orthogonal with respect to the functional $u(t)$ associated with $\Gamma(t)$ satisfy an Appell type property*

$$\Delta_t R_n(x;t) = \alpha_2^n(t)\, R_{n-2}(x;t)\,, \tag{18}$$

where

$$\alpha_2^n(t) \;=\; \frac{\big\langle u(t+1), x^n R_n(x;t+1)\big\rangle}{(1+u_2(t+1))\,\big\langle u(t), x^{n-2} R_{n-2}(x;t)\big\rangle} = \frac{\eta_{n-1}(t)}{1+\gamma_1(t+1)}\,, \tag{19}$$

for $n = 2, \ldots,$ and $\eta_n(t)$ was defined in (13).

Proof: (1) ⇒ (2). By induction it can be proved that

$$\Delta_t \Gamma^n(t) = B(t)\,\Gamma^n(t) - \Gamma^n(t+1)\,B(t)\,, \tag{20}$$

where $B(t)$ is defined in (12). By using (2)

$$e_0^\top\,\Delta_t \Gamma^n(t)\,e_0 = \Delta_t\big(e_0^\top\,\Gamma^n(t)\,e_0\big) = \Delta_t u_n(t)\,,$$

where $e_0^\top = (1, 0, \ldots)$. Moreover, from (20) we have

$$\begin{aligned} e_0^\top\,\Delta_t \Gamma^n(t)\,e_0 \;&=\; \gamma_1(t+1)\,\Gamma^n_{1,1}(t) - \big(\Gamma^n_{1,1}(t+1)\,\gamma_1(t+1) + \Gamma^n_{1,3}(t+1)\,\eta_1(t)\big)\\ &=\; u_2(t+1)u_n(t) - u_{n+2}(t+1)\,, \end{aligned}$$

because $\gamma_1(t+1) = \Gamma^2_{1,1}(t+1) = u_2(t+1)$ and as a consequence of the product of matrices $\eta_1(t) = \Gamma^2_{3,1}(t+1) = \gamma_1(t+1)\gamma_2(t+1)$, which completes the proof.

Moreover, from (10) we obtain

$$\eta_n(t) = \frac{\gamma_{n+1}(t+1)}{\gamma_{n-1}(t)}\eta_{n-1}(t)\,, \quad n = 2, \ldots\,,$$

and

$$\begin{cases} \big(1+\gamma_1(t+1)\big)\Delta_t\gamma_1(t) = -\eta_1(t) \\ \big(1+\gamma_1(t+1)\big)\Delta_t\gamma_n(t) = \eta_{n-1}(t) - \eta_n(t)\,, \quad n = 2, \ldots, \end{cases} \tag{21}$$

what leads to the Δ-Volterra lattice (14).

(2) $\Rightarrow$ (3). From (15), then

$$\Delta_t S(z;t) = \sum_{n=0}^{\infty} \frac{\Delta_t u_n(t)}{z^{n+1}} = -z^2 \sum_{n=0}^{\infty} \frac{u_{n+2}(t+1)}{z^{n+3}} + u_2(t+1) \sum_{n=0}^{\infty} \frac{u_n(t)}{z^{n+1}}$$

where we have used that $u_0(t+1) = 1$ and $u_1(t+1) = 0$. As a consequence, we obtain (16).

(3) $\Rightarrow$ (4). By using

$$S(z;t) = \big\langle u(t), \frac{1}{z-x} \big\rangle,$$

and (5), if we apply the Δ_t operator, we have that the equation (16) reads as

$$\begin{aligned} \Delta_t S(z;t) &= \big\langle \Delta_t u(t), \frac{1}{z-x} \big\rangle \\ &= -z^2 \big\langle u(t+1), \frac{1}{z-x} \big\rangle + u_2(t+1) \big\langle u(t), \frac{1}{z-x} \big\rangle + z \\ &= \big\langle u(t+1), \frac{-z^2}{z-x} + z + x \big\rangle + u_2(t+1) \big\langle u(t), \frac{1}{z-x} \big\rangle \\ &= \big\langle u(t+1), \frac{-x^2}{z-x} \big\rangle + u_2(t+1) \big\langle u(t), \frac{1}{z-x} \big\rangle\,, \end{aligned}$$

which implies

$$\big\langle \Delta_t u(t) + x^2\, u(t+1) - u_2(t+1) u(t), \frac{1}{z-x} \big\rangle = 0\,,$$

and so, all the moments for the linear functional $\Delta_t u(t) + x^2\, u(t+1) - u_2(t+1)\, u(t)$ are zero, and (17) is obtained.

Moreover, we have

$$\big(1+u_2(t+1)\big)\Delta_t u(t) = \big(-x^2 + u_2(t+1)\big)u(t+1)\,.$$

(4) ⇒ (5). First of all, let us show that a symmetric regular linear functional $u(t)$ satisfying (15), is such that $1+u_2(t+1)=1+\gamma_1(t+1)\neq 0$. Let us assume that $u_2(t+1)=-1$. Then, from (15) for $n=2$, we obtain that $u_4(t+1)=1$ which yields

$$\det(H_3(t+1))=\begin{vmatrix} u_0(t+1) & u_1(t+1) & u_2(t+1)\\ u_1(t+1) & u_2(t+1) & u_3(t+1)\\ u_2(t+1) & u_3(t+1) & u_4(t+1)\end{vmatrix}=\begin{vmatrix} 1 & 0 & -1\\ 0 & -1 & 0\\ -1 & 0 & 1\end{vmatrix}=0\,,$$

in contradiction with being $u(t)$ a regular linear functional (*cf.* for instance [7]).

Let $\{R_n(x;t)\}_{n\in\mathbb{N}}$ be the sequence of monic symmetric orthogonal polynomials with respect to the linear functional $u(t)$, *i.e.*

$$R_n(-x;t)=(-1)^nR_n(x;t)\,,\quad n\in\mathbb{N}\,.$$

Since $\{R_n(x;t)\}_{n\in\mathbb{N}}$ is a basis in the space of polynomials of degree n, we have

$$\nabla_t R_n(x;t+1)=\sum_{k=1}^{n}\alpha_k^n(t)\,R_{n-k}(x;t)\,. \tag{22}$$

By convention we shall assume that $\alpha_1^0=0$. For $n=1$, from (22) it is easy to check that $\alpha_1^1(t)=0$ because $R_0(x;t)=1$. Now, if we suppose that we already have tested that

$$\Delta_t R_n(x;t)=\alpha_1^n(t)\,R_{n-1}(x;t)+\alpha_2^n(t)\,R_{n-2}(x;t)\,,$$

by comparison of the coefficients in x we have that $\alpha_1^n(t)=0$, using that $R_n(x;t)$ is symmetric.

We shall prove for $n=1,\ldots$ that $\alpha_k^n=0$ for $k=3,\ldots,n$ and $\alpha_2^n\neq 0$.

From (22) we can write

$$R_n(x;t+1)=R_n(x;t)+\sum_{k=1}^{n}\alpha_k^n(t)\,R_{n-k}(x;t)\,, \tag{23}$$

and by using the orthogonality of $R_n(x;t)$ it holds

$$\big\langle u(t),R_n(x;t+1)\big\rangle=\alpha_n^n(t)\,\big\langle u(t),1\big\rangle\,.$$

Moreover, since $\big\langle u(t+1),x^2R_n(x;t+1)\big\rangle=0$, for $n=3,\ldots$, we have

$$\alpha_n^n(t)\,\big\langle u(t),1\big\rangle\;=\;\alpha_n^n(t)\,\big\langle u(t),R_0(x;t)\big\rangle=\sum_{k=1}^{n}\alpha_k^n(t)\,\big\langle u(t),R_{n-k}(x;t)\big\rangle$$

$$\begin{aligned} &= \left\langle u(t+1), \nabla_t R_n(x;t+1)\right\rangle = -\left\langle \Delta_t(u(t)), R_n(x;t+1)\right\rangle \\ &= \left\langle u(t+1), x^2 R_n(x;t+1)\right\rangle - u_2(t+1)\left\langle u(t), R_n(x;t+1)\right\rangle \\ &= -u_2(t+1)\left\langle u(t), R_n(x;t+1)\right\rangle . \end{aligned}$$

We now obtain $(1+u_2(t+1))\alpha_n^n(t)\left\langle u(t),1\right\rangle = 0$. Assuming that $1+u_2(t+1) \neq 0$ and since $\left\langle u(t),1\right\rangle \neq 0$, we have $\alpha_n^n(t) = 0$.

In the next step we shall prove that $\alpha_{n-1}^n(t) = 0$. From

$$\begin{aligned} \sum_{k=1}^{n-1} \alpha_k^n(t)\left\langle u(t), xR_{n-k}(x;t)\right\rangle &= \alpha_{n-1}^n(t)\left\langle u(t), xR_1(x;t)\right\rangle \\ &= \left\langle u(t), x\nabla_t R_n(x;t+1)\right\rangle = -\left\langle \Delta_t(u(t)), xR_n(x;t+1)\right\rangle \\ &= \left\langle u(t+1), x^3 R_n(x;t+1)\right\rangle - u_2(t+1)\left\langle u(t), xR_n(x;t+1)\right\rangle \\ &= -u_2(t+1)\left\langle u(t), R_n(x;t+1)\right\rangle = -u_2(t+1)\alpha_{n-1}^n(t)\left\langle u(t), xR_1(x;t)\right\rangle , \end{aligned}$$

using that $\left\langle u(t+1), x^3 R_n(x;t+1)\right\rangle = 0$, for $n = 4, \ldots$ and (23), we obtain

$$\big(1+u_2(t+1)\big)\alpha_{n-1}^n(t)\left\langle u(t), xR_1(x;t)\right\rangle = 0 .$$

Since $1+u_2(t+1) \neq 0$ and $\left\langle u(t), xR_1(x;t)\right\rangle \neq 0$ by orthogonality, we conclude that $\alpha_{n-1}^n(t) = 0$.

Repeating this process we obtain that $\alpha_k^n(t) = 0$ for $k = 4, \ldots, n$. Let us prove in the last step that $\alpha_3^n(t) = 0$. From

$$\begin{aligned} \sum_{k=1}^{3} \alpha_k^n(t)\left\langle u(t), x^{n-3}R_{n-k}(x;t)\right\rangle &= \alpha_3^n(t)\left\langle u(t), x^{n-3}R_{n-3}(x;t)\right\rangle \\ &= \left\langle u(t), x^{n-3}\nabla_t R_n(x;t+1)\right\rangle = -\left\langle \Delta_t(u(t)), x^{n-3}R_n(x;t+1)\right\rangle \\ &= \left\langle u(t+1), x^{n-1}R_n(x;t+1)\right\rangle - u_2(t+1)\left\langle u(t), xR_n(x;t+1)\right\rangle \\ &= -u_2(t+1)\left\langle u(t), x^{n-3}R_n(x;t+1)\right\rangle \\ &= -u_2(t+1)\alpha_3^n(t)\left\langle u(t), x^{n-3}R_{n-3}(x;t)\right\rangle , \end{aligned}$$

using that $\left\langle u(t+1), x^{n-1}R_n(x;t+1)\right\rangle = 0$ and (23), we obtain

$$(1+u_2(t+1))\alpha_3^n(t)\left\langle u(t), x^{n-3}R_{n-3}(x;t)\right\rangle = 0 .$$

Since $1+u_2(t+1) \neq 0$ and $\left\langle u(t), x^{n-3}R_{n-3}(x;t)\right\rangle \neq 0$ by orthogonality, we conclude that $\alpha_3^n(t) = 0$.

Therefore, we have obtained that

$$\nabla_t R_n(x;t+1) = \alpha_2^n(t)\, R_{n-2}(x;t)\,.$$

Finally, we will determine α_2^n explicitly:

$$\begin{aligned}\alpha_2^n(t)\,\big\langle u(t), x^{n-2}R_{n-2}(x;t)\big\rangle &= \big\langle u(t), x^{n-2}\nabla_t(R_n(x;t+1))\big\rangle \\ &= -\big\langle \Delta_t(u(t)), x^{n-2}R_n(x;t+1)\big\rangle \\ &= \big\langle u(t+1), x^n R_n(x;t+1)\big\rangle - u_2(t+1)\,\big\langle u(t), x^{n-2}R_n(x;t+1)\big\rangle \\ &= \big\langle u(t+1), x^n R_n(x;t+1)\big\rangle \\ &\qquad - u_2(t+1)\,\big\langle u(t), x^{n-2}\big(R_n(x;t) + \alpha_2^n(t)R_{n-2}(x;t)\big)\big\rangle\,,\end{aligned}$$

using (17). Hence

$$\big(1+u_2(t+1)\big)\alpha_2^n(t)\,\big\langle u(t), x^{n-2}R_{n-2}(x;t)\big\rangle = \big\langle u(t+1), x^n R_n(x;t+1)\big\rangle\,,$$

which gives the value of $\alpha_2^n(t)$ given in (19). Moreover, when $n=2$, we can obtain easily that

$$\alpha_2^2(t) = -\Delta_t\gamma_1(t)\,,$$

taking into account that $R_2(x;t) = x^2 - \gamma_1(t)$ and $\alpha_2^2(t) = \alpha_2^2(t)R_0(x;t) = \Delta_t R_2(x;t)$.

(5) ⇒ (1) If we apply Δ_t to the recurrence relation

$$xR_n(x;t) = R_{n+1}(x;t) + \gamma_n(t)\,R_{n-1}(x;t)\,,\quad n=1,\ldots\,, \tag{24}$$

with $R_{-1}(x;t)=0$ and $R_0(x;t)=1$, we get

$$\begin{aligned}\alpha_2^n xR_{n-2}(x;t) = \Delta_t R_{n+1}(x;t) &+ \Delta_t\gamma_n(t)\,R_{n-1}(x;t) \\ &+ \gamma_n(t+1)\,\Delta_t R_{n-1}(x;t)\,.\end{aligned} \tag{25}$$

If we use again the recurrence relation to expand

$$xR_{n-2}(x;t) = R_{n-1}(x;t) + \gamma_{n-2}(t)\,R_{n-3}(x;t)\,,$$

and $\Delta_t R_{n+1}(x;t) = \alpha_2^{n+1}\,R_{n-1}(x;t)$, by equating in (25) the coefficients in $R_{n-1}(x,t)$ and $R_{n-3}(x;t)$, we get the equations

$$\alpha_2^{n+1}(t)\,\gamma_{n-1}(t) = \alpha_2^n(t)\,\gamma_{n+1}(t+1)\,,\quad \alpha_2^n(t) = \alpha_2^{n+1}(t) + \Delta_t\gamma_n(t)\,,\ n=2,\ldots\,.$$

As a consequence, using (19) we obtain (21) and

$$\alpha_2^n(t) = \frac{1}{1+\gamma_1(t+1)}\frac{\gamma_n(t+1)\gamma_{n-1}(t+1)\cdots\gamma_2(t+1)\gamma_1(t+1)}{\gamma_{n-2}(t)\gamma_{n-3}(t)\cdots\gamma_1(t)}\,.$$

Thus, we have

$$\begin{aligned}\big(1+\gamma_1(t+1)\big)\Delta_t\gamma_n(t) &= \alpha_2^n(t)-\alpha_2^{n+1}(t)\\ &= \frac{\gamma_n(t+1)\gamma_{n-1}(t+1)\cdots\gamma_2(t+1)\gamma_1(t+1)}{\gamma_{n-1}(t)\gamma_{n-2}(t)\cdots\gamma_1(t)}\big(\gamma_{n-1}(t)-\gamma_{n+1}(t+1)\big)\,,\end{aligned}$$

and we get the desired result. ∎

Theorem 2. *Assume that the normalized symmetric functional $u(t)$ verifies*

$$u(t)=\kappa\big(1+x^2\big)^{1-t}v\,,$$

where κ is the normalizing constant and v is a positive definite linear functional. Then, the coefficients $\{\gamma_n(t)\}_{n\in\mathbb{N}}$ of the Jacobi matrix $\Gamma(t)$ associated to the functional $u(t)$ are solution of the Δ-Volterra lattice (14).

Proof: Let

$$f(x,t)=\big(1+x^2\big)^{1-t}\,, \tag{26}$$

and the moments

$$\langle v,x^n\rangle=\int x^n\,d\varrho(x)\,,\quad n=0,1,\ldots\,.$$

Let $u_n(t)$ the moments of the linear functional $u(t)$,

$$u_n(t)=\frac{\int f(x,t)x^n\,d\varrho(x)}{\int f(x,t)\,d\varrho(x)}\,.$$

Since

$$\Delta_t(f(t)/g(t))=\frac{\Delta_t f(t)\,g(t)-f(t)\,\Delta_t g(t)}{g(t)g(t+1)}\,,$$

then

$$\Delta_t u_n(t)=\frac{\int\Delta_t f(x,t)\,x^n\,d\varrho(x)}{\int f(x,t+1)\,d\varrho(x)}-\frac{\big(\int f(x,t)\,x^n\,d\varrho(x)\big)\big(\int\Delta_t f(x,t)\,d\varrho(x)\big)}{\big(\int f(x,t)\,d\varrho(x)\big)\big(\int f(x,t+1)\,d\varrho(x)\big)}\,.$$

By using $\Delta_t f(x,t)=-x^2\,f(x,t+1)$, we obtain

$$\Delta_t u_n(t)=-u_{n+2}(t+1)+u_2(t+1)\,u_n(t)\,,$$

which completes the proof. ∎

Remark 1. Let us consider the difference operator

$$\Delta_{t,h} f(x,t) = \frac{f(x,t+h) - f(x,t)}{h}, \qquad \lim_{h\to 0} \Delta_{t,h} f(x,t) = \frac{\partial}{\partial t} f(x,t)\,.$$

In this case, the function $f_h(x,t)$ to be considered analogue of (26) is

$$f_h(x,t) = \left(1 + hx^2\right)^{1-t/h}.$$

It yields,

$$\lim_{h\to 0} f_h(x,t) = \exp(-x^2 t)\,,$$

which is the evolution (6) associated to the continuous case [25].

Next, we prove a Lax-type theorem [17, Theorem 3, p. 270].

Theorem 3. *Let $\lambda(t)$ be a spectral point of the Jacobi matrix $\Gamma(t)$,* i.e.

$$\Gamma(t)\,\mathcal{P}(\lambda(t)) = \lambda(t)\,\mathcal{P}(\lambda(t)); \tag{27}$$

then, $\Gamma(t)$ satisfies (10) if, and only if, $\Delta_t \lambda(t) = 0$.

Proof: If we apply the Δ_t operator to (27) we obtain

$$\Delta_t \Gamma(t) \mathcal{P}(\lambda(t)) + \Gamma(t+1) \Delta_t \mathcal{P}(\lambda(t)) = \Delta_t \lambda(t) \mathcal{P}(\lambda) + \lambda(t+1) \Delta_t \mathcal{P}(\lambda(t)).$$

Then,

$$\begin{aligned} B(t)\lambda(t)\mathcal{P}(\lambda(t)) - \Gamma(t+1)B(t)\mathcal{P}(\lambda(t)) \\ + \left(\Gamma(t+1) - \lambda(t+1)\,\mathcal{I}\right) \Delta_t \mathcal{P}(\lambda(t)) = \left(\Delta_t \lambda(t)\right) \mathcal{P}(\lambda(t)), \end{aligned}$$

and so,

$$\begin{aligned} \big(\Gamma(t+1) - \lambda(t+1)\big)\big(\Delta_t \mathcal{P}(\lambda(t)) - B(t)\,\mathcal{P}(\lambda(t))\big) \\ = \big(\Delta_t \lambda(t)\, I - (\lambda(t+1) - \lambda(t)) B(t)\big) \mathcal{P}(\lambda(t)), \end{aligned}$$

with $B(t)$ defined by (12), or equivalently,

$$\big(\Gamma(t+1) - \lambda(t+1)\big)\big(\Delta_t \mathcal{P}(\lambda(t)) - B(t)\,\mathcal{P}(\lambda(t))\big) = \Delta_t \lambda(t) (I - B(t))\, \mathcal{P}(\lambda(t)).$$

From this we get, as $1 + \gamma_1(t+1) \neq 0$, that $\Delta_t \lambda(t) = 0$ if and only if

$$\big(\Gamma(t+1) - \lambda(t+1)\big)\big(\Delta_t \mathcal{P}(\lambda(t)) - B(t)\,\mathcal{P}(\lambda(t))\big) = 0,$$

or, what is equivalent, there exists $s \in \mathbb{R}$ such that

$$\Delta_t \mathcal{P}(\lambda(t)) = B(t)\,\mathcal{P}(\lambda(t)) + s\mathcal{P}(\lambda(t+1)),$$

which is equation (18) in vector notation, as $s = \gamma_1(t+1)$. ∎

3. Bäcklund or Miura transformations and sequences of polynomials

Bäcklund or Miura transformations are equations that relate different solutions of the same nonlinear evolution equation [11, 18, 21]. In this section we give a simple connection between Δ-Volterra and Δ-Toda lattices by using background knowledge of the theory of orthogonal polynomials [7].

Lemma 1. *Let $\{\gamma_n(t)\}_{n\in\mathbb{N}}$ be a solution of the Δ-Volterra lattice (14). Then $\{a_n(t)\}_{n\in\mathbb{N}}$ and $\{b_n(t)\}_{n\in\mathbb{N}}$ defined by $a_0(t)=1$ and*

$$a_n(t) = \gamma_{2n}(t)\gamma_{2n-1}(t)\,, \quad b_n(t) = \gamma_{2n+1}(t) + \gamma_{2n}(t) + c\,, \quad n = 1, \ldots\,, \tag{28}$$

are solution of the Δ-Toda lattice (9). Moreover, the sequences $\{\tilde{a}_n(t)\}_{n\in\mathbb{N}}$ and $\{\tilde{b}_n(t)\}_{n\in\mathbb{N}}$ defined by $\tilde{a}_0(t)=1$ and

$$\tilde{a}_n(t) = \gamma_{2n+1}(t)\gamma_{2n}(t)\,, \quad \tilde{b}_n(t) = \gamma_{2n+2}(t) + \gamma_{2n+1}(t) + c\,, \quad n = 1, \ldots\,, \tag{29}$$

are also solution of the Δ-Toda lattice (9), assuming that $\gamma_0(t)=1$.

Proof: If we apply the Δ_t operator to the first equation of (28) we obtain

$$\Delta_t a_n(t) = \Delta_t \gamma_{2n}(t)\,\gamma_{2n-1}(t) + \gamma_{2n}(t+1)\,\Delta_t\gamma_{2n-1}(t)\,.$$

From (14) it yields

$$\begin{aligned}\Delta_t a_n(t) &= \frac{\gamma_{2n}(t+1)\cdots\gamma_1(t+1)}{\big(1+\gamma_1(t+1)\big)\gamma_{2n-1}(t)\cdots\gamma_1(t)}\big(\gamma_{2n-1}(t)-\gamma_{2n+1}(t+1)\big)\gamma_{2n-1}(t)\\ &\quad + \gamma_{2n}(t+1)\frac{\gamma_{2n-1}(t+1)\cdots\gamma_1(t+1)}{\big(1+\gamma_1(t+1)\big)\gamma_{2n-2}(t)\cdots\gamma_1(t)}\big(\gamma_{2n-2}(t)-\gamma_{2n}(t+1)\big)\\ &= \frac{\gamma_{2n}(t+1)\cdots\gamma_1(t+1)\big(\gamma_{2n-1}(t)+\gamma_{2n-2}(t)-\gamma_{2n+1}(t+1)-\gamma_{2n}(t+1)\big)}{\big(1+\gamma_1(t+1)\big)\gamma_{2n-2}(t)\cdots\gamma_1(t)}\end{aligned}$$

where by using (28) we finally obtain

$$\Delta_t a_n(t) = \frac{a_n(t+1)\cdots a_1(t+1)}{\big(1+\gamma_1(t+1)\big)a_{n-1}(t)\cdots a_1(t)}\big(b_{n-1}(t)-b_n(t+1)\big)\,.$$

Moreover, if we apply the Δ_t operator to the second equation of (28) we obtain

$$\Delta_t b_n(t) = \Delta_t\gamma_{2n+1}(t) + \Delta_t\gamma_{2n}(t)\,,$$

where by using (28) the result follows.

The results for $\{\tilde{a}_n(t)\}_{n\in\mathbb{N}}$ and $\{\tilde{b}_n(t)\}_{n\in\mathbb{N}}$ follow in a similar way. ■

Given a family of tridiagonal matrices $\{J(t), t \in \mathbb{R}\}$, as in (3), we consider the sequence of polynomials $\{P_n(x;t)\}_{n\in\mathbb{N}}$ defined in (4). It is well-known [7] that, if $a_n(t) \neq 0$ for $n = 1, 2, \ldots$, then the sequence $\{P_n(x;t)\}_{n\in\mathbb{N}}$ is orthogonal with respect to some quasi-definite moment functional.

Lemma 2. *Let $\{a_n(t)\}_{n\in\mathbb{N}}$ and $\{b_n(t)\}_{n\in\mathbb{N}}$ be solution of the Δ-Toda lattice (9), and $\{P_n(x;t)\}_{n\in\mathbb{N}}$ be the sequence of orthogonal polynomials with Jacobi matrix (3). Let $c \in \mathbb{C}$ such that $P_n(c;t) \neq 0$, for each $n \in \mathbb{N}$ and for all $t \in \mathbb{R}$. Then the sequence $\{\gamma_n(t)\}_{n\in\mathbb{N}}$ defined in (28) is solution of the Δ-Volterra lattice (14), assuming that $\gamma_0(t) = 1$.*

Proof: From [7, Exercise 9.6, page 49] we have that the coefficients $\gamma_n(t)$ have the following representation

$$\gamma_{2n+1}(t) = -\frac{P_{n+1}(c;t)}{P_n(c;t)}, \qquad \gamma_{2n+2}(t) = -a_{n+1}(t)\frac{P_n(c;t)}{P_{n+1}(c;t)}, \quad n = 0, 1, \ldots,$$

for the odd and even cases.

If we apply the Δ_t operator to the first equation, we obtain

$$\Delta_t\gamma_{2n+1}(t) = -\frac{\Delta_t P_{n+1}(c;t)}{P_n(c;t)} + \frac{P_{n+1}(c;t+1)\,\Delta_t P_n(c;t)}{P_n(c;t)P_n(c;t+1)}.$$

In [4] we have proved that a necessary and sufficient condition for $\{a_n(t)\}_{n\in\mathbb{N}}$ and $\{b_n(t)\}_{n\in\mathbb{N}}$ be a solution of a Δ-Toda lattice (9) is that $\{P_n(x;t)\}_{n\in\mathbb{N}}$ satisfy an Appell property

$$\Delta_t P_n(x;t) = \alpha_1^n(t)\,P_{n-1}(x;t), \qquad \alpha_1^n(t) = \frac{1}{1+\gamma_1(t+1)}\prod_{k=1}^{n}\frac{a_k(t+1)}{a_{k-1}(t)},$$

assuming that $1+\gamma_1(t+1) \neq 0$ and $a_0(t) = 1$. Therefore,

$$\begin{aligned}
\Delta_t\gamma_{2n+1}(t) &= -\alpha_1^{n+1}(t) - \frac{\gamma_{2n+1}(t+1)\alpha_1^n(t)P_{n-1}(c;t)}{P_n(c;t)}\\
&= -\alpha_1^{n+1}(t) + \frac{\gamma_{2n+1}(t+1)\gamma_{2n}(t)\alpha_1^n(t)}{a_n(t)} = \alpha_1^{n+1}(t)\Big(-1+\frac{\gamma_{2n+1}(t+1)\gamma_{2n}(t)}{a_{n+1}(t+1)}\Big)\\
&= \frac{\alpha_1^{n+1}(t)}{\gamma_{2n+2}(t+1)}\big(-\gamma_{2n+2}(t+1)+\gamma_{2n}(t)\big),
\end{aligned}$$

which yields the odd part of (14). The even part can be proved in a similar way. ∎

As a consequence, if $\{a_n(t)\}_{n\in\mathbb{N}}$ and $\{b_n(t)\}_{n\in\mathbb{N}}$ are solution of the Δ-Toda lattice defined in (9), then from Lemma 2 we construct a solution of the Δ-Volterra lattice (14) denoted by $\{\gamma_n(t)\}_{n\in\mathbb{N}}$. Now, from Lemma 1 and these coefficients $\{\gamma_n(t)\}_{n\in\mathbb{N}}$ we construct another solution $\{\tilde{a}_n(t)\}_{n\in\mathbb{N}}$ and $\{\tilde{b}_n(t)\}_{n\in\mathbb{N}}$ of the Δ-Toda lattice defined in (9).

Let us denote by $\Gamma_n(t)$ the finite submatrix formed by the first n rows and columns of $\Gamma(t)$. We may summarize these results as follows, which is a Δ_t-analogue of [6, Theorem 1.3], where the full Toda and Volterra hierarchy has been considered.

Theorem 4. *Let us consider the family $\{\Gamma(t), t\in\mathbb{R}\}$, of tridiagonal infinite matrices defined in (11) and let $c\in\mathbb{C}$ be such that $\det(\Gamma_n(t)-c\mathcal{I}_n)\neq 0$, for each $n\in\mathbb{N}$ and for all $t\in\mathbb{R}$. Then there exists a sequence $\{\gamma_n(t)\}_{n\in\mathbb{N}}$, $t\in\mathbb{R}$, solution of (14) and there exists a pair of two sequences $\{a_n(t)\}_{n\in\mathbb{N}}$, $\{b_n(t)\}_{n\in\mathbb{N}}$, and $\{\tilde{a}_n(t)\}_{n\in\mathbb{N}}$, $\{\tilde{b}_n(t)\}_{n\in\mathbb{N}}$, $t\in\mathbb{R}$, solutions of (9) such that (28) and (29) hold.*
Moreover, for each $c\in\mathbb{C}$ in the above conditions, the sequences $\{\gamma_n(t)\}_{n\in\mathbb{N}}$, $\{a_n(t)\}_{n\in\mathbb{N}}$, $\{b_n(t)\}_{n\in\mathbb{N}}$, and $\{\tilde{a}_n(t)\}_{n\in\mathbb{N}}$, $\{\tilde{b}_n(t)\}_{n\in\mathbb{N}}$ are the unique sequences verifying (28) and (29).

Notice that the condition $\det(\Gamma_n(t)-c\mathcal{I}_n)\neq 0$, is equivalent to $P_n(c;t)\neq 0$ for the monic polynomials $P_n(x;t)$ defined by (4) [7, 24, 27].

4. Example: Modified Legendre functional

Let us consider

$$\langle v, p(x)\rangle = \frac{1}{2i}\int_{-i}^{i} p(x)\,dx\,,$$

the Legendre linear functional on $[-i,i]$ normalized to have first moment equal to one, and let us consider the functional

$$u(t) = \frac{2\,\Gamma\left(\frac{5}{2}-t\right)}{\sqrt{\pi}\,\Gamma(2-t)}(1+x^2)^{1-t}\,v\,.$$

Then, the even moments are explicitly given by

$$u_{2n}(t) = \frac{(-1)^n\,\Gamma\left(n+\frac{1}{2}\right)\,\Gamma\left(\frac{5}{2}-t\right)}{\sqrt{\pi}\,\Gamma\left(n-t+\frac{5}{2}\right)}$$

and $u_{2n+1}(t)=0$, $n\in\mathbb{N}$ due to the symmetry of $u(t)$.

Let us consider the sequence $\{\gamma_n(t)\}_{n\in\mathbb{N}}$ defined by

$$\gamma_n(t) = -\frac{n(n-2t+2)}{(2n-2t+1)(2n-2t+3)}\,, \ n=1,\dots\,, \quad \gamma_0(t)=1\,,\ t\neq 1\,,$$

which is solution of the Δ-Volterra equations (14).

The sequence of monic symmetric polynomials $\{R_n(x;t)\}_{n\in\mathbb{N}}$ which satisfy the three term recurrence relation (24) are explicitly given, for $n\in\mathbb{N}$, by

$$\begin{aligned} R_n(x;t) &= \frac{\left(-\frac{i}{2}\right)^n n!\, C_n^{\left(\frac{3}{2}-t\right)}(ix)}{\left(\frac{3}{2}-t\right)_n} \\ &= x^n\, {}_2F_1\Big(\frac{1}{2}\Big((-1)^n - 2\Big[\frac{n}{2}\Big]\Big), -\Big[\frac{n}{2}\Big]; t+\frac{(-1)^n}{2} - 2\Big[\frac{n}{2}\Big] - 1; -\frac{1}{x^2}\Big)\,, \quad t<\frac{3}{2}\,, \end{aligned}$$

where $C_n^{(\lambda)}(x)$ are the Gegenbauer (or ultraspherical) polynomials defined in [16, (9.8.19)] and $\big[x\big]$ gives the integer part of x.

These polynomials are orthogonal with respect to the normalized linear functional

$$u(t) = -\frac{i\left(x^2+1\right)^{1-t}\Gamma\left(\frac{5}{2}-t\right)}{\sqrt{\pi}\,\Gamma(2-t)} = \tilde{\kappa}\left(1+x^2\right)^{1-t}\,,$$

i.e., for all $n,m\in\mathbb{N}$,

$$\int_{-i}^{i} \tilde{\kappa}\left(1+x^2\right)^{1-t} R_n(x;t)R_m(x;t)\,dx = \frac{\left(-\frac{1}{4}\right)^n\Gamma(n+1)\left(3-2t\right)_n}{\left(\frac{3}{2}-t\right)_n\left(\frac{5}{2}-t\right)_n}\,\delta_{nm},\ t<\frac{3}{2}.$$

The sequence of orthogonal polynomials $\{R_n(x;t)\}_{n\in\mathbb{N}}$ coincides with the monic orthogonal polynomials sequence defined in [20, (17)] for $r=4-2t$, $s=0$, $p=1$ and $q=1$ or with the polynomials defined in [20, (86)] with $a=0$ and $b=t-1$. It is shown that they are also finitely orthogonal with respect to the second kind of beta weight function $x^{-2a}(1+x^2)^{-b}$ on $(-\infty,\infty)$.

Observe that the difference equation (18) can be written as

$$\Delta_t R_n(x;t) = \frac{(n-1)n}{(-2n+2t-1)(-2n+2t+1)}\,R_{n-2}(x;t)\,, \quad n=2,\dots\,.$$

Using the Miura transformations (28) we obtain explicitly the sequences

$$a_n(t) = \frac{4n(2n-1)(2n-2t+1)(n-t+1)}{(4n-2t-1)(4n-2t+1)^2(4n-2t+3)}\,, \quad n = 1, \dots\,,$$
$$b_n(t) = \frac{-4n(2n-2t+3)+2t-1}{(4n-2t+1)(4n-2t+5)}\,, \quad n \in \mathbb{N}\,, \quad t \neq 1\,,$$

which are solutions of the Δ-Toda lattice defined in (9).

The sequence of monic polynomials $\{P_n(x;t)\}_{n\in\mathbb{N}}$ defined by the three term recurrence relation (4) can be identified in terms of monic shifted Jacobi polynomials

$$G_n^{(\alpha,\beta)}(x) = \frac{(-1)^n(\beta+1)_n}{(\alpha+\beta+n+1)_n}\,{}_2F_1(-n,\alpha+\beta+n+1;\beta+1;x), \alpha,\beta > -1, \tag{30}$$

as

$$P_n(x;t) = (-1)^n\, G_n^{(1-t,-1/2)}(-x)\,, \quad n \in \mathbb{N}\,,$$

and moreover $y(x) = P_n(x;t)$ obey the following second order differential equation

$$x(1+x)y''(x) + \frac{1}{2}((5-2t)x+1)y'(x) - \frac{1}{2}n(2n-2t+3)y(x) = 0\,.$$

Thus, the following orthogonality relation holds,

$$\int_{-1}^{0} \frac{(1+x)^{1-t}}{\sqrt{-x}} P_n(x;t)P_m(x;t)\,dx$$
$$= \frac{\sqrt{\pi}\,4^{-2n+t-1}\Gamma(2n+1)\Gamma\big(\frac{5}{2}-t\big)\Gamma(2n-2t+3)}{\Gamma(2-t)\Gamma\big(2n-t+\frac{3}{2}\big)\Gamma\big(2n-t+\frac{5}{2}\big)}\,\delta_{n,m}\,,$$

$n, m \in \mathbb{N}$, for $t < \frac{3}{2}$. It is easy to verify that in this case

$$R_{2n}(x;t) = P_n(x^2;t), \quad R_{2n+1}(x;t) = x\,Q_n(x^2;t)\,, \quad n \in \mathbb{N}\,,$$

where the polynomials $Q_n(x;t)$ are obtained from the polynomials $P_n(x;t)$ by the Christoffel transformation, [7]:

$$Q_n(x;t) = \frac{P_{n+1}(x;t) - \psi_n(t)\,P_n(x;t)}{x}\,, \tag{31}$$

where

$$\psi_n(t) = \frac{P_{n+1}(0;t)}{P_n(0;t)} = -\frac{(2n+1)(2t-2n-3)}{(2t-4n-3)(2t-4n-5)}\,, \quad n \in \mathbb{N}\,.$$

In a similar way, using the transformations (29) we obtain the new recurrence coefficients

$$\tilde{a}_n(t) = \frac{4n(2n+1)(2n-2t+3)(n-t+1)}{(4n-2t+1)(4n-2t+3)^2(4n-2t+5)}\,, \quad n = 1, \dots\,,$$

$$\tilde{b}_n(t) = \frac{-4n(2n-2t+5)+6t-9}{(4n-2t+3)(4n-2t+7)}\,, \quad n \in \mathbb{N}\,, \quad t \neq 1\,,$$

which also satisfy the chain of difference equations (9) for the Δ-Toda lattice. The monic polynomials $\{\tilde{P}_n(x;t)\}_{n\in\mathbb{N}}$ generated by $\tilde{P}_{-1}(x;t) = 0, \tilde{P}_0(x;t) = 1$,

$$\tilde{P}_n(x;t) = (x - \tilde{b}_{n-1}(t))\tilde{P}_{n-1}(x;t) - \tilde{a}_{n-1}(t)\tilde{P}_{n-2}(x;t),\ n = 1, \dots,$$

can be identified in terms of monic shifted Jacobi polynomials (30) as

$$\tilde{P}_n(x;t) = (-1)^n\, G_n^{(1-t,1/2)}(-x)\,, \quad n \in \mathbb{N}\,.$$

Thus, $y(x) = \tilde{P}_n(x;t)$ is solution of the equation of hypergeometric type

$$x(1+x)\,y''(x) + \frac{1}{2}((7-2t)x+3)\,y'(x) - \frac{1}{2}n(2n-2t+5)\,y(x) = 0\,,$$

and their polynomial solutions have the orthogonality property

$$\int_{-1}^{0}(1+x)^{1-t}\sqrt{-x}\,\tilde{P}_n(x;t)\,\tilde{P}_m(x;t)\,dx$$
$$= \frac{\sqrt{\pi}\,2^{-4n+2t-3}\Gamma(2n+2)\Gamma\big(\frac{7}{2}-t\big)\Gamma(2n-2t+4)}{\Gamma(2-t)\Gamma\big(2n-t+\frac{5}{2}\big)\Gamma\big(2n-t+\frac{7}{2}\big)}\,\delta_{n,m}\,,$$

$n, m \in \mathbb{N}$, for $t < \frac{3}{2}$. It is easy to verify that in this case the new solution $\tilde{P}_n(x;t)$ coincides with the monic kernel polynomials corresponding to $\{P_n(x;t)\}_{n\in\mathbb{N}}$ defined in (31), *i.e.*

$$\tilde{P}_n(x;t) = Q_n(x;t), \qquad n \in \mathbb{N}\,.$$

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I. Area
Departamento de Matemática Aplicada II, E.E. Telecomunicación, Universidade de Vigo, 36310-Vigo, Spain.

E-mail address: area@uvigo.es

A. Branquinho
CMUC, Department of Mathematics, University of Coimbra, Apartado 3008, EC Santa Cruz, 3001-501 COIMBRA, Portugal.

E-mail address: ajplb@mat.uc.pt

A. Foulquié Moreno
CIDMA, Department of Mathematics, University of Aveiro, Campus de Santiago, 3810–193 Aveiro, Portugal.

E-mail address: foulquie@ua.pt

E. Godoy
Departamento de Matemática Aplicada II, E.E. Industrial, Universidade de Vigo, Campus Lagoas-Marcosende, 36310 Vigo, Spain.

E-mail address: egodoy@dma.uvigo.es