Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 16–16

LOCALIC MAPS CONSTRUCTED FROM OPEN AND CLOSED PARTS

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Dedicated to Bernhard Banaschewski on the occasion of his 90th birthday

Abstract: Assembling a localic map $f\colon L \to M$ from localic maps $f_i\colon S_i \to M$, $i \in J$, defined on closed resp. open sublocales (J finite in the closed case) follows the same rules as in the classical case. The corresponding classical facts immediately follow from the behavior of preimages but for obvious reasons such a proof cannot be imitated in the point-free context. Instead, we present simple proofs based on categorical reasoning. There are some related aspects of localic preimages that are of interest, though. They are investigated in the second half of the paper.

Keywords: frame, locale, sublocale, sublocale lattice, open sublocale, closed sublocale, localic map, preimage, Boolean frame, linear frame.

Math. Subject Classification (2010): 06D22, 54C05.

Introduction

In classical topology one has the useful two facts that

> *if $A_1, \ldots, A_n$ are closed subspaces (resp. if A_i, $i \in J$, are open subspaces, J arbitrary) of a space X such that $\bigcup_i A_i = X$ and if $f_i\colon A_i \to Y$ are continuous maps such that for all i, j,*
> $$f_i|(A_i \cap A_j) = f_j|(A_i \cap A_j)$$
> *then the map $f\colon X \to Y$ defined by $f(x) = f_i(x)$ for $x \in A_i$ is continuous.*

The proof is extremely simple: $f^{-1}[B] = \bigcup_i f_i^{-1}[B]$ and hence if B is closed (resp. open) in Y then $f^{-1}[B]$ is closed (resp. open) in X. Since continuous

Received April 8, 2016.
Support from the projects P202/12/G061 (Grant Agency of the Czech Republic) and MTM2015-63608-P (Ministry of Economy and Competitiveness of Spain) and from the Centre for Mathematics of the University of Coimbra (UID/MAT/00324/2013 funded by the Portuguese Government through FCT/MCTES and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020) is gratefully acknowledged.

maps are characterized among the general ones by sending the open (resp. closed) subsets to open (resp. closed) ones by preimages, this is all we need.

This reasoning cannot be imitated in the point-free setting, but the statement has an exact point-free counterpart nevertheless. Indeed, if localic maps $f_i\colon S_i \to M$ are defined on a system $(S_i)_{i\in J}$ of open resp. closed sublocales (J is finite in the closed case), if they agree on the intersections $S_i \cap S_j$, and if $\bigvee_{i\in J} S_i = L$ then there is precisely one localic map $f\colon L \to M$ restricting to f_i on S_i. Proofs are presented in Sections 2 and 3.

The classical and point-free facts have a common categorical background; namely, they can be viewed as pushing out. But while in the classical case we have the simple fact that can be categorically interpreted, in the point-free modification we have the categorical facts first, afterwards translated into the desired statements (a genuine application of categorical reasoning, hopefully pleasing the category minded reader). In particular in the statement on the closed sublocales we have a fairly simple categorical proof but no reasonably simple direct one, not to speak of something resembling the classical pointy one (the proof in the open case is slightly more direct, but even there the categorical view is essential).

We work with sublocales of a locale (frame) L as with subobjects naturally carried by (some of the) subsets, that is, locales that are subsets S of L embedded by inclusion maps $j\colon S \subseteq L$ that are localic (thus, in particular, in the statement above we have the f_i actual restrictions $f|S_i$). Therefore we can speak of preimages of sublocales (in particular, of the closed and open ones) under general maps $f\colon L \to M$. If f is a localic one we have closed preimages of closed sublocales, and after a certain modification (see 1.4 below) also open preimages of open sublocales. As it is to be expected, this does not characterize the localic maps among the general $f\colon L \to M$ but such information on f is of interest. The associated questions are discussed in Section 4 which we then conclude comparing the set-theoretical preimage with the localic one (the modification mentioned above) in some cases.

1. Preliminaries

1.1. The category of frames. Recall that a *frame* is a complete lattice L satisfying the distributivity rule

$$(\bigvee A) \wedge b = \bigvee\{a \wedge b \mid a \in A\} \tag{frm}$$

for all $A \subseteq L$ and $b \in L$, and that a *frame homomorphism* $h\colon L \to M$ preserves all joins and all finite meets. The resulting category is denoted by **Frm**.

A *coframe* satisfies (frm) with the roles of joins and meets reversed.

1.1.1. The equality (frm) states, in other words, that for every $b \in L$ the mapping $- \wedge b = (x \mapsto x \wedge b) : L \to L$ preserves all joins (suprema). Hence every $- \wedge b$ has a right Galois adjoint resulting in a *Heyting operation* $\to$ with

$$a \wedge b \leq c \quad \text{iff} \quad a \leq b \to c.$$

Thus, each frame is a Heyting algebra (note that, however, the frame homomorphisms do not coincide with the Heyting ones so that **Frm** differs from the category of complete Heyting algebras). The operation $\to$ and some of its basic properties (e.g. $a \to a = 1$, $a \to b = 1$ iff $a \leq b$, $1 \to a = a$, and $a \to (b \to c) = (a \wedge b) \to c$) will be often used in the sequel.

1.2. The concrete category Loc. The functor $\Omega\colon \mathbf{Top} \to \mathbf{Frm}$ from the category of topological spaces into that of frames ($\Omega(f)$ sending an open set $U \subseteq Y$ to $f^{-1}[U]$ for a continuous map $f\colon X \to Y$ in **Top**) is a full embedding on an important and substantial part of **Top**, the subcategory of sober spaces. This justifies to regard frames as a natural generalization of spaces. Since Ω is contravariant, one introduces the *category of locales* **Loc** as the dual of the category of frames. Often one just considers the formal $\mathbf{Frm}^{\mathrm{op}}$ but it is of advantage to represent it as a concrete category with specific maps as morphisms. For this purpose one defines a *localic map* $f\colon L \to M$ as the right Galois adjoint of a frame homomorphism $h = f^*\colon M \to L$. This can be done since frame homomorphisms preserve suprema; but of course not every mapping preserving infima is a localic one. Here is a characterization (see [6] or [5]):

1.2.1. *Let $f\colon L \to M$ have a left adjoint $f^*\colon M \to L$. Then it is a localic map iff*
(a) $f[L \smallsetminus \{1\}] \subseteq M \smallsetminus \{1\}$, *and*
(b) $f(f^*(a) \to b) = a \to f(b)$.

1.3. The coframe of sublocales. A *sublocale* of a frame L is a subset $S \subseteq L$ such that

(1) $M \subseteq S$ implies $\bigwedge M \in S$, and
(2) if $a \in L$ and $s \in S$ then $a \to s \in S$.

If we require only (1) we speak of a *meet-subset*.

The set of all sublocales ordered by inclusion, denoted by

$$\mathcal{S}(L),$$

is a co-frame, with the lattice operations

$$\bigwedge_{i\in J} S_i = \bigcap_{i\in J} S_i \quad \text{and} \quad \bigvee_{i\in J} S_i = \{\bigwedge A \mid A \subseteq \bigcup_{i\in J} S_i\}.$$

The top of $\mathcal{S}(L)$ is L and the bottom is the set $\mathsf{O} = \{1\}$ (the *empty sublocale*).

We have the closed resp. open sublocales

$$\mathfrak{c}(a) = {\uparrow}a \quad \text{resp.} \quad \mathfrak{o}(a) = \{x \mid a \to x = x\} = \{a \to x \mid x \in L\}$$

modelling closed resp. open subspaces (and corresponding precisely to the *closed* resp. *open parts* in [1]). They are complements of each other, and the $\mathfrak{o}(a)$ are in a natural one-one correspondence with the elements of L, preserving joins and finite meets. We have (see e.g. [6]):

- $\mathfrak{o}(0) = \mathsf{O}, \mathfrak{o}(1) = L, \mathfrak{o}(a \wedge b) = \mathfrak{o}(a) \cap \mathfrak{o}(b), \mathfrak{o}(\bigvee a_i) = \bigvee \mathfrak{o}(a_i)$,
- $\mathfrak{c}(0) = L, \mathfrak{c}(1) = \mathsf{O}, \mathfrak{c}(a \wedge b) = \mathfrak{c}(a) \vee \mathfrak{c}(b), \mathfrak{c}(\bigvee a_i) = \bigcap \mathfrak{c}(a_i)$.

1.4. Images and preimages. For a localic map $f\colon L \to M$ and for any sublocale $S \subseteq L$ we have the image $f[S]$ which is a sublocale again. On the other hand, the set-theoretic preimage (briefly, *set-preimage*) $f^{-1}[S]$ of a sublocale S is not necessarily a sublocale. It is a meet-subset, though, and hence (see the formula for the join of sublocales above) there is the largest sublocale

$$f_{-1}[S] = \bigvee\{T \mid T \in \mathcal{S}(L),\ T \subseteq f^{-1}[S]\}$$

contained in $f^{-1}[S]$. It will be referred to as the *localic preimage.* We have the Galois adjunction

$$f[S] \subseteq T \quad \text{iff} \quad S \subseteq f_{-1}[S].$$

For closed sublocales we have $f_{-1}[\mathfrak{c}(a)] = f^{-1}[\mathfrak{c}(a)] = \mathfrak{c}(f^*(a))$. For open sublocales the localic and set-preimages do not necessarily coincide (see Section 4 below), but we do have $f_{-1}[\mathfrak{o}(a)] = \mathfrak{o}(f^*(a))$.

For more about frames see e.g. [2, 6]. For basic facts from category theory see [4] (or the Appendix in [6]), and for the basics of classical topology see e.g. [3].

2. Assembling a localic map from open parts

2.1. Setting. We have a cover $(a_i)_{i\in J}$ of a frame L, in the sublocale language, a cover $(\mathfrak{o}(a_i))_{i\in J}$ of L by open sublocales. Further, we have localic maps $f_i\colon \mathfrak{o}(a_i)\to M$ and assume that

$$\forall i,j\in J,\ f_i|(\mathfrak{o}(a_i)\cap\mathfrak{o}(a_j)) = f_j|(\mathfrak{o}(a_i)\cap\mathfrak{o}(a_j)).$$

Since $\mathfrak{o}(a_i)\cap\mathfrak{o}(a_j)=\mathfrak{o}(a_i\wedge a_j)$ this amounts to the system of equalities

$$\forall i,j\in J\ \forall x\in L,\quad f_i((a_i\wedge a_j)\to x) = f_j((a_i\wedge a_j)\to x).$$

2.2. Let $h_i\colon M\to\mathfrak{o}(a_i)$ be the frame homomorphisms adjoint to f_i.

Lemma. *For all i,j we have $h_i(x)\wedge a_i\wedge a_j = h_j(x)\wedge a_i\wedge a_j$.*

Proof: For all y, $h_i(x)\wedge a_i\wedge a_j\le y$ iff $h_i(x)\le(a_i\wedge a_j)\to y$ iff $x\le f_i((a_i\wedge a_j)\to y) = f_j((a_i\wedge a_j)\to y)$ iff $h_j(x)\le(a_i\wedge a_j)\to y$ iff $h_j(x)\wedge a_i\wedge a_j\le y$. ■

2.3. Define a mapping $f\colon L\to M$ by setting

$$f(x)=\bigwedge_{i\in J} f_i(a_i\to x)$$

and a mapping $h\colon M\to L$ by

$$h(x)=\bigvee_{i\in J}(h_i(x)\wedge a_i).$$

2.4. Lemma. *If $x\in\mathfrak{o}(a_k)$ then $f(x)=f_k(x)$.*

Proof: If $x\in\mathfrak{o}(a_k)$ then for every i,

$$\begin{aligned} f_i(a_i\to x) = f_i(a_i\to(a_k\to x)) = f_i((a_i\wedge a_k)\to x) = \\ = f_k((a_i\wedge a_k)\to x)\ge f_k(a_k\to x)\end{aligned}$$

and $f_k(x)=f_k(a_k\to x)$ is among the factors in the definition of $f(x)$. ■

2.5. Lemma. *The mapping $h\colon M\to L$ preserves binary meets.*

Proof: By 2.2 we have

$$\begin{aligned} h(x) \wedge h(y) &= \bigvee_{i,j} h_i(x) \wedge h_j(y) \wedge a_i \wedge a_j = \bigvee_{i,j} h_i(x) \wedge h_i(y) \wedge a_i \wedge a_j = \\ &= \bigvee_{i,j} h_i(x \wedge y) \wedge a_i \wedge a_j = \bigvee_{i} h_i(x \wedge y) \wedge a_i \wedge \bigvee_{j} a_j = \\ &= \bigvee_{i} h_i(x \wedge y) \wedge a_i = h(x \wedge y). \quad \blacksquare \end{aligned}$$

2.6. Theorem. *If localic maps* $f_i \colon \mathfrak{o}(a_i) \to M$ *agree on the intersections* $\mathfrak{o}(a_i) \cap \mathfrak{o}(a_j)$ *and if* $\bigvee_i \mathfrak{o}(a_i) = L$ *then there exists precisely one localic map* $f \colon L \to M$ *such that* $f|\mathfrak{o}(a_i) = f_i$ *for all* i*, namely the* f *from* 2.3*.*

Proof: By definition of $\bigvee S_i$ in $\mathcal{S}(L)$ and taking into account that localic maps preserve meets we see that there is at most one such f.

Now observe that the maps from 2.3 are adjoint. Indeed, we have

$$\begin{aligned} h(x) \le y \quad &\text{iff} \quad \forall i \in J,\ h_i(x) \le a_i \to y \\ &\text{iff} \quad \forall i \in J,\ x \le f(a_i \to y) \qquad \text{iff} \quad x \le \bigwedge_{i \in J} f_i(a_i \to y) = f(y). \end{aligned}$$

Hence, first, h preserves all joins. By 2.5 it preserves binary meets; since also $h(1) = \bigvee h_i(1) \wedge a_i = \bigvee a_i = 1$, h is a frame homomorphism and f is a localic map. $\blacksquare$

2.7. Note. Needless to say, we have here the diagram

$$\begin{tikzcd} & \mathfrak{o}(a_i) \arrow[dr, "\subseteq"] & \\ \mathfrak{o}(a_i) \cap \mathfrak{o}(a_j) \arrow[ur, "\subseteq"] \arrow[dr, "\subseteq"'] \arrow[r, phantom, "(i, j \in J)"] & {} & L \\ & \mathfrak{o}(a_j) \arrow[ur, "\subseteq"'] & \end{tikzcd}$$

about which we have proved that it is a *multiple pushout*, that is, that

$$\begin{tikzcd}
\mathfrak{o}(a_i) \arrow[dr, "\subseteq"] & \\
(i,j \in J) & L \\
\mathfrak{o}(a_j) \arrow[ur, "\subseteq"'] &
\end{tikzcd}$$

is the colimit of the rest of the diagram. In the next section it will be of advantage to reverse the reasoning, namely considering the colimit first and then deduce the required result.

3. Assembling a localic map from closed parts

3.1. Setting. This time we have a finite closed cover of L, that is, closed sublocales $\mathfrak{c}(a_1), \dots, \mathfrak{c}(a_n)$ such that $\bigvee_{i=1}^{n} \mathfrak{c}(a_i) = L$ in $\mathcal{S}(L)$. Further, we have localic maps $f_i \colon \mathfrak{c}(a_i) \to M$ such that

$$\forall i,j,\ \ f_i|(\mathfrak{c}(a_i) \cap \mathfrak{c}(a_j)) = f_j|(\mathfrak{c}(a_i) \cap \mathfrak{c}(a_j)).$$

Since $\mathfrak{c}(a_i) \cap \mathfrak{c}(a_j) = \mathfrak{c}(a_i \vee a_j) = {\uparrow}(a_i \vee a_j)$, this amounts to the requirement that

$$\forall i,j, \quad x \geq a_i \vee a_j \ \Rightarrow\ f_i(x) = f_j(x).$$

We are looking for an $f \colon L \to M$ such that $f|\mathfrak{c}(a_i) = f_i$ for all i.

3.2. Consider the diagram of frame homomorphisms

$$\begin{tikzcd}
L \arrow[r, "\beta"] \arrow[d, "\alpha"'] & {\uparrow}b \arrow[d, "\alpha'"] \\
{\uparrow}a \arrow[r, "\beta'"] & {\uparrow}(a \vee b)
\end{tikzcd} \tag{$*$}$$

with $\alpha(x) = \alpha'(x) = a \vee x$ and $\beta(x) = \beta'(x) = b \vee x$. It is a well known (and almost obvious) fact that this diagram is a pushout. But we also have the following:

Proposition. *If $a \wedge b = 0$ then $(*)$ is a pullback.*

Proof: Extend $(*)$ to

$$\begin{tikzcd} L \arrow[rr, "\beta"] \arrow[dr, "\varepsilon"] \arrow[dd, "\alpha"'] & & {\uparrow}b \arrow[dd, "\alpha'"] \\ & {\uparrow}a \times {\uparrow}b \arrow[ur, "p_2"] \arrow[dl, "p_1"] & \\ {\uparrow}a \arrow[rr, "\beta'"] & & {\uparrow}(a \vee b) \end{tikzcd}$$

with p_i the product projections and ε the mapping given by

$$\varepsilon(x) = (a \vee x, b \vee x).$$

By the standard construction of pullback it suffices to prove that ε is the equalizer of $\beta' p_1$ and $\alpha' p_2$ (the equalities $\alpha = p_1 \varepsilon$ and $\beta = p_2 \varepsilon$ are trivial).

First, obviously ε is a frame homomorphism, and $\beta' p_1 \varepsilon(x) = a \vee b \vee x = \alpha' p_2 \varepsilon(x)$. It is one-to-one: if $a \vee x = a \vee y$ and $b \vee x = b \vee y$ then

$$x = (a \wedge b) \vee x = (a \vee x) \wedge (b \vee x) = (a \vee y) \wedge (b \vee y) = y.$$

Now let $h\colon M \to {\uparrow}a \times {\uparrow}b$ be a frame homomorphism such that $\beta' p_1 h = \alpha' p_2 h$. That is, for $h_i = p_i h$ we have $h_1(x) \vee b = h_2(x) \vee a$. We need to show that there is a unique k that completes the diagram

$$\begin{tikzcd} L \arrow[r, "\varepsilon"] & {\uparrow}a \times {\uparrow}b \arrow[r, shift left, "\beta' p_1"] \arrow[r, shift right, "\alpha' p_2"'] & {\uparrow}(a \vee b) \\ M \arrow[u, dotted, "k"] \arrow[ur, bend right, "h"'] & & \end{tikzcd}$$

Define

$$k\colon M \to L \text{ by setting } k(x) = h_1(x) \wedge h_2(x).$$

Then

$$\begin{aligned} \varepsilon k(x) &= (a \vee (h_1(x) \wedge h_2(x)), b \vee (h_1(x) \wedge h_2(x))) = \\ &= ((a \vee h_1(x)) \wedge (a \vee h_2(x)), (b \vee h_1(x)) \wedge (b \vee h_2(x))) = \\ &= ((a \vee h_1(x)) \wedge (b \vee h_1(x)), (a \vee h_2(x)) \wedge (b \vee h_2(x))) = \\ &= ((a \wedge b) \vee h_1(x), (a \wedge b) \vee h_2(x)) = (h_1(x), h_2(x)) = h(x). \end{aligned}$$

Since h, ε are homomorphisms and ε is one-to-one it follows that k is a homomorphism and that it is unique such that $h = \varepsilon k$. ∎

3.3. Consider the embedding (localic) maps

$$j_1\colon \mathfrak{c}(a) \hookrightarrow L \quad \text{and} \quad j_2\colon \mathfrak{c}(b) \hookrightarrow L.$$

They are the right adjoints of the α resp. β above and hence the pullback $(*)$ in **Frm** from 3.2 translates to the pushout in **Loc**

$$\begin{array}{ccc} L & \xleftarrow{j_2} & \mathfrak{c}(b) \\ {\scriptstyle j_1}\uparrow & & \uparrow{\scriptstyle j_1'} \\ \mathfrak{c}(a) & \xleftarrow{j_2'} & \mathfrak{c}(a \vee b) \end{array} \qquad (**)$$

with j_1' and j_2' the inclusion maps.

3.4. Theorem. *Let $\mathfrak{c}(a_i)$, $i = 1, 2, \dots, n$, be closed sublocales of L such that $\bigvee_{i=1}^{n} \mathfrak{c}(a_i) = L$ and let $f_i\colon \mathfrak{c}(a_i) \to M$ be localic maps such that*

$$\text{for all } i, j, \quad f_i|(\mathfrak{c}(a_i) \cap \mathfrak{c}(a_j)) = f_j|(\mathfrak{c}(a_i) \cap \mathfrak{c}(a_j)).$$

Then there exists precisely one localic map $f\colon L \to M$ such that $f|\mathfrak{c}(a_i) = f_i$ for all i.

Proof: It suffices to prove the statement for $n = 2$. Set $a = a_1$ and $b = a_2$. Since $L = \mathfrak{c}(a) \vee \mathfrak{c}(b) = {\uparrow}(a \wedge b)$ we have $a \wedge b = 0$ and can use the pullback from 3.2, and consequently the pushout $(**)$. The equality $f_1|(\mathfrak{c}(a) \cap \mathfrak{c}(b)) = f_2|(\mathfrak{c}(a) \cap \mathfrak{c}(b))$ reads in the notation from 3.3 $f_1 j_2' = f_2 j_1'$ and hence we have a localic map $f\colon L \to M$ such that $f j_i = f_i$, that is, $f|\mathfrak{c}(a) = f_1$ and $f|\mathfrak{c}(b) = f_2$. The unicity follows from f preserving meets (the same as the unicity in 2.6). ∎

4. Maps $f\colon L \to M$ and preimages

4.1. In this section we will, first, discuss preserving closed and open sublocales by preimages. That is, we have frames L and M and ask what mappings $f\colon L \to M$ are characterized by the requirement(s) that

$$f^{-1}[S] \text{ is closed for closed } S \quad \text{resp.} \quad f^{-1}[S] \text{ is open for open } S. \qquad (4.1)$$

For a localic map f we have $f_{-1}[\mathfrak{c}(a)] = f^{-1}[\mathfrak{c}(a)] = \mathfrak{c}(f^*(a))$ and $f_{-1}[\mathfrak{o}(a)] = \mathfrak{o}(f^*(a)$. The question is what maps we obtain if one or both of the conditions (4.1) are assumed.

4.2. Proposition. *Preimages $f^{-1}[\mathfrak{c}(a)]$ of closed sublocales are closed iff f has a left adjoint (that is, iff f preserves all meets).*

Proof: Let f^* exist. Then for any $a \in M$,

$$f^{-1}[\mathfrak{c}(a)] = \{x \in L \mid a \leq f(x)\} = \{x \in L \mid f^*(a) \leq x\} = \mathfrak{c}(f^*(a)).$$

On the other hand, if for each a there is a $b = \phi(a)$ such that $f^{-1}[\mathfrak{c}(a)] = \mathfrak{c}(\phi(a))$ then $a \leq f(x)$ iff $\phi(a) \leq x$. ∎

4.2.1. Note. Here we have a characteristics of meet-preserving maps among all the $f\colon L \to M$ akin to that of continuous maps in classical topology. Hence, the reader may expect at least a proof of assembling a meet-preserving map $f\colon L = {\uparrow}a_1 \vee \cdots \vee {\uparrow}a_n \to M$ from meet-preserving $f_i\colon {\uparrow}a_i \to M$ following precisely the trivial reasoning about assembling a continuous map. But even here the translation is not quite straightforward ($\bigvee S_i$ is not $\bigcup S_i$), and a value of such a result is meagre: meet-preserving maps do not have much geometric sense, taking the ${\uparrow}a$ for something like closed subobjects is only a weak analogy, there are no reasonable opens complementing them, etc. .

4.3. The set- and localic preimages of closed sublocales under localic maps coincide. This is, however, not the case for open sublocales. We will discuss preserving open sublocales by set-preimages for general $f\colon L \to M$ (we have to: $f_{-1}[S]$ makes sense for localic maps only). Thus we have to keep in mind that the condition $f^{-1}[\mathfrak{o}(a)] = \mathfrak{o}(b)$ is not automatic even for a localic f (while $f_{-1}[\mathfrak{o}(a)] = \mathfrak{o}(b)$ is).

The coincidence of $f_{-1}[\mathfrak{o}(a)]$ and $f^{-1}[\mathfrak{o}(a)]$ will be discussed in the second part of this section.

4.4. Observation. *Let L, M be frames and let $f\colon L \to M$ be a mapping. Then all the $f^{-1}[S]$ with open S are open iff there is a mapping $\phi\colon M \to L$ such that*

$$\phi(x) \to y = y \quad \textit{iff} \quad x \to f(y) = f(y). \tag{4.4}$$

(Indeed, (4.4) is just a reformulation of $\mathfrak{o}(\phi(x)) = f^{-1}[\mathfrak{o}(x)]$.)

4.4.1. Remarks. (1) Note that even if f is a right adjoint, such ϕ may exist without coinciding with the f^*. Consider the following trivial example:

Take $f = \mathrm{const}_1$, the right adjoint of which is $f^* = \mathrm{const}_0$. Now for $\phi = \mathrm{const}_1 \neq f^*$ one has always $\phi(x) \to y = 1 \to y = y$ and also $x \to f(y) = x \to 1 = 1$.

(2) For each localic map one has $f(f^*(x) \to y) = x \to f(y)$ (recall 1.2.1). Thus, for $\phi = f^*$ (and f localic) the implication "$\Rightarrow$" in (4.4) is automatic.

4.5. The linear case. The situation is simpler in the case of linearly ordered frames L, M. We have

Proposition. *Let L, M be linearly ordered frames and $f\colon L \to M$ a mapping. Then the following statements are equivalent.*

(1) *f has a left adjoint f^* such that $f^*(1) = 1$.*
(2) *f is a localic map.*
(3) *$f^{-1}[S]$ is closed for each closed S and satisfies (4.4) with $\phi = f^*$ (and hence in particular $f^{-1}[S]$ is open for each open S).*

Proof: (1)≡(2) is trivial, since $x \wedge y = \min\{x, y\}$ is preserved by any monotone map.

(2)⇒(3): In view of 4.2 it suffices to prove the statement about (4.4). In a linearly ordered frame we have $x \to y = 1$ if $x \leq y$ and $x \to y = y$ otherwise. Furthermore, if f^* is a frame homomorphism, $f(x) = 1$ implies $x = 1$. Thus we have for $y = 1$, both $f^*(x) \to y = y$ and $x \to f(y) = f(y)$ for any x, and for $y \neq 1$, $f^*(x) \to y = y$ iff $f^*(x) \not\leq y$ iff $x \not\leq f(y)$ iff $x \to f(y) = f(y)$.

(3)⇒(1): By 4.2 f is a right adjoint. Now to prove that $f^*(1) = 1$ we have to show that $f(y) = 1$ implies $y = 1$: if $f(y) = 1$ we have $x \to f(y) = f(y)$ and hence $f^*(x) \to y = y$ and also $x \leq f(y)$, so that $f^*(x) \leq y$ and hence $y = f^*(x) \to y = 1$. ■

Remark. Note that, by the example in 4.4.1, we cannot replace (3) by the statement that $f^{-1}[S]$ is closed resp. open for each closed resp. open S. Thus, even in the simple case of linearly ordered frames, preserving both closed and open sublocales by f's preimages does not make f a localic map.

4.6. Let $f\colon L \to M$ be a localic map. We have

Proposition. *$f^{-1}[\mathfrak{o}(a)]$ is a sublocale for each $a \in M$ iff*

$$(a \to f(y) = f(y),\ f^*(a) \leq x \to y) \quad \Rightarrow \quad x \to y = 1 \ (\textit{that is, } x \leq y).$$

Proof: Evidently $f^{-1}[\mathfrak{o}(a)]$ is a sublocale of L iff

$$\forall x, y \in L,\ (a \to f(y) = f(y) \ \Rightarrow\ a \to f(x \to y)).$$

But $a \to f(x \to y) = \bigvee\{w \mid w \wedge a \leq f(x \to y)\}$ and therefore $a \to f(x \to y) = f(x \to y)$ iff

$$w \wedge a \leq f(x \to y) \ \Rightarrow\ w \leq f(x \to y).$$

This formula is obviously equivalent to

$$x \wedge f^*(w) \wedge f^*(a) \leq y \ \Rightarrow\ x \wedge f^*(w) \leq y$$

and this, in turn, to

$$x \wedge f^*(a) \leq y \ \Rightarrow x \leq y$$

(by setting in particular $w = 1$). ∎

4.7. One extreme case of the behavior of the $f\colon L \to M$ with respect to the preimages of open sublocales is the case of linear L and general M. Since in L, $x \to y = 1$ or y, $f^{-1}[\mathfrak{o}(a)]$ which is always a meet-set is automatically a sublocale. In fact, an L such that for all $f\colon L \to M$, $f^{-1}[\mathfrak{o}(a)]$ is a sublocale has typically only trivial pseudocomplements and hence "is not far from being linear" (an element x with $0 < x^* < 1$ and a localic map $f\colon L \to M$ with linear $M \ncong \{0,1\}$ and $f(x) = 0$ would contradict 4.6: set $a = f(x)$).

4.8. We know, however, more about the other extreme case, that of a Boolean M.

Theorem. *$f^{-1}[\mathfrak{o}(a)]$ is a sublocale for each $f\colon L \to M$ and $a \in M$ if and only if M is Boolean.*

Proof: "$\Leftarrow$" follows from 4.6, but also directly: $a \to x = a^* \vee x = x$ iff $x \geq a^*$; hence $f^{-1}[\mathfrak{o}(a)] = f^{-1}[\mathfrak{c}(a^*)]$ which is a sublocale.

"$\Rightarrow$": Let the statement on the f hold. Take $L = \mathcal{S}(M)^{\mathrm{op}}$ and $f\colon L \to M$ the right adjoint to the frame embedding $h = (x \mapsto \mathfrak{c}(x))\colon M \to L$. Thus in particular $f(\mathfrak{c}(x)) = x$.

Now for a fixed $y \in M$ consider $a = y \vee y^*$ so that in particular $a \to 0 = a^* = 0$ and $0 \in \mathfrak{o}(a)$. We have $f(0_M) = 0$ and hence $0_M \in f^{-1}[\mathfrak{o}(a)]$. Since $f^{-1}[\mathfrak{o}(a)]$ is a sublocale, each complemented element of L, in particular each $\mathfrak{c}(x)$, is in $f^{-1}[\mathfrak{o}(a)]$. Thus, $a \to x = a \to f(\mathfrak{c}(x)) = x$ for each x, and $1 = a \to a = a$. ∎

4.9. Remark. Examples of localic maps with (in our notation) $f_{-1}[S] = \mathsf{O} \neq f^{-1}[S]$ can be found in [7]:

(a) In Example 4.2 we find a localic surjection $f\colon L \to \mathbb{Q}$ such that for any nonzero pointless sublocale S of $\mathbb{Q}$, its localic preimage $f_{-1}[S]$ is zero. Of course, since f is a surjection, the set-theoretical preimage $f^{-1}[S]$ must be nonzero.

(b) Example 4.10 yields another localic surjection $f\colon L \to \mathbb{Q}$ that satisfies the identity $f_{-1}[S] = f_{-1}[X \smallsetminus (X \smallsetminus S)]$ for any sublocale S of $\mathbb{Q}$ such that $f_{-1}[S]$ is closed. Therefore, it suffices to take such an S for which $X \smallsetminus (X \smallsetminus S)$ (the *double supplement* of S) differs from S: their set-theoretical preimages will be certainly different. Furthermore, Proposition 4.9 in [7] shows that the same happens with any surjection $f\colon L \to M$ which is a regular epimorphism.

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