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INTERSECTION OF A DOUBLE CONE AND A LINE IN THE SPLIT-QUATERNIONS CONTEXT

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Abstract: This is a work on an application of split-quaternions to Analytic Geometry. Concretely, the intersection of a double cone and a line is studied in the real split-quaternions setting and it can be the empty set, a point, two points or a line. The distinction of these cases, and the determination of the intersection, has use in computer graphics, motion planning and collision detection. Three powerful concepts are most useful for this purpose: the norm of a real split-quaternion and the Lorentz, cross and inner, products of vectors in the $(2+1)$-Minkowski space.

Keywords: double cone, line, intersection, split-quaternion, event, direction.
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1. Introduction

Among the problems that can be approached in Spatial Analytic Geometry, some concern intersections of geometric 3D objects. For instance, the intersection of a cone and a line is one of them. Beyond its intrinsic mathematical interest, the latter problem is also relevant in other areas such as computer graphics, motion planning and collision detection.

There is a series of books, called *Graphics Gems*, that provides the graphics community with tools to face real programming issues. In [3], the intersection of a line and a cylinder was treated. An extension of this work appeared in [1], where the intersection of a line and a cone was addressed.

In the present text, we study the intersection of a double cone and a line in the real split-quaternions context. Through the norm of the split-quaternion that gives the line direction, we characterize the conditions for the intersection to occur. This can be the empty set, a point, two points or a line. Moreover, when it happens, we present an explicit formula for computing the intersection.

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In section 2 we present the real split-quaternion algebra and recall its identification with a real matrix algebra. In section 3 we characterize the intersection of a double cone with a line, giving explicit expressions for the intersection when it occurs.

2. The real split-quaternions

In this section we collect some definitions, notations and results that are needed in the next section.

Let F be an arbitrary field and U be a finite-dimensional vector space over F.

The vector space U over F is a *semi-Euclidean* (or *pseudo-Euclidean*) *space* if $F = \mathbb{R}$ and U is equipped with a non-degenerate indefinite bilinear form. As in the positive definite case, the mentioned form is called the *inner product* and we denote it by $\langle \cdot, \cdot \rangle$.

If U is a semi-Euclidean space, then an element $x \in U$ is said to be *spacelike*, *lightlike* or *timelike* if $\langle x, x \rangle > 0, \langle x, x \rangle = 0$ or $\langle x, x \rangle < 0$, respectively. The *light cone* is the set of all lightlike-elements. As in the definite case, two elements $x, y \in U$ are said to be *orthogonal* if $\langle x, y \rangle = 0$. So, the light cone consists of all elements that are orthogonal to themselves.

A vector space homomorphism $\varphi : U \to U$ is called an *involution* of U if, for all $u, v \in U$, $\varphi(\varphi(u)) = u$ and $\varphi(uv) = \varphi(v)\varphi(u)$.

The vector space U over F is an *algebra over* F if U is equipped with a bilinear map $s : U^2 \to U$, usually called *multiplication*. Given an algebra U, with multiplication denoted by juxtaposition, we now recall a few more concepts related to composition algebras assuming, from now on, that $\mathrm{ch}(F) \neq 2$.

U is a *composition algebra* over F if it is endowed with a nondegenerate quadratic form (the *norm*) $n : U \to F$ (that is, the associated symmetric bilinear form $(x, y) = \frac{1}{2}(n(x + y) - n(x) - n(y))$ is nondegenerate) which is multiplicative, *i.e.*, for any $x, y \in U$,

$$n(xy) \;=\; n(x)n(y).$$

Let U be a composition algebra over F. An element $z \in U$ is *isotropic* if $z \neq 0$ and $n(z) = 0$. A unital composition algebra U, that is, a composition algebra with identity e is a *Hurwitz algebra*. Furthermore, the involution $x \mapsto \overline{x} = (x, e)e - x$ of U, called the *standard conjugation*, satisfies $\overline{x}x = x\overline{x} = n(x)e$ and $x + \overline{x} = tr(x)e$, being $tr(x) \in F$ the *trace* of x. An element

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two latter cases, the intersection points of the line with the double cone will be given by

$$s \;=\; a + \frac{\langle a,b\rangle \pm \sqrt{-n(a\wedge b)}}{n(b)}b.$$

If $n(b)=0$, then from equation (2), we obtain

$$-2\lambda\,\langle a,b\rangle + n(a) \;=\; 0. \tag{3}$$

If the event a belongs to the separation plane of b, then $\langle a,b\rangle = 0$. In this case, the points of the line will belong to the double cone if, and only if, $n(a)=0$. In this case, all points of the line will belong to the double cone. If this is not the case, there will be no intersection.

If the event does not belong to the separation plane of b, then $\langle a,b\rangle \neq 0$ and the solution of equation (3) will be

$$\lambda \;=\; \frac{n(a)}{2\,\langle a,b\rangle}.$$

In this case, the intersection point of the line with the double cone will be given by

$$s \;=\; a + \frac{n(a)}{2\,\langle a,b\rangle}b. \quad \blacksquare$$

Observe that if the event a or the direction b are timelike, then, by Proposition 3.5, $n(a\wedge b)<0$ and the line will always intersect at least once the double cone.

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$x\in U$ is *invertible* if and only if $n(x)\neq 0$. Moreover, $x^{-1}=\dfrac{\overline{x}}{n(x)}$ if x is invertible.

By the generalized Hurwitz theorem in [4], a 4-dimensional Hurwitz algebra over F is a (*generalized*) *quaternion algebra*, that is, an algebra over F with two generators i and j satisfying the relations $i^2=a$, $j^2=b$ and $ij=-ji$, with $a,b\in F\backslash\{0\}$. This algebra can be denoted as in [6], using the Hilbert symbol, by $\left(\frac{a,b}{F}\right)$. Setting $ij=k$, we have that $\left(\frac{a,b}{F}\right)$ is 4-dimensional over F, with basis $\{1,i,j,k\}$. Moreover, $k^2=-ab$ and any two of the elements in $\{i,j,k\}$ anticommute.

Taking $F=\mathbb{R}$ and $a=-1$, $b=1$, we obtain the real split quaternion algebra $\left(\frac{-1,1}{\mathbb{R}}\right)$. In what follows, throughout the work, we denote this associative algebra and its multiplication by $\widehat{\mathbb{H}}$ and juxtaposition, respectively, and its identity 1 will be omitted most times. Notice that, for instance, $i+j$ is an isotropic element of $\widehat{\mathbb{H}}$.

	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	1	$-i$
k	k	j	i	1

TABLE 1. Multiplication table of $\widehat{\mathbb{H}}$.

Let $p=p_0+p_1i+p_2j+p_3k\in\widehat{\mathbb{H}}$. If $p_0=0$ then p is called *pure split-quaternion*. The *scalar part* and the *vector part* of p are p_0 and $V_p=p_1i+p_2j+p_3k$, respectively. The conjugate of p is $\overline{p}=p_0-V_p$ and the norm of p is

$$n(p) \;=\; p_0^2+p_1^2-p_2^2-p_3^2.$$

In this work, whenever convenient, the subspace $\widehat{\mathbb{H}}_0$, of the pure split-quaternions, of $\widehat{\mathbb{H}}$ is identified with the $(2+1)$-Minkowski space $\mathbb{R}^{2+1}$. This is the inner product space consisting of the real vector space $\mathbb{R}^3$ equipped with the Lorentz inner product

$$\langle x,y\rangle_L \;=\; -x_1y_1+x_2y_2+x_3y_3.$$

Notice that $\langle x, x\rangle_L = -n(x)$. The vector space may be regarded as a semi-normed vector space, provided that the (semi-) norm is given by

$$\|x\| = \sqrt{|\langle x, x\rangle_L|}.$$

The Lorentz cross product of $x, y \in \mathbb{R}^3$ is defined as follows, [2]:

$$x \wedge_L y = \begin{vmatrix} -i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

In the cited reference, it is proved that the identity

$$\langle x \wedge_L y, z \wedge_L w\rangle_L = -\begin{vmatrix} \langle x, z\rangle_L & \langle x, w\rangle_L \\ \langle y, z\rangle_L & \langle y, w\rangle_L \end{vmatrix}$$

holds in $\mathbb{R}^3$. In particular, taking $z = x$ and $w = y$ leads to the identity

$$n(x \wedge_L y) = n(x)n(y) - \langle x, y\rangle_L^2. \tag{1}$$

The multiplication of two split-quaternions $p = p_0 + V_p$ and $q = q_0 + V_q$ can be related to the Lorentz inner product and to the Lorentz cross product as follows: $pq = p_0q_0 + \langle V_p, V_q\rangle_L + p_0V_q + q_0V_p + V_p \wedge_L V_q$. In particular, if p and q are pure then

$$pq = \langle V_p, V_q\rangle_L + V_p \wedge_L V_q.$$

Lastly but importantly, for computational purposes, the algebra $\widehat{\mathbb{H}}$ can be identified with the real algebra $\mathbb{M}_{2\times2}(\mathbb{R})$, this one equipped with the usual matrix multiplication. The identification is due to the known isomorphism $\psi : \widehat{\mathbb{H}} \to \mathbb{M}_{2\times2}(\mathbb{R})$ defined by, [6]:

$$a + bi + cj + dk \mapsto \begin{bmatrix} a+d & b+c \\ c-b & a-d \end{bmatrix}.$$

3. Intersection of a double cone and a line

In this section, we will take a line and a double cone and determine the conditions for their intersection to occur. We will consider a double cone to be a geometric figure made up of two right circular infinite cones placed apex to apex such that both share the same axis of symmetry. Of course, this double cone could be in any place of the three dimensional space, but for the sake of convenience, we will consider that the cone apexes are at the origin of the coordinates and that the axis of symmetry is the vertical axis.

(i) If a doesn't belong to the separation plane of b, then the line $\mathfrak{L}$ will intersect the double cone once at

$$s = a + \frac{n(a)}{2 < a, b >} b.$$

(ii) If a belongs to the separation plane of b

(α) and $n(a) = 0$, then the line $\mathfrak{L}$ will intersect the double cone an infinite number of times, i.e., all the points of the line will belong to the double cone;

(β) and $n(a) \neq 0$, then the line $\mathfrak{L}$ will not intersect the double cone.

Proof: Consider the line that passes through an event a and has direction $b \neq 0$. The generic point of this line can be represented by $X = a + \lambda b$, with $\lambda \in \mathbb{R}$. The intersection of this line with the double cone will consist of the points of this line that belong to the double cone. But all the points of the double cone are lightlike. Hence, if s is such a point, then $\langle s, s\rangle = 0$. But

$$\begin{aligned} \langle s, s\rangle &= \langle a + \lambda b, a + \lambda b\rangle \\ &= \langle a, a\rangle + 2\lambda \langle a, b\rangle + \lambda^2 \langle b, b\rangle . \end{aligned}$$

Since $\langle a, a\rangle = -n(a)$ and $\langle b, b\rangle = -n(b)$, we obtain

$$\lambda^2 n(b) - 2\lambda \langle a, b\rangle + n(a) = 0. \tag{2}$$

If $n(b) \neq 0$, then

$$\begin{aligned} \lambda &= \frac{2\langle a, b\rangle \pm \sqrt{4\langle a, b\rangle^2 - 4n(a)n(b)}}{2n(b)} \\ &= \frac{\langle a, b\rangle \pm \sqrt{\langle a, b\rangle^2 - n(a)n(b)}}{n(b)}. \end{aligned}$$

But $\langle a, b\rangle^2 - n(a)n(b) = -n(a \wedge b)$. Hence,

$$\lambda = \frac{\langle a, b\rangle \pm \sqrt{-n(a \wedge b)}}{n(b)},$$

from where we conclude that the line will intersect the double cone 0, 1 or 2 times whenever $n(a \wedge b)$ is positive, zero and negative, respectively. In the

Proposition 3.5. *Let u, v be two events such that at least one of them is timelike. Then $n(u \wedge v) < 0$.*

Proof: Let us suppose, without loose of generality, that u is timelike. Then, $n(u) > 0$. We can decompose v such that $v = \alpha u + \beta u_\perp$, where $u_\perp$ is a convenient vector belonging to the separation plane of u. Hence,

$$\begin{aligned} n(v) &= -\langle v, v\rangle \\ &= -\langle \alpha u + \beta u_\perp, \alpha u + \beta u_\perp\rangle \\ &= -\alpha^2\langle u, u\rangle - \beta^2\langle u_\perp, u_\perp\rangle \\ &= \alpha^2 n(u) + \beta^2 n(u_\perp), \end{aligned}$$

and

$$\begin{aligned} \langle u, v\rangle &= \langle u, \alpha u + \beta u_\perp\rangle \\ &= \alpha\langle u, u\rangle \\ &= -\alpha n(u). \end{aligned}$$

From (1), we obtain

$$\begin{aligned} n(u \wedge v) &= n(u)n(v) - \langle u, v\rangle^2 \\ &= n(u)\left(\alpha^2 n(u) + \beta^2 n(u_\perp)\right) - \left(-\alpha n(u)\right)^2 \\ &= \alpha^2 n^2(u) + \beta^2 n(u)n(u_\perp) - \alpha^2 n^2(u) \\ &= \beta^2 n(u)n(u_\perp). \end{aligned}$$

But by hypothesis, $n(u) > 0$. From Lemma 3.3, $n(u_\perp) < 0$ and so we conclude that $n(u \wedge v) < 0$. ∎

In the following theorem, we characterize the conditions for a line to intersect a double cone and give an explicit formula for the intersections.

Theorem 3.6. *Let $\mathfrak{L}$ be a line passing at an event a and with direction $b \neq 0$.*

(1) If $n(b) \neq 0$, then the line $\mathfrak{L}$ will intersect the double cone at 0, 1 or 2 points whenever $n(a \wedge b)$ is positive, zero or negative, respectively. The intersecting points are given by

$$s = a + \frac{< a, b > \pm\sqrt{-n(a \wedge b)}}{n(b)} b.$$

(2) If $n(b) = 0$, then the line $\mathfrak{L}$ will intersect the double cone at 0, 1 or infinite points.

If this is not the case, we can always translate and/or rotate the cone such that these conditions are met.

In order to study the intersection, we will use the pure split-quaternions and find out that they form a convenient framework.

Consider the $(2+1)$-Minkowski space with the two horizontal axis chosen to be spatial dimensions while the vertical axis is time. The split-quaternions i, j, k are the unit vectors in the t, x, y axis, respectively (Figure 1). For the sake of simplicity, in what follows, we write $\langle\cdot,\cdot\rangle$ and $\wedge$ instead of $\langle\cdot,\cdot\rangle_L$ and $\wedge_L$, respectively.

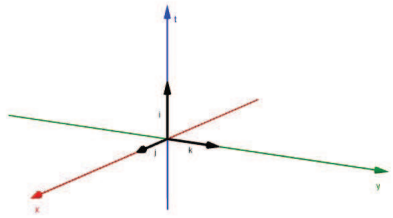

FIGURE 1. The $(2+1)$-Minkowski space with the unit vectors i, j, k represented.

Each point in the Minkowski space is usually called an *event*. In what follows, through a vectorization of the affine Minkowski space, the end-point of a position vector (with respect to the origin of the coordinates) is identified with that vector.

Definition 3.1. *A* **line** *is the set of events (pure split-quaternions) $\{a + \lambda b \ : \ a, b \in \widehat{\mathbb{H}}_0, b \neq 0, \lambda \in \mathbb{R}\}$, where b gives the line direction and a is an event in the Minkowski space.*

Definition 3.2. *The* **double cone** *is the set of events (pure split-quaternions) $\{ti + xj + yk \ : \ x^2 + y^2 = t^2, \ x, y, t \in \mathbb{R}\}$.*

Observe that any double cone that we consider can fit this definition after an appropriate scaling. For this reason, we will consider only double cones that satisfy this definition. Thus, we will be considering always this double cone and an arbitrary line.

The upper cone is known as the *future light cone* and the lower one as the *past light cone*. If an event is inside the future light cone, we will call it a *future event* and if it is inside the past light cone, we will call it a *past event* (Figure 2). Thus if a is a future event, then $tr(a) > 0$ and $n(a) > 0$. Likewise, if it is a past event, then $tr(a) < 0$ but still $n(a) > 0$. In any case, if a line passes through a future or past event, then it must intersect the double cone.

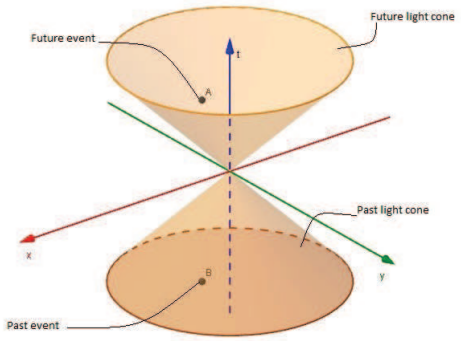

FIGURE 2. A light cone diagram.

Given a vector b, the set of all events orthogonal to b will form a plane known as a *separation* plane. Given an event a, if b is either timelike or lightlike or spacelike, then the inner product of a with b will be negative, zero or positive when a is above, over or below the separation plane of b, respectively. Else, if b is spacelike, then the inner product of a with b will be negative if a and b are on the same side with respect to the separation plane of b, zero if a is over the separation plane of b, and positive otherwise.

If a line is timelike, i.e, with a timelike vector direction, its separation plane contains no lightlike event, [5]. In other words, a timelike vector will never be orthogonal to a lightlike one.

Lemma 3.3. *Let $u \in \widehat{\mathbb{H}}_0$ be timelike. If $u_\perp \neq 0$ is orthogonal to u, then $u_\perp$ is spacelike.*

Proof: Let $u \in \widehat{\mathbb{H}}_0$ be timelike and $u_\perp$ belong to the separation plane of u. Then $n(u) > 0$ and $\langle u, u_\perp \rangle = 0$.

For each $u_\perp$ considered, there exists an α such that $v = u_\perp + \alpha u$ is lightlike, i.e, $n(v) = 0$.

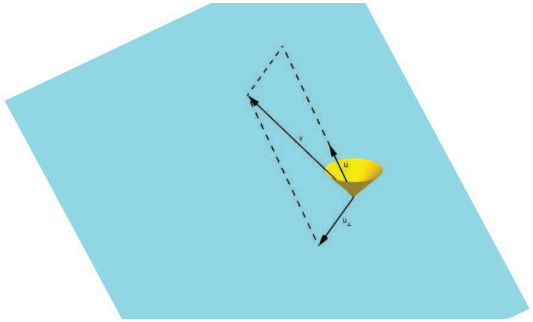

FIGURE 3. Representation of u, $u_\perp$ and the lightlike vector v.

Hence,

$$\begin{aligned} n(v) &= -\langle v, v \rangle \\ &= -\langle u_\perp + \alpha u, u_\perp + \alpha u \rangle \\ &= -\langle u_\perp, u_\perp \rangle - 2\alpha \langle u_\perp, u \rangle - \alpha^2 \langle u, u \rangle \\ &= n(u_\perp) + \alpha^2 n(u). \end{aligned}$$

But $n(v) = 0$. Thus, $n(u_\perp) + \alpha^2 n(u) = 0$ which implies that $n(u_\perp) = -\alpha^2 n(u)$. Since $n(u) > 0$, we conclude that $n(u_\perp) < 0$ and $u_\perp$ is spacelike. ■

Lemma 3.4. *Let $u \in \widehat{\mathbb{H}}_0 \backslash \{0\}$ be lightlike. If $u_\perp \neq 0$ is orthogonal to u, then $u_\perp$ will never be timelike, i.e., $n(u_\perp) \leq 0$.*

Proof: Let us demonstrate by contradiction. Suppose that given $u \in \widehat{\mathbb{H}}_0$ lightlike there exists a timelike $u_\perp$ orthogonal to u. If this is the case, then u is orthogonal to $u_\perp$ and, by Lemma 3.3, u must be spacelike, which is a contradiction since u is by hypothesis lightlike. This completes the proof. ■

Observe that if u is spacelike, then an orthogonal vector $u_\perp$ may be spacelike, lightlike or timelight. For example, given the spacelike vector $u = (1, 1, 1)$, any orthogonal vector $u_\perp = (x, y, t)$ must satisfy the equation $t = x + y$. The vectors $v_1 = (1, 1, 2)$, $v_2 = (1, -1, 0)$ and $v_3 = (1, 0, 1)$ satisfy the referred equation but $n(v_1) > 0$, $n(v_2) < 0$ and $n(v_3) = 0$, which implies that v_1, v_2 and v_3 is timelike, spacelike and lightlike, respectively.