Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 16–21

TWO CHARACTERISATIONS OF GROUPS AMONGST MONOIDS

ANDREA MONTOLI, DIANA RODELO AND TIM VAN DER LINDEN

Abstract: The aim of this paper is to solve a problem proposed by Dominique Bourn: to provide a categorical-algebraic characterisation of groups amongst monoids and of rings amongst semirings. In the case of monoids, our solution is given by the following equivalent conditions:

(i) G is a group;
(ii) G is a *Mal'tsev object*, i.e., the category $\mathsf{Pt}_G(\mathsf{Mon})$ of points over G in the category of monoids is unital;
(iii) G is a *protomodular object*, i.e., all points over G are *stably strong*, which means that any pullback of such a point along a morphism of monoids $Y \to G$ determines a split extension

$$0 \longrightarrow K \overset{k}{\rhd\!\longrightarrow} X \underset{f}{\overset{s}{\leftleftarrows}} Y \longrightarrow 0$$

in which k and s are jointly strongly epimorphic.

We similarly characterise rings in the category of semirings.

On the way we develop a *local* or *object-wise* approach to certain important conditions occurring in categorical algebra. This leads to a basic theory involving what we call *unital* and *strongly unital* objects, *subtractive* objects, *Mal'tsev* objects and *protomodular* objects. We explore some of the connections between these new notions and give examples and counterexamples.

Keywords: Fibration of points; strongly epimorphic pair; (strongly) unital, subtractive, Mal'tsev, protomodular category.
Math. Subject Classification (2010): 18D35, 20J15, 18E99, 03C05, 08C05.

1. Introduction

The concept of *abelian object* plays a key role in categorical algebra. In the study of categories of non-abelian algebraic structures—such as groups, Lie algebras, loops, rings, crossed modules, etc.—the "abelian case" is usually seen

Received May 16, 2016.

This work was partially supported by the Centre for Mathematics of the University of Coimbra—UID/MAT/00324/2013, funded by the Portuguese Government through FCT/MCTES and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020 and by the FCT grant number SFRH/BPD/69661/2010.

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as a basic starting point, often simpler than the general case, or sometimes even trivial. Most likely there are known results which may or may not be extended to the surrounding non-abelian setting. Part of categorical algebra deals with such generalisation issues, which tend to become more interesting precisely where this extension is not straightforward. Abstract commutator theory for instance, which is about *measuring non-abelianness*, would not exist without a formal interplay between the abelian and the non-abelian worlds, enabled by an accurate definition of abelianness.

Depending on the context, several approaches to such a conceptualisation exist. Relevant to us are those considered in [3]; see also [21, 31, 29] and the references in [3]. The easiest is probably to say that an **abelian object** is an object which admits an internal abelian group structure. This makes sense as soon as the surrounding category is *unital*—a condition introduced in [5], see below for details—which is a rather weak additional requirement on a pointed category implying that an object admits at most one internal abelian group structure. So that, in this context, "being abelian" becomes a property of the object in question.

The full subcategory of a unital category $\mathbb{C}$ determined by the abelian objects is denoted $\mathsf{Ab}(\mathbb{C})$ and called the **additive core** of $\mathbb{C}$. The category $\mathsf{Ab}(\mathbb{C})$ is indeed additive, and if $\mathbb{C}$ is a finitely cocomplete regular [2] unital category, then $\mathsf{Ab}(\mathbb{C})$ is a reflective [3] subcategory of $\mathbb{C}$. If $\mathbb{C}$ is moreover Barr exact [2], then $\mathsf{Ab}(\mathbb{C})$ is an abelian category, and called the **abelian core** of $\mathbb{C}$.

For instance, in the category Lie_K of Lie algebras over a field K, the abelian objects are K-vector spaces, equipped with a trivial (zero) bracket; in the category Gp of groups, the abelian objects are the abelian groups, so that $\mathsf{Ab}(\mathsf{Gp}) = \mathsf{Ab}$; in the category Mon of monoids, the abelian objects are abelian groups as well: $\mathsf{Ab}(\mathsf{Mon}) = \mathsf{Ab}$; etc. In all cases the resulting commutator theory behaves as expected.

Beyond abelianness: weaker conditions. The concept of an abelian object has been well studied and understood. For certain applications, however, it is too strong: the "abelian case" may not just be *simple*, it may be *too simple*. Furthermore, abelianness may "happen too easily". As explained in [3], the Eckmann–Hilton argument implies that any internal monoid in a unital category is automatically a *commutative* object. For instance, in the category of monoids any internal monoid is commutative, so that in particular an internal

group is always abelian: $\mathsf{Gp}(\mathsf{Mon}) = \mathsf{Ab}$. Amongst other things, this fact is well known to account for the abelianness of the higher homotopy groups.

If we want to capture groups amongst monoids, avoiding abelianness turns out to be especially difficult. One possibility would be to consider gregarious objects [3], because the "equation"

$$\text{commutative} + \text{gregarious} = \text{abelian}$$

holds in any unital category. But this notion happens to be too weak, since examples were found of gregarious monoids which are not groups. On the other hand, as explained above, the concept of an internal group is too strong, since it gives us abelian groups. Whence the subject of our present paper: to find out how to

$$\text{characterise } \textit{non-abelian} \text{ groups inside the category of monoids}$$

in categorical-algebraic terms. That is to say, is there some weaker concept than that of an abelian object which, when considered in Mon, gives the category Gp?

This question took quite a long time to be answered. As explained in [13, 14], the study of monoid actions, where an **action** of a monoid B on a monoid X is a monoid homomorphism $B \to \mathrm{End}(X)$ from B to the monoid of endomorphisms of X, provided a first solution to this problem: a monoid B is a group if and only if all split epimorphisms with codomain B correspond to monoid actions of B. However, this solution is not entirely satisfactory, since it makes use of features which are typical for the category of monoids, and thus cannot be exported to other categories.

Another approach to this particular question is to consider the concept of $\mathscr{S}$-protomodularity [13, 15, 9], which allows to single out a protomodular [4] subcategory $\mathscr{S}(\mathbb{C})$ of a given category $\mathbb{C}$, depending on the choice of a convenient class $\mathscr{S}$ of points in $\mathbb{C}$—see below for details. Unlike the category of monoids, the category of groups is protomodular. And indeed, when $\mathbb{C} = \mathsf{Mon}$, the class $\mathscr{S}$ of so-called *Schreier points* [26] does characterise groups in the sense that $\mathscr{S}(\mathsf{Mon}) = \mathsf{Gp}$. A similar characterisation is obtained through the notion of $\mathscr{S}$-Mal'tsev categories [9]. However, this characterisation is "relative", in the sense that it depends on the choice of a class $\mathscr{S}$. Moreover, the definition of the class $\mathscr{S}$ of Schreier points is ad-hoc, given that it again crucially depends on $\mathbb{C}$ being the category of monoids. So the problem is somehow shifted to another level.

The approach proposed in our present paper is different because it is *local* and *absolute*, rather than *global* and *relative*. "Local" here means that we consider

conditions defined object by object: *protomodular* objects, *Mal'tsev* objects, *(strongly) unital* objects and *subtractive* objects. While $\mathscr{S}$-protomodularity deals with the protomodular subcategory $\mathscr{S}(\mathbb{C})$ as a whole. "Absolute" means that there is no class $\mathscr{S}$ for the definitions to depend on.

More precisely, we show in Theorem 7.7 that the notions of a protomodular object and a Mal'tsev object give the desired characterisation of groups amongst monoids—whence the title of our paper. Moreover, we find suitable classes of points which allow us to establish the link between our absolute approach and the relative approach of $\mathscr{S}$-protomodularity and the $\mathscr{S}$-Mal'tsev condition (Proposition 7.15 and Proposition 6.17).

The following table gives an overview of the classes of objects we consider, and what they amount to in the category of monoids Mon and in the category of semirings SRng. Here GMon denotes the category of gregarious monoids mentioned above.

TABLE 1. Special objects in the categories Mon and SRng

all objects	unital objects	subtractive objects	strongly unital objects	Mal'tsev objects	protomodular objects
$\mathbb{C}$	$\mathsf{U}(\mathbb{C})$	$\mathsf{S}(\mathbb{C})$	$\mathsf{SU}(\mathbb{C})$	$\mathsf{M}(\mathbb{C})$	$\mathsf{P}(\mathbb{C})$
Mon SRng	Mon SRng	GMon Rng	GMon Rng	Gp Rng	Gp Rng

In function of the category $\mathbb{C}$ it is possible to separate all classes of special objects occurring in Table 1. Indeed, a given category is unital, say, precisely when all of its objects are unital; while there exist examples of unital categories which are not subtractive, Mal'tsev categories which are not protomodular, and so on.

The present paper is the starting point of an exploration of this new object-wise approach, which is being further developed in ongoing work.

Example: protomodular objects. Let us, as an example of the kind of techniques we use, briefly sketch the definition of a protomodular object. Given an object B, a **point over** B is a pair of morphisms $(f\colon A \to B, s\colon B \to A)$ such that $fs = 1_B$. A category with finite limits is said to be **protomodular** [4,

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Acknowledgements

We are grateful to Dominique Bourn for proposing the problem that led to the present paper. We would also like to thank Alan Cigoli and Xabier García-Martínez for fruitful discussions.

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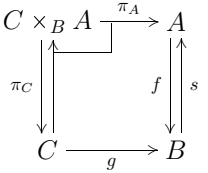
3] when for every pullback



of a point (f, s) over B along some morphism g with codomain B, the morphisms π_A and s are **jointly strongly epimorphic**: they do not both factor through a given proper subobject of A. In a pointed context, this condition is equivalent to the validity of the *split short five lemma* [4]. This observation gave rise to the notion of a **semi-abelian** category—a pointed, Barr exact, protomodular category with finite coproducts [22]—which plays a fundamental role in the development of a categorical-algebraic approach to homological algebra for non-abelian structures; see for instance [11, 19, 1, 18, 30].

A point (f, s) satisfying the condition mentioned above (that π_A and s are jointly strongly epimorphic) is called a **strong point**. When also all of its pullbacks satisfy this condition, it is called a **stably strong point**. We shall say that B is a **protomodular object** when all points over B are stably strong points. Writing $\mathsf{P}(\mathbb{C})$ for the full subcategory of $\mathbb{C}$ determined by the protomodular objects, we clearly have that $\mathsf{P}(\mathbb{C}) = \mathbb{C}$ if and only if $\mathbb{C}$ is a protomodular category. In fact, $\mathsf{P}(\mathbb{C})$ is *always* a protomodular category, as soon as it is closed under finite limits in $\mathbb{C}$. We study some of its basic properties in Section 7, where we also prove one of our main results: if $\mathbb{C}$ is the category of monoids, then $\mathsf{P}(\mathbb{C})$ is the category of groups (Theorem 7.7). This is one of two answers to the question we set out to study, the other being a characterisation of groups amongst monoids as the so-called *Mal'tsev objects* (essentially Theorem 6.13).

Structure of the text. Since the concept of a (stably) strong point plays a key role in our work, we recall its definition and discuss some of its basic properties in Section 2. Section 3 recalls the definitions of $\mathscr{S}$-Mal'tsev and $\mathscr{S}$-protomodular categories in full detail.

In Section 4 we introduce the concept of *strongly unital* object. We show that these coincide with the *gregarious* objects when the surrounding category is regular. We prove stability properties and characterise rings amongst semirings as the strongly unital objects (Theorem 4.3).

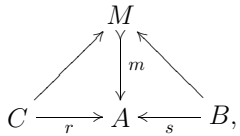
Section 5 is devoted to the concepts of *unital* and *subtractive* object. Our main result here is Proposition 5.14 which, mimicking Proposition 3 in [23], says that an object of a pointed regular category is strongly unital if and only if it is unital and subtractive.

In Section 6 we introduce *Mal'tsev* objects and prove that any Mal'tsev object in the category of monoids is a group (Theorem 6.13). Section 7 treats the concept of a *protomodular* object. Here we prove our paper's main result, Theorem 7.7: a monoid is a group if and only if it is a protomodular object, and if and only if it is a Mal'tsev object. We also explain in which sense the full subcategory determined by the protomodular objects is a protomodular core [15].

2. Stably strong points

We start by recalling some notions that occur frequently in categorical algebra, focusing on the concept of a *strong point*.

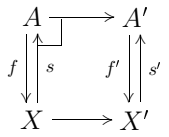
2.1. Jointly strongly epimorphic pairs. A cospan $(r\colon C \to A, s\colon B \to A)$ in a category $\mathbb{C}$ is said to be **jointly extremally epimorphic** when it does not factor through a monomorphism, which means that for any commutative diagram where m is a monomorphism



the monomorphism m is necessarily an isomorphism. If $\mathbb{C}$ is finitely complete, then it is easy to see that the pair (r, s) is jointly epimorphic. In fact, in a finitely complete category the notions of extremal epimorphism and strong epimorphism coincide. Therefore, we usually refer to the pair (r, s) as being **jointly strongly epimorphic**. Recall that, if $\mathbb{C}$ is moreover a regular category [2], then extremal epimorphisms and strong epimorphisms coincide with the regular epimorphisms.

2.2. The fibration of points. A **point** $(f\colon A \to B, s\colon B \to A)$ in $\mathbb{C}$ is a split epimorphism f with a chosen splitting s. Considering a point as a diagram in $\mathbb{C}$, we obtain the category of points in $\mathbb{C}$, denoted $\mathsf{Pt}(\mathbb{C})$: morphisms between

if and only if it is the pullback



of some point (f', s') in $\mathsf{P}(\mathbb{C})$. Note that $\mathscr{P}$ is a class of strong points, since they are pullbacks of stably strong points (the codomain X' is a protomodular object). The class $\mathscr{P}$ is also a pullback-stable class since any pullback of a point (f, s) in $\mathscr{P}$ is also a pullback of a point in $\mathsf{P}(\mathbb{C})$. The class $\mathscr{P}$ is not closed under finite limits in $\mathsf{Pt}(\mathbb{C})$, in general. So, strictly speaking, it does not give rise to an $\mathscr{S}$-protomodular category. However, as we observed for the Mal'tsev case, the fact (which follows immediately from the definition of $\mathscr{P}$) that all points in $\mathsf{P}(\mathbb{C})$ belong to $\mathscr{P}$ allows us to apply the same arguments as in the proof of Proposition 6.2 in [15] (and its generalisation to the non-pointed case, given in [9]) to conclude that $\mathsf{P}(\mathbb{C})$ is a protomodular category. Indeed, as we now show, it is the protomodular core $\mathscr{P}(\mathbb{C})$ of $\mathbb{C}$ relative to the class of points $\mathscr{P}$. In other words, it is the category of $\mathscr{P}$-special objects of $\mathbb{C}$.

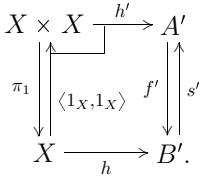
Proposition 7.15. *If $\mathbb{C}$ is a pointed finitely complete category, and the subcategory $\mathsf{P}(\mathbb{C})$ of protomodular objects is closed under finite limits in $\mathbb{C}$, then it coincides with the protomodular core $\mathscr{P}(\mathbb{C})$ consisting of the $\mathscr{P}$-special objects of $\mathbb{C}$.*

Proof: If X is a protomodular object, it is obviously $\mathscr{P}$-special, since the point

$$(\pi_1\colon X \times X \to X, \quad \Delta_X = \langle 1_X, 1_X\rangle\colon X \to X \times X)$$

belongs to the subcategory $\mathsf{P}(\mathbb{C})$, which is closed under binary products.

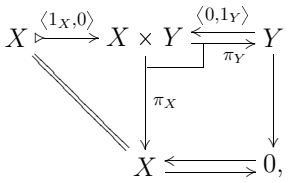
Conversely, suppose that X is $\mathscr{P}$-special. Then the point (π_1, Δ_X) is a pullback of a point (f', s') in $\mathsf{P}(\mathbb{C})$



But then X, which is the kernel of π_2, is also the kernel of f', and hence it belongs to $\mathsf{P}(\mathbb{C})$. ∎

Proposition 7.12. *If $\mathbb{C}$ is a pointed finitely complete category, then the zero object is protomodular if and only if $\mathbb{C}$ is unital.*

Proof: The zero object 0 is protomodular if and only if every point over it is stably strong. This means that, for any X, $Y \in \mathbb{C}$, in the diagram



the morphisms $\langle 1_X, 0\rangle$ and $\langle 0, 1_Y\rangle$ are jointly strongly epimorphic. This happens if and only if $\mathbb{C}$ is unital. ∎

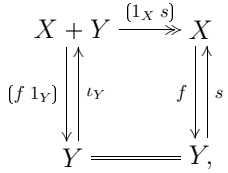
Proposition 7.13. *If $\mathbb{C}$ is a regular category with binary coproducts, then the following conditions are equivalent:*

(i) *Y is a protomodular object;*
(ii) *for every morphism $f\colon X \to Y$, the point*

$$([f\ 1_Y]\colon X + Y \to Y, \quad \iota_Y\colon Y \to X + Y)$$

is stably strong.

Proof: This follows from Proposition 2.11 applied to the morphism of points

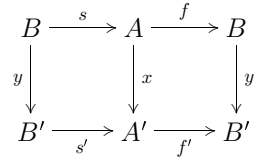


for any given point $(f\colon X \to Y, s\colon Y \to X)$. ∎

7.14. $\mathsf{P}(\mathbb{C})$ is a protomodular core. Similarly to what we did for Mal'tsev objects, we now show that the subcategory $\mathsf{P}(\mathbb{C})$ of protomodular objects is a protomodular core with respect to a suitable class $\mathscr{P}$ of points, provided that $\mathsf{P}(\mathbb{C})$ is closed under finite limits in $\mathbb{C}$.

Let $\mathbb{C}$ be a finitely complete category such that $\mathsf{P}(\mathbb{C})$ is closed under finite limits. We define the class $\mathscr{P}$ in the following way: a point (f, s) belongs to $\mathscr{P}$

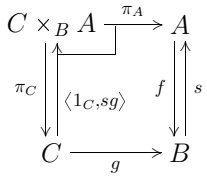
points are pairs $(x, y)\colon (f, s) \to (f', s')$ of morphisms in $\mathbb{C}$ making the diagram



commute. If $\mathbb{C}$ has pullbacks of split epimorphisms, then the functor $\mathrm{cod}\colon \mathsf{Pt}(\mathbb{C}) \to \mathbb{C}$, which associates with every split epimorphism its codomain, is a fibration, usually called the **fibration of points** [4]. Given an object B of $\mathbb{C}$, we denote the fibre over B by $\mathsf{Pt}_B(\mathbb{C})$. An object in this category is a point with codomain B, and a morphism is of the form $(x, 1_B)$.

2.3. Strong points. We now assume $\mathbb{C}$ to be a finitely complete category.

Definition 2.4. We say that a point $(f\colon A \to B, s\colon B \to A)$ is a **strong point** when for every pullback



(**A**)

along any morphism $g\colon C \to B$, the pair (π_A, s) is jointly strongly epimorphic.

Strong points were already considered in [27], under the name of *regular points*, and independently in [8], under the name of *strongly split epimorphisms*.

Many algebraic categories have been characterised in terms of properties of strong points (see [5, 3]), some of which we recall throughout the text. For instance, by definition, a finitely complete category is **protomodular** [4] precisely when all points in it are strong. For a pointed category, this condition is equivalent to the validity of the split short five lemma [4]. Examples of protomodular categories are the categories of groups, of rings, of Lie algebras (over a commutative ring with unit) and, more generally, every *variety of Ω-groups* in the sense of Higgins [20]. Protomodularity is also a key ingredient in the definition of a *semi-abelian category* [22].

On the other hand, in the category of sets, a point (f, s) is strong if and only if f is an isomorphism. To see this, it suffices to pull it back along the unique morphism from the empty set $\varnothing$.

2.5. Pointed categories. In a pointed category, we denote the kernel of a morphism f by $\ker(f)$. In the pointed case, the notion of strong point mentioned above coincides with the one considered in [28]:

Proposition 2.6. *Let $\mathbb{C}$ be a pointed finitely complete category.*

(1) *A point (f, s) in $\mathbb{C}$ is strong if and only if the pair $(\ker(f), s)$ is jointly strongly epimorphic.*
(2) *Any split epimorphism f in a strong point (f, s) is a normal epimorphism.*

Proof: (1) If (f, s) is a strong point, then $(\ker(f), s)$ is jointly strongly epimorphic: to see this, it suffices to take the pullback of f along the unique morphism with domain the zero object. Conversely, if we take an arbitrary pullback as in (**A**), then $\ker(f) = \pi_A\langle 0, \ker(f)\rangle$. We conclude that (π_A, s) is jointly strongly epimorphic because $(\ker(f), s)$ is.

(2) Since (f, s) is a strong point, the pair $(\ker(f), s)$ is jointly strongly epimorphic; thus it is jointly epimorphic. It easily follows that f is the cokernel of its kernel $\ker(f)$. ∎

In a pointed finitely complete context, asking that certain product projections are strong points gives rise to the notions of unital and strongly unital categories. In fact, when for all objects X, Y in $\mathbb{C}$ the point

$$(\pi_X \colon X \times Y \to X, \quad \langle 1_X, 0\rangle \colon X \to X \times Y)$$

is strong, $\mathbb{C}$ is said to be a **unital** category [5]. The category $\mathbb{C}$ is called **strongly unital** ([5], see also Definition 1.8.3 and Theorem 1.8.15 in [3]) when for all objects X in $\mathbb{C}$ the point

$$(\pi_1 \colon X \times X \to X, \quad \Delta_X = \langle 1_X, 1_X\rangle \colon X \to X \times X)$$

is strong. Observe that we could equivalently ask the point (π_2, Δ_X) to be strong. It is well known that every strongly unital category is necessarily unital [3, Proposition 1.8.4].

Examples 2.7. As shown in [3, Theorem 1.2.15], a variety in the sense of universal algebra is a unital category if and only if it is a **Jónsson–Tarski variety**. This means that the corresponding theory contains a unique constant 0 and a binary operation + subject to the equations $0 + x = x = x + 0$.

In particular, the categories of monoids and of semirings are unital. Moreover, every protomodular category is strongly unital.

Example 7.6. $\mathsf{P}(\mathsf{SRng}) = \mathsf{M}(\mathsf{SRng}) = \mathsf{SU}(\mathsf{SRng}) = \mathsf{S}(\mathsf{SRng}) = \mathsf{Rng}$. If X is a protomodular semiring, then it is obviously a strongly unital semiring, thus a ring by Theorem 4.3. We already mentioned that if X is a ring, then every point over it in SRng is stably strong, since it is a Schreier point by [13, Proposition 6.1.6]. In particular, the category $\mathsf{P}(\mathsf{SRng}) = \mathsf{Rng}$ is closed under finite limits and it is protomodular. Thanks to Propositions 7.2 and 6.3, we also have that $\mathsf{M}(\mathsf{SRng}) = \mathsf{Rng}$.

Theorem 7.7. *If $\mathbb{C}$ is the category Mon of monoids, then $\mathsf{P}(\mathbb{C}) = \mathsf{M}(\mathbb{C}) = \mathsf{Gp}$, the category of groups. In other words, the following conditions are equivalent, for any monoid M:*

(i) *M is a group;*
(ii) *M is a Mal'tsev object, i.e., $\mathsf{Pt}_M(\mathsf{Mon})$ is a unital category;*
(iii) *M is a protomodular object, i.e., all points over M in the category of monoids are stably strong.*

Proof: If M is a group, then every point over it is stably strong, because it is a Schreier point by Proposition 3.4 in [14]. This proves that (i) implies (iii). (iii) implies (ii) by Proposition 7.2, and (ii) implies (i) by Theorem 6.13. ∎

Remark 7.8. Note that, in particular, $\mathsf{P}(\mathsf{Mon})$ is closed under finite limits in the category Mon.

Remark 7.9. The proof of Theorem 6.13 may be simplified to obtain a direct proof that (iii) implies (i) in Theorem 7.7. Instead of the pullback diagram (**G**), we may consider the simpler pullback of $\begin{pmatrix}1_M & 1_M\end{pmatrix} \colon M + M \to M$ along $m \colon \mathbb{N} \to M$.

Remark 7.10. As recalled in Example 4.9, there are gregarious monoids that are not groups. Hence, in Mon, the subcategory $\mathsf{P}(\mathsf{Mon})$ is strictly contained in $\mathsf{SU}(\mathsf{Mon})$.

Example 7.11. In the category $\mathsf{Cat}_X(\mathbb{C})$ of internal categories over a fixed base object X in a finitely complete category $\mathbb{C}$, any internal groupoid over X is a protomodular object. This follows from results in [9]: any pullback of any split epimorphism over such an internal groupoid "has a fibrant splitting", which implies that it is a strong point. So, over a given internal groupoid over X, all points are stably strong, which means that this internal groupoid is a protomodular object.

Similarly to the Mal'tsev case, we also have:

Proposition 7.2. *Let $\mathbb{C}$ be a finitely complete category. Every protomodular object is a Mal'tsev object.*

Proof: Let Y be a protomodular object and consider the following pullback of split epimorphisms:

$$\begin{array}{ccc} A \times_Y C & \underset{\pi_C}{\overset{\langle sg, 1_C\rangle}{\rightleftarrows}} & C \\ {\scriptstyle \pi_A}\downarrow\uparrow{\scriptstyle \langle 1_A, tf\rangle} & & {\scriptstyle g}\downarrow\uparrow{\scriptstyle t} \\ A & \underset{f}{\overset{s}{\rightleftarrows}} & Y. \end{array}$$

Since Y is protomodular, the point (g, t) is stably strong and, consequently, $(\pi_A, \langle 1_A, tf\rangle)$ is a strong point. Moreover, the pullback of s along π_A is precisely $\langle sg, 1_C\rangle$, so that the pair $(\langle 1_A, tf\rangle, \langle sg, 1_C\rangle)$ is jointly strongly epimorphic, as desired. Observe that this proof is a simplified version of that of Theorem 3.2.1 in [15]. ∎

Note that, in the regular case, the above result follows from Proposition 6.2 via Lemma 2.12.

The inclusion $\mathsf{P}(\mathbb{C}) \subset \mathsf{M}(\mathbb{C})$ is strict, in general, by the following proposition, Proposition 6.9 and the fact that there exist Mal'tsev categories which are not protomodular.

Proposition 7.3. *If $\mathbb{C}$ is a finitely complete category, then $\mathsf{P}(\mathbb{C}) = \mathbb{C}$ if and only if $\mathbb{C}$ is protomodular.*

Proof: By definition, a finitely complete category is protomodular if and only if all points in it are strong. When this happens, automatically all of them are stably strong. ∎

Corollary 7.4. *If $\mathbb{C}$ is a finitely complete category and $\mathsf{P}(\mathbb{C})$ is closed under finite limits in $\mathbb{C}$, then $\mathsf{P}(\mathbb{C})$ is a protomodular category.*

Proof: Apply Proposition 7.3 to $\mathsf{P}(\mathbb{C})$. ∎

Observe that this hypothesis is satisfied when $\mathbb{C}$ is the category Mon of monoids, or the category SRng of semirings, as can be seen as a consequence of Example 7.6 and Theorem 7.7 below.

Proposition 7.5. *If $\mathbb{C}$ is regular, then $\mathsf{P}(\mathbb{C})$ is closed under quotients in $\mathbb{C}$.*

Proof: This follows immediately from Proposition 2.11. ∎

2.8. Stably strong points. We are especially interested in those points for which the property of being strong is pullback-stable.

Definition 2.9. We say that a point (f, s) is **stably strong** if every pullback of it along any morphism is a strong point. More explicitly, for any morphism g, the point $(\pi_C, \langle 1_C, sg\rangle)$ in Diagram (**A**) is strong.

Note that a stably strong point is always strong (it suffices to pull it back along the identity morphism) and that the collection of stably strong points determines a subfibration of the fibration of points.

In a protomodular category, *all* points are stably strong (since all points are strong). In the category of sets, all strong points are stably strong (since isomorphisms are preserved by pullbacks).

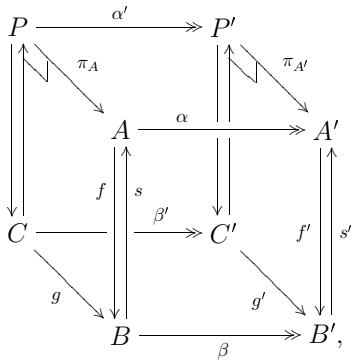
2.10. The regular case. In the context of regular categories [2], (stably) strong points are **closed under quotients**: this means that in any commutative diagram

$$\begin{array}{ccc} A & \overset{\alpha}{\twoheadrightarrow} & A' \\ {\scriptstyle f}\downarrow\uparrow{\scriptstyle s} & & {\scriptstyle f'}\downarrow\uparrow{\scriptstyle s'} \\ B & \underset{\beta}{\twoheadrightarrow} & B', \end{array}$$

where α and β are regular epimorphisms and (f, s) is (stably) strong, also (f', s') is (stably) strong.

Proposition 2.11. *In a regular category, (stably) strong points are closed under quotients.*

Proof: Let us first prove that the quotient of a strong point is always strong. So let (f, s) be a strong point, and consider the diagram



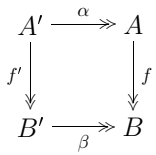
where P' is the pullback of f' along an arbitrary morphism g', C is the pullback of g' along β, and P is the pullback of f along g. By pullback cancelation, the upper square is a pullback too, hence by regularity α' is a regular epimorphism. Since α is a regular epimorphism, we have that $\alpha\pi_A$ and αs are jointly strongly epimorphic. Then it easily follows that $\pi_{A'}$ and s' are jointly strongly epimorphic, so that the point (f', s') is a strong point.

If now (f, s) is stably strong, then the point $P \leftrightarrows C$ is strong. As its quotient, the point $P' \leftrightarrows C'$ is strong. Hence (f', s') is stably strong.

Note that the first claim still holds if the category is just finitely complete; we only use the regularity of α' in the second claim. ∎

As a consequence, in a regular category, a point (f, s) is stably strong if and only if the point $(\pi_1, \langle 1_A, sf\rangle)$ induced by its kernel pair is stably strong. Equivalently one could consider the point $(\pi_2, \langle sf, 1_A\rangle)$.

Certain pushouts involving strong points are necessarily **regular pushouts**. Recall that such is a commutative square of regular epimorphisms



where also the comparison arrow $\langle f', \alpha\rangle \colon A' \to B' \times_B A$ is a regular epimorphism. It is well known that every regular pushout is a pushout.

By the commutativity of the diagram on the left, we see that $h_1 = x = h_2$. Consequently, h (and h' as well) is a monomorphism, thus an isomorphism, so that $X \cong B'$ is a Mal'tsev object. ∎

Proposition 6.17. *If $\mathbb{C}$ is a finitely complete category and the subcategory $\mathsf{M}(\mathbb{C})$ of Mal'tsev objects is closed under finite limits in $\mathbb{C}$, then it coincides with the subcategory $\mathscr{M}(\mathbb{C})$ of $\mathscr{M}$-special objects of $\mathbb{C}$.*

Proof: If X is a Mal'tsev object, it is obviously $\mathscr{M}$-special, since the point

$$(\pi_1 \colon X \times X \to X, \quad \Delta_X = \langle 1_X, 1_X\rangle \colon X \to X \times X)$$

belongs to the subcategory $\mathsf{M}(\mathbb{C})$, which is closed under binary products.

The converse implication follows from Lemma 6.16. ∎

Strictly speaking, we cannot apply Proposition 4.3 in [9] to conclude that $\mathsf{M}(\mathbb{C})$ is the Mal'tsev core of $\mathbb{C}$ relatively to $\mathscr{M}$, since the class $\mathscr{M}$ we are considering does not satisfy all the conditions of Definition 3.2. Indeed, our class $\mathscr{M}$ is not stable under pullbacks, neither need it to be closed in $\mathsf{Pt}(\mathbb{C})$ under finite limits, in general. However, all the arguments of the proof of Proposition 4.3 in [9] are still applicable to our context, since, by definition of the class $\mathscr{M}$, we know that every point between objects in $\mathsf{M}(\mathbb{C})$ belongs to $\mathscr{M}$. So, we can conclude that, if $\mathsf{M}(\mathbb{C})$ is closed in $\mathbb{C}$ under finite limits, then it is a Mal'tsev category, being the Mal'tsev core of $\mathbb{C}$ relatively to the class $\mathscr{M}$. Observe that we could also conclude that $\mathsf{M}(\mathbb{C})$ is a Mal'tsev category simply by Corollary 6.10.

7. Protomodular objects

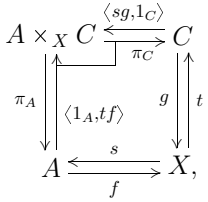
In this final section we introduce the (stronger) concept of a protomodular object and prove our paper's main result, Theorem 7.7: a monoid is a group if and only if it is a protomodular object, and if and only if it is a Mal'tsev object.

Definition 7.1. Given an object Y of a finitely complete category $\mathbb{C}$, we say that Y is **protomodular** if every point with codomain Y is stably strong.

We write $\mathsf{P}(\mathbb{C})$ for the full subcategory of $\mathbb{C}$ determined by the protomodular objects.

Obviously, every protomodular object is strongly unital. Hence it is also unital and subtractive (Proposition 5.14). We also have:

Proposition 6.15. *Let $\mathbb{C}$ be a finitely complete category. Given any pullback of split epimorphisms with (f, s) a point in $\mathscr{M}$*



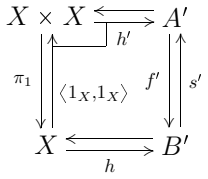
the pair $(\langle 1_A, tf\rangle, \langle sg, 1_C\rangle)$ is jointly strongly epimorphic.

Proof: Since (f, s) is a pullback of a point in $\mathsf{M}(\mathbb{C})$ as in (**I**), we see that the pair $(\langle 1_A, tf\rangle a, \langle sg, 1_C\rangle)$ is jointly strongly epimorphic. It easily follows that also $(\langle 1_A, tf\rangle, \langle sg, 1_C\rangle)$ is jointly strongly epimorphic. ∎

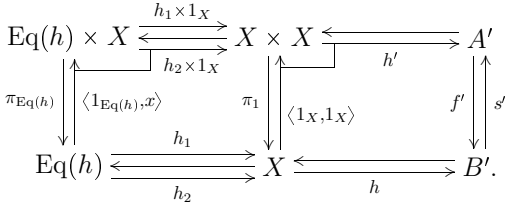
Note that the property above already occured in Definition 3.2(1).

Lemma 6.16. *In a finitely complete category $\mathbb{C}$, if X is an $\mathscr{M}$-special object, then it is a Mal'tsev object.*

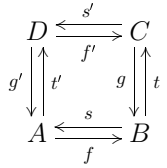
Proof: By definition there is a point (f', s') in $\mathsf{M}(\mathbb{C})$ and a point $X \leftrightarrows B'$ in $\mathbb{C}$ such that the square



is a pullback. Taking the kernel pairs to the left of h and h', since $\mathrm{Eq}(h') \cong \mathrm{Eq}(h) \times X$ we obtain a diagram of the shape



A **double split epimorphism** in a category $\mathbb{C}$ is a point in the category of points in $\mathbb{C}$, so a commutative diagram

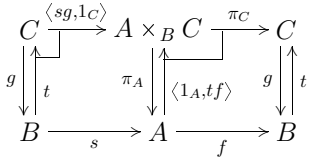


(**B**)

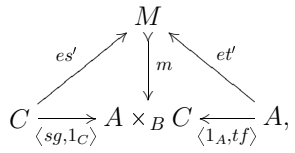
where the four "obvious" squares commute.

Lemma 2.12. *In a regular category, every double split epimorphism as in* (**B**)*, in which (g, t) is a stably strong point, is a regular pushout.*

Proof: Take the pullback $A \times_B C$ of f and g, consider the comparison morphism $\langle g', f'\rangle \colon D \to A \times_B C$ and factor it as a regular epimorphism $e \colon D \to M$ followed by a monomorphism $m \colon M \to A \times_B C$. Since (g, t) is a stably strong point, its pullback $(\pi_A, \langle 1_A, tf\rangle)$ in the diagram



is a strong point. As a consequence, the pair $(\langle sg, 1_C\rangle, \langle 1_A, tf\rangle)$ is jointly strongly epimorphic. They both factor through the monomorphism m as in the diagram



so that m is an isomorphism. ∎

Lemma 2.13. *In a regular category, consider a commutative square of regular epimorphisms with horizontal kernel pairs*

$$\begin{array}{ccc} \mathrm{Eq}(g) \rightrightarrows A' & \xrightarrow{g} & A \\ {\scriptstyle f''}\downarrow & \downarrow{\scriptstyle f'} & \downarrow{\scriptstyle f} \\ \mathrm{Eq}(h) \rightrightarrows B' & \xrightarrow[h]{} & B. \end{array}$$

If any of the commutative squares on the left is a regular pushout (and so, in particular, f'' is a regular epimorphism), then the square on the right is also a regular pushout.

Proof: The proof is essentially the same as the one of Proposition 3.2 in [7]. ∎

Proposition 2.14. *In a regular category, every regular epimorphism of points*

$$\begin{array}{ccc} D & \twoheadrightarrow & C \\ \downarrow\uparrow & & \downarrow\uparrow \\ A & \twoheadrightarrow & B, \end{array}$$

where the point on the left (and hence also the one on the right) is stably strong, is a regular pushout.

Proof: This follows immediately from Lemma 2.12 and Lemma 2.13. ∎

3. $\mathscr{S}$-Mal'tsev and $\mathscr{S}$-protomodular categories

As mentioned in Section 2, a finitely complete category $\mathbb{C}$ in which all points are (stably) strong defines a protomodular category. If such an "absolute" property fails, one may think of protomodularity in "relative" terms, i.e., with respect to a class $\mathscr{S}$ of stably strong points. We also recall the absolute and relative notions for the Mal'tsev context.

Recall that a finitely complete category $\mathbb{C}$ is called a **Mal'tsev category** [16, 17] when every internal reflexive relation in $\mathbb{C}$ is automatically symmetric or, equivalently, transitive; thus an equivalence relation. Protomodular categories are always Mal'tsev categories [5]. Moreover, Mal'tsev categories admit a well-known characterisation through the fibration of points:

Proposition 3.1. [5, Proposition 10] *A finitely complete category $\mathbb{C}$ is a Mal'tsev category if and only if every fibre $\mathsf{Pt}_Y(\mathbb{C})$ is (strongly) unital.* ∎

Case $s = 2$, $s' = 2$. For the equality

$$\overline{m} = \underline{l}_1 \blacksquare \overline{r}_1 \blacksquare \underline{l}_2 \blacksquare \overline{r}_2 \blacksquare \underline{ml'_1} \blacksquare \overline{r}'_1 \blacksquare \underline{l}'_2 \blacksquare \overline{r}'_2$$

to hold, we must have one of the "inner" elements on the right side of the equality equal to e_M.

- If $\underline{ml'_1} = e_M$, then m is right invertible.
- If $\overline{r}_1 = e_M$ or $\underline{l}_2 = e_M$, then the word $\underline{l}_1 \blacksquare \overline{r}_1 \blacksquare \underline{l}_2 \blacksquare \overline{r}_2$ is not reduced.
- If $\overline{r}'_1 = e_M$ or $\underline{l}'_2 = e_M$, then the word $\underline{l}'_1 \blacksquare \overline{r}'_1 \blacksquare \underline{l}'_2 \blacksquare \overline{r}'_2$ is not reduced.
- If $\overline{r}_2 = e_M$, then $\overline{m} = \underline{l}_1 \blacksquare \overline{r}_1 \blacksquare \underline{l_2 m l'_1} \blacksquare \overline{r}'_1 \blacksquare \underline{l}'_2 \blacksquare \overline{r}'_2$. Again, $\underline{l_2 m l'_1} = e_M$ as in the second case, and (**H**) implies that m is invertible.

We see that the last case reduces to one of the previous ones and it is straightforward to check that the same happens for general s, $s' \geqslant 2$. ∎

Below, in Theorem 7.7, we shall prove that groups are precisely the Mal'tsev monoids: $\mathsf{M}(\mathsf{Mon}) = \mathsf{Gp}$.

6.14. $\mathsf{M}(\mathbb{C})$ is a Mal'tsev core. As we already recalled in Section 3, if $\mathbb{C}$ is an $\mathscr{S}$-Mal'tsev category, then the subcategory of $\mathscr{S}$-special objects $\mathscr{S}(\mathbb{C})$ is a Mal'tsev category, called the Mal'tsev core of $\mathbb{C}$ relatively to $\mathscr{S}$. We now show that the subcategory $\mathsf{M}(\mathbb{C})$ of Mal'tsev objects is a Mal'tsev core with respect to a suitable class $\mathscr{M}$ of points, provided that $\mathsf{M}(\mathbb{C})$ is closed under finite limits in $\mathbb{C}$.

Let $\mathbb{C}$ be a finitely complete category such that $\mathsf{M}(\mathbb{C})$ is closed under finite limits. We define $\mathscr{M}$ as the class of points (f, s) in $\mathbb{C}$ for which there exists a pullback of split epimorphisms

$$\begin{array}{ccc} A & \underset{s}{\overset{a}{\rightleftarrows}} & A' \\ {\scriptstyle f}\downarrow\uparrow & & \downarrow\uparrow{\scriptstyle s'} \\ X & \underset{}{\rightleftarrows} & X', \end{array} \qquad (\mathbf{I})$$

for some point (f', s') in $\mathsf{M}(\mathbb{C})$. Note that the class $\mathscr{M}$ is obviously stable under pullbacks along split epimorphisms. Moreover, all points in $\mathsf{M}(\mathbb{C})$ belong to $\mathscr{M}$.

simplification is possible, besides perhaps when $\underline{l}_1$, $\overline{r}_s$, $\underline{l}'_1$ or $\overline{r}'_{s'}$ happens to be equal to e_M. Computing the second component, we see that

$$\begin{aligned}\overline{m} &= \underline{l}_1 \blacksquare \overline{r}_1 \blacksquare \cdots \blacksquare \underline{l}_s \blacksquare \overline{r}_s \blacksquare \underline{m} \blacksquare \underline{l}'_1 \blacksquare \overline{r}'_1 \blacksquare \cdots \blacksquare \underline{l}'_{s'} \blacksquare \overline{r}'_{s'} \\ &= \underline{l}_1 \blacksquare \overline{r}_1 \blacksquare \cdots \blacksquare \underline{l}_s \blacksquare \overline{r}_s \blacksquare \underline{ml'_1} \blacksquare \overline{r}'_1 \blacksquare \cdots \blacksquare \underline{l}'_{s'} \blacksquare \overline{r}'_{s'}.\end{aligned}$$

This leads to a proof that m is right invertible. Indeed, for such an equality to hold, certain cancellations must be possible so that the overlined elements can get together on the right. Next we study four basic cases which all others reduce to.

Case $s = s' = 1$. For the equality

$$\overline{m} = \underline{l}_1 \blacksquare \overline{r}_1 \blacksquare \underline{ml'_1} \blacksquare \overline{r}'_1$$

to hold, we must have $\overline{r}_1 = e_M$ or $\underline{ml'_1} = e_M$. In the latter situation, m is right invertible. If, on the other hand, $\overline{r}_1 = e_M$, then $\underline{l}_1 = e_M$ by (**H**). The equality $\overline{m} = \underline{ml'_1} \blacksquare \overline{r}'_1$ implies that $\underline{ml'_1} = e_M$.

Case $s = 2$, $s' = 1$. For the equality

$$\overline{m} = \underline{l}_1 \blacksquare \overline{r}_1 \blacksquare \underline{l}_2 \blacksquare \overline{r}_2 \blacksquare \underline{ml'_1} \blacksquare \overline{r}'_1$$

to hold, we must have one of the "inner" elements on the right side of the equality equal to e_M.

- If $\underline{ml'_1} = e_M$, then m is right invertible.
- If $\overline{r}_1 = e_M$ or $\underline{l}_2 = e_M$, then the word $\underline{l}_1 \blacksquare \overline{r}_1 \blacksquare \underline{l}_2 \blacksquare \overline{r}_2$ is not reduced.
- If $\overline{r}_2 = e_M$, then $\overline{m} = \underline{l}_1 \blacksquare \overline{r}_1 \blacksquare \underline{l_2 ml'_1} \blacksquare \overline{r}'_1$. Since $\overline{r}_1$ is different from e_M, we have that $l_2 ml'_1 = e_M$, so that l_2 admits an inverse on the right and l'_1 admits one on the left. From (**H**), we also know that l_2 is admits an inverse on the left and l'_1 admits one on the right. Thus, they are both invertible elements, and hence so is m.

Case $s = 1$, $s' = 2$. For the equality

$$\overline{m} = \underline{l}_1 \blacksquare \overline{r}_1 \blacksquare \underline{ml'_1} \blacksquare \overline{r}'_1 \blacksquare \underline{l}'_2 \blacksquare \overline{r}'_2$$

to hold, we must have one of the "inner" elements on the right side of the equality equal to e_M.

- If $\underline{ml'_1} = e_M$, then m is right invertible.
- If $\overline{r}'_1 = e_M$ or $\underline{l}'_2 = e_M$, then the word $\underline{l}'_1 \blacksquare \overline{r}'_1 \blacksquare \underline{l}'_2 \blacksquare \overline{r}'_2$ is not reduced.
- If $\overline{r}_1 = e_M$, then $\underline{l}_1 = e_M$ by (**H**), so that $\overline{m} = \underline{ml'_1} \blacksquare \overline{r}'_1 \blacksquare \underline{l}'_2 \blacksquare \overline{r}'_2$. This is impossible, since $\overline{r}'_1$ and $\underline{l}'_2$ are non-trivial.

More precisely, for every pullback of split epimorphisms

$$\begin{array}{ccc} A \times_Y C & \underset{\pi_C}{\overset{\langle sg, 1_C\rangle}{\rightleftarrows}} & C \\ {\scriptstyle \pi_A}\downarrow\uparrow{\scriptstyle \langle 1_A, tf\rangle} & & {\scriptstyle g}\downarrow\uparrow{\scriptstyle t} \\ A & \underset{f}{\overset{s}{\rightleftarrows}} & Y \end{array} \qquad (\mathbf{C})$$

(which is a binary product in $\mathsf{Pt}_Y(\mathbb{C})$), the morphisms $\langle 1_A, tf\rangle$ and $\langle sg, 1_C\rangle$ are jointly strongly epimorphic.

Let $\mathbb{C}$ be a finitely complete category, and $\mathscr{S}$ a class of points which is stable under pullbacks along any morphism.

Definition 3.2. Suppose that the full subcategory of $\mathsf{Pt}(\mathbb{C})$ whose objects are the points in $\mathscr{S}$ is closed in $\mathsf{Pt}(\mathbb{C})$ under finite limits. The category $\mathbb{C}$ is said to be:

(1) **$\mathscr{S}$-Mal'tsev** [9] if, for every pullback of split epimorphisms (**C**) where the point (f, s) is in the class $\mathscr{S}$, the morphisms $\langle 1_A, tf\rangle$ and $\langle sg, 1_C\rangle$ are jointly strongly epimorphic;
(2) **$\mathscr{S}$-protomodular** [13, 15, 9] if every point in $\mathscr{S}$ is strong.

The notion of $\mathscr{S}$-protomodular category was introduced to describe, in categorical terms, some convenient properties of *Schreier split epimorphisms* of monoids and of semirings. Such split epimorphisms were introduced in [26] as those points which correspond to classical monoid actions and, more generally, to actions in every category of *monoids with operations*, via a semidirect product construction.

In [13, 14] it was shown that, for Schreier split epimorphisms, relative versions of some properties of all split epimorphisms in a protomodular category hold, like for instance the *split short five lemma*.

In [15] it is proved that every category of monoids with operations, equipped with the class $\mathscr{S}$ of Schreier points, is $\mathscr{S}$-protomodular, and hence an $\mathscr{S}$-Mal'tsev category. Indeed, as shown in [15, 9], every $\mathscr{S}$-protomodular category is an $\mathscr{S}$-Mal'tsev category. Later, in [25] it was proved that every Jónsson–Tarski variety is an $\mathscr{S}$-protomodular category with respect to the class $\mathscr{S}$ of Schreier points. An example of an $\mathscr{S}$-Mal'tsev category which is not $\mathscr{S}$-protomodular, given in [10], is the category of quandles.

The following definition first appeared in [15, Definition 6.1] for pointed $\mathscr{S}$-protomodular categories, then it was extended in [9] to $\mathscr{S}$-Mal'tsev categories.

Definition 3.3. Let $\mathbb{C}$ be a finitely complete category and $\mathscr{S}$ a class of points which is stable under pullbacks along any morphism. An object X in $\mathbb{C}$ is **$\mathscr{S}$-special** if the point

$$(\pi_1 \colon X \times X \to X, \quad \Delta_X = \langle 1_X, 1_X \rangle \colon X \to X \times X)$$

belongs to $\mathscr{S}$ or, equivalently, if the point (π_2, Δ_X) belongs to $\mathscr{S}$. We write $\mathscr{S}(\mathbb{C})$ for the full subcategory of $\mathbb{C}$ determined by the $\mathscr{S}$-special objects.

According to Proposition 6.2 in [15] and its generalisation [9, Proposition 4.3] to $\mathscr{S}$-Mal'tsev categories, if $\mathbb{C}$ is an $\mathscr{S}$-Mal'tsev category, then the subcategory $\mathscr{S}(\mathbb{C})$ of $\mathscr{S}$-special objects of $\mathbb{C}$ is a Mal'tsev category, called the **Mal'tsev core** of $\mathbb{C}$ relatively to the class $\mathscr{S}$. When $\mathbb{C}$ is $\mathscr{S}$-protomodular, $\mathscr{S}(\mathbb{C})$ is a protomodular category, called the **protomodular core** of $\mathbb{C}$ relatively to the class $\mathscr{S}$.

Proposition 6.4 in [15] shows that the protomodular core of the category Mon of monoids relatively to the class $\mathscr{S}$ of Schreier points is the category Gp of groups; similarly, the protomodular core of the category SRng of semirings is the category Rng of rings, also with respect to the class of Schreier points.

Our main problem in this work is to obtain a categorical-algebraic characterisation of groups amongst monoids, and of rings amongst semirings. Based on the previous results, one direction is to look for a suitable class $\mathscr{S}$ of stably strong points in a general finitely complete category $\mathbb{C}$ such that the full subcategory $\mathscr{S}(\mathbb{C})$ of $\mathscr{S}$-special objects gives the category of groups when $\mathbb{C}$ is the category of monoids and gives the category of rings when $\mathbb{C}$ is the category of semirings: $\mathscr{S}(\mathsf{Mon}) = \mathsf{Gp}$ and $\mathscr{S}(\mathsf{SRng}) = \mathsf{Rng}$.

We explore different possible classes in the following sections as well as the outcome for the particular cases of monoids and semirings. A first "obvious" choice is to consider $\mathscr{S}$ to be the class of *all* stably strong points in $\mathbb{C}$. Then an $\mathscr{S}$-special object is precisely what we call a strongly unital object in the next section. We shall see that the subcategory $\mathscr{S}(\mathbb{C})$ of $\mathscr{S}$-special objects is the protomodular core (namely Rng) in the case of semirings, but not so in the case of monoids. Moreover, we propose an alternative "absolute" solution to our main problem, not depending on the choice of a class $\mathscr{S}$ of points, and we compare it with this "relative" one.

for $l, r \in M$. We use essentially the same notations for the elements of $M + \mathbb{N}$, writing a generic element as $\underline{m}_1 \blacksquare \overline{n}_1 \blacksquare \cdots \blacksquare \underline{m}_t \blacksquare \overline{n}_t$.

We see that the pullback P consists of pairs

$$(\underline{m}_1 \blacksquare \overline{n}_1 \blacksquare \cdots \blacksquare \underline{m}_t \blacksquare \overline{n}_t,\ \underline{l}_1 \blacksquare \overline{r}_1 \blacksquare \cdots \blacksquare \underline{l}_s \blacksquare \overline{r}_s) \in (M + \mathbb{N}) \times (M + M)$$

such that $m_1 m^{n_1} \cdots m_t m^{n_t} = l_1 r_1 \cdots l_s r_s$. We also know that

$$\begin{aligned} i_1(\underline{m}_1 \blacksquare \overline{n}_1 \blacksquare \cdots \blacksquare \underline{m}_t \blacksquare \overline{n}_t) &= (\underline{m}_1 \blacksquare \overline{n}_1 \blacksquare \cdots \blacksquare \underline{m}_t \blacksquare \overline{n}_t,\ \underline{m_1 m^{n_1} \cdots m_t m^{n_t}}), \\ i_2(\underline{l}_1 \blacksquare \overline{r}_1 \blacksquare \cdots \blacksquare \underline{l}_s \blacksquare \overline{r}_s) &= (\underline{l_1 r_1 \cdots l_s r_s},\ \underline{l}_1 \blacksquare \overline{r}_1 \blacksquare \cdots \blacksquare \underline{l}_s \blacksquare \overline{r}_s). \end{aligned}$$

Note that $(\overline{1}, \overline{m})$ belongs to P, where $\overline{1}$ is our way to view $1 \in \mathbb{N}$ as an element of $M + \mathbb{N}$. Since by assumption i_1 and i_2 are jointly strongly epimorphic, we have

$$\begin{aligned} (\overline{1}, \overline{m}) = (&\underline{m}_1^1 \blacksquare \overline{n}_1^1 \blacksquare \cdots \blacksquare \underline{m}_{t_1}^1 \blacksquare \overline{n}_{t_1}^1, \underline{m_1^1 m^{n_1^1} \cdots m_{t_1}^1 m^{n_{t_1}^1}}) \\ &\blacksquare (\underline{l_1^1 r_1^1 \cdots l_{s_1}^1 r_{s_1}^1},\ \underline{l}_1^1 \blacksquare \overline{r}_1^1 \blacksquare \cdots \blacksquare \underline{l}_{s_1}^1 \blacksquare \overline{r}_{s_1}^1) \\ &\vdots \\ &\blacksquare (\underline{m}_1^k \blacksquare \overline{n}_1^k \blacksquare \cdots \blacksquare \underline{m}_{t_k}^k \blacksquare \overline{n}_{t_k}^k, \underline{m_1^k m^{n_1^k} \cdots m_{t_k}^k m^{n_{t_k}^k}}) \\ &\blacksquare (\underline{l_1^k r_1^k \cdots l_{s_k}^k r_{s_k}^k},\ \underline{l}_1^k \blacksquare \overline{r}_1^k \blacksquare \cdots \blacksquare \underline{l}_{s_k}^k \blacksquare \overline{r}_{s_k}^k) \end{aligned}$$

for some $m_j^i,\ l_j^i,\ r_j^i \in M$ and $n_j^i \in \mathbb{N}$. Computing the first component we get that $\overline{1}$ is equal to

$$\underline{m}_1^1 \blacksquare \overline{n}_1^1 \blacksquare \cdots \blacksquare \underline{m}_{t_1}^1 \blacksquare \overline{n}_{t_1}^1 \blacksquare \underline{l_1^1 r_1^1 \cdots l_{s_1}^1 r_{s_1}^1} \blacksquare \cdots \blacksquare \underline{m}_1^k \blacksquare \overline{n}_1^k \blacksquare \cdots \blacksquare \underline{m}_{t_k}^k \blacksquare \overline{n}_{t_k}^k \blacksquare \underline{l_1^k r_1^k \cdots l_{s_k}^k r_{s_k}^k}.$$

Since 1 cannot be written as a sum $n_1^1 + \cdots + n_{t_k}^k$ in $\mathbb{N}$ unless all but one of the n_j^i is zero, we see that the equality above reduces to $(\overline{1}, \overline{m})$ being equal to

$$(\underline{l_1 r_1 \cdots l_s r_s},\ \underline{l}_1 \blacksquare \overline{r}_1 \blacksquare \cdots \blacksquare \underline{l}_s \blacksquare \overline{r}_s) \blacksquare (\overline{1}, \underline{m}) \blacksquare (\underline{l'_1 r'_1 \cdots l'_{s'} r'_{s'}},\ \underline{l'_1} \blacksquare \overline{r'_1} \blacksquare \cdots \blacksquare \underline{l'_{s'}} \blacksquare \overline{r'_{s'}}).$$

Equality of the first components gives us

$$\overline{1} = \underline{l_1 r_1 \cdots l_s r_s} \blacksquare \overline{1} \blacksquare \underline{l'_1 r'_1 \cdots l'_{s'} r'_{s'}}$$

from which we deduce that

$$l_1 r_1 \cdots l_s r_s = e_M = l'_1 r'_1 \cdots l'_{s'} r'_{s'}. \tag{\textbf{H}}$$

This means that $\underline{l}_1 \blacksquare \overline{r}_1 \blacksquare \cdots \blacksquare \underline{l}_s \blacksquare \overline{r}_s$ and $\underline{l'_1} \blacksquare \overline{r'_1} \blacksquare \cdots \blacksquare \underline{l'_{s'}} \blacksquare \overline{r'_{s'}}$ are in the kernel of $\begin{pmatrix} 1_M & 1_M \end{pmatrix} \colon M + M \to M$. Without loss of generality we may assume that these two products are written in their reduced form, meaning that no further

Corollary 6.10. *If $\mathbb{C}$ is a finitely complete category and $\mathsf{M}(\mathbb{C})$ is closed under finite limits in $\mathbb{C}$, then $\mathsf{M}(\mathbb{C})$ is a Mal'tsev category.*

Proof: Apply Proposition 6.9 to $\mathsf{M}(\mathbb{C})$. ∎

Proposition 6.11. *If $\mathbb{C}$ is a regular category, then $\mathsf{M}(\mathbb{C})$ is closed under quotients in $\mathbb{C}$.*

Proof: Given a Mal'tsev object X and a regular epimorphism $f\colon X \to Y$, any double split epimorphism over Y may be pulled back to a double split epimorphism over X, which is a regular pushout by assumption. It is straightforward to check that the given double split epimorphism over Y is then a regular pushout. ∎

Example 6.12. As a consequence of Example 7.6 below, in the category of semirings the Mal'tsev objects are precisely the rings: $\mathsf{M}(\mathsf{SRng}) = \mathsf{SU}(\mathsf{SRng}) = \mathsf{Rng}$.

Theorem 6.13. *If $\mathbb{C}$ is the category Mon of monoids, then $\mathsf{M}(\mathbb{C})$ is contained in the subcategory Gp of groups. In other words, if the category $\mathsf{Pt}_M(\mathsf{Mon})$ is unital then the monoid M is a group.*

Proof: Let M be a Mal'tsev object in the category of monoids. Given any element $m \neq e_M$ of M, we are going to prove that it is right invertible. This suffices for the monoid M to be a group.

Consider the pullback diagram

$$\begin{array}{ccc} P & \underset{\pi_2}{\overset{i_2}{\leftrightarrows}} & M+M \\ {\scriptstyle \pi_1}\downarrow\uparrow{\scriptstyle i_1} & & \downarrow\uparrow{\scriptstyle \iota_1} \\ M+\mathbb{N} & \underset{(1_M\ m)}{\overset{\iota_M}{\leftrightarrows}} & M \end{array} \qquad (\mathbf{G})$$

where $m\colon \mathbb{N} \to M$ is the morphism sending 1 to m.

Recall that $M + M$ may be seen as the set of words of the form

$$\underline{l}_1 \blacksquare \overline{r}_1 \blacksquare \cdots \blacksquare \underline{l}_s \blacksquare \overline{r}_s$$

for $\underline{l}_i$, $\overline{r}_i \in M$, subject to the rule that we may multiply underlined with underlined elements and overlined with overlined ones, or any of such with the neutral element e_M. The two coproduct inclusions can be described as

$$\iota_1(l) = \underline{l} \qquad \iota_2(r) = \overline{r}$$

4. Strongly unital objects

The aim of this section is to introduce the concept of a strongly unital object. We characterise rings amongst semirings as the strongly unital objects (Theorem 4.3). We prove stability properties for strongly unital objects and show that, in the regular case, they coincide with the *gregarious* objects of [3].

Let $\mathbb{C}$ be a pointed finitely complete category.

Definition 4.1. Given an object Y of $\mathbb{C}$, we say that Y is **strongly unital** if the point

$$(\pi_1\colon Y \times Y \to Y, \quad \Delta_Y = \langle 1_Y, 1_Y\rangle\colon Y \to Y \times Y)$$

is stably strong.

Note that we could equivalently ask that the point (π_2, Δ_Y) is stably strong. We write $\mathsf{SU}(\mathbb{C})$ for the full subcategory of $\mathbb{C}$ determined by the strongly unital objects.

Remark 4.2. An object Y in $\mathbb{C}$ is strongly unital if and only if it is $\mathscr{S}$-special, when $\mathscr{S}$ is the class of all stably strong points in $\mathbb{C}$.

Theorem 4.3. *If $\mathbb{C}$ is the category SRng of semirings, then $\mathsf{SU}(\mathbb{C})$ is the category Rng of rings. In other words, a semiring X is a ring if and only if the point*

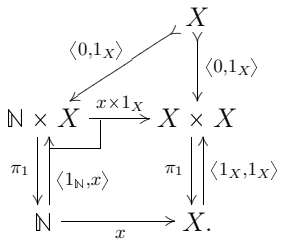
$$(\pi_1\colon X \times X \to X, \quad \Delta_X = \langle 1_X, 1_X\rangle\colon X \to X \times X)$$

is stably strong in SRng.

Proof: If X is a ring, then every point over it is stably strong, since by Proposition 6.1.6 in [13] it is a Schreier point. Hence, it suffices to show that any strongly unital semiring is a ring. Suppose that the point

$$(\pi_1\colon X \times X \to X, \quad \Delta_X = \langle 1_X, 1_X\rangle\colon X \to X \times X)$$

is stably strong. Given any element $x \neq 0_X$ of X, consider the pullback of π_1 along the morphism $x\colon \mathbb{N} \to X$ sending 1 to x:



Consider the element $(1, 0_X) \in \mathbb{N} \times X$. Since the morphisms $\langle 1_{\mathbb{N}}, x\rangle$ and $\langle 0, 1_X\rangle$ are jointly strongly epimorphic, $(1, 0_X)$ can be written as the product of chain of elements of the form $(0, \bar{x})$ and (n, nx). Using the fact that $0 \in \mathbb{N}$ is absorbing for the multiplication in $\mathbb{N}$ and that in every semiring the sum is commutative and the multiplication is distributive with respect to the sum, we get that $(1, 0_X)$ can be written as

$$(1, 0_X) = (0, y) + (1, x)$$

for a certain $y \in X$. Then $y + x = 0_X$ and hence the element x is invertible for the sum. Thus we see that X is a ring. ∎

Remark 4.4. Note that, in particular, $\mathsf{SU}(\mathsf{SRng}) = \mathsf{Rng}$ is a protomodular category, so that Rng is the protomodular core of SRng with respect to the class $\mathscr{S}$ of all stably strong points. As such, it is necessarily the largest protomodular core of SRng induced by some class $\mathscr{S}$.

Recall from [3, 6] that a **split right punctual span** is a diagram of the form

$$X \underset{f}{\overset{s}{\rightleftarrows}} Z \underset{g}{\overset{t}{\rightleftarrows}} Y \qquad (\mathbf{D})$$

where $fs = 1_X$, $gt = 1_Y$ and $ft = 0$.

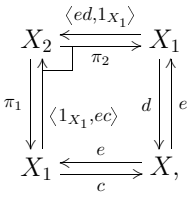
Proposition 4.5. *If $\mathbb{C}$ is a pointed finitely complete category, then the following conditions are equivalent:*

(i) *Y is a strongly unital object of $\mathbb{C}$;*
(ii) *for every morphism $f\colon X \to Y$, the point*

$$(\pi_X\colon X \times Y \to X, \quad \langle 1_X, f\rangle\colon X \to X \times Y)$$

is stably strong;

Considering the pullback



we see that $\langle 1_{X_1}, ec\rangle$ and $\langle ed, 1_{X_1}\rangle$ are jointly (strongly) epimorphic, because X is a Mal'tsev object. Then there is at most one morphism m satisfying the equalities (**F**). ∎

Proposition 6.7. *In a finitely complete category, any reflexive relation on a Mal'tsev object is transitive.*

Proof: The proof is essentially the same as that of [15, Proposition 5.3]. ∎

Example 6.8. Unlike the case of Mal'tsev categories, it is not true that every internal category with a Mal'tsev object of objects is a groupoid. Neither is it true that every reflexive relation on a Mal'tsev object is symmetric. The category Mon of monoids provides counterexamples. Indeed, as we show below in Theorems 6.13 and 7.7, the Mal'tsev objects in Mon are precisely the groups. As a consequence of Propositions 2.2.4 and 3.3.2 in [13], in Mon an internal category over a group is a groupoid if and only if the kernel of the domain morphism is a group. Similarly, a reflexive relation on a group is symmetric if and only if the kernels of the two projections of the relation are groups. A concrete example of a (totally disconnected) internal category which is not a groupoid is the following. If M is a commutative monoid and G is a group, consider the reflexive graph

$$M \times G \underset{\pi_G}{\overset{\pi_G}{\underset{\langle 0,1_G\rangle}{\rightleftarrows}}} G.$$

It is an internal category by Proposition 3.2.3 in [13], but in general it is not a groupoid, since the kernel of π_G, which is M, need not be a group.

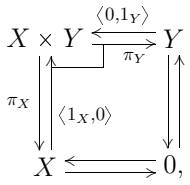
Proposition 6.9. *If $\mathbb{C}$ is a finitely complete category, then $\mathsf{M}(\mathbb{C}) = \mathbb{C}$ if and only if $\mathbb{C}$ is a Mal'tsev category.*

Proof: By Proposition 3.1. ∎

Proposition 6.4. *If $\mathbb{C}$ is a pointed finitely complete category, then the zero object is a Mal'tsev object if and only if $\mathbb{C}$ is unital.*

Proof: The zero object 0 is a Mal'tsev object if and only if, for any $X, Y \in \mathbb{C}$, in the diagram

$$\begin{tikzcd} X \times Y \arrow[r, shift right, "\pi_Y"'] \arrow[d, shift right, "\pi_X"'] & Y \arrow[l, shift right, "\langle 0,1_Y\rangle"'] \arrow[d, shift right] \\ X \arrow[u, shift right, "\langle 1_X,0\rangle"'] \arrow[r, shift right] & 0, \arrow[l, shift right] \arrow[u, shift right] \end{tikzcd}$$



the morphisms $\langle 1_X, 0\rangle$ and $\langle 0, 1_Y\rangle$ are jointly strongly epimorphic. This happens if and only if $\mathbb{C}$ is unital. ∎

Remark 6.5. By Proposition 3.1, $\mathbb{C}$ is a Mal'tsev category if and only if all fibres $\mathsf{Pt}_Y(\mathbb{C})$ are unital if and only if they are strongly unital. For a Mal'tsev object Y in a category $\mathbb{C}$ the fibre $\mathsf{Pt}_Y(\mathbb{C})$ is unital, but not strongly unital in general. The previous proposition provides a counterexample: if $\mathbb{C} = \mathsf{Mon}$ and $Y = 0$, then Y is a Mal'tsev object, but the category $\mathsf{Pt}_Y(\mathsf{Mon}) = \mathsf{Mon}$ is not strongly unital [3, Example 1.8.2].

Next we see that some well-known properties which hold for Mal'tsev categories are still true for Mal'tsev objects.

Proposition 6.6. *In a finitely complete category, a reflexive graph whose object of objects is a Mal'tsev object admits at most one structure of internal category.*

Proof: Given a reflexive graph

$$\begin{tikzcd} X_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & X \arrow[l, "e" description] \end{tikzcd}$$

where X is a Mal'tsev object, let $m\colon X_2 \to X_1$ be a multiplication, where X_2 is the object of composable arrows. If this multiplication endows the graph with a structure of internal category, then it must be compatible with the identities, which means that

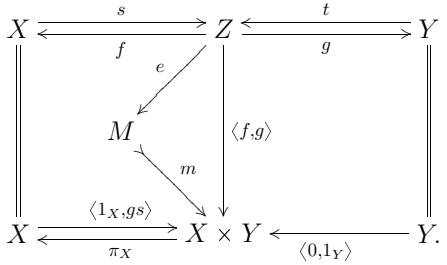
$$m\langle 1_{X_1}, ec\rangle = m\langle ed, 1_{X_1}\rangle = 1_{X_1}. \qquad (\mathbf{F})$$

(iii) *for every $f\colon X \to Y$, the point $(\pi_X, \langle 1_X, f\rangle)$ is strong;*
(iv) *given any split right punctual span* (**D**)*, the map $\langle f, g\rangle\colon Z \to X \times Y$ is a strong epimorphism.*

Proof: The equivalence between (i), (ii) and (iii) hold since any pullback of the point (π_1, Δ_Y) is of the form $(\pi_X, \langle 1_X, f\rangle)$ and any pullback of $(\pi_X, \langle 1_X, f\rangle)$ is also a pullback of (π_1, Δ_Y).

To prove that (iii) implies (iv), consider a split right punctual span as in (**D**). By assumption, the point $(\pi_X\colon X \times Y \to X, \langle 1_X, gs\rangle\colon X \to X \times Y)$ is strong. Suppose that $\langle f, g\rangle$ factors through a monomorphism m

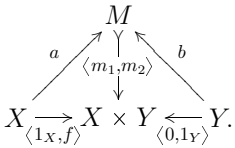
$$\begin{tikzcd} X \arrow[rr, shift left, "s"] \arrow[dd, equal] & & Z \arrow[ll, shift left, "f"] \arrow[rr, shift right, "g"'] \arrow[dl, "e"'] \arrow[dd, "\langle f,g\rangle"] & & Y \arrow[ll, shift right, "t"'] \arrow[dd, equal] \\ & M \arrow[dr, tail, "m"] & & & \\ X \arrow[rr, shift left, "\langle 1_X,gs\rangle"] & & X \times Y \arrow[ll, shift left, "\pi_X"] & & Y. \arrow[ll, "\langle 0,1_Y\rangle"] \end{tikzcd}$$



Both $\langle 1_X, gs\rangle$ and $\langle 0, 1_Y\rangle$ factor through m, indeed $\langle 1_X, gs\rangle = mes$ and $\langle 0, 1_Y\rangle = met$. Since $\langle 1_X, gs\rangle$ and $\langle 0, 1_Y\rangle$ are jointly strongly epimorphic, m is an isomorphism.

To prove that (iv) implies (iii), we must show that $\langle 0, 1_Y\rangle\colon Y \to X \times Y$ and $\langle 1_X, f\rangle\colon X \to X \times Y$ are jointly strongly epimorphic. Suppose that they factor through a monomorphism $m = \langle m_1, m_2\rangle\colon M \to X \times Y$:

$$\begin{tikzcd} & M \arrow[d, tail, "\langle m_1,m_2\rangle"] & \\ X \arrow[ur, "a"] \arrow[r, "\langle 1_X,f\rangle"'] & X \times Y & Y. \arrow[ul, "b"'] \arrow[l, "\langle 0,1_Y\rangle"] \end{tikzcd}$$



Then we have $m_1 a = 1_X$, $m_1 b = 0$ and $m_2 b = 1_Y$. Hence we get a diagram

$$\begin{tikzcd} X \arrow[r, shift left, "a"] & M \arrow[l, shift left, "m_1"] \arrow[r, shift right, "m_2"'] & Y \arrow[l, shift right, "b"'] \end{tikzcd}$$

as in (**D**). By assumption, the monomorphism $\langle m_1, m_2\rangle$ is also a strong epimorphism, so it is an isomorphism. ∎

In general, a given point (π_1, Δ_Y) can be strong without being stably strong. Nevertheless, if all such points are strong (so that $\mathbb{C}$ is strongly unital), then they are stably strong (by Propositions 1.8.13 and 1.8.14 in [3] and Proposition 4.5). This gives:

Corollary 4.6. *If $\mathbb{C}$ is a pointed finitely complete category, then $\mathsf{SU}(\mathbb{C}) = \mathbb{C}$ if and only if $\mathbb{C}$ is strongly unital.* ■

Corollary 4.7. *If $\mathbb{C}$ is a pointed finitely complete category and $\mathsf{SU}(\mathbb{C})$ is closed under finite limits in $\mathbb{C}$, then $\mathsf{SU}(\mathbb{C})$ is a strongly unital category.*

Proof: The category $\mathsf{SU}(\mathbb{C})$ is obviously pointed. Its inclusion into $\mathbb{C}$ preserves monomorphisms and binary products and it reflects isomorphisms. ■

Proposition 4.8. *If $\mathbb{C}$ is a pointed regular category, then $\mathsf{SU}(\mathbb{C})$ is closed under quotients in $\mathbb{C}$.*

Proof: This follows immediately from Proposition 2.11. ■

When $\mathbb{C}$ is a regular unital category, an object Y satisfying condition (iv) of Proposition 4.5 is called a **gregarious** object (Definition 1.9.1 and Theorem 1.9.7 in [3]). So, in that case, $\mathsf{SU}(\mathbb{C})$ is precisely the category of gregarious objects in $\mathbb{C}$.

Example 4.9. $\mathsf{SU}(\mathsf{Mon}) = \mathsf{GMon}$, the category of gregarious monoids. A monoid Y is gregarious if and only if for all $y \in Y$ there exist u, $v \in Y$ such that $uyv = 1$ (Proposition 1.9.2 in [3]). Counterexample 1.9.3 in [3] provides a gregarious monoid which is not a group: the monoid Y with two generators x, y and the relation $xy = 1$. Indeed $Y = \{y^n x^m \mid n, m \in \mathbb{N}\}$ and $x^n(y^n x^m)y^m = 1$.

For monoids and the class $\mathscr{S}$ of all stably strong points of monoids, we have $\mathscr{S}(\mathsf{Mon}) = \mathsf{SU}(\mathsf{Mon}) = \mathsf{GMon} \neq \mathsf{Gp}$ as explained in Remark 4.2. In particular, there are in Mon stably strong points which are not Schreier. Since $\mathscr{S}(\mathsf{Mon})$ is not protomodular, it is not a protomodular core with respect to the class $\mathscr{S}$. Hence for the case of monoids, such a class $\mathscr{S}$ does not meet our purposes. The major issue here concerns the closedness of the class $\mathscr{S}$ in $\mathsf{Pt}(\mathbb{C})$ under finite limits. To avoid this difficulty, in the next sections our work focuses more on objects rather than classes.

We obtain a double split epimorphism over Y given by

$$\begin{array}{ccc} M & \underset{\pi_C m}{\overset{c}{\rightleftarrows}} & C \\ {\scriptstyle \pi_A m}\downarrow\uparrow{\scriptstyle a} & & {\scriptstyle g}\downarrow\uparrow{\scriptstyle t} \\ A & \underset{f}{\overset{s}{\rightleftarrows}} & Y, \end{array}$$

whose comparison morphism to the pullback of f and g is $m\colon M \to A \times_Y C$. By assumption, m is a regular epimorphism, hence it is an isomorphism. ■

Proposition 6.3. *Let $\mathbb{C}$ be a pointed regular category. Every Mal'tsev object in $\mathbb{C}$ is a strongly unital object.*

Proof: Let Y be a Mal'tsev object. By Proposition 4.5, given a split right punctual span

$$X \underset{f}{\overset{s}{\rightleftarrows}} Z \underset{g}{\overset{t}{\leftrightarrows}} Y$$

we need to prove that the the morphism $\langle f, g\rangle\colon Z \to X \times Y$ is a strong epimorphism. Consider the commutative diagram on the right

$$\begin{array}{ccccc} \mathrm{Eq}(f) & \underset{\pi_1}{\overset{\pi_1}{\rightrightarrows}} & Z & \xrightarrow{\ f\ } & X \\ {\scriptstyle g'}\downarrow\uparrow{\scriptstyle t'} & & {\scriptstyle g}\downarrow\uparrow{\scriptstyle t} & & \downarrow\uparrow \\ \mathrm{Eq}(!_Y) & \underset{\pi_1}{\overset{\pi_1}{\rightrightarrows}} & Y & \xrightarrow[!_Y]{} & 0 \end{array}$$

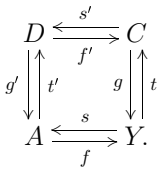
and take kernel pairs to the left. Note that the square on the right is a regular epimorphism of points. Since Y is a Mal'tsev object, by Proposition 6.2 the double split epimorphism of first (or second) projections on the left is a regular pushout. Lemma 2.13 tells us that the square on the right is a regular pushout as well, which means that the morphism $\langle f, g\rangle\colon Z \to X \times Y$ is a regular, hence a strong, epimorphism. ■

For a pointed finitely complete category $\mathbb{C}$, the category $\mathsf{SU}(\mathbb{C})$ obviously contains the zero object. By the following proposition we see that the zero object is not necessarily a Mal'tsev object. Hence if $\mathbb{C}$ is pointed and regular, but not unital, then $\mathsf{M}(\mathbb{C})$ is strictly contained in $\mathsf{SU}(\mathbb{C})$.

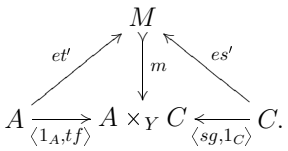
Proposition 6.2. *Let $\mathbb{C}$ be a regular category. For any object Y in $\mathbb{C}$, the following conditions are equivalent:*

(i) *Y is a Mal'tsev object;*
(ii) *every double split epimorphism (**B**) over Y is a regular pushout;*
(iii) *every double split epimorphism (**B**) over Y, in which f' and g' are jointly monomorphic, is a pullback.*

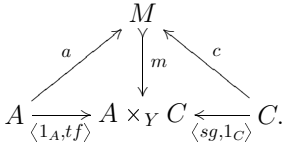
Proof: The equivalence between (ii) and (iii) is immediate.
(i) ⇒ (ii). Consider a double split epimorphism over Y



We want to prove that the comparison morphism $\langle g', f' \rangle \colon D \to A \times_Y C$ is a regular epimorphism. Suppose that $\langle g', f' \rangle = me$ is its factorisation as a regular epimorphism followed by a monomorphism. We obtain the commutative diagram



By assumption $(\langle 1_A, tf \rangle, \langle sg, 1_C \rangle)$ is jointly strongly epimorphic, which proves that m is an isomorphism and, consequently, $\langle g', f' \rangle$ is a regular epimorphism.
(ii) ⇒ (i). Consider a pullback of split epimorphisms (**C**) and a monomorphism m such that $\langle 1_A, tf \rangle$ and $\langle sg, 1_C \rangle$ factor through m



5. Unital objects and subtractive objects

It is known that a pointed finitely complete category is strongly unital if and only if it is unital and subtractive [23, Proposition 3]. Having introduced the notion of a strongly unital object, we now explore analogous notions for the unital and subtractive cases. Our aim is to prove that the equivalence above also holds "locally" for objects in any pointed regular category.

Let $\mathbb{C}$ be pointed and finitely complete.

Definition 5.1. Given an object Y of $\mathbb{C}$, we say that Y is **unital** if the point

$$(\pi_1 \colon Y \times Y \to Y, \quad \langle 1_Y, 0 \rangle \colon Y \to Y \times Y)$$

is stably strong.

Note that we could equivalently ask that the point $(\pi_2, \langle 0, 1_Y \rangle)$ is stably strong. We write $\mathsf{U}(\mathbb{C})$ for the full subcategory of $\mathbb{C}$ determined by the unital objects.

The following results are proved similarly to the corresponding ones obtained for strongly unital objects. Recall from [3, 6] that a **split punctual span** is a diagram of the form

$$X \underset{f}{\overset{s}{\rightleftarrows}} Z \underset{g}{\overset{t}{\rightleftarrows}} Y \qquad (\mathbf{E})$$

where $fs = 1_X$, $gt = 1_Y$, $ft = 0$ and $gs = 0$.

Proposition 5.2. *If $\mathbb{C}$ is a pointed finitely complete category, then the following conditions are equivalent:*

(i) *Y is a unital object of $\mathbb{C}$;*
(ii) *for every object X, the point $(\pi_X \colon X \times Y \to X, \langle 1_X, 0 \rangle \colon X \to X \times Y)$ is stably strong;*
(iii) *for every object X, the point $(\pi_X, \langle 1_X, 0 \rangle)$ is strong;*
(iv) *given any split punctual span (**E**), the map $\langle f, g \rangle \colon Z \to X \times Y$ is a strong epimorphism.* ∎

Just as any strongly unital category is always unital, we also have:

Corollary 5.3. *In a pointed finitely complete category, a strongly unital object is always unital.*

Proof: By Propositions 4.5 and 5.2. ∎

Corollary 5.4. *If $\mathbb{C}$ is a pointed finitely complete category, then $\mathsf{U}(\mathbb{C}) = \mathbb{C}$ if and only if $\mathbb{C}$ is unital.* ∎

Examples 5.5. Mon and SRng are not strongly unital, but they are unital, being Jónsson–Tarski varieties (see Examples 2.7). So, $\mathsf{U}(\mathsf{Mon}) = \mathsf{Mon}$ and $\mathsf{U}(\mathsf{SRng}) = \mathsf{SRng}$.

Corollary 5.6. *If $\mathbb{C}$ is a pointed finitely complete category and $\mathsf{U}(\mathbb{C})$ is closed under finite limits in $\mathbb{C}$, then $\mathsf{U}(\mathbb{C})$ is a unital category.*

Proof: Apply Corollary 5.4 to $\mathsf{U}(\mathbb{C})$. ∎

Proposition 5.7. *If $\mathbb{C}$ is a pointed regular category, then $\mathsf{U}(\mathbb{C})$ is closed under quotients in $\mathbb{C}$.* ∎

5.8. Subtractive categories, subtractive objects. We recall the definition of a subtractive category from [23]. A relation $r = \langle r_1, r_2\rangle\colon R \to X \times Y$ in a pointed category is said to be **left (right) punctual** [6] if $\langle 1_X, 0\rangle\colon X \to X \times Y$ (respectively $\langle 0, 1_Y\rangle\colon Y \to X \times Y$) factors through r. A pointed finitely complete category $\mathbb{C}$ is said to be **subtractive**, if every left punctual reflexive relation on an object X in $\mathbb{C}$ is right punctual. It is equivalent to asking that right punctuality implies left punctuality—which is the implication we shall use to obtain a definition of subtractivity for objects.

Examples 5.9. A variety of universal algebras is said to be **subtractive** [32] when the corresponding theory contains a unique constant 0 and a binary operation s, called a **subtraction**, subject to the equations $s(x, 0) = x$ and $s(x, x) = 0$. Of course this happens if and only if the condition of 5.8 is satisfied (see [23]). We write Sub for the subtractive variety of **subtraction algebras**, which are triples $(X, s, 0)$ where X is a set, s a subtraction on X and 0 the corresponding constant.

It is shown in [24] that a pointed regular category $\mathbb{C}$ is subtractive if and only if every span $\langle s_1, s_2\rangle\colon A \to B \times C$ is **subtractive**: written in set-theoretical terms, its induced relation $r = \langle r_1, r_2\rangle\colon R \to B \times C$, where $\langle s_1, s_2\rangle = rp$ for r a monomorphism and p a regular epimorphism, satisfies the condition

$$(b, c),\ (b, 0) \in R \quad \Rightarrow \quad (0, c) \in R.$$

Proposition 5.10. *In a pointed regular category, consider a split right punctual span* (**D**). *The span $\langle g, f\rangle$ is subtractive if and only if $f\ker(g)$ is a regular epimorphism.*

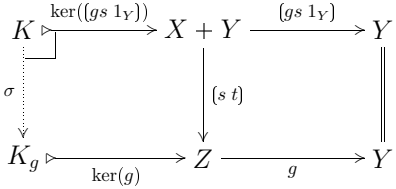
Proof: Thanks to the Barr embedding theorem [2], in a regular context it suffices to give a set-theoretical proof (see Metatheorem A.5.7 in [3], for instance).

is such that $\delta_f = \begin{bmatrix}1_X & 0\end{bmatrix}\ker(\begin{bmatrix}f & 1_Y\end{bmatrix})$ is a regular epimorphism.

Proof: The implication (i) ⇒ (ii) is obvious. Conversely, given any split right punctual span (**D**), we have a morphism $gs\colon X \to Y$, so for the split right punctual span

$$X \underset{\begin{bmatrix}1_X & 0\end{bmatrix}}{\overset{\iota_X}{\rightleftarrows}} X + Y \underset{\begin{bmatrix}gs & 1_Y\end{bmatrix}}{\overset{\iota_Y}{\rightleftarrows}} Y$$

we have that $\delta_{gs} = \begin{bmatrix}1_X & 0\end{bmatrix}\ker(\begin{bmatrix}gs & 1_Y\end{bmatrix})$ is a regular epimorphism. The induced morphism σ between kernels in the diagram



is such that $f\ker(g)\sigma = f\begin{bmatrix}s & t\end{bmatrix}\ker(\begin{bmatrix}gs & 1_Y\end{bmatrix}) = \delta_{gs}$ is a regular epimorphism; consequently, $f\ker(g)$ is a regular epimorphism as well. ∎

6. Mal'tsev objects

Even though the concept of a strongly unital object is strong enough to characterise rings amongst semirings as in Theorem 4.3, it fails to give us a characterisation of groups amongst monoids. For that purpose we need a stronger concept. The aim of the present section is two-fold: first to introduce Mal'tsev objects, then to prove that any Mal'tsev object in the category of monoids is a group (Theorem 6.13). In fact, also the opposite inclusion holds: groups are precisely the Mal'tsev monoids. This follows from the results in the next section, where the even stronger concept of a protomodular object is introduced.

Definition 6.1. We say that an object Y of a finitely complete category $\mathbb{C}$ is a **Mal'tsev object** if the category $\mathsf{Pt}_Y(\mathbb{C})$ is unital.

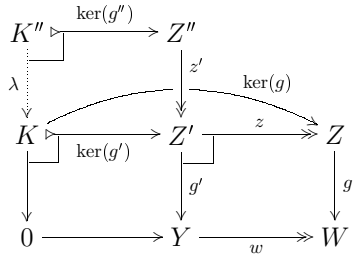
As explained after Proposition 3.1, this means that for every pullback of split epimorphisms over Y as in (**C**), the morphisms $\langle 1_A, tf\rangle$ and $\langle sg, 1_C\rangle$ are jointly strongly epimorphic.

We write $\mathsf{M}(\mathbb{C})$ for the full subcategory of $\mathbb{C}$ determined by the Mal'tsev objects.

$t'' \colon Y \to Z''$ such that $\langle f'', g'' \rangle t'' = \langle 0, 1_Y \rangle$ and $zz't'' = tw$. So, we get a split right punctual span

$$X' \underset{f''}{\overset{s''}{\rightleftarrows}} Z'' \underset{g''}{\overset{t''}{\leftrightarrows}} Y,$$

so that $f'' \ker(g'')$ is a regular epimorphism, by assumption. Since g' is a pullback of g and $g'' = g'z'$, we have the commutative diagram



between their kernels. Finally, the morphism $xf'' \ker(g'')$ is a regular epimorphism (since both x and $f'' \ker(g'')$ are) and from

$$xf'' \ker(g'') = fzz' \ker(g'') = fz \ker(g')\lambda = f \ker(g)\lambda$$

we conclude that $f \ker(g)$ is a regular epimorphism, as desired. ∎

In the presence of binary coproducts, a pointed regular category $\mathbb{C}$ is subtractive if and only if any split right punctual span of the form

$$X \underset{(1_X\ 0)}{\overset{\iota_1}{\rightleftarrows}} X + X \underset{(1_X\ 1_X)}{\overset{\iota_2}{\leftrightarrows}} X$$

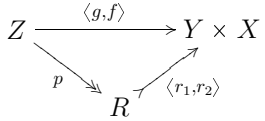
is such that $\delta_X = (1_X\ 0) \ker((1_X\ 1_X))$ is a regular epimorphism (see Theorem 5.1 in [12]). This result leads us to the following characterisation, where an extra morphism f appears as in Proposition 4.5, to be compatible with the pullback-stability in the definitions of unital and strongly unital objects.

Proposition 5.18. *In a pointed regular category $\mathbb{C}$ with binary coproducts the following conditions are equivalent:*

(i) *an object Y in $\mathbb{C}$ is subtractive;*
(ii) *for any morphism $f \colon X \to Y$, the split right punctual span*

$$X \underset{(1_X\ 0)}{\overset{\iota_X}{\rightleftarrows}} X + Y \underset{(f\ 1_Y)}{\overset{\iota_Y}{\leftrightarrows}} Y$$

Consider the factorisation



of $\langle g, f \rangle$ as a regular epimorphism p followed by a monomorphism $\langle r_1, r_2 \rangle$. Then $(y, x) \in R$ if and only if $y = g(z)$ and $x = f(z)$, for some $z \in Z$.

Suppose that $\langle g, f \rangle$ is subtractive. Given any $x \in X$, we have $(gs(x), x) \in R$ for $z = s(x)$ and $(gs(x), 0) \in R$ for $z = tgs(x)$. Then $(0, x) \in R$ by assumption, which means that $0 = g(z)$ and $x = f(z)$, for some $z \in Z$. Thus $f \ker(g)$ is a regular epimorphism.

The converse implication easily follows since $(0, x) \in R$, for any $x \in X$, because $f \ker(g)$ is a regular epimorphism. ∎

This result leads us to the following "local" definition:

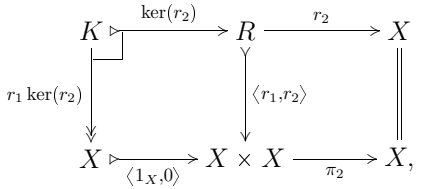
Definition 5.11. Given an object Y of a pointed regular category $\mathbb{C}$, we say that Y is **subtractive** when for every split right punctual span (**D**), the morphism $f \ker(g)$ is a regular epimorphism.

We write $\mathsf{S}(\mathbb{C})$ for the full subcategory of $\mathbb{C}$ determined by the subtractive objects.

Proposition 5.12. *If $\mathbb{C}$ is a pointed regular category, then $\mathbb{C}$ is subtractive if and only if all of its objects are subtractive.*

Proof: As recalled above, if $\mathbb{C}$ is subtractive, then every span is subtractive. Then every object is subtractive by Proposition 5.10.

Conversely, consider a right punctual reflexive relation $\langle r_1, r_2 \rangle \colon R \to X \times X$. By assumption, $r_1 \ker(r_2)$ is a regular epimorphism. In the commutative diagram between kernels



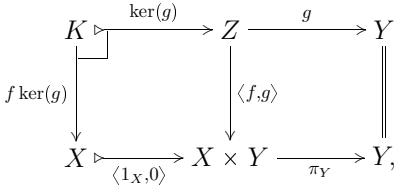
the left square is necessarily a pullback. So, the regular epimorphism $r_1 \ker(r_2)$ is also a monomorphism, thus an isomorphism. The morphism $\ker(r_2)$ gives the factorisation of $\langle 1_X, 0\rangle$ needed to prove that R is a left punctual relation. ∎

Corollary 5.13. *If $\mathbb{C}$ is a pointed regular category and $\mathsf{S}(\mathbb{C})$ is closed under finite limits in $\mathbb{C}$, then $\mathsf{S}(\mathbb{C})$ is a subtractive category.*

Proof: Apply the above proposition to $\mathsf{S}(\mathbb{C})$. ∎

Proposition 5.14 ($\mathsf{S}(\mathbb{C}) \cap \mathsf{U}(\mathbb{C}) = \mathsf{SU}(\mathbb{C})$)**.** *Let $\mathbb{C}$ be a pointed regular category. An object Y of $\mathbb{C}$ is strongly unital if and only if it is unital and subtractive.*

Proof: We already observed that a strongly unital object is unital (Corollary 5.3). To prove that Y is subtractive, we consider an arbitrary split right punctual span (**D**). In the commutative diagram between kernels



the left square is necessarily a pullback. By Proposition 4.5, $\langle f, g\rangle$ is a regular epimorphism, hence so is $f \ker(g)$.

Conversely, given a subtractive unital object Y and a split right punctual span (**D**), by Proposition 4.5 we must show that the middle morphism $\langle f, g\rangle$ of the diagram above is a regular epimorphism. Let mp be its factorisation as a regular epimorphism p followed by a monomorphism m. The pair $(\langle 1_X, 0\rangle, \langle 0, 1_Y\rangle)$ being jointly strongly epimorphic and $f \ker(g)$ being a regular epimorphism, we see that the pair $(\langle 1_X, 0\rangle f \ker(g), \langle 0, 1_Y\rangle)$ is jointly strongly epimorphic; moreover it factors through the monomorphism m. Consequently, m is an isomorphism. ∎

Corollary 5.15. $\mathsf{S}(\mathsf{Mon}) = \mathsf{GMon}$, $\mathsf{S}(\mathsf{CMon}) = \mathsf{Ab}$ *and* $\mathsf{S}(\mathsf{SRng}) = \mathsf{Rng}$.

Proof: This is a combination of Examples 2.7 with, respectively, Example 4.9; [3, Example 1.9.4] with Proposition 4.5 and the remark following Proposition 4.8; and Theorem 4.3. ∎

Example 5.16. Groups are (strongly) unital objects in the category Sub of subtraction algebras. In fact, if for every $y \in Y$ there is a $y^* \in Y$ such that

$s(0, y^*) = y$, then Y is a unital object; in particular, any group is unital. To see this, we must prove that for any subtraction algebra X, the pair

$$(\langle 1_X, 0\rangle \colon X \to X \times Y, \quad \langle 0, 1_Y\rangle \colon Y \to X \times Y)$$

is jointly strongly epimorphic. This follows from the fact that

$$s((x, 0), (0, y^*)) = (s(x, 0), s(0, y^*)) = (x, y)$$

for all $x \in X$ and $y \in Y$. Note that the inclusion $\mathsf{Gp} \subset \mathsf{SU}(\mathsf{Sub})$ is strict, because the three-element subtraction algebra

s	0	1	2
0	0	1	2
1	1	0	0
2	2	0	0

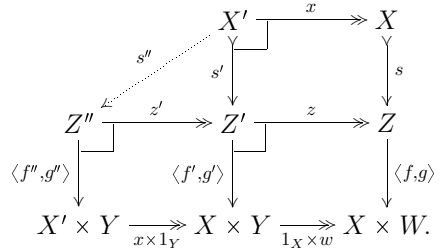
satisfies the condition on the existence of y^*. However, it is not a group, since the unique group of order three has a different induced subtraction.

Proposition 5.17. *Let $\mathbb{C}$ be a pointed regular category. Then $\mathsf{S}(\mathbb{C})$ is closed under quotients in $\mathbb{C}$.*

Proof: Suppose that Y is a subtractive object in $\mathbb{C}$ and consider a regular epimorphism $w \colon Y \to W$. To prove that W is also subtractive, consider a split right punctual span

$$X \underset{f}{\overset{s}{\rightleftarrows}} Z \underset{g}{\overset{t}{\rightleftarrows}} W;$$

we must prove that $f \ker(g)$ is a regular epimorphism. Consider the following diagram where all squares are pullbacks:



Note that from the bottom right pullback we can deduce that the pullback of g along w is g'. Since $f's' = x$, there is an induced morphism $s'' \colon X' \to Z''$ such that $\langle f'', g''\rangle s'' = \langle 1_{X'}, g's'\rangle$ and $z's'' = s'$. There is also an induced morphism